Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

No.88

ON THE GEOMETRY BEHIND THE FAIRNESS CONCEPTS A LA RAWLS AND A LA KOLM FOR LOCATION CONFLICTS

by

AXEL OSTMANN and MARTIN STRAUB

July 1979

H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse / Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

SUMMARY

In this paper we consider the geometry behind the fairness à la RAWLS and à la KOLM for location conflicts. This geometry is taken to be a middleline geometry generated by a pseudodistance. In part I we state the differences to ordinary geometry. Part II contains a standard example and a pathological one. The third part on degeneracy gives the conditions upon the middleline geometry to be the euclidian one. It turns out that in this case the pseudodistance is generated by a scalarproduct.
These results make it possible to generalize the construction of fairness solutions and the classification of location conflicts, both given in OSTMANN 2.

RÉSUMÉE

Dans ce papier nous découvrons la géomètrie cachée derrière les solutions de "fairness" à la RAWLS et à la KOLM pour des conflits de localisation. Cette géomètrie est modelée comme une géomètrie des lignes intermédiaires centrales produits par une pseudodistance. Dans le premier part nous expliquons les différences à la géomètrie ordinaire. Des examples, l'un standardisant, l'autre pathologique, se trouvent dans le part prochain . Le troisième part contient des conditions pour garantir l'euclidicité de la géomètrie. La condition plus suggestive est peut-être que la pseudodistance soit produite par un produit scalaire.
Ces résultats rendent possible la généralisation du construction pour les solutions "fairness" et du classification des conflits de localisation qui se trouvent dans OSTMANN 2.

VORWORT

Diese Arbeit ist ein Kolletivprodukt im Projekt "Standortspiele".

Der Kritik Wolfram Richters an OSTMANN 2 (an der geringen Relevanz des dort betrachteten Spezialfalles euklidischer Nutzenfunktionen) ist es zu danken, daß, nach der nur kurzlebigen Vermutung, die pseudodistanzerzeugten Mittelliniengeometrien seien eh' alle zur euklidischen isomorph, also entzerrbar, vorliegende Arbeit in Angriff genommen wurde.

Besonders hilfreich waren uns die Diskussionen im Anfangsstadium der Arbeit mit Joachim Rosenmüller und Götz Huttel. Für den weiteren Klärungsprozeß und die Frage der Darstellung der Ergebnisse waren uns die Geduld und die Fragen und Einwürfe der Teilnehmer des Kolloquiums zur Spieltheorie am IMW eine wichtige Hilfe. Besonderen Dank sei an Reinhard Selten und Wulf Albers gerichtet. Gudrun Dräger sei herzlichen Dank für die zur Drucklegung nötige Arbeit.

Wir hoffen, daß die Genannten mit dem vorliegenden Papier zufrieden sein können, und natürlich auch die Leser.

Die Autoren beglückwünschen sich zu ihrer ausgewogenen Zusammenarbeit.

TABLE OF CONTENT

PART I

ELEMENTS

§ 0 AN EARLIER ATTEMPT AND INTERPRETATIONS

0.1 THE EARLIER ATTEMPT

In OSTMANN 1 and 2 the following optimizations are considered:

$$\begin{cases} (\text{R-M}) & \max\limits_{x \in C_A} \min\limits_{b \in B} d_b(x) \\ (\text{K-M}) & \text{L-}\max\limits_{x \in C_A} \pi(d_b(x)_{b \in B}) \end{cases}$$

with $\pi : y \to y' :: i \leq j \Rightarrow y_i' \leq y_j'$ and $\{y_i\} = \{y_j\}$;
A being a finite subset of an $\mathbb{R}^m$, $\emptyset \neq B \subset A$, C_A the convex hull of A, d_b the euclidian distance from $b \in B$, L-max the lexicographic maximum operator.

The dual problems, easier to handle, were considered too:

$$\begin{cases} (\text{R-m}) & \min\limits_{x \in C_A} \max\limits_{b \in B} d_b(x) \\ (\text{K-m}) & \text{L-}\min\limits_{x \in C_A} \pi^*(d_b(x)_{b \in B}) \end{cases}$$

with $\pi^* : y \to y' :: i \leq j \Rightarrow y_j' \leq y_i'$ and $\{y_i\} = \{y_j\}$.

For our design we will remember two results:

1. construction by decomposition (see 0.3) and
2. classification of (A,B) by (local) continuity and equality of the solution concepts "R" and "K" (see 0.4).

But first let us give some remarks on the interpretations.

0.2 INTERPRETATIONS ECONOMISTS ARE INTERESTED IN

For all such interpretations more general functions than the euclidian distance are suggested. The d_b's are seen as utility or disutility functions on the space of alternatives $\mathbb{R}^m$. b represents an individual's best or worst result.

A somewhat sophisticated interpretation is that one discussed in ZECKHAUSER/WEINSTEIN and ROSENMÜLLER .

For $m=2$ we have the most simple interpretations as
(-M) repulsive euclidian location conflicts and
(-m) attractive euclidian location conflicts,
with fair solutions à la RAWLS ((R-), "make the worst-off best-off") and à la KOLM ((K-), "justice pratique" or "lexicographic Rawls"). We represent individuals or groups by their standpoint. In the repulsive case we see the individual interests as to get the object of planning as far as possible. We assume that there is no other conflict than that on the location. Examples are given by necessary facilities that burden the environment, planned by an ideal society of common planning and decision on the base of an individual calculus on utilities.

If the reader does not agree with this abstract model there are three ways of going through:

1. We think that there are many real situations where people behave like that (maybe because they have no other arguments or ideas or informations, maybe because they do not like to invest more time and energy into the conflict);
2. the reader searches for other interpretations;
3. the reader is interested in the mathematical structure we will design (we hope he will get a new look on some geometry we usually use).

Let us go back to the location conflict.

In the attractive case, examples of objects of planning are parks (cf. ZECKHAUSER), hospitals, say simply "attractive supply services".

Let us now consider a similar but more technocratic interpretation, a society where Planner Allmighty is calculating for reasons of peace. He likes to calculate ...

...the best supplied point		...the worst supplied point	
(R-m)	(K-m)	(R-M)	(K-M)
	there are different measures of the group efficiency of supply there is no summing-up		

...the point of smallest infection		...the point of strongest infection	
(R-M)	(by all sources) (K-M)	(R-m)	(by all sources) (K-m)
	R takes into account only the next (M) or the farest (m)-source K: all sources simultaneously		

(If we have a summing-up of supply or infection potentials we have to deal with the FERMAT-WEBER-problem, but if there are equal weights, we can use the following concept of decomposition to get regions where the b-component ist the most important.)

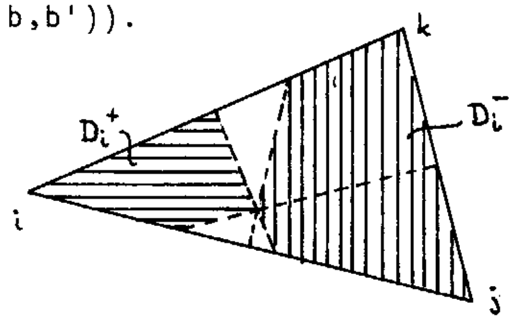
0.3 DECOMPOSITION OR "SPHERES OF COMMAND"

We can decompose the optimization into $|B|$ optimizations on b's sphere of cammand, $b \in B$, defined as:

$$D_b^+ := \{x \in C_A \; ; \; d_b(x) = \min_{b' \in B} d_{b'}(x)\} \qquad \text{resp.}$$

$$D_b^- := \{x \in C_A \; ; \; d_b(x) = \max_{b' \in B} d_{b'}(x)\} \quad .$$

These D_b^+ are polytopes constructable by the middle-hyperplanes $B(b,b')$. Let us denote the generated halfplane that contains b (resp. b') by $H_b(B(b,b'))$ resp. $H_{b'}(B(b,b'))$.



$A = B = \{i,j,k\}$

Then $$D_b^+ = \bigcap_{\substack{b' \in B \\ b' \neq b}} H_b(B(b,b')) \quad \text{and}$$

$$D_b^- = \bigcap_{\substack{b' \in B \\ b' \neq b}} H_{b'}(B(b,b')) \; .$$

Optimization is finite because we only have to take into account the vertices of the D_b-regions. In this paper we like to give the geometrical structure to enable the same procedure for d_b being a pseudodistance.

0.4 CLASSIFICATION OF LOCATION STRUCTURES

In OSTMANN 2, § 2, the space Π of location structures (A,B) is defined.

It was proved that:

For attracting conflicts (-m)-solution correspondence is continuous and single-valued, and for repulsive conflicts (-M)-solution correspondence for (R-M) is upper hemicontinuous; moreover the situation can be symbolized by the following diagram (§ 4):

Π = [upper hemicontinuous R]

[upper hemicontinuous K]
= [R = K]

[single-valued K]

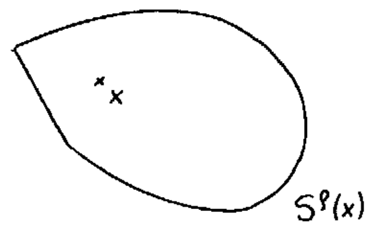
[single-valued R]
= [lower hemicontinuous K]
= [lower hemicontinuous R]

where [ℓ] is the subspace of Π where property ℓ holds for its elements (locally). Moreover all these subspaces are dense in Π .

The main auxiliary for these theorems was the decomposition. So the geometry we will construct is also the tool to generalize these theorems.

§ 1 THE GENERALIZED MODEL TO DEAL WITH AND SOME TRIVIAL PROPERTIES

We are accustomed to look at pictures like this:



when there is reported on noxious emissions. The sphere $S^\rho(x)$ means equal density of infection and it is suggested that stretched spheres also represent equal density. The unsymmetric form may be caused by integration of winds by their frequency.

We give a family of utility/disutility functions that formalizes that suggestion. We turn from "backward measurement" (measures from the emission source) like above to "forward measurement" (measures from the individual affected) in the definition below.

DEFINITION 1.1: $u_i : \mathbb{R}^2 \to \mathbb{R}$ is called pseudodistance of an individual $i \in \mathbb{R}^2$ iff it has the form $u_i(x) = f\varphi(x-i)$ with $f: \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ monotone, φ a positively homogenuous continuous functional with strictly convex unit sphere $S'(0) = \{x; \varphi(x) = 0\}$. $u : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} : u(i,x) = u_i(x)$ is called pseudonorm.

REMARK 1.2: In general measuring forward $(u_i(x))$ is not equal to measuring backwards $(u_x(i))$. There is equality iff φ is a (strict) norm.

<u>REMARK AND DEFINITION 1.3</u>: If u is pseudonorm then u^* with $u_i^*(x) = f\varphi^*(x-i)$, $\varphi^*(y) = \varphi(-y)$ is a pseudonorm, too. u equals u^* iff φ is norm. * turns forward measuring into backward measuring (and backward into forward). The *-concepts are called the dual ones.

So, as euclidian norm induces (elementary) euclidian geometry, we now define the induced geometrical objects for u.

<u>DEFINITION 1.4</u>: $B(i,j) := \{x \in \mathbb{R}^2;\ u_i(x) = u_j(x)\}$ is called <u>middleline</u> for i and j. $\mathcal{B} := \{B(i,j);\ i,j \in \mathbb{R}^2,\ i \neq j\}$ is the family of middlelines or shortly "the lines".

$\mathcal{B}_x = \{B(i,j) \in \mathcal{B};\ x \in B(i,j)\}$ is called pencil in x,

$\mathcal{B}^x = \{B(x,j);\ j \in \mathbb{R}^2\}$ is called anti-pencil for x,

$S^\rho(i) = \{x \in \mathbb{R}^2;\ u_i(x) = \rho\}$ is the ρ-sphere around i,

$S_*^\rho(x) = \{i \in \mathbb{R}^2;\ u_i(x) = \rho\}$ is the dual ρ-sphere.

$E^\rho(i) = \{x \in \mathbb{R}^2;\ u_i(x) \leq \rho\}$ and $E_*^\rho(x)$ the corresponding closed balls.

<u>REMARK 1.5</u>: Let us look at the euclidian objects. $\mathcal{G}$ be the family of euclidian lines. For u the euclidian distance we have $\mathcal{G} = \mathcal{B} = \mathcal{B}_x \cup \mathcal{B}^x$. Topologically $\mathcal{B}$ is a Moebius strip.

The following remarks give some trivial properties of the objects just defined.

<u>REMARK 1.6</u>: Two spheres intersect at most two times. $S^\rho(i) = \rho S'(0) + i$. The triangle unequality holds: $\varphi(x - y) + \varphi(y - z) \leq \varphi(x - z)$; there is equality iff there is (euclidian) collinearity.

<u>REMARK 1.7</u>: For all f the $B(i,j)$ generated by $u_\nu = f\varphi(\cdot-\nu)$, $\nu = i,j$ are identical. For the following we set w.l.o.g. $f = \mathrm{id}$. $B(\lambda i,\lambda j) = \lambda B(i,j)$, $B(i + a, j + a) = a + B(i,j)$: that is, generation of lines commutes with translations and (centered) stretchings.

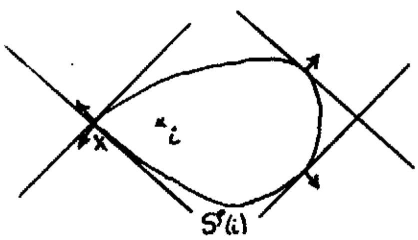
In general the generators of a middleline are not uniquely determined (look at the euclidian case; cf. part III).

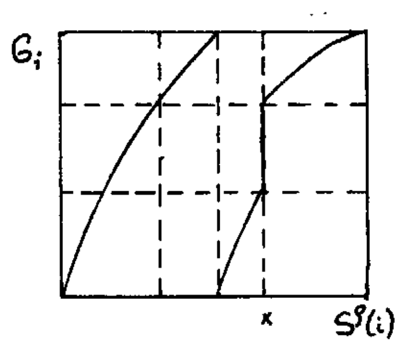
§ 2 ON TANGENTS

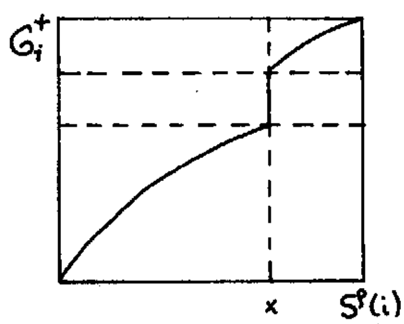
<u>DEFINITION 2.1</u>: Let G_i be the euclidian pencil in i and G_i^+ the oriented euclidian pencil in i (pencil of rays). For any $S^\rho(i)$, $\rho \in \mathbb{R}^+$ the correspondence $T_i : S^\rho(i) \to\!\!\cdot\; G_i$ is called <u>tangent correspondence</u> iff T_i assigns a point $x \in S^\rho(i)$ all euclidian lines out of G_i that intersect $S^\rho(i)$ after the translation $i \to x$ in exactly one point, that is in x. $T_i^+ : S^\rho(i) \to\!\!\cdot\; G_i^+$ is defined in the corresponding way (we can identify rays with halfplanes).

<u>REMARK 2.2</u>: G_i and G_i' is homeomorphic to the circle.

The following sketch shows T_i and T_i^+. The horizontal coordinate shows $S^\rho(i)$: we have to identify right and left limit of the interval. The vertical coordinate shows G_i resp. G_i' : identify upper and lower limit.







