Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 108

Wolfram F. Richter

FROM ABILITY TO PAY TO CONCEPTS OF EQUAL SACRIFICE

May 1981

H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

From Ability to Pay to Concepts of Equal Sacrifice

Wolfram F. Richter

Abstract:

The ability-to-pay approach to taxation in Mill's tradition is given a rigorous treatment in a social choice theoretic framework. Two model variants are discussed. The first allows 1) to formalize ability-to-pay, 2) to axiomatize concepts of equal sacrifice, and 3) to derive extensive statements on progressivity. However, there is no such axiomatized concept that implies moderate progressivity irrespective of taxpayers' utility function. For the second model variant the situation is then reversed. Here a sacrifice concept can be defined that generally leads to moderately progressive tax schedules. But no convincing axiomatization along the line of ability-to-pay is known.

0. INTRODUCTION

This paper resumes the classical question how and whether progressive taxation can be normatively justified along the line of *ability to pay* by making all taxpayers incur *equal sacrifice*. The "ability-to-pay approach" in its subjective interpretation as originated by J.St.Mill (1921, p. 804) is given a rigorous treatment.

The subject has a most venerable tradition in public finance which is not to be reviewed here. Instead, the interested reader is asked to consult Musgrave's (1959) standard work on public finance which includes a highly recommendable presentation of the problem.

When attempting to formalize such an old branch of economic theory two major difficulties have to be faced. The first comes from the use of language. It is certainly desirable to make associative definitions that refer to well established terms. However, as the expressions have to be chosen in accordance with the formal model - and not *vice-versa* - the habit can become misleading.

The term of *equal sacrifice prescription* gives a vivid example. Below, the classical equal sacrifice concepts will uniformly be justified and recovered as first-order conditions to minimizations of appropriately aggregated personal sacrifice profiles. As such they lend themselves to straightforward generalizations - to "non-classical sacrifice concepts". This latter term - though being very much catchy and associative - is actually misleading. What we are going to generalize is not the involved notion of subjective sacrifice but the social aggregation rule, the *equality* prescription. Nevertheless, we shall continue to talk of "sacrifice concepts" when optimality conditions for the minimization of aggregate sacrifice - i.e. for "least sacrifice" - are meant.

The second difficulty comes from the need to interpret verbal texts before formalizing expressed ideas. Here, a clear warning is in order. No attempt is being made to formalize just the ideas of any particular economist. Instead, a mathematical model is presented which is said to allow a consistent interpretation in the sacrifice theoretic *spirit*. An example may again help to make the point clear.

The model will draw a sharp line between the ability-to-pay *principle* and equal sacrifice *prescriptions*. The principle is formulated and logically treated as an axiom that in connection with others allows to characterize concepts of equal sacrifice. The literature, on the other hand, tends to make no such distinction. "Since John Stuart Mill, the ability-to-pay rule has been viewed in terms of an equal sacrifice prescription." (R.A. Musgrave and P.B. Musgrave, 1873, p. 198). That leads to the question what the "correct" formulation of an ability-to-pay *axiom* might be? Below, we shall propound one formalization that allows to draw the conclusions we like to draw. By way of contrast we shall not discuss which different formalization would equally - or even better - be justifiable in regard to the literature.

Mill's approach raises two basic questions. The first is whether any sacrifice concept *exists at all* which upon application demands a moderately progressive tax schedule irrespective of the specific taxpayers' utility function. (We require *moderate* progressivity to rule out the trivial but controversal case when marginal tax rates equal one as under the equal-marginal-sacrifice prescription.) And secondly, which sacrifice concept(s) allow(s) some convincing *axiomatization* in the light of the ability-to-pay principle? These are the questions of existence (Ex) and axiomatization (Ax).

The classics solved neither (Ex) nor (Ax). Below, partial answers are given only. For this purpose, two model variants are considered. The first is to demonstrate how, principally, problems like (Ax) can satisfactorily be attacked by the methods of social choice theory. However, existence (Ex) will be violated which supports Mill's (often recited) remark that progressivity is "too disputable altogether". The second variant provides a solution to (Ex) leaving (Ax) an open problem. More precisely, a non-classical sacrifice concept is formulated that implies moderate progressivity for all utility functions under rather mild conditions. It is noteworthy that this sacrifice concept allows to justify tax schedules with constant residual progression. Unfortunately, an axiomatic characterization is still missing that is convincing on the grounds of public finance and that distinctly exploits the idea of taxation according to ability to pay. (Some game-theoretic "meta-axiomatization" does in fact exist. Cf. Richter (1981b)).