LEMMA 2.3: (without proof): T_i and T_i^+ are strictly locally monotone (define this in the obvious way), since $S^\rho(i)$ is strictly convex. Let us consider the inverse correspondences $T_i^{\leftarrow}$ and $T_i^{+\leftarrow}$: since $S^\rho(i)$ is strictly convex, $T_i^{+\leftarrow}$ is single-valued (or function) and $T_i^{\leftarrow}$ is two-valued.

DEFINITION 2.4: Two spheres $S^\rho(i)$ and $S^\sigma(j)$ are said to be tangent to each other iff $S^\rho(i) \cap S^\sigma(j) \neq \emptyset$ and $T_i(x) \times T_j(x)$ contains a pair of parallels whenever $x \in S^\rho(i) \cap S^\sigma(j)$.

REMARK 2.5: For these x the pair of parallels carries two opposite orientations. The parallel through x is a dividing hyperplane.

DEFINITION 2.6: $C(i,j)$ be the set of maximal elements of $\mathbb{R}^2$ with respect to $(-u_i, -u_j)$ and the natural halforder on $\mathbb{R}^2$. For attracting conflicts the interpretation of $C(i,j)$ is that it is the set of PARETO-points.

REMARK 2.7: $C(i,j)$ is homeomorphic to a closed real interval. Its boundary is $\{i,j\}$. If φ is norm, then $C(i,j)$ is the euclidian interval. (For a proof cf. MC.KELVEY/WENDELL and substitute T for their ∇.) When we parametrize $C(i,j)$ by i-distances, we get the other distance as a strictly antitone function.

§ 3 ON TOPOLOGY

Looking at $(\mathbb{R}^2, \mathcal{B})$ as a topological object, there are mainly three simple ways to introduce the topology on $\mathcal{B}$. For this paragraph cf. SALZMANN, p. 4.

DEFINITION 3.1: $\mathcal{B}_1$ be the topological space over $\mathcal{B}$ induced by generation:

$\mathbb{R}^2 \times \mathbb{R}^2 \setminus \text{diag} \to \mathcal{B} : (i,j) \to B(i,j)$.

A set A is open in $\mathcal{B}_1$ iff A can be generated by an open set in $\mathbb{R}^2 \times \mathbb{R}^2 \setminus \text{diag}$.

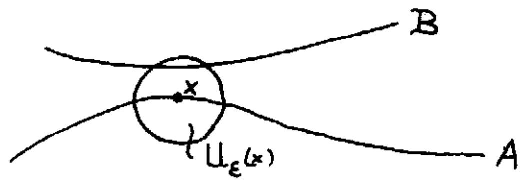
Neighbourhoods of A can be written as

$V_\varepsilon(A) = \{B \in \mathcal{B}$; there exists $(i',j') \in U_\varepsilon(i) \times U_\varepsilon(j)$ with $B = B(i',j')\}$ with $U_\varepsilon(\cdot)$ the ordinary, euclidian ε-neighbourhoods. You might say: there are "representations" in the neighbourhood...

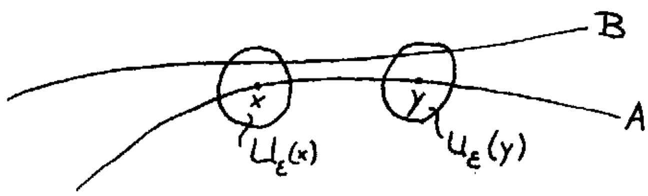
REMARK 3.2: In the definition above we used the natural topology on $\mathbb{R}^4$. It is the same as that one generated by φ or φ^*. All pseudonorms generate the same topology. With the above definition we can write $\mathcal{B}_1$ as $(\mathbb{R}^2 \times \mathbb{R}^2 \setminus \text{diag})_{/\sim}$ with $(i,j) \sim (i',j')$ iff $B(i,j) = B(i',j')$. In the euclidian case the equivalence classes are generated by translations in a fixed direction (that one of the line) and stretchings with center on the line. So dimension reduces to 2.

For the non-euclidian case see part III, § 1.

<u>DEFINITION 3.3</u>: $\mathcal{B}_2$ be the uniform space over $\mathcal{B}$ of pointwise convergence. The topology can be constructed by the $V_\varepsilon^x(A) := \{B \in \mathcal{B} ; B \cap U_\varepsilon(x) \neq \emptyset\}$, $x \in A$ and $U_\varepsilon(x)$ the ordinary ε-neighbourhoods of x in $\mathbb{R}^2$.



<u>DEFINITION 3.4</u>: $\mathcal{B}_3$ be the topological space over $\mathcal{B}$ induced "geometrically". This topology can be constructed by $V_\varepsilon^{x,y}(A) := \{B \in \mathcal{B} ; B \cap U_\varepsilon(x) \neq \emptyset \neq B \cap U_\varepsilon(y)\}$, $x,y \in A$.



<u>REMARK 3.5</u>: If there are $A,B \in \mathcal{B}$, $A \neq B$, $x,y \in A \cap B$, we can seperate A and B in $\mathcal{B}_2$ resp. $\mathcal{B}_3$ by a neighbourhood of type $V_\varepsilon^z(A)$ resp. $V_\varepsilon^{z_1,z_2}(A)$ with $z_i \notin A \cap B$.

<u>LEMMA 3.6</u>: $\mathcal{B}_2$ and $\mathcal{B}_3$ are identical.

<u>Proof</u>. Since $V_\varepsilon^{x,y}(A) = V_\varepsilon^x(A) \cap V_\varepsilon^y(A)$, the topology of $\mathcal{B}_3$ is coarser. Since $V_\varepsilon^{x,y}(A) \subseteq V_\varepsilon^x(A)$ the topology of $\mathcal{B}_2$ is coarser.

LEMMA 3.7: $\mathcal{B}_1$ and $\mathcal{B}_3$ are identical.

Proof. Let $A = B(i,j)$. $V_\varepsilon(A) \subseteq V_\varepsilon^{x,y}(A)$ is easy to get. We choose ρ such that $S^\rho(i) \cap S^\rho(j) = : \{x,y\}$. Since φ is continuous, there is an appropriate ε' for any ε . To get $B \in V_\varepsilon^{x,y}(A) \subseteq V_\varepsilon(A)$ we have to construct some i',j' with $B = B(i',j')$. This is done by the backwards distance. Let $x',y' \in B$ with $x' \in U_\varepsilon(x)$, $y' \in U_\varepsilon(y)$, $\{i',j'\} : = S_*^{\varphi*(x-i)}(x) \cap S_*^{\varphi*(y-i)}(y)$. The work is done by using the following property of the correspondence $\rho \to S_*^\rho(x) \cap S_*^\rho(y)$: in a neighbourhood of a two-valued ρ two-valuedness holds.

Summing up, we have:

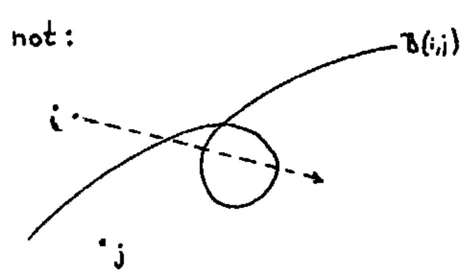
THEOREM 3.8: All the defined topologies on $\mathcal{B}$ are the same one.

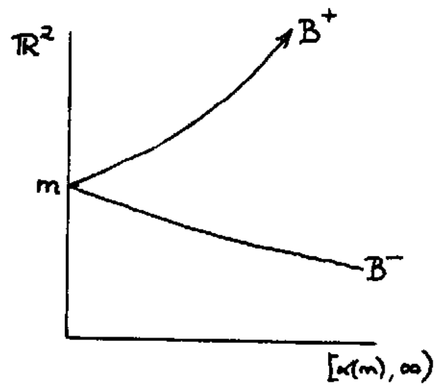
§ 4 MIDDLELINES

The basic idea is given in the following theorem:

<u>THEOREM 4.1</u>: There is one and only one point $m = m(i,j)$ on $B = B(i,j)$ that has minimal pseudodistance from i and j. B is closed and homeomorphic to the real line. m devides B into halflines B^+ and B^-. On the halflines the common pseudodistance $\alpha(z) := \varphi(z-i) = \varphi(z-j)$ is monotone. B devides the plane into two components $H_i(B)$ and $H_j(B)$. These components are defined by the inequalities $\varphi(z-i) \gtrless \varphi(z-j)$. Both of them are unbounded.

<u>Sketch of the proof</u>: At first we can get existence and uniqueness of m by looking at $B(i,j) \cap C(i,j)$ (cf. 2.3). m is the only element of this intersection. For closedness of B, we remember the continuity of $\Psi(z) := \varphi(z-i) - \varphi(z-j)$. Next we parametrize B by means of the continuous correspondence $\phi: [\alpha(m),\infty) \to\!\bullet\ B : \rho \to S^\rho(i) \cap S^\rho(j)$. We note that the intersection is two-elemented for $\rho \in (\alpha(m),\infty)$, or in other words: the curve B has no loops.

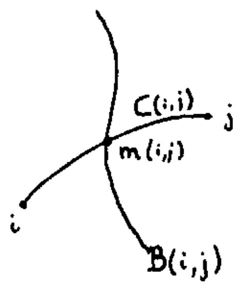




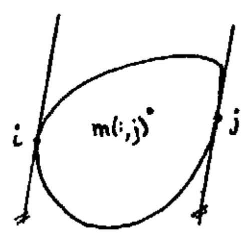
We have m in the interior of $E^{\rho}(i) \cap E^{\rho}(j)$ because of the stretchings. By strict convexity of the spheres we know that there are two and only two elements of $S^{\rho}(i) \cap S^{\rho}(j)$.

Because there are no loops we have halflines B^{+} and B^{-} with monotone pseudodistance. Further we have at most two components of $\mathbb{R}^2 \setminus B$. If there would be only one there would be a path from i to j so that the inequality $\varphi(.-i) > \varphi(.-j)$ (w.l.o.g.) would hold for all its points. That is a contradiction. If $H_i(B)$ would be bounded, there would be a loop.

REMARK 4.2: We state two ways of constructing $m(i,j)$:
1) $m(i,j)$ is the unique point of intersection of $B(i,j)$ and $C(i,j)$.



2) $m(i,j)$ is the center of the unique "antisphere" (compare § 7) $\{x ; \varphi(m-x) = \alpha\}$ having parallel tangents in i and j .

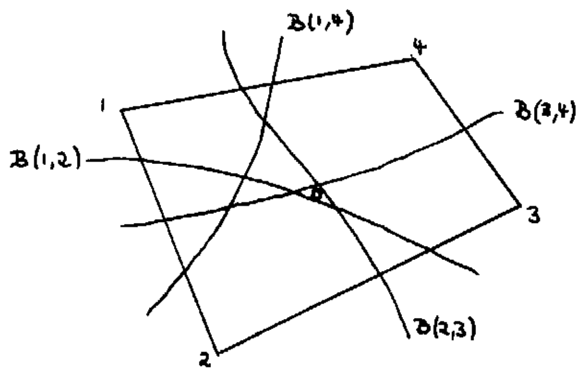


REMARK 4.3: If φ is norm, then $m(i,j)$ is the euclidian middle $\frac{i+j}{2}$ and $B(i,j)$ is symmetrical w.r.t. the pointreflection in $m(i,j)$. The proof follows by the symmetry of $\varphi : \varphi(x) = \varphi(-x) = \varphi^*(x)$, and the translation invariance.

REMARK 4.4: The generated halfplanes are convex iff middlelines are euclidian (i. e. $\mathcal{B} \subseteq G$) . Any strict convexity would cause strict concavity on the other side. Cf. part III § 2.

§ 5 CONFIGURATIONS OR DECOMPOSITIONS

REMARK 5.1: By construction the halfplanes $H_i(B(i,j))$ hold the inequality $\varphi(\cdot - i) < \varphi(\cdot - j)$. This way some configurations become impossible. We give an example:

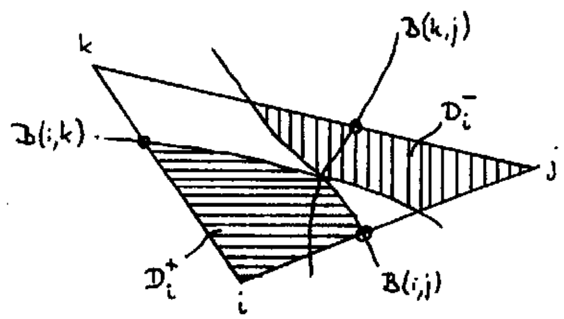


In the region signed by o , we get

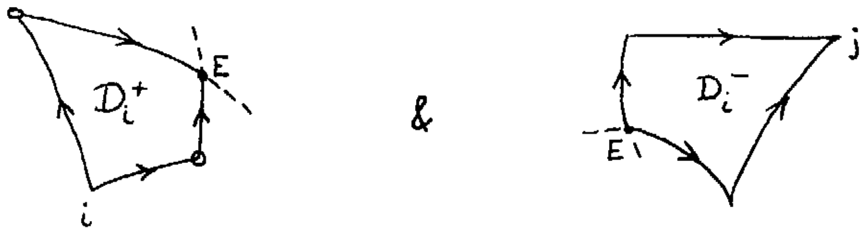
$\varphi(z-1) < \varphi(z-2) < \varphi(z-3) < \varphi(z-4) < \varphi(z-1)$.

That is a contradiction.

REMARK 5.2: The halfplanes are in general not convex, and so are the D_b's (take the definitions in 0.3). To generalize the principle of construction there is no need for convexity since we have inequalities and half-monotony. For the latter regard the following sketch of a triangle:



If we sketch D_i^+ and D_i^- with halfmonotony on the middlelines and monotony on the euclidian segments, we get:



It is clear that KOLM- and RAWLS-solutions have to be vertices, in our example: E.

REMARK 5.3: In contrary to the real affine plane two different lines can intersect infinitely often. We give an example in part II § 1.6. But this property does not disturb the construction, because we have strict half-monotony and a bounded region C_A we optimize in. In the sketch of 5.2 the B's are assumed to intersect in one point. This question is handled in § 7 where we deal with antipencils

$B^i := \{B(i,j)\ ;\ j \in \mathbb{R}^2,\ i \neq j\}$.

§ 6 ASYMPTOTES

For the following it will be convenient to use another parametrization of B :

LEMMA 6.1: Any ray in i (or j) intersects B(i,j) in at most one point. (We omit the elementary proof: compare Th. 4.1.)

COROLLAR 6.2: B(i,j) can be parametrized by the angles $\varphi : B \to S^1$, $\varphi(x) := \measuredangle(\overline{im(i,j)}, \overline{ix})$; $\psi : B \to S^1$, $\psi(x) := \measuredangle(\overline{j\, m(i,j)}, \overline{jx})$, φ and ψ are continuous unimodular functions of $x \in B$.

Since B has no loops, φ and ψ are not surjective, and hence, since $B \simeq \mathbb{R}$, $\varphi^{-1}(B)$ and $\psi^{-1}(B)$ are open intervals.

DEFINITION 6.3: The union of the rays corresponding to the two endpoints of $\overline{\varphi^{-1}(B)}$ resp. $\overline{\psi^{-1}(B)}$ are called the asymptotes of B in i resp. j , $As_i(B) = As_i(B(i,j))$ resp. $As_j(B) = As_j(B(i,j))$.

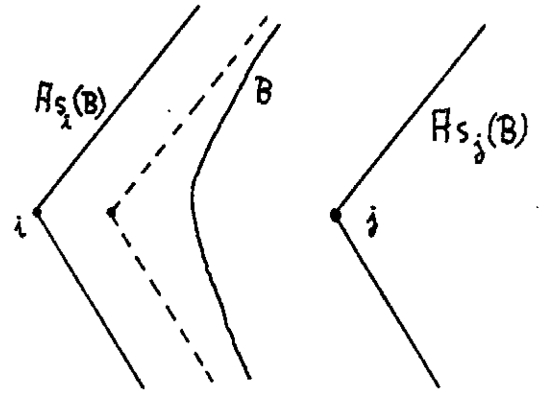
By definition the asyptotes are "cracked" lines:

$As_i(B), As_j(B) \in K := \{\vec{xy} \cup \vec{xz} \; ; \; x \neq y, x \neq z\}$.

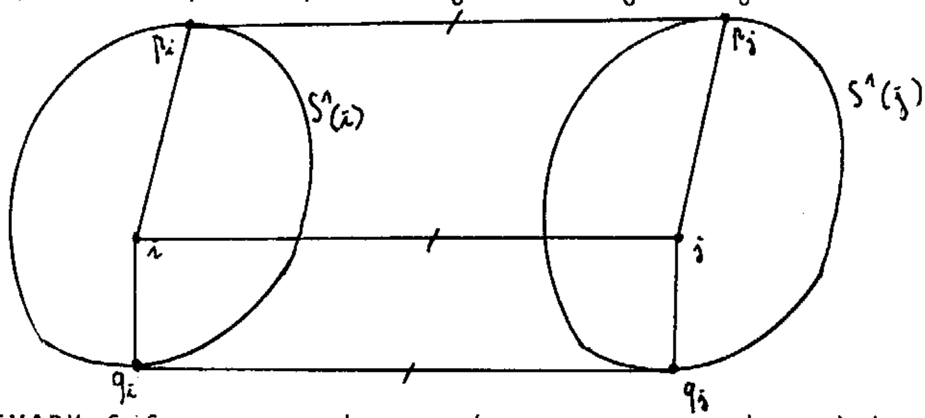
THEOREM 6.4: $As_i(B) \parallel As_j(B)$, B lies in the strip bounded by $As_i(B), As_j(B)$.

(" $\parallel$ " being defined for cracked lines in the obvious manner.)

(Consider $\lim\varphi(x) = \lim(2\pi - \Psi(x)) = 2\pi - \lim\Psi(x) \quad (\alpha(x) \to \infty)$.)



THEOREM 6.5 (Construction of $As_i(B)$ and $As_j(B)$)
There exist exactly four points $p_i, q_i \in S^1(i)$, $p_j, q_j \in S^1(j)$ with $T_i^+(p_i) \parallel T_j^+(p_j) \parallel \overline{ij}$, $T_i^+(q_i) \parallel T_j^+(q_j) \parallel \overline{ij}$ and
$As_i(B) = \overrightarrow{ip_i} \cup \overrightarrow{iq_i}$, $As_j(B) = \overrightarrow{jp_j} \cup \overrightarrow{jq_j}$.



REMARK 6.6: p_i and p_j (resp. q_i and q_j) have maximal distance from $\overline{ij}$ measured "parallel the corresponding ray" of the asymptote.

Proof of Theorem 6.5: It suffices to show: $p_i \in \partial\varphi^{-1}(B)$.

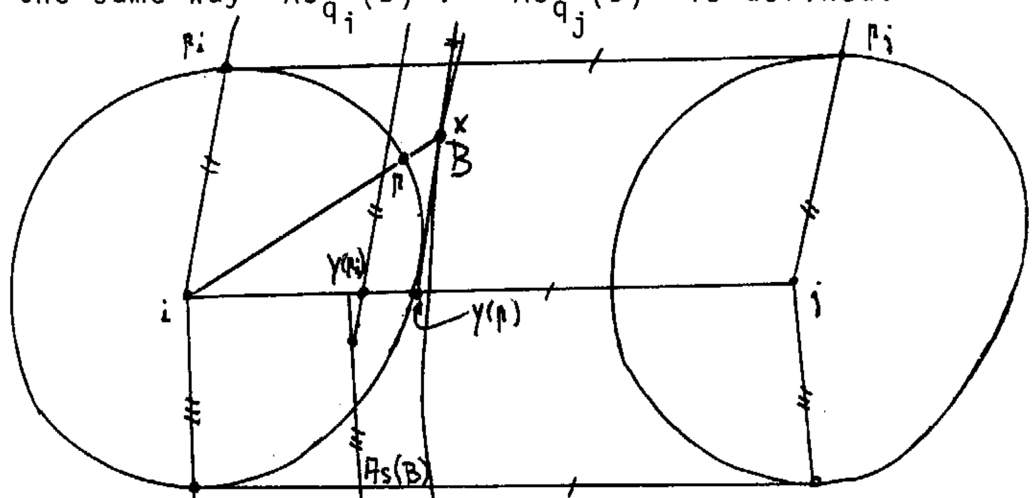
1) $p_i \notin \varphi^{-1}(B)$:
   If $p_i \in \varphi^{-1}(B)$, then there exists $y \in \overrightarrow{ip_i} \cap B$, and by similarity follows $d_i(y) < d_j(y)$ which contradicts $y \in B$.

2) $p_i \in \partial\varphi^{-1}(B)$:
   In any neighbourhood of p_i there exists $p \in S^1(i)$ and $q \in S^1(j)$ with the same "parallel" distance from $\overline{ij}$, such that $\overrightarrow{ip} \cap \overrightarrow{jq} \neq \emptyset$. By similarity $\overrightarrow{ip} \cap \overrightarrow{jq} \in B$ and thus $p \in \varphi^{-1}(B)$, i. e. $p_i \in \partial\varphi^{-1}(B)$.

<u>DEFINITION 6.7</u>: $k \in K$ is called the axis of B : $k = A(B)$, if $k \parallel As_i(B)$ $(\parallel As_j(B))$ and k cracked in $m(i,j)$ (i. e. $k = \overrightarrow{m(i,j)y} \cup \overrightarrow{m(i,j)z}$).

By Th. 6.4 we know that B lies in the strip between $As_i(B)$, $As_j(B)$. In the sequel we ask whether B would even converge towards a cracked line, when the distance $d_i(x) = d_j(x) = \alpha$ grows to infinity.

<u>DEFINITION 6.8</u>: We consider the situation of the proof of Th. 6.5, part 2):
$\overrightarrow{ip}$ and $\overrightarrow{jq}$ intersect in a point $x \in B$, $x = x(p)$. Take the ray $A(p)$ through $x \parallel \overrightarrow{ip_i}$ and start in the unique point $y = y(p) \in \overline{ij} \cap A(p)$. If $y(p)$ converges towards a point called $y(p_i)$ for $p \to p_i$ (from one side), then $As_{p_i}(B) := As_{p_j}(B)$ is defined as the ray starting in $y(p_i) \parallel \overrightarrow{ip_i}$.
In the same way $As_{q_i}(B) := As_{q_j}(B)$ is defined.



Because of the symmetry of the definition of $As_{p_i}(B)$, $As_{q_i}(B)$, we restrict our further considerations on $As_{p_i}(B)$.

<u>THEOREM 6.9</u>: If $As_{p_i}(B)$ exists, then

$d(x(\alpha), As_{p_i}(B)) \to 0 \quad (\alpha \to \infty)$

(where $x(\alpha) \in B_{p_i} : d_i(x(\alpha)) = \alpha$; B_{p_i} being the appropriate half of B, and $d(.,.)$ the euclidian distance) .

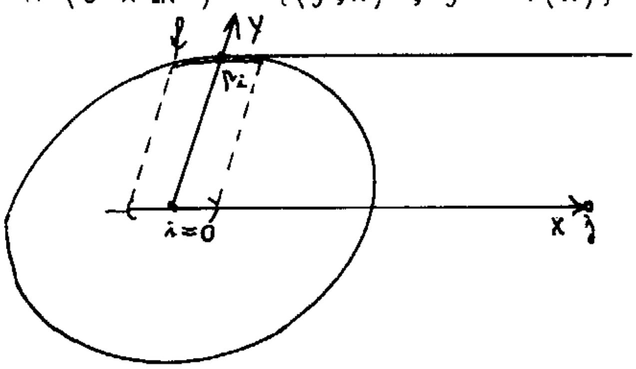
<u>Proof</u>. $\alpha = \alpha(p) = d_i(x(p))$ being a continuous parametrization of B_{p_i}, $\alpha \to \infty$ includes $p \to p_i$, and by supposition $y(p) \to y(p_i)$, thus

$d(x(\alpha), As_{p_i}(B)) = d(A\ (p), As_{p_i}(B)) = d(y(p), y(p_i)) \to 0$.

The following theorem gives an easy sufficient condition for the existence of $As_{p_i}(B)$ and explains its construction.
For the formulation of the theorem we introduce a coordinate system by: $0 := i$, x-axis $:= \overrightarrow{ij}$, y-axis $:= \overrightarrow{ip_i}$.

Be the "upper part" of $S^1(i)$ represented in a neighbourhood U of $i = 0$ by a function of x:

$S^1(i) \cap (U \times \mathbb{R}^+) = \{(y,x)\ ;\ y = f(x)\}$.



f has one-sided derivatives f'_+, f'_- (as a concave function) .

<u>THEOREM 6.10</u>:

1) $f'_+(0) \neq f'_-(0)$ (i. e. $f'_+(0) \neq 0$ or $f'_-(0) \neq 0$) implies the existence of $As_{p_i}(B)$ given by $y \in As_{p_i}(B) \cap \overleftrightarrow{ij}$ with $(\overleftrightarrow{ij}, \overleftrightarrow{yj}) = \lambda(-f'_-(0), f'_+(0)) \quad (\lambda \in \mathbb{R}^+)$.

2) Be f n-times continuously differentiable (in U, $n \in \mathbb{N}$) with $f'(0) = f''(0) = \dots = f^{(n-1)}(0) = 0$, $f^{(n)}(0) \neq 0$, then $As_{p_i}(B)$ exists and is given by $y \in As_{p_i}(B) \cap \overline{ij}$, with y being the euclidian middlepoint of $\overline{ij}$.

Proof.

1) By similarity

$$|\overline{iy(p)}| : |\overline{y(p)j}| = |\overline{rp}| : |\overline{qs}|$$
$$= -\text{ Ascent }(\overline{qp_j}) : \text{Ascent }(\overline{pp_i}),$$

and by $p \to p_i$ 1) follows.

2) As in 1) with $\Theta_p, \Psi_p \in (0,1)$ by the Mean-Value-Theorem

$$|\overline{iy(p)}| : |\overline{y(p)j}| = -\text{ Ascent }(\overline{qp_j}) : \text{Ascent }(\overline{pp_i})$$
$$= \frac{f'(x_q + \Psi_p(x_{pj} - x_q))}{f'(\Theta_p x_p)}$$
$$\to 1 \qquad (p \to p_i),$$

Applying l'Hôpital's rule in the last step.

REMARK 6.11: It is easy to see that not every B has an asymptote $As(B)$ (see part II,§2), but naturally the question arises "how often" this may occur.

Consider (for fixed φ) $S^1(0)$ and all $B \in \mathcal{B}_0^1 := \{B(0,j); j \in S^1(0)\}$, then all $B \in \mathcal{B}$ result from those by translations and dilatations. Because the asymptotes are invariant under these operations, we may restrict ourselves to asking for asymptotes of $B(0,j)$ ($j \in S^1(0)$). By the Lebesque-(L-)measure on $S^1(0)$ we get a natural measure on $\mathcal{B}_0^1$, making precise expressions like "how often" there are asymptotes in $\mathcal{B}$.

Because T_i^+ is a locally monotone and, up to denumerably many x, single-valued correspondence, f" exists (L-)almost everywhere and one might argue an asymptote to exist (L-)almost everywhere, too. But this is not true, moreover, an example in part II, §2 shows that for any $\varepsilon > 0$ there exists φ and a measurable set $J \subseteq S^1(0)$ with $L(S^1(0) \setminus J) < \varepsilon$ such that for any $j \in J$ $B(0,j)$ has no asymptote.

§ 7 PENCILS AND ANTIPENCILS

DEFINITION 7.1: $\mathcal{B}^x := \{B \in \mathcal{B} \; ; \; B = B(x,j), \; j \in \mathbb{R}^2 \setminus \{x\}\}$ is called antipencil in x $(x \in \mathbb{R}^2)$.

THEOREM 7.2 (Theorem of the Three Points)
Two different middlelines $B(x,i)$ and $B(x,j)$ from an antipencil $\mathcal{B}^x$ have at most one point in common which lies on $B(i,j)$, too.
If i,x,j are not collinear (in the sense of the Real Affine Geometry), then $B(x,i)$ and $B(x,j)$ have exactly one point in common.
In other words: Any triangle has at most one middlepoint, any non-degenerate one has exaclty one.

Proof. Be $z \in B(x,i) \cap B(x,j)$, then $d_i(z) = d_x(z) = d_j(z)$, thus $z \in B(i,j)$. Be $z_1, z_2 \in B(x,i) \cap B(x,j)$, then, using the dual distance $d^*(\varphi^*)$:

$$d_i(z_1) = d_x(z_1) = d_j(z_1) = d^*_{z_1}(i) = d^*_{z_1}(x) = d^*_{z_1}(j) =: \rho_1 ,$$

$$d_i(z_2) = d_x(z_2) = d_j(z_2) = d^*_{z_2}(i) = d^*_{z_2}(x) = d^*_{z_2}(j) =: \rho_2 ,$$

thus the antispheres $S_*^{\rho_1}(z_1)$ and $S_*^{\rho_2}(z_2)$ intersect in the three different points i,x,j , thus $z_1 = z_2$. If i,x,j are not collinear, $As_i(B(x,i)) \parallel As_x(B(x,i)) \nparallel As_x(B(x,j)) \parallel As_j(B(x,j))$, thus the strips formed by $As_i(B(x,i))$, $As_x(B(x,i))$ resp. $As_x(B(x,j))$, $As_j(B(x,j))$, in which $B(x,i)$ resp. $B(x,j)$ is lying, have a compact intersection, in which $B(x,i)$ must intersect $B(x,j)$.

<u>COROLLARY 7.3</u>: $B(x,i)$ and $B(x,j)$ are different for $i \neq j$.

<u>COROLLARY 7.4</u>: $\mathcal{B}^x$ is homeomorphic to the (open) torus, especially it is two-dimensional.

<u>DEFINITION 7.5</u>: $\mathcal{B}_i := \{B \in \mathcal{B} \; ; \; i \in B\}$ is called pencil in i.

We consider

$\beta : \mathcal{B}^i \to \mathcal{B}_i : B(i,j) \to i - m(i,j) + B(i,j)$,

that is a continuous bijective application (translation) translating $m(i,j)$ into i . Thus we have a one-to-one correspondence between elements of $\mathcal{B}^i$ and $\mathcal{B}_i$.

If $\mathcal{B} = \mathcal{G}$, then $\mathcal{B} = \mathcal{B}_o \cup \mathcal{B}^o$ and $\mathcal{B}_o \sim S^1$.

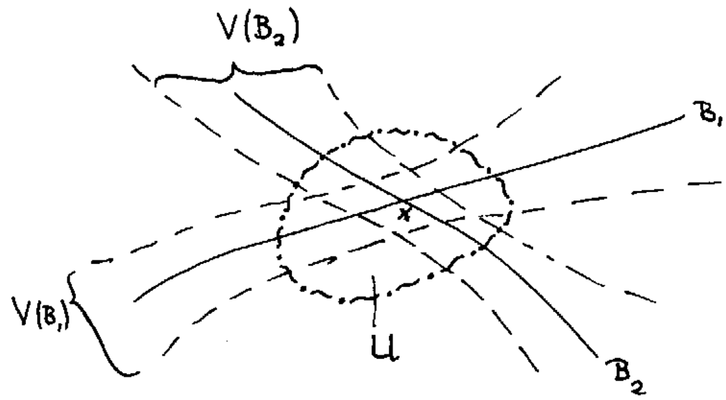
Since the pencil has no great meaning for the solution concepts, it will not be discussed further.