1. ON THE DETERMINATION OF EQUITABLE TAX SHARES

1.1 Environments

There are n taxpayers, $N = \{1,\ldots,i,\ldots,n\}$ with incomes $y_i \geq 0$ before tax. (It is convenient to talk of "income" though "index of ability to pay" would be more correct.) Individual $i \in N$ derives utility from income after tax $y_i - t_i$ according to $u_i(\cdot)$ which exhibits decreasing positive marginal increments: $u_i' > 0$, $u_i'' \leq 0$. We make use of vector notation: $y = (y_1, \ldots, y_n)$, $t = (t_1, \ldots, t_n)$ and $u = (u_1, \ldots, u_n) : \mathbb{R} \to \mathbb{R}^n$. We write $y \in \mathbb{R}^n_+$ if $y \geq 0$, i.e. $y_i \geq 0$ for all $i \in N$. If $y_i > 0$ for all $i \in N$ then $y >> 0$ or $y \in \mathbb{R}^n_{++}$.

Public expenditure is exogenously fixed at $g \geq 0$. Let $C \subseteq N$ denote the subset of taxpayers that are to share the tax burden $g \in \mathbb{R}_+$.

A *feasible environment* is then specified by

$$e = (u, y, g, C) \in E := \underline{\underline{U}} \times \mathbb{R}^{n+1}_+ \times 2^N$$

with

$$\underline{\underline{U}} := \{u : \mathbb{R} \to \mathbb{R}^n \mid u_i' > 0, u_i'' \leq 0\} \text{ and}$$

$$2^N := \{C \subseteq N\}.$$

The set of all *feasible tax distributions* is assumed to be

$$T(e) := \{t \in \mathbb{R}^n_+ \mid \sum_{j \in C} t_j = g\}.$$

(The notational dependence of T on e is actually redundant. $T = T(g,C)$ would be more precise but less convenient).

The parametization of feasible environments by u, y, and g is in line with the literature. This does certainly not hold for C, the set of taxpayers that are exogenously determined to pay for public services g. Why should we consider the case C$\neq$N at all? There seems to be no cogent reason from the point of interpretation. On the other hand the theory becomes more powerful if we do, if we force ourselves to fix ideas about distributive justice for such an *extended* domain of feasible environments. So the primary question is not which reasons could make one discriminate between taxpayers? The question is how to allocate tax shares among N if C$\neq$N is exogenously given by reasons whatsoever.

The exogenous determination of y and g is obviously debatable for its partial equilibrium character. As a matter of fact, there have been made great efforts to go beyond the classics by explaining y and g within the model. (The endogenous determination of public expenditure was persued by Schäffle, Pigon and Dalton. Cf. Musgrave (1959, p. 113). The effects of taxation on work effort and hence on incomes was extensively studied by Mirrlees (1971).) However, the obtained results did hardly meet prior expectations. The case for progression turned out to be even more disputable.

By fixing y and g, exogenously, one could say we ignored the efficiency aspect of taxation, neglected welfare considerations and that we restricted attention to distributive justice. Our view is even further narrowed by the definition of T(e). The nonnegativity condition of tax shares renders impossible subsidies to income. The scope for redistributional targets is hence rather limited. It is not so much a correction of the personal income distribution that we aim at but an equitable allocation of tax shares according to individual abilities to pay.

An *equitable* allocation is denoted by $t^*(e) = t^e \in T(e)$.

As there seems to be no prior reason why $t^*(e)$ should be unique we consider *choice correspondences:*

$$T^* : E \to \mathbb{R}^n_+ , \quad \emptyset \neq T^*(e) \subseteq T(e) \; .$$

($t^*(e) = t^e$ is always meant to imply $t^e \in T^*(e)$).
The crucial question is which properties of T^* can be justified on purely normative grounds? This is a matter of *social choice* theory. We have to aggregate taxpayers' preferences to a socially acceptable outcome. There is some hope (which will actually be deceived) that progressivity is a derivative and instrinsic property of choice correspondences T^* that meet fundamental principles of social choice and distributive equity. (Pfähler's (1978) remarkable approach to the determination of an equitable tax system is not too far in spirit.)