§ 8 THE GEOMETRICAL OPERATIONS

DEFINITION 8.1: The correspondence "joining" is defined by
$\cup : (\mathbb{R}^2 \times \mathbb{R}^2 \setminus \text{diag}) \to\!\!\cdot\; \mathcal{B} : (x,y) \to x \cup y := \{B \in \mathcal{B} \,;\, x,y \in B\}$.
The correspondence "intersecting" is defined by
$\cap : (\mathcal{B} \times \mathcal{B} \setminus \text{diag}) \to\!\!\cdot\; \mathbb{R} : (B_1,B_2) \to B_1 \cap B_2$, seen as set-theoretical intersection.
In the following we restrict $\cap$ to where $B_1 \cap B_2 \neq \emptyset$.

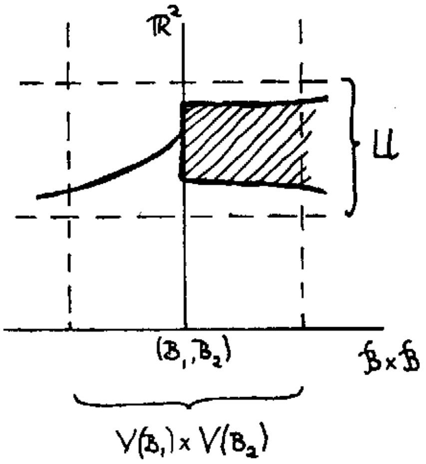
REMARK 8.2: In usual geometry $\cup$ and $\cap$ are single-valued, and so we have to modify the definitions of the continuity of the geometrical operations.

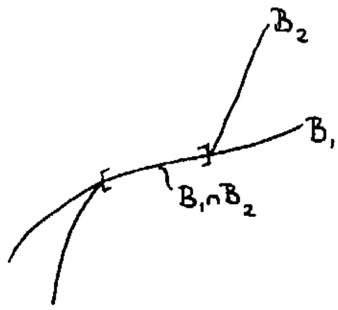
Usually by the continuity of intersection we mean the following: If $B_1 \cap B_2 = x$ and U is a neighbourhood of x , then there are disjoint neighbourhoods of $V(B_1)$ and $V(B_2)$ of the B_i's such that each line in $V(B_1)$ meets each line in $V(B_2)$ at a point of U (cf. SALZMANN, pp. 4, 11).



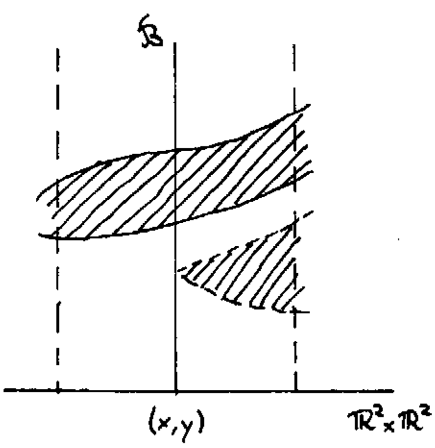
This definition gives rise to the following modification:
Let $B_1 \cap B_2 \neq \emptyset$ and U a neighbourhood of $B_1 \cap B_2$, ...
This definition is equivalent to upper hemicontinuity of the

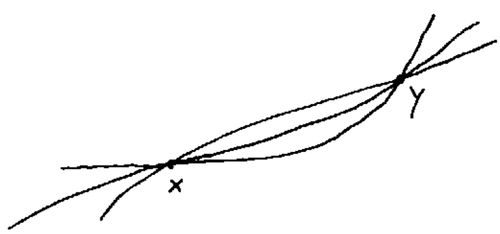
correspondence $\cap$ (e. g. cf. HILDENBRAND, p. 22). For a simple intuition we can say there are no new ones appearing by small moves.





The usual definition for continuity of joining is as follows: If U is an open set of lines, the set of pairs of points (x,y), such that $x \cup y$ exists and belongs to U , is open (cf. SALZMANN, pp.4,11). Modification is given by:... such that $x \cup y$ is not empty and there is some connecting line (i. e. element of $x \cup y$) that belongs to U . This definition is equivalent to lower hemicontinuity of the correspondence $\cup$. For a simple intuition we can say that there are no collapses of the image by small moves, or more exactly: For any element of $x \cup y$ and any sequences $x^{\nu} \to x$, $y^{\nu} \to y$ there is convergence in the image: $B^{\nu} \to B$ with $B^{\nu} \in x^{\nu} \cup y^{\nu}$.





Upper hemicontinuity for $\cup$ and lower hemicontinuity for $\cap$ are defined in the obvious way. An example where intersection is not lower hemicontinuous is given in part II.

<u>THEOREM 8.3</u>: Intersection is upper hemicontinuous.

<u>Proof</u>. Let U be an (w. l. o. g.) open neighbourhood of $B_1 \cap B_2$. So B_1 and B_2 differ on ∂U and we can give a separation by separating ε-neighbourhoods of $\mathcal{B}_2$: $V_\varepsilon^y(B_1)$ and $V_\varepsilon^y(B_2)$ with $y \in \partial U$. By the shape of the middlelines we know that pairs of lines of

$(V_\varepsilon^y(B_1) \cap V_\varepsilon^x(B_1)) \times (V_\varepsilon^y(B_2) \cap V_\varepsilon^x(B_2))$, $x \in B_1 \cap B_2$

must intersect in U .

In other words: topology $\mathcal{B}_2$ is the topology to guarantee upper hemicontinuity of $\cap$.

By the next lemma we give a construction for $x \cup y$.

<u>LEMMA 8.4</u>: If there is more than one intersecting point of $S^\alpha(x)$ and $S^\beta(y)$, say i and j , so $B(i,j)$ is connecting line for x and y . All elements of $x \cup y$ are obtained this way. $x \cup y$ is not empty and is closed (α and β arbitrary).

<u>Proof</u>. The first statements follow by simple calculation. Non-emptyness follows from strict convexity and positive homogenity. For closedness consider $B^\nu \to B$, $B^\nu \in x \cup y$: $\mathcal{B}_2$-topology guarantees that $x,y \in B$.

The positive reals α and β of Lemma 8.4 are used to parametrize $x \cup y$. As a first step we figure out:

LEMMA AND DEFINITION 8.5:

$J^1_{xy} := \{(\alpha,\beta) \in \mathbb{R}^+ \times \mathbb{R}^+ ; S^{\alpha}_{*}(x) \cap S^{\beta}_{*}(y) \text{ is a singleton}\}$

is the boundary of

$J^2_{xy} := \{(\alpha,\beta) \in \mathbb{R}^+ \times \mathbb{R}^+ ; S^{\alpha}_{*}(x) \cap S^{\beta}_{*}(y) \text{ is two-elemented}\}$.

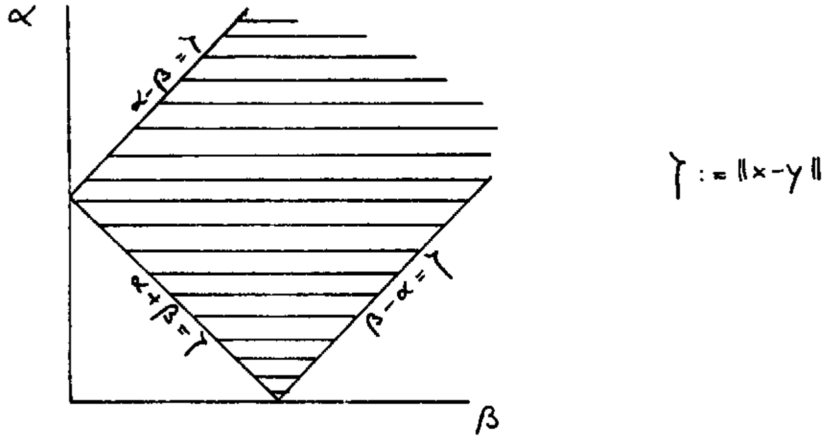
Description. Let J_{xy} be the curve of intersecting points generated by J^1_{xy} . For any sphere $S^{\alpha}_{*}(x)$ we take the tangent spheres $S^{\beta}_{*}(y)$. There are exactly two of them ($\alpha \neq 0$), the radii of whom we write $\sigma_1(\alpha)$ and $\sigma_2(\alpha)$. The one has empty intersection with the inner of $S^{\alpha}_{*}(x)$, the other one envelops $S^{\alpha}_{*}(x)$. $\sigma_1(\alpha) = \sigma_2(\alpha)$ holds iff $\alpha = 0$.
J_{xy} has no loops and divides into halfplanes. Different pairs from J^2_{xy} can generate the same line.

We resume:

LEMMA 8.6: $x \cup y$ is generated by the intersecting points of the spheres $S^{\alpha}_{*}(x)$ and $S^{\beta}_{*}(y)$ with $\alpha \in \mathbb{R}^+$ and $\beta \in (\sigma_1(\alpha),\sigma_2(\alpha))$. Especially we have: $x \cup y$ is connected.

COROLLARY 8.7: If φ is norm, then J_{xy} is the euclidian connecting line.

This case is illustrated by:

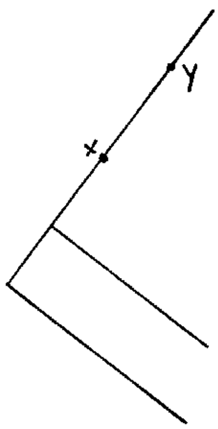


In general, $x \cup y$ is two-dimensional. If middlelines have a special shape, dimension becomes smaller.

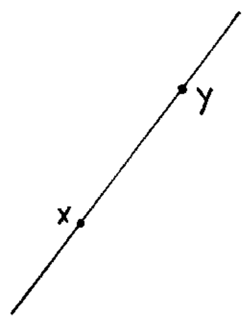
$x \cup y \subset K \setminus G$: one-dimensional

$x \cup y \subset G$: one-elemented

This becomes clear if we look at the transformations on these lines:



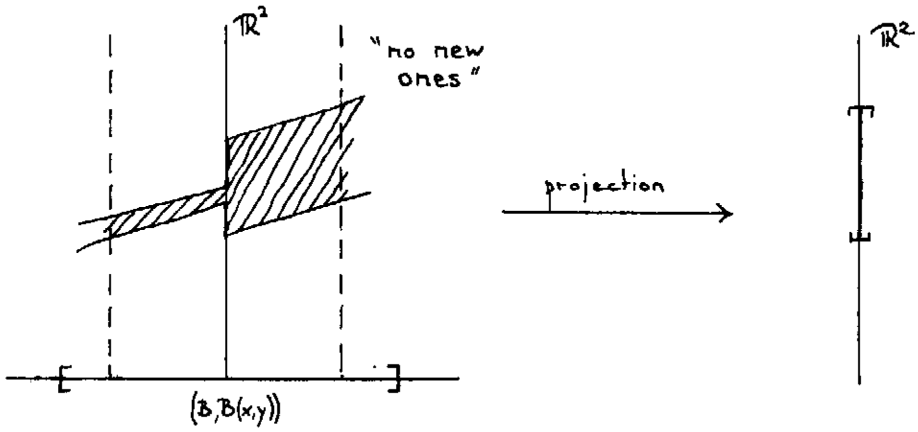
Lines of shape "K" are transformed into themselves by stretchings in m .



Lines of shape "G" are transformed into themselves by stretchings and translations into its own direction.

THEOREM 8.8: $x \cup y$ cuts an interval $I_{x,y}$ out of $B(x,y)$.

$I_{x,y}$ is short-hand for $\{B \cap B(x,y)\ ;\ B \in x \cup y\}$. Theorem 8.3 gives u.h.c. of $\cap$. Lemmata 8.4 and 8.6 give closedness and connectivity of $x \cup y$. So $I_{x,y}$ is closed and connected, too.



Because of the shape of $B(x,y)$ (Theorem 4.1) dimension reduces and $I_{x,y}$ is homeomorphic to the interval.

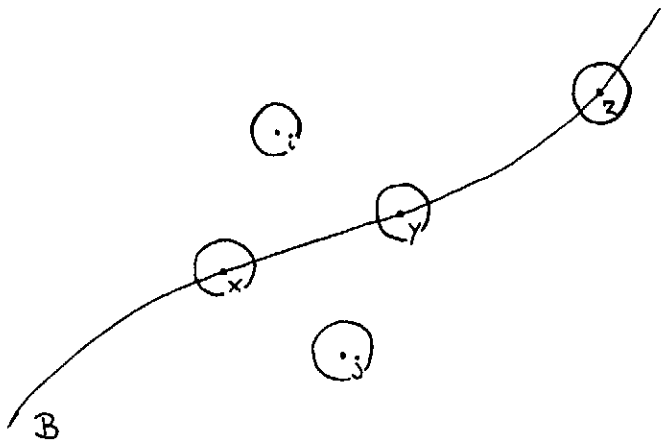
COROLLARY 8.9: If $x \cup y \subset K$, then $I_{x,y}$ is a singleton and equals $\{m(x,y)\}$.

THEOREM 8.10: Joining is continuous.

Proof. First: It is lower hemicontinuous:
Let A be an open set of lines. W. l. o. g. $A = V_\varepsilon^{x,y}(A)$ (we refer to topology $\mathcal{B}_3$). Then $\{(x',y')\ ;\ x' \cup y' \subset A\}$ equals the $U_\varepsilon(x) \times U_\varepsilon(y)$ defining $V_\varepsilon^{x,y}(A)$. So topology $\mathcal{B}_3$ is the topology "to make joining l.h.c.".

Next: It is upper hemicontinuous, too:
It is to prove that for any neighbourhood V of $x \cup y$, there are neighbourhoods U_x and U_y of x resp. y , so that for any pair $(x',y') \in U_x \times U_y$ the join $x' \cup y'$ intersects V .

Neighbourhoods of $x \cup y$ contain a neighbourhood of any fixed $B \in x \cup y$. W. l. o. g. we refer to the basis of topology $\mathcal{B}_2$: $V(B) = V_\varepsilon^z(B)$. Then by continuity of φ and the equivalence of $\mathcal{B}_2$ and $\mathcal{B}_3$ we know about the existence of appropriate neighbourhoods of x and y
$((x' \cup y') \cap V_\varepsilon^z(B) \neq \emptyset)$:



$B \in x' \cup y'$ can be generated by (i',j') with i' and j' in such neighbourhoods of i and j that
$\varphi(z' - i') = \varphi(z' - j') = \alpha$ with $z' \in U_\varepsilon(z)$ and
$\alpha = \varphi(z - i) = \varphi(z - j)$.

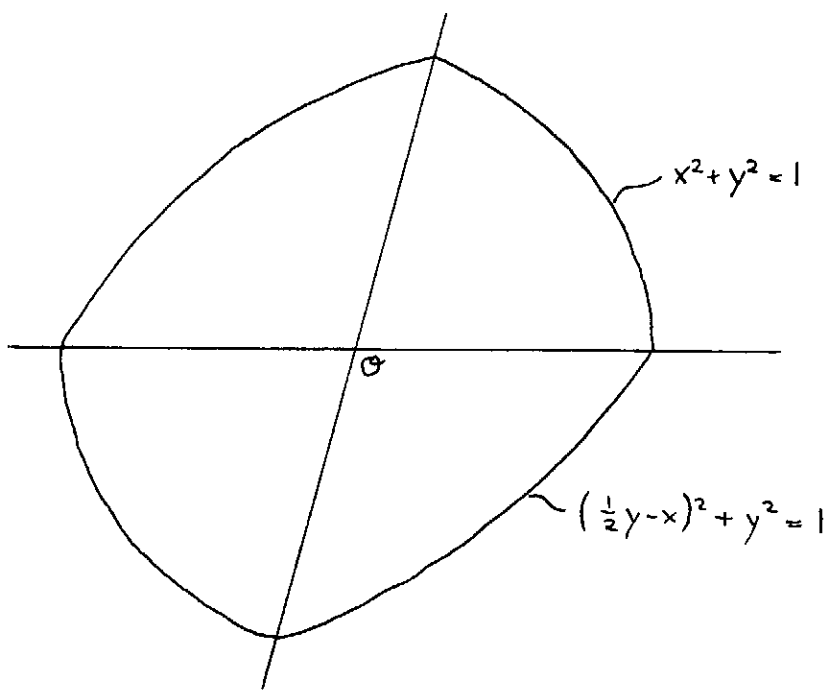
PART II

EXAMPLES

§ 1 A STANDARD EXAMPLE

1.1 THE UNIT SPACE

φ is norm and the unit sphere is given by the boundary line of the intersection of the ball of a circle $x^2 + y^2 = 1$ and of that of an ellipse $(\frac{1}{2} y - x)^2 + y^2 = 1$.
The ellipse can be generated by $(x,y) \to ((\frac{1}{2}y + x), y)$ and the circle.



1.2 MIDDLELINES

We classify $\mathcal{B}$ by that segments of the spheres that determine the line in a neighbourhood of m . This depends on the ascent for the generating pair (i,j). (We define the ascent in a homogenuous form.)

Type of $B = B(i,j)$	Ascent of (i,j)	Determining segments
H_1	$(1,4)$ i. e. where $S^1(0)$ is kinked	one elliptic one circular
Z_2	$\{(\alpha,4) ; \alpha \leq 1\}$	both elliptic
H_2	$(1,0)$ i. e. where $S^1(0)$ is kinked	one circular one elliptic
Z_1	$\{(1,\alpha) ; \alpha \in (0,4)\}$	both circular

1.3 TYPE H_r

We restrict our attention to $r = 1$. The other case can be handled in the corresponding way.

Let $H_1 \ni H = B(0, \frac{1}{\sqrt{17}}(1,4)) = H^+ \cup H^{+*}$ with

$$\begin{cases} H^+ = \{(x,y) ; (\frac{1}{2}y-x)^2+y^2 = (x-\frac{1}{\sqrt{17}})^2+(y-\frac{4}{\sqrt{17}})^2 \text{ and } y \geq \frac{2}{\sqrt{17}}\} \\ \text{and } * \text{ the point reflection at } m = \frac{1}{\sqrt{17}}(1,4) \end{cases}.$$

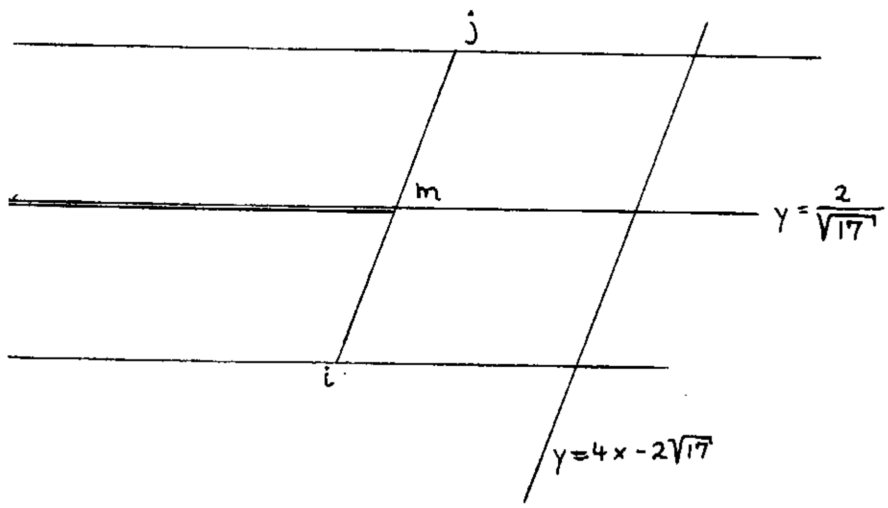
For the H^+-generating quadratic equation

$$\begin{cases} Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \\ A = 0,\ B = -\frac{1}{2},\ C = \frac{1}{4},\ D = \frac{1}{\sqrt{17}},\ E = \frac{4}{\sqrt{17}},\ F = -1 \end{cases}$$

we get the shape-determing determinants

$$\Delta = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = 0 \qquad \text{and} \qquad \delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = -\frac{1}{4} < 0 \quad .$$

So we know that the quadratic equation gives a pair of intersecting (euclidian) lines.
These can be calculated as $y = \frac{2}{\sqrt{17}}$ and $y = 4x - 2\sqrt{17}$.

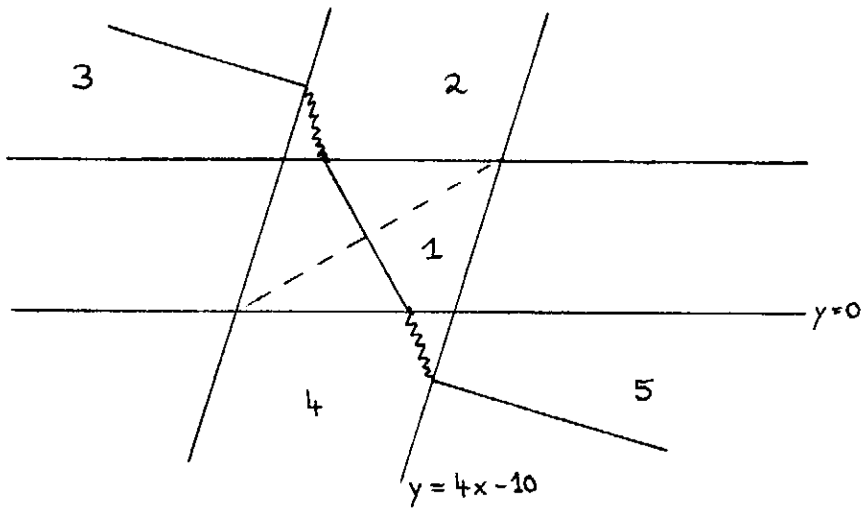


H^+ is marked by a doubleline.

It follows: $H = Ax(H) = \{(x,y) \; ; \; y = \frac{2}{\sqrt{17}}\}$.
$As_i(H)$ and $As_j(H)$ are the parallels through i resp. j .
$H = As_{p_i}(H)$ (cf. part I, § 6.3 f.).
$\mathcal{H}_1 = \{H^\tau ; \; \tau$ a translation$\}$ = all parallels of H .

1.4 TYPE Z_r

Let us restrict to Z_1 . Let $Z_1 \ni Z = B(o,(3,2))$. We give a scetch of the regions of equal determining segments.

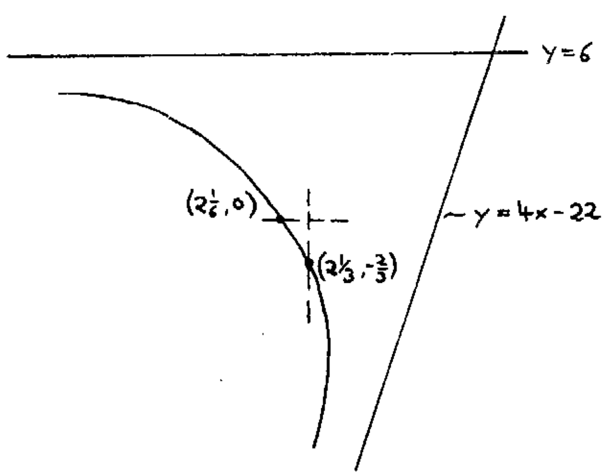


Because of the symmetry we restrict our computations to regions 1, 4 and 5. In region 1 and 5 we will get euclidian line segments, since the determining parts of the spheres are both circular resp. both elliptic. In region 1 we get the (euclidian) perpendicular bisector: $y = -\frac{3}{2}x + \frac{13}{4}$.

The segment in region 4 is calculated as fullfilling

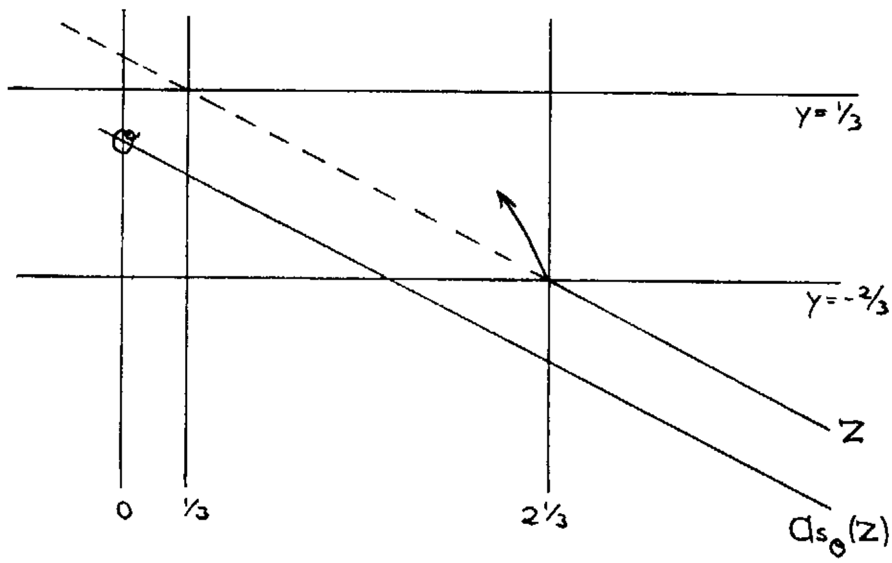
$\frac{1}{4}y^2 - xy + 6x + 4y - 13 = 0$.

By $\Delta \neq 0$, $\delta < 0$ we know that it is a hyperbolic segment. The hyperbola has asymptotes $y = 6$ and $y = 4x - 22$. The intersection with the boundary of region 4 is $\{(2\frac{1}{6}, 0), (\frac{7}{3}, -\frac{2}{3})\}$.



Let us go back now to region 5.

Euclidian lines of ascent (2,-3) are transformed by $(x,y) \to (x + \frac{1}{2} y\ ,\ y)$ into lines of ascent (2,-1), that is the ascent of the line segment in region 5.



1.5 THE JOIN

To classify types of the join $x \cup y$ for pairs (x,y) we have to examine the ascents of the pairs (x,y) ; $x,y \in B \in \mathcal{B}$. By asymptotic considerations we get:

$$\bigwedge_{x,y\in B} \text{Ascent } (x,y) \in \{(\alpha,4)\ ;\ \alpha < 1\} \quad \Leftrightarrow \quad B \in Z_1$$

$$\bigwedge_{x,y\in B} \text{Ascent } (x,y) \in \{(1,\alpha)\ ;\ \alpha \in (0,4)\} \quad \Leftrightarrow \quad B \in Z_2 \quad .$$

From 1.3 we know:

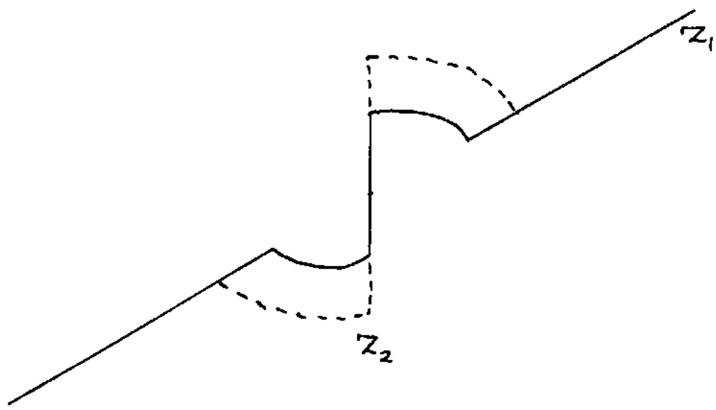
$$\bigwedge_{x,y\in B} \text{Ascent } (x,y) = 0 \quad \Leftrightarrow \quad B \in H_1$$

$$\bigwedge_{x,y\in B} \text{Ascent } (x,y) = 4 \quad \Leftrightarrow \quad B \in H_2$$

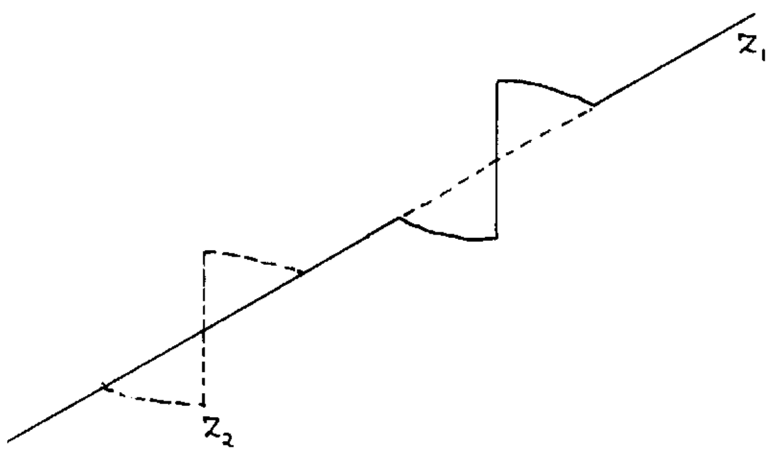
Thus $x \cup y$ contains only lines of one type.

1.6 THE INTERSECTION

In § 5 we remarked that there are intersections with a continuum of points. The figures give two examples.



In this example Z_2 is a stretched version of Z_1 .



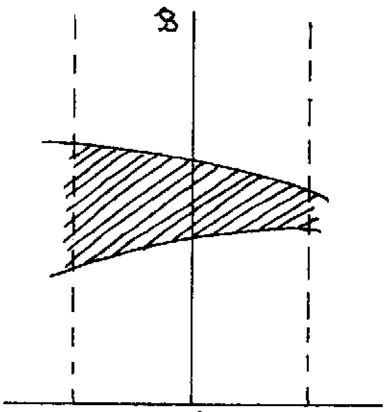
Z_2 is a translated version of Z_1 .

1.7 JOIN IS NOT COMPACT

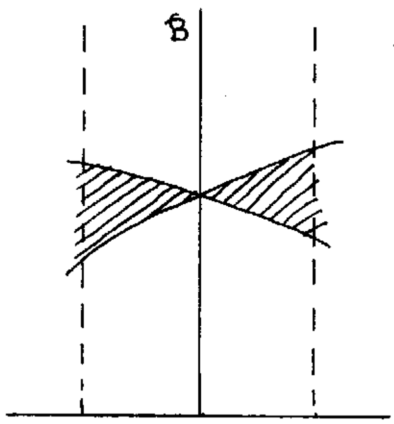
Consider Z_1 in the upper sketch and select a pair (x,y), $x,y \in Z_1$, such that they lay in the left (unbounded) line segment. We use translations τ_ν of length $\nu \in \mathbb{N}$ in direction of that line segment to the right. The sequence $(Z_1^{\tau_\nu})_{\nu \in \mathbb{N}}$ does not converge (the euclidian line in this direction is not in $\mathcal{B}$) . So the join $x \cup y$ is not compact.