1.2 Taxation according to ability to pay

The classical concepts of equal sacrifice give rise to at least two points at assue. a) How should one measure the subjective sacrifice incurred by the individual taxpayer? And b) how should one compare the sacrifice of different people, a question that amounts to asking for an admissible aggregation rule? We thus like to break up the classical sacrifice prescription into a) its underlying *notion of subjective sacrifice* and b) an *aggregation rule*.

The subjective sacrifice of taxpayer $i \in N$ will be a function of u_i, y_i, and t_i: $s_i = \sigma(u_i, y_i, t_i)$. For symmetry reasons $\sigma(\cdot)$ is carrying no index i. The denomination of the classical sacrifice concepts is suggesting three different specifications:

absolute sacrifice $\quad s_i = u_i(y_i) - u_i(y_i - t_i)$

proportional sacrifice $\quad s_i = \dfrac{u_i(y_i) - u_i(y_i - t_i)}{u_i(y_i)}$

marginal sacrifice $\quad s_i = u_i'(y_i - t_i)$.

Let (AS), (PS), (MS) denote the respective classical prescriptions of equalizing sacrifice. Thus (AS) is to stand for

$$u_i(y_i) - u_i(y_j - t_i^e) = \text{const} \quad (i \in N).$$

Prima facie the classical sacrifice concepts seem to apply the same aggregation rule, namely equalization, to different sacrifice notions. However, a marginal sacrifice is a little convincing notion. If the equal-marginal-sacrifice prescription (MS) has strong appeal than less in itself as an equity rule but primarily as first-order condition to the utilitarian recommendation

$$\sum_{i \in N} u_i(y_i - t_i) \rightarrow \max , \quad \sum_{i \in N} t_i = g. \qquad (1)$$

Note that (1) is equivalent to

$$\sum_{i \in N} [u_i(y_i) - u_i(y_i - t_i)] \rightarrow \min , \quad \sum_{i \in N} t_i = g \qquad (1')$$

which supports the view - stressed by theorem 1, below - that (AS) and (MS) are primarily not differing with respect to the employed sacrifice notion (a) but with respect to the chosen aggregation rule (b). (1') indicates that we do not have to rely on the dubious notion of marginal sacrifice to justify (MS).

Even if we reject the marginal sacrifice *notion* we are still left with (at least) two sacrifice specifications which both make good sense. The conflict of choice seems to call for an axiomatic procedure.

However, this route can and will not be followed, here. Later we shall see that the axiomatic basis of section 1 is incompatible with the proportional sacrifice notion. Its further discussion is therefore deferred to section 2. For the course of section 1 we just convene to *define sacrifice in absolute terms*. With the shortened notation

$$s_i^e = \sigma(u_i, y_i, t_i^e) = u_i(y_i) - u_i(y_i - t_i^e) \qquad (i \in N)$$

we now formalize the idea of taxation according to ability to pay:

t^e *taxes according to ability to pay* (ATP) if

$$y_j \leq y_k \quad (j,k \in N) \quad \text{implies} \quad s_j^e \leq s_k^e .$$

Note that this axiom is not demanding *equal* sacrifice. (Requiring $s_j^e = s_k^e$ for all j,k would us leave with (AS), alone.)

It is widely held that tax fomulas should display marginal rates that do not exceed one. This requirement is weakened by:

t^e *preserves the order of incomes* - or t^e is *order preserving* (OP) if

$$y_j - t_j^e \leq y_k - t_k^e \quad \text{whenever} \quad y_j < y_k .$$

(OP) is a condition of vertical distributional equity. It is stated in units of income and does not refer to utility functions. Note that equal treatment in the form of $y_j = y_k$

implying $y_j - t_j^e = y_k - t_k^e$ would only follow if (OP) were strengthened to $y_j \leq y_k$ implying $y_j - t_j^e \leq y_k - t_k^e$.