1.8 ON CONTINUITY

Our version of continuity of joining as developed in 1.2 - 1.4 is intuitively represented by the following sketches:

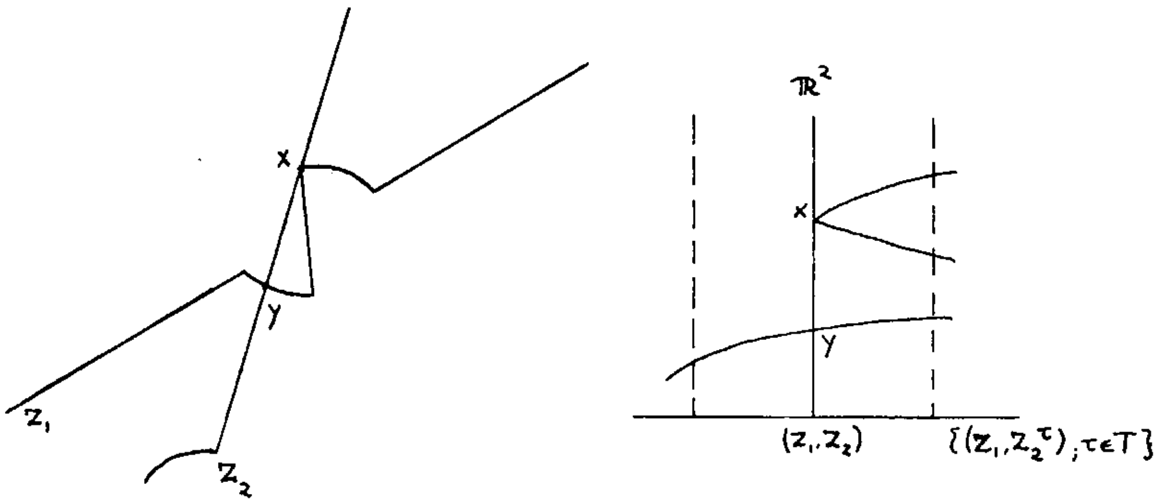


$(x,y) \in \mathcal{Z}_\nu$



$(x,y) \in \mathcal{H}_\nu$

To get an example against lower hemicontinuity for intersection, we have to intersect lines of type Z_ν.



T are translations in arrowdirection.

If we transfer Z_2 by an ε-translation the image may collapse.

§ 2 AN EXAMPLE FOR PATHOLOGICAL ASYMPTOTIC BEHAVIOUR

As announced in part I, § 6, we construct for arbitrary $\varepsilon > 0$ a φ such that the unit circle $S^1(0)$ contains a measurable set J_ε with $L(S^1(0)\setminus J_\varepsilon) < \varepsilon$ and $B(0,j)$ has no asymptote for $j \in J_\varepsilon$:

Be $\mathcal{J}^N := \{ \sum_{n=1}^{N} \frac{i_n}{2^n} ; i_n \in \{0,1\}; (n = 1,\dots,N)\}$,

define by induction

$$J^1_\varepsilon := \bigcup_{j_1 \in \mathcal{J}^1} (j_1 - \tfrac{\varepsilon}{4}, j_1 + \tfrac{\varepsilon}{4}) \cap [0,1) \quad (= [0,\tfrac{\varepsilon}{4}) \cup (\tfrac{1}{2}-\tfrac{\varepsilon}{4}, \tfrac{1}{2}+\tfrac{\varepsilon}{4}) \text{ for } \varepsilon < 1)$$

$$J^N_\varepsilon := \{ \bigcup_{j_N \in \mathcal{J}^N} (j_N - \tfrac{\varepsilon}{2^{2N}}, j_N + \tfrac{\varepsilon}{2^{2N}}) \cup \bigcup_{n=1}^{N-1} J^n_\varepsilon \} \cap [0,1) \quad ,$$

and set

$$J_\varepsilon := \bigcup_{N \in \mathbb{N}} J^N_\varepsilon \quad ,$$

then

$$J^1_\varepsilon \subseteq J^2_\varepsilon \subseteq \dots \subseteq J_\varepsilon \subseteq [0,1)$$

and

$$L(J^N_\varepsilon) < \sum_{n=1}^{N} \sum_{k=1}^{2^n} \frac{2\varepsilon}{2^{2k}} = \sum_{n=1}^{N} \frac{\varepsilon}{2^n} < \varepsilon \quad ,$$

thus $L(J_\varepsilon) < \varepsilon$.

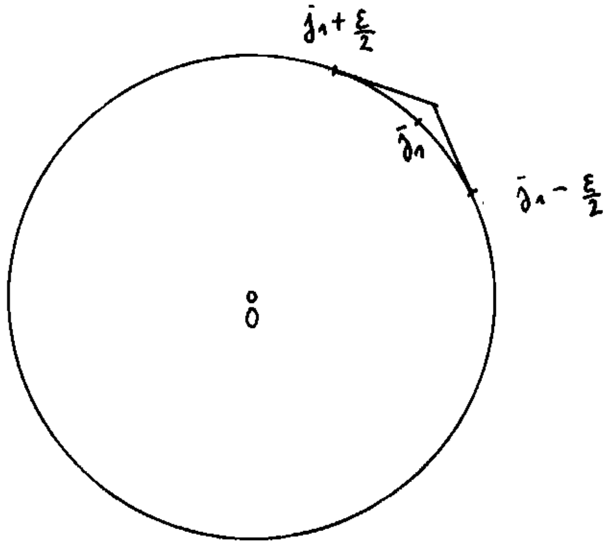
Furthermore, for $\varepsilon < \frac{1}{2}$, $x \in [0,1)\setminus J_\varepsilon$, we have:

$$* \begin{cases} \text{For any } M \in \mathbb{N} \text{ there exists } N \geq M \text{ and } j_N, k_N \in \mathcal{J}^N \text{ with} \\ |x - j_N| = \min_{i_N \in \mathcal{J}^N} |x - i_N| , \ |x - k_N| = \min_{i_N \in \mathcal{J}^N \setminus \{j_N\}} |x - i_N| \\ \text{and} \\ y \in (j_N - \frac{\varepsilon}{2^{2N+1}}, j_N + \frac{\varepsilon}{2^{2N+1}}) \Rightarrow z := |2x-y| (\text{mod } 1) \notin (k_N - \frac{\varepsilon}{2^{2N+1}}, k_N - \frac{\varepsilon}{2^{2N+1}}) . \end{cases}$$

(z is the reflection of y at x , therefore
$z \notin (k_N - \frac{\varepsilon}{2^{2N+1}} , k_N + \frac{\varepsilon}{2^{2N+1}})$ would imply $x \in J_\varepsilon^{N+1} \subset J_\varepsilon$
in contradiction to $x \notin J_\varepsilon$.)

We are choosing now as starting figure for the construction of $S^1(0)$ the euclidian cicle $C^{\frac{1}{2\pi}}(0)$, the points of which may be homeomorphically identified with $[0,1)$ (the neighbourhoods of 0 and 1 $(0 = 1)$ being properly defined). Thus we may regard J_2^N , J_ε as subsets of $C^{\frac{1}{2\pi}}(0)$.

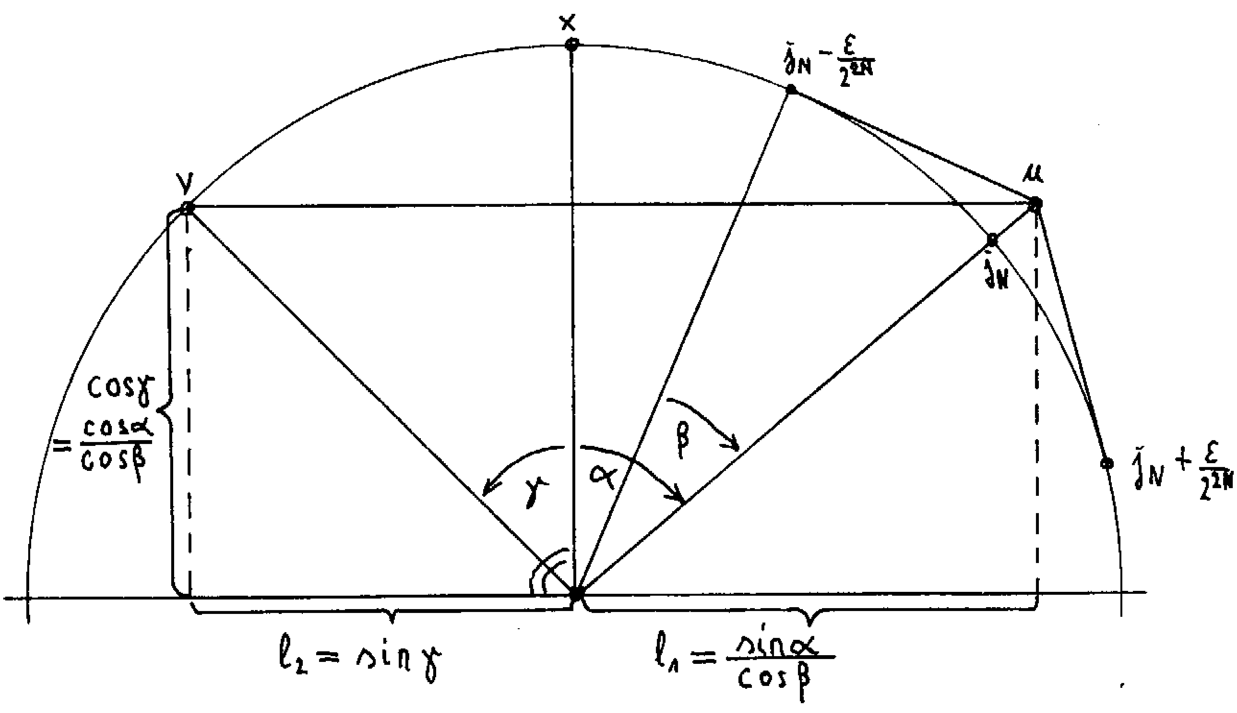
In J_ε^1 we modify $C^{\frac{1}{2\pi}}(0)$ by drawing in any $j_1 - \frac{\varepsilon}{4}$, $j_1 + \frac{\varepsilon}{4}$ $(j_1 \in J^1)$ the tangents to $C^{\frac{1}{2\pi}}(0)$. The new figure is convex, but not strictly; the latter may be repaired by replacing the tangents by very flat arcs tangent to $C^{\frac{1}{2\pi}}(0)$ in $j_1 - \frac{\varepsilon}{4}$, $j_1 + \frac{\varepsilon}{4}$. This modification being entirely trivial, it would complicate the considerations and computations, and so we work with the tangents nevertheless.



In J_ε^2 we modify $C^{\frac{1}{2\pi}}(0)$ in those intervals $(j_2-\frac{\varepsilon}{8}, j_2+\frac{\varepsilon}{8})$ with $(j_2-\frac{\varepsilon}{8}, j_2+\frac{\varepsilon}{8}) \cap J_\varepsilon^1 = \emptyset$ in the same manner, and do the same with J_ε^3 ... .

For $S^1(0)$ - constructed in this way - we show with (*) by direct computation that for any direction $\vec{ox}$ with $x \in S^1(0) \setminus J_\varepsilon$ the corresponding B (i. e. all B up to a set of measure $\leq \varepsilon$) has no asymptote:

Be $x \in S^1(0) \setminus J_\varepsilon$, then for any $M \in \mathbb{N}$ we may choose j_N, k_N according to * :



We only need to show $\frac{l_2}{l_1} \nrightarrow 1$ $(M \to \infty)$, and by *

$$v \notin \bigcup_{k_N \in J^N} (k_N - \frac{\varepsilon}{2^{2N+1}}, k_N + \frac{\varepsilon}{2^{2N+1}}) ,$$

and, because, if eventually

$$V \in \bigcup_{k_N \in J} N\left(k_N - \frac{\varepsilon}{2^{2N}},\ k_N + \frac{\varepsilon}{2^{2N}}\right) \setminus \left(k_N - \frac{\varepsilon}{2^{2N+1}},\ k_N + \frac{\varepsilon}{2^{2N+1}}\right),$$

V "deviates much less from $C^{\frac{1}{2\pi}}(0)$ than u" , we may even assume $V \in C^{\frac{1}{2\pi}}(0)$.

Now, by $\cos\gamma = \frac{\cos\alpha}{\cos\beta}$ we get

$$\begin{aligned}
\frac{l_2}{l_1} &= \frac{\sin\gamma \cdot \cos\beta}{\sin\alpha} \\
&= \sqrt{1 - \cos^2\gamma} \cdot \frac{\cos\beta}{\sin\alpha} \\
&= \sqrt{\frac{\cos^2\beta}{\sin^2\alpha} - \frac{\cos^2\alpha \cdot \cos^2\beta}{\cos^2\beta \cdot \sin^2\alpha}} \\
&= \sqrt{\frac{\cos^2\beta - \cos^2\alpha}{\sin^2\alpha}} \\
&= \sqrt{\frac{\sin^2\alpha - \sin^2\beta}{\sin^2\alpha}} \\
&= \sqrt{1 - \frac{\sin^2\beta}{\sin^2\alpha}} \quad .
\end{aligned}$$

By construction $\frac{\sin\beta}{\sin\alpha} \to \varepsilon > 0 \quad (M \to \infty)$, thus $\frac{l_2}{l_1} \not\to 1 \quad (M \to \infty)$.

PART III

DEGENERACY

§ 1 ON UNIQUE GENERATORS

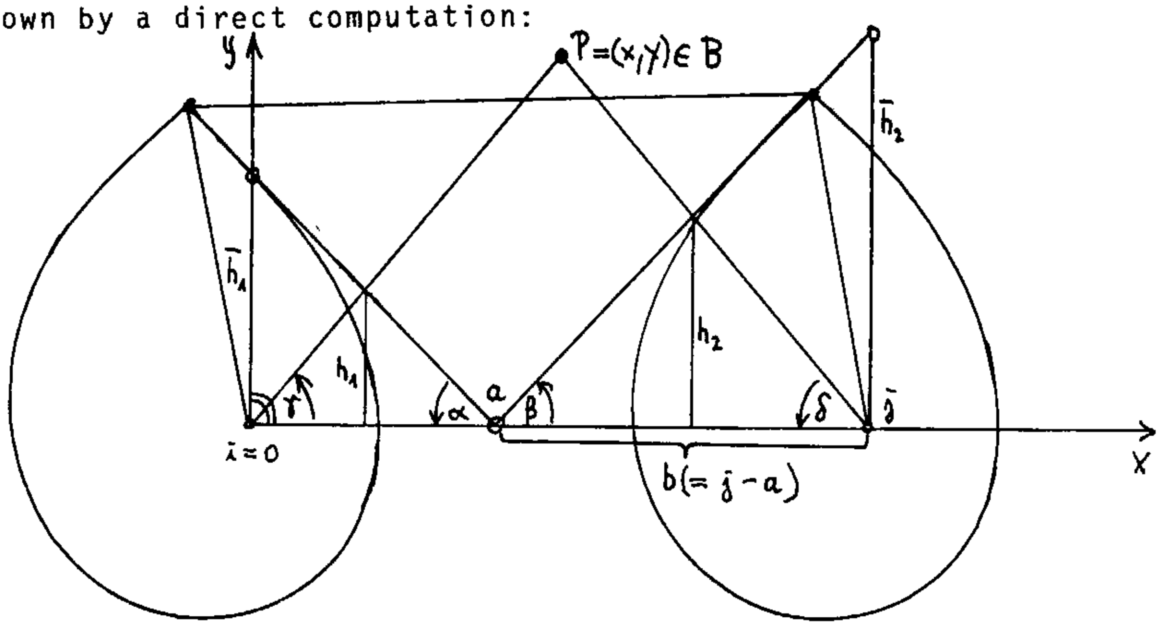
In this paragraphe we characterize degenerate middlelines $B \in \mathcal{G} \cap \mathcal{B}$ by the fact that their generating points i,j are not uniquely determined. In other words: For any nondegenerate $B \in \mathcal{B} \setminus \mathcal{G}$ there are two unique generating points $i,j \in \mathbb{R}^2$, $i \neq j$, with $B = B(i,j)$.

THEOREM 1.1:

1) If $B = B(i,j) \in \mathcal{B} \cap \mathcal{G}$, then $B = B(k,l)$ with $k \in \mathbb{R}^2 \setminus B$ arbitrary and $\overline{kl} \parallel \overline{ij}$, $\mathcal{B}$ deviding $\overline{kl}$ in the same proportions as $\overline{ij}$.
   (Thus l may be called the "reflected image" of k - "reflected at B, parallel to $\overline{ij}$ and with the appropriate proportions".)
2) If $B = B(i,j) \in \mathcal{B} \setminus \mathcal{G}$, then $B = B(k,l)$ iff $(i,j) = (k,l)$ (resp. $(i,j) = (l,k)$).

Proof. The first part being trivial by similarity, we only have to show the second one:
First we show that $B(i,j) = B(k,l)$ includes $\overline{ij} \parallel \overline{kl}$:
Clearly $As_i(B) \parallel As_k(B)$, since otherwise the strips of the asymptotes would have a compact intersection including B, hence B would be compact which is impossible. If $S^1(0)$ has no cracks in the asymptotic directions, i. e. in $y,z \in S^1(0)$ with $\overrightarrow{0y}, \overrightarrow{0z} \parallel As_i(B)$, then by construction of the asymptotes the direction of the generating points is uniquely determined. If there are cracks, then $As(B)$ exists and $\overline{ij} \nparallel \overline{kl}$ would cause a different asymptotic behaviour of $B(i,j)$ and $B(k,l)$ as can be

shown by a direct computation:



$$h_1 : h_2 = \bar{h}_1 : \bar{h}_2 = \frac{a}{b}\,\frac{\tan\,\alpha}{\tan\,\beta} \qquad (\bar{h}_1 = a\,\tan\alpha\;,\;\bar{h}_2 = b\,\tan\beta\;),$$

$$h_1 \mathrm{ctg}\gamma + h_1 \mathrm{ctg}\alpha = a\;,\; h_1 = \frac{a}{\mathrm{ctg}\gamma + \mathrm{ctg}\alpha}$$

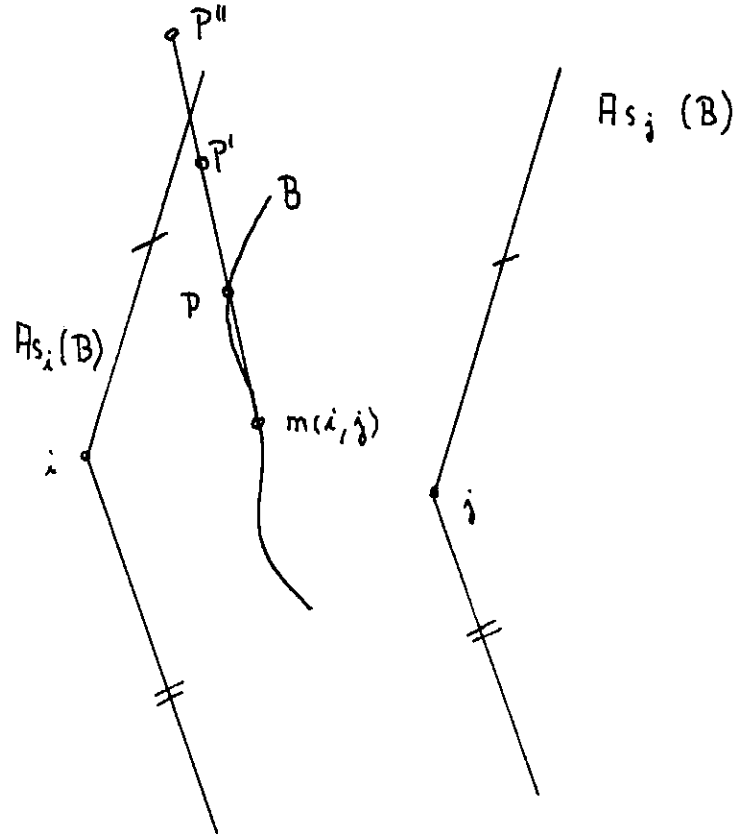
$$h_2 \mathrm{ctg}\beta + h_2 \mathrm{ctg}\delta = b\;,\; h_2 = \frac{b}{\mathrm{ctg}\delta + \mathrm{ctg}\beta}$$

$$h_1 : h_2 = \frac{a}{b}\,\frac{\mathrm{ctg}\delta + \mathrm{ctg}\beta}{\mathrm{ctg}\gamma + \mathrm{ctg}\alpha} = \frac{a}{b}\,\frac{\tan\alpha}{\tan\beta}$$

$$\mathrm{ctg}\gamma = \frac{\tan\beta}{\tan\alpha}\,(\mathrm{ctg}\delta + \mathrm{ctg}\beta) - \frac{\tan\beta}{\tan\alpha}\,\mathrm{ctg}\alpha$$

$$= \frac{\tan\beta}{\tan\alpha}\,\mathrm{ctg}\delta + \frac{1 - \frac{\tan\beta}{\tan\alpha}}{\tan\alpha}$$

Since for $\overline{kl} \nparallel \overline{ij}$ $\frac{\tan\beta}{\tan\alpha}$ have different values $(\alpha\uparrow \Rightarrow \beta\downarrow)$, we get a different asymptotic behaviour for $B(i,j)$ and $B(k,l)$.
If $ij \parallel kl$, $(i,j) \neq (k,l)$ (and $(i,j) \neq (l,k)$) and $P \in B$, $\overline{m(i,j)P} \nparallel As_i(B)$, then we can iteratively construct by similarity points $P', P'', \dots, P^{(n)} \in \overline{m(i,j)P}$, $P^{(n)}$ no more lying int the strip formed by $As_i(B)$, $As_j(B)$, contradicting $P^{(n)} \in B$:


P''
P'
B
P
As_i (B)
m(i,j)
i
As_j (B)
j

§ 2 CONDITIONS FOR DEGENERATION

In this paragraph we give some geometric equivalent conditions for the degenerate case, where all middlelines are straight. The next paragraphe will give an easy analytical characterization of this case.

The first theorem characterizes the degenerate case by the operations "$\cap$" , "$\cup$" being (at most) singlevalued.

<u>THEOREM 2.1:</u> $\mathcal{B} = \mathcal{G}$ is equivalent to

1) $B = A(B) \quad (B \in \mathcal{B})$, or

2) $\chi(B_1 \cap B_2) \leq 1 \quad (B_1 \neq B_2 \ ; \ B_1, B_2 \in \mathcal{B})$, or

3) $\chi(x \cup y) = 1 \quad (x \neq y \ ; \ x,y \in \mathbb{R}^2)$.

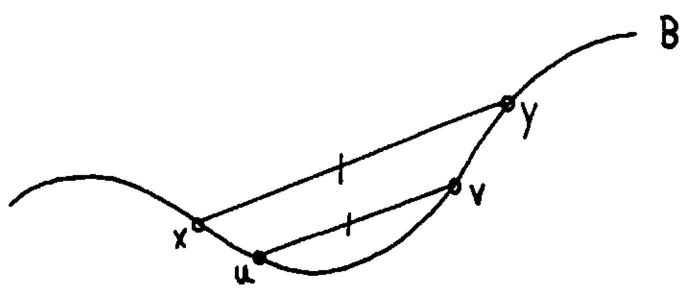
<u>Proof</u>. We show: $\mathcal{B} = \mathcal{G} \Rightarrow 1) \Rightarrow 2) \Rightarrow 3) \Rightarrow \mathcal{B} = \mathcal{G}$.

$\mathcal{B} = \mathcal{G} \Rightarrow 1)$: The statement is trivial, because $B = A(B) \quad (B \in \mathcal{G})$.

1) $\Rightarrow$ 2) : From 1) $\mathcal{B} \subseteq \mathcal{G}$ and hence $\chi(B_1 \cap B_2) \leq 1$ follows.

2) $\Rightarrow$ 3) : If $\chi(x \cup y) > 1$, there are $B_1 \neq B_2$, $B_1, B_2 \in \mathcal{B}$ and $x,y \in B_1, B_2$, hence $x,y \in B_1 \cap B_2$, hence (I,8.4) $(x \neq y)$ $\chi(B_1 \cap B_2) > 1$, hence 2) is violated.

3) $\Rightarrow \mathcal{B} = \mathcal{G}$: If $\mathcal{B} \nsubseteq \mathcal{G}$, there is $B \in \mathcal{B} \setminus \mathcal{G}$ and on B there are points x,y and u,v such that $\overline{xy} \parallel \overline{uv}$, but $\overline{xy} \neq \overline{uv}$:



By similarity ($u \to x$, $v \to y$) we can get $\bar{B} \in \mathcal{B}$ from B such that $\bar{B} \neq B$, $x,y \in \bar{B}$, hence $B,\bar{B} \in x \cup y$, and 3) is violated, hence $\mathcal{B} \subseteq G$. From $\mathcal{B} \subseteq G$ 1) follows trivially, and because we can construct $B \in \mathcal{B}$ with asymptotes in any direction, $\mathcal{B} = G$ follows.

The next theorem characterizes the case $\mathcal{B} = G$ by the compactness of the pencil $\mathcal{B}_x$ and of $x \cup y$.

<u>THEOREM 2.2</u>: $\mathcal{B} = G$ is equivalent to

1) $x \cup y$ compact ($x \neq y$, $x,y \in \mathbb{R}^2$) , or
2) $\mathcal{B}_x$ compact for some $x \in \mathbb{R}^2$, or
3) $\mathcal{B}_x$ compact ($x \in \mathbb{R}^2$) .

<u>Proof</u>. We show: $\mathcal{B} = G \Leftrightarrow$ 1) , $\mathcal{B} = G \Leftrightarrow$ 2) , 2) $\Leftrightarrow$ 3) .

$\mathcal{B} = G \Leftrightarrow$ 1) : "$\Rightarrow$" being trivial we have only to show "$\Leftarrow$" :

We first state a lemma:

<u>LEMMA 2.3</u>: If $\bar{B} \in \mathcal{B} \cap G$ and $S^1(0)$ has no cracks in the points $u,v \in S^1(0)$ with $\overline{uv} = \overline{ou} = \overline{ov} \parallel \bar{B}$, then any $B \in \mathcal{B}$ with asymptotes parallel $\bar{B}$ belongs to G .

<u>Proof of the Lemma.</u> Be $\bar{B} = B(\bar{i},\bar{j})$, then for any $B = B(i,j) \in \mathcal{B}$ with $As_i(B) \parallel As_j(B) \parallel \bar{B}$, $\overline{ij} \parallel \overline{\bar{i}\bar{j}}$ (by construction of $As_i(B)$, $As_j(B)$, because $S^1(0)$ is not cracked in u,v) , and by this B is similar to $\bar{B}$, hence $B \in G$.

Proof of the theorem (continued):

(1) $\Rightarrow \mathcal{B} = \mathcal{G}$:) If $\mathcal{B} \not\subseteq \mathcal{G}$, there is $B = B(i,j) \in \mathcal{B} \setminus \mathcal{G}$, and, because $S^1(0)$ has atmost denumerable many cracks, by continuity we may assume that $S^1(0)$ is not cracked in $u,v \in S^1(0)$, $\overline{ou} = \overline{ov} = \overline{uv} \parallel As_i(B) \parallel As_j(B)$, and thus by lemma 2.3 there is no $\bar{B} \in \mathcal{B} \cap \mathcal{G}$, $\bar{B} \parallel As_i(B) \parallel As_j(B)$. We can choose $x,y \in B$, $x \neq y$, $\overline{xy} \parallel As_i(B)$, and by dilatations and translations we get i_n, j_n: $\overline{i_n j_n} \parallel \overline{ij}$, $|i_n - j_n| \to \infty$, $x,y \in B(i_n,j_n) =: B^n$, hence $B^n \to \overline{xy} \in \mathcal{B}$, contradicting 1) , hence $\mathcal{B} \subseteq \mathcal{G}$, hence condition 1) of theorem 2.1 is fullfilled, thus $\mathcal{B} = \mathcal{G}$.

$\mathcal{B} = \mathcal{G} \Leftrightarrow$ 2): "$\Rightarrow$" being trivial, we show "$\Leftarrow$" :
If $\mathcal{B} \neq \mathcal{G}$, $x \cup y$ is not campact by 1) for some $x \neq y$, hence $\mathcal{B}_x = \bigcup_{z \neq x} x \cup z$ is not compact, in contradiction to 2) .

2) $\Rightarrow$ 3): The statement is trivial using translations.

For the next theorem we need the notion of a "pencil of cracked lines".

DEFINITION 2.4: We call $K_x \subseteq \{\overrightarrow{xy} \cup \overrightarrow{xz} \,;\, y,z \in \mathbb{R}^2 , y \neq x , z \neq x\}$ a "pencil of in x cracked lines", if for any $y \in \mathbb{R}^2$, $y \neq x$, there is $z \in \mathbb{R}^2$, $z \neq x$,such that $\overrightarrow{xy} \cup \overrightarrow{xz} \in K_x$.

THEOREM 2.5: $\mathcal{B} = \mathcal{G}$ is equivalent to

1) $\mathcal{B} \supseteq K_x$ for some $x \in \mathbb{R}^2$, or
2) $\mathcal{B} \supseteq K_x$ for all $x \in \mathbb{R}^2$.

Proof.

1) $\Leftrightarrow$ 2) being trivial by translations, and

$\mathcal{B} = \mathcal{G} \Rightarrow$ 1) being trivial, too, we need only to show

1) $\Rightarrow \mathcal{B} = \mathcal{G}$. For this sake we need an analogue to lemma 2.3:

LEMMA 2.3* : If $\bar{B} \in \mathcal{B}$ contains a ray $\overrightarrow{xy}$ $(x \neq y)$ and $S^1(0)$ has no crack in $u \in S^1(0)$, $\overrightarrow{ou} \parallel \overrightarrow{xy}$, then any $B \in \mathcal{B}$ with asymptotes containing a ray $\parallel \overrightarrow{xy}$ contain themselves a ray $\parallel \overrightarrow{xy}$.

The proof of lemma 2.3* is analogue to that of lemma 2.3.