<u>Remark:</u> Let $u_i = U$ for all $i \in N$, $U'' < o$. Then (ATP) together with (OP) is equivalent to

$$u'_j(y_j - t_j^e) > u'_k(y_k - t_k^e) \quad \text{implying} \quad s_j^e \leq s_k^e \; . \tag{2}$$

The proof is straightforward:

Under the stated assumptions (2) is equivalent to

$$y_j - t_j^e < y_k - t_k^e \quad \text{implying} \quad s_j^e \leq s_k^e \; . \tag{3}$$

Assume first $y_j - t_j^e < y_k - t_k^e$. (OP) implies $y_j \leq y_k$. Hence (3) follows from (ATP). For the reversal assume (3). Suppose that in contradiction to (ATP) $y_j \leq y_k$ implied $s_j^e > s_k^e$. By $U' > o$ $U(y_j) \leq U(y_k)$. Hence $U(y_j - t_j^e) < U(y_k - t_k^e)$ or $y_j - t_j^e < y_k - t_k^e$ which contradicts (3).

Finally, for (OP) assume $y_j < y_k$. If we had $y_j - t_j^e > y_k - t_k^e$ then by $U' > o$ $s_j^e < s_k^e$ would follow, which contradicts (3).

When we come to the axiomatization of social aggregation rules for sacrifice profiles we shall assume (2) and not (ATP) together with (OP). This would not really matter if we could restrict attention to the case where all taxpayers have identical utility functions. When progressivity is discussed, later, identical utility functions have to be considered, anyway. However, for the purpose of axiomatizing aggregation rules the assumption of identical utility functions would be too special. The set of feasible environments E has to be "sufficiently rich" to allow strong conclusions.

Clearly, (ATP) is more in line with the traditional understanding of taxation according to ability to pay. However, (2) is not too far off, either. Under (ATP) the notion of sacrifice has been subjectivized whereas the index of ability to pay, namely y_i, is still an objective concept. Under (2) the subjective conceptualization - initiated by Mill - is carried to its end. Marginal utility from income after tax becomes the measuring rod for subjective ability to pay. The smaller marginal utility is the larger the (subjective) ability to pay.

Condition (2) is related to the notion of *weighting equity* which was defined by A.K. Sen (1974). See axiom A1), below, and the corresponding discussion in Richter (1981a).

1.3 Axioms

We now state the axioms which we are going to demand for T^*, i.e. for equitable allocations of tax shares. They may appear to be rather strong. However, the reader should bear in mind that they are all met by an infinity of (generalized) sacrifice concepts - including (the adapted version of) (AS) and (MS). (The classical conditions need to be restated for "boundary solutions", i.e. when $C \neq N$ or $t^e \notin \mathbb{R}^n_{++}$.)

Axioms A1) to A3) are to hold for all $e \in E$. Fix any such $e = (u,y,g,C) \in E$ with $t^e \in T^*(e)$. Recall the notation

$$s_i^e = \sigma(u_i, y_i, t_i^e) = u_i(y_i) - u_i(y_i - t_i^e) .$$

As $t \geq o$ $s^e = (s_1^e, \ldots, s_n^e) \geq o$.

A1) *Weighting Equity*

a) $s^e = o$ if $C = \emptyset$ and

$$\max_{i \in N \setminus C} s_i^e \leq \min_{i \in C} s_i^e \quad \text{if} \quad \emptyset \neq C \neq N .$$

b) $u_j'(y_j - t_j^e) > u_k'(y_k - t_k^e)$ $(j,k \in C)$

implies $s_j^e \leq s_k^e$.

A1) is the generalization of condition (2) we need to cope with the case when $C \neq N$. According to A1a) taxpayers who are not to share the public burden g should not incur larger sacrifice than those who share.

If nothing else were stated the notational dependence of T^* on u would imply the strong informational requirement that utilities are cardinally measurable and interpersonally fully comparable. Clearly, the other extreme is desirable, namely, ordinal measurability and interpersonal incomparability. However, Arrow's impossibility theorem indicates that we cannot demand that much without risk of violating other principles of equity, collective rationality, etc.

Hence we compromise on what is called *cardinal unit comparability* in A.K. Sen's (1977) terminology. This informational basis is justifiable by the observation that it is met by (AS) and (MS) - though not by (PS). A weaker informational requirement is considered in section 2.