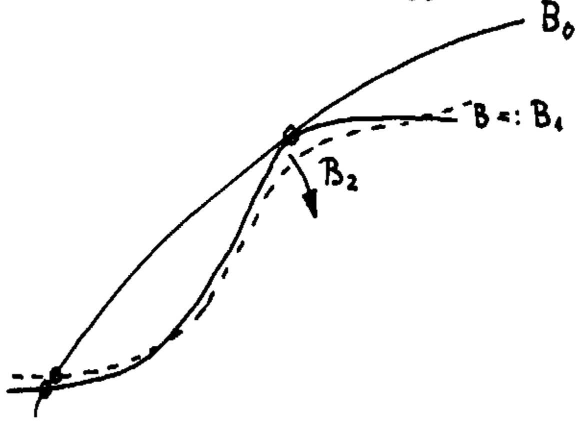
Proof of the theorem (continued): For $B \in \mathcal{B}$ let the directions of the asymptotes be given by two rays $\overrightarrow{xy}$, $\overrightarrow{xz}$ $(x \neq y\ ,\ x \neq z)$. Assuming - by continuity w.l.g. - the regularity condition on $S^1(0)$ of lemma 2,3* to be fulfilled, and choosing $K_1, K_2 \in K_x$ with $\overrightarrow{xy} \in K_1$, $\overrightarrow{xz} \in K_2$, we conclude (by lemma 2.3*) $B = \overrightarrow{xy} \cup \overrightarrow{xz}$. It remains only to show that any B of this form belongs to $\mathcal{G}$, which will be proved in the next paragraph.

We have proved in Part I the operation $\cap$ to be upper hemicontinuous (u.h.c.), and given an example for $\cap$ not to be lower hemicontinuous (l.h.c.) in Part II. Indeed the next theorem shows that, if $\cap$ l.h.c., then $\mathcal{B} = \mathcal{G}$.

THEOREM 2.5: $\mathcal{B} = \mathcal{G}$ is equivalent to "$\cap$ l.h.c." .

Proof. If $\mathcal{B} = \mathcal{G}$, then trivially $\cap$ is l.h.c., thus only the converse needs to be shown:

If $\mathcal{B} \nsubseteq \mathcal{G}$, then we construct a contradiction to "$\cap$ l.h.c." as shown in the following figure with some $B \in \mathcal{B} \setminus \mathcal{G}$:



($P \in B_0 \cap B = B_0 \cap B_1$ disappears when changing $B = B_1$ "a little bit" into B_2 , i. e. $\cap$ is not l.h.c. in (B_0, B_1).) Thus from "$\cap$ l.h.c." follows $\mathcal{B} \subseteq \mathcal{G}$ and hence (by asymptotes in any direction) $\mathcal{B} = \mathcal{G}$ follows, too.

The next theorem gives two conditions for $\mathcal{B} = \mathcal{G}$ taken from Topological Geometry (there stated as lemmata).

<u>THEOREM 2.6</u>: $\mathcal{B} = \mathcal{G}$ is equivalent to

1) (Compactness Lemma, cf. SALZMANN 2.13)
   $\mathcal{A} \subseteq \mathcal{B}$ is compact iff there is a compact subset $K \subset \mathbb{R}^2$ such that for any $A \in \mathcal{A}$ $A \cap K \neq \emptyset$, or
2) (Transversality Lemma, cf. SALZMANN 2.8)
   Let be $A, B \in \mathcal{B}$, $A \cap B = \emptyset$, then $H_1(A) \cap B \neq \emptyset$ and $H_2(A) \cap B = \emptyset$, $\mathbb{R}^2$ being divided into too open parts $H_1(A), H_2(A)$ by A ($\mathbb{R}^2 = H_1(A) \cup A \cup H_2(A)$) .

<u>REMARK 2.7</u>: If $\mathcal{A} \subseteq \mathcal{B}$ compact, then the conclusion of 1) is anyway valid.

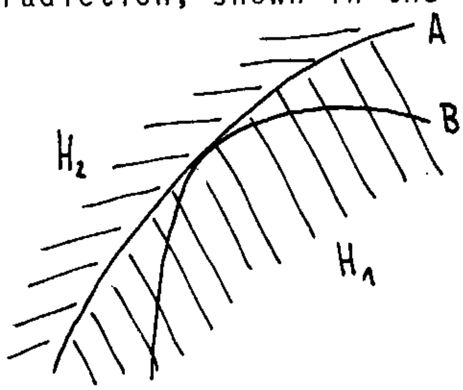
<u>Proof.</u>

Remark: If the remark would not be true, we could find outside any compact ball $B \subset \mathbb{R}^2$ $A \in \mathcal{A}$ with $A \cap B = \emptyset$, thus constructing a sequence $A_n \in \mathcal{A}$ without a point of accumulation. Thus A would not be sequentially compact, hence not compact.

$\mathcal{B} = G \Leftrightarrow$ 1): "$\Rightarrow$" being trivial, we show "$\Leftarrow$" :

$A := x \cup y$ fulfills the condition of 1) with $K = \{x,y\}$ for any $x \neq y$, thus $x \cup y$ being compact for any $x \neq y$, thus $\mathcal{B} = G$ by TH.2.1).

$\mathcal{B} = G \Leftrightarrow$ 2) : "$\Rightarrow$" being trivial, from $\mathcal{B} \not\subseteq G$ we may construct a contradiction, shown in the following figure:



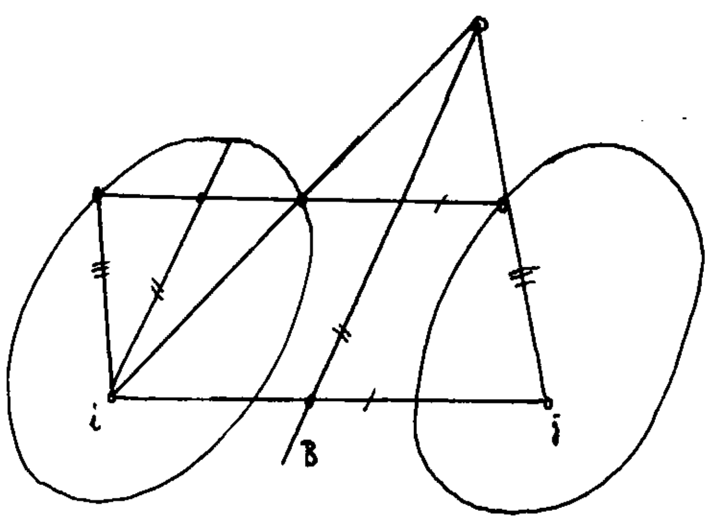
Thus from 2) $\mathcal{B} \subseteq G$, hence $\mathcal{B} = G$ follows.

§ 3 CHARACTERIZATION BY SCALAR PRODUCTS

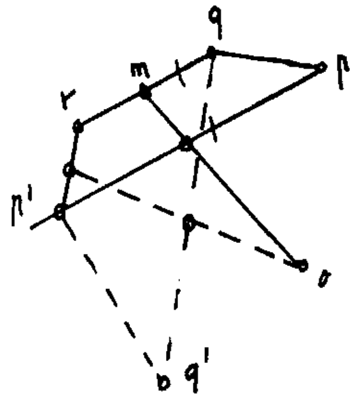
Having given in § 2 some geometric and analytical (equivalent) conditions for the degenerate case $\mathcal{B} = \mathcal{G}$, we state here a simple explicit analytical characterization.

THEOREM 3.1: $\mathcal{B} = \mathcal{G}$ is equivalent to the existence of a scalar product $<.,.>$ such that $\varphi(x) = \sqrt{<x,x>}$ $(x \in \mathbb{R}^2)$, or equivalent to $S^1(0)$ being elliptic.

Proof. Since $\mathcal{B} = \mathcal{G}$ any $B = B(i,j) \in \mathcal{B}$ is its own asymptote, and by Part I, TH. 6.10, we know that - up to denumerable many directions - $B \cap \overline{ij}$ contains exactly the euclidian middlepoint of $\overline{ij}$. We deal furthermore only with such $B \in \mathcal{B}$.



By similarity (see figure), we can construct from any point $p \in S^1(i)$ a point $p' \in S^1(i)$ by reflecting p parallel to $\overline{ij}$ at the parallel to B in i . Because this holds for any direction of B - up to denumerable many ones - , we may construct from three points $p,q,r \in S^1(0)$ other points $\in S^1(0)$:



By reflecting p at $\overline{om}$ parallel to $\overline{rq}$, m being the euclidian middlepoint of $\overline{rq}$, we get $p' \in S^1(0)$, and in the same way $q' \in S^1(0)$, and so on.

Taking p,q,r "close" (i. e. more precisely, a point p , the escent in p and the curvature in p , supposing w.l.g. both to exist in p) , we get $S^1(0)$.

On the other hand, by affine geometry for three points p,q,r with 0 lying in the inner angle formed by $\overline{pq}$ and $\overline{qr}$, there exists exactly one ellipse $E^1(0)$ with center 0 containing p,q,r . The same construction principle as for $S^1(0)$ being valid for $E^1(0)$, too, and any $p,q,r \in S^1(0)$ (taken in the appropriate order) having 0 in the inner angle, $S^1(0) = E^1(0)$ follows. Thus having shown one direction of the second part of the theorem, one direction of the first part is an easy consequence. Any nondegenerate $E^1(0)$ being of the form $E^1(0) = \{x \; ; \; x^TAx = 1\}$ with a positive definite symmetric matrix A , we get $\varphi(x) = \sqrt{<x,x>}$ $(x \in \mathbb{R}^2)$ by setting $<x,x> := x^TAx$.

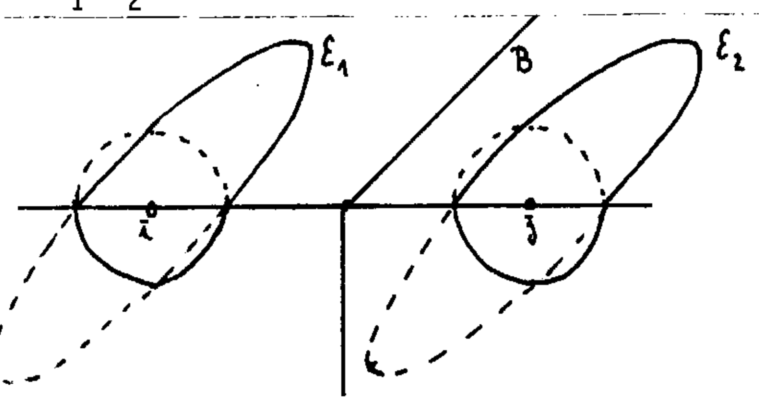
If we have on the other hand $\varphi(x) = \sqrt{<x,x>}$ $(x \in \mathbb{R}^2)$ we get (by a wellknown theorem of Riesz) a positive definite symmetric matrix A with $<x,x> = x^TAx$, resp. if we have $S^1(0) = E^1(0)$

being elliptic we have $S^1(0) = \mathcal{E}^1(0) = \{x \; ; \; x^T A x = 1\}$ with such a matrix A . Now we simply compute:

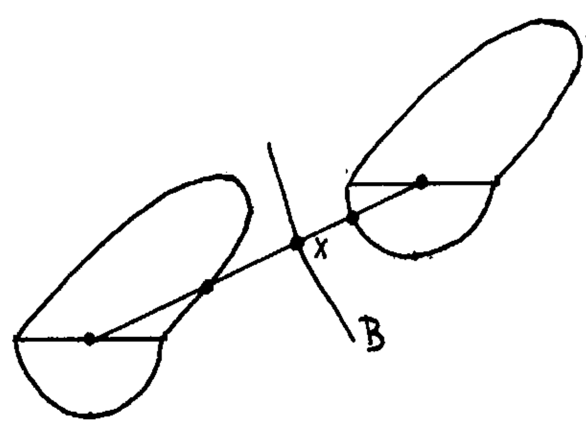
$$\begin{aligned} B(i,j) &= \{x \; ; \; \varphi(x-i) = \varphi(x-j)\} \\ &= \{x \; ; \; (x-i)^T A (x-i) = (x-j)^T A (x-j)\} \\ &= \{x \; ; \; x^T A x - 2i^T A x + i^T A i = x^T A x - 2j^T A x + j^T A j\} \\ &= \{x \; ; \; 2(j^T - i^T) A x = j^T A j - i^T A i\} \in \mathcal{G} \quad (i \neq j) \end{aligned} ,$$

the theorem being completely proved.

REMARK 3.2: We can now easily complete the proof of TH. 4 in § 2 . Having shown any $B = B(i,j) \in \mathcal{B}$ to be of the form $B = \overrightarrow{xy} \cup \overrightarrow{xz}$, we can conclude $S^1(0)$ to consist of two elliptic parts $\mathcal{E}_1, \mathcal{E}_2$:



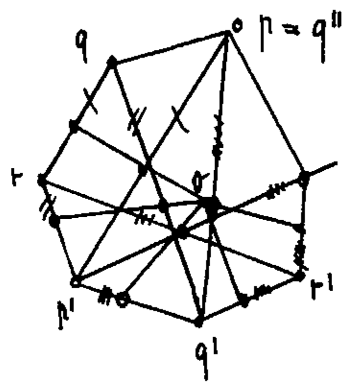
But then there are $\bar{j}$ such that parts of $B(i,\bar{j})$ result from different ellipses in contradiction to the form of $B(i,j)$, unless the ellipses are equal, i. e. $\mathcal{B} = \mathcal{G}$:



<u>REMARK 3.3</u>: By the proof of Th. 3.1 it is shown, too, how to characterize degenerate directions not resulting from cracks in $S^1(0)$, i. e. directions with (all or only one, see § 2, L.3) B, having asymptotes parallel to, to be straight, $B \in G$:

$S^1(0)$ must be symmetric with respect to all degenerate directions and with respect to reflection in the corresponding $\overline{IJ}$ - direction.

Either there are finitely many such directions only, and then by successive generation of new points by reflection, starting with three points p,q,r, $\overline{om_{pq}}$, $\overline{om_{qr}}$ (m_{pq}, m_{qr} being the euclidian middlepoints of $\overline{pq}$, $\overline{qr}$) being degenerate directions, we get only points, the euclidian middlepoints, corresponding to degenerate directions (not resulting from cracks in $S^1(0)$), and furthermore we come back to an old point after finitely many steps:



or there are infinitely many degenerate directions, becoming dense on $S^1(0)$ by construction, $S^1(0)$ then being elliptic, and any direction being degenerate.

For the case of degenerate directions resulting from cracks in $S^1(0)$ we can proceed in a similar manner, replacing the middlepoints in the reflection procedure by points of other varying proportions. We leave the details to the reader.

REFERENCES

HILDENBRAND : Core and Equilibria of a Large Economy
Princeton, N. Y. 1974

KOLM, S-Ch. : Justice et Equité
Paris, CNRS 1972

Mc.KELVEY/WENDELL: Voting equilibria in Multidimensional Choice Spaces
Math. of OR 1, 2, pp. 144-158 , 1976

OSTMANN, A (1): Fair Play und Standortparadigma
Inst. of Statistics and Math. Economics, Karlsruhe 1978

OSTMANN, A (2): On Location Conflicts and their Fair Solution Concepts
Inst. of Math. Economics, Bielefeld 1978

RAWLS, J.: Theory of Justice
Cambridge, Mass. 1971

ROSENMOLLER, J.: On Values, Location Conflicts and Public Goods
Inst. of Math. Economics, Bielefeld 1979

SALZMANN: Topological Planes
Adv. in Math. 2, pp. 1-60, 1967

ZECKHAUSER: Determining the Qualities of a Public Good ...
W. Econ. II, 1, pp. 39-60, 1973

ZECKHAUSER/WEINSTEIN: The Topology of Pareto-optimal Regions with Public Goods
Econometrica 42, 4, pp. 643-666, 1974

For further reading:

ALBERS/OSTMANN/RICHTER/ROSENMOLLER: Projekt Standortspiele
Inst. of. Math. Economics, Bielefeld, 1978

RICHTER, W.: A Game-theoretic Approach to Location-Allocation Conflicts
Inst. of Statistics and Math. Economics, Karlsruhe 1978