A2) *Cardinal unit comparability*

$$T^*(u,y,g,C) = T^*(au+v,\ y,g,C)$$
$$\text{for all } a \in \mathbb{R}_{++}\ ,\ v \in \mathbb{R}^n\ .$$

(Let us demonstrate in passing that the proportional sacrifice notion is incompatible with A1) and A2). Choose any $e \in E$ with

$C = N$, $u_i(Y) = \alpha_i Y + \beta_i$ $(i=1,2)$, $\alpha_2 > \alpha_1$, $t^e \in T^*(e) \cap \mathbb{R}^n_{++}$. A2) implies $t^e \in T^*(u+v,y,g,C)$ for all $v \in \mathbb{R}^n$. By A1b)

$$s_2^e \;/\; (u_2(y_2) + v_2) \leq s_1^e \;/\; (u_1(y_1) + v_1)$$

had to follow for all $v \in \mathbb{R}^n$ which is obviously impossible.)

A3) $T^*(e)$ is compact.

This is a regularity condition which is trivially fulfilled for point-valued T^*. If we only considered environments $e \in E$ with $C = N$ then $T(e)$ would be bounded and it would suffice to require closedness of $T^*(e)$.

Let $E^{lin} := \{e \in E \mid u_i(Y) = \alpha_i Y + \beta_i$ for all $i \in N\}$ be the set of environments where taxpayers have linear utilities.

A4) *Surjectivity:* There exists a mapping

$$\mathbb{R}^n_+ \rightarrow E \times \mathbb{R}^n_+ \;,\; s \mapsto (e^s, t^s) \in E^{lin} \times T^*(e^s) \quad \text{s.t.}$$

$$s_i = u_i^s(y_i^s) - u_i^s(y_i^s - t_i^s) \quad \text{for all } i \in N.$$

A4) ensures that every logically conceivable sacrifice profile s is in fact eligible under T^* - "is some s^e" - for appropriate choice of $e^s \in E^{lin}$. (One might conjecture that A4) could be relaxed by replacing E^{lin} by E .

However, the theorem in Richter (1981a) on which we are going to rely would no longer be directly applicable.)

At the first glance A4) might appear an innocuous assumption. This is certainly not true. Only note that the choice correspondence T^* which is induced by (AS) (as defined above) does not satisfy A4). Under (AS) only equally distributed sacrifice profiles are eligible. As a matter of fact A4) is difficult to reconcile with the very spirit of imposing equal absolute sacrifice.

On the other hand (AS) is an appealing equity rule only when *all* taxpayers are liable to pay for public services. When $C \neq N$ (AS) is highly debatable. Why should $i \in N \setminus C$ and $j \in C$ incur equal sacrifice? As a matter of fact this theory will replace (AS) by some "$\|\cdot\|_\infty$-sacrifice concept". The latter meets axiom A4) and still demands an equal absolute sacrifice like (AS) when $C=N$.

Our last axiom admits a more convenient statement if some further notation is introduced before.

$$S^e := \{s \in \mathbb{R}^n_+ \mid \exists t \in T(e) : s_i \geq u_i(y_i) - u_i(y_i - t_i) \quad \text{for all } i \in N\},$$

$$F^e := \{s^e \in S^e \mid \exists t^e \in T^*(e) : s^e_i = u_i(y_i) - u_i(y_i - t^e_i) \quad (i \in N)\}.$$

(Up to the inequality sign) S^e is the set of feasible sacrifice profiles with respect to e. F^e is the set of sacrifice profiles that are considered to be socially equitable for this environment.

A5) *Strong axiom of revealed preference* (SARP)

For all $r \in \mathbb{N}$, $e^1,\ldots,e^r \in E$,

$s^\rho \in F^{e^\rho}$ $(\rho=1,\ldots,r)$,

$s^\rho \in S^{e^{\rho+1}} \setminus F^{e^{\rho+1}}$ $(\rho=1,\ldots,r-1)$

the implication is $s^r \notin S^{e^1}$.

This famous axiom is most easily understood when $r = 2$, i.e. when it reduces to its "weak" version. If s^1 is deemed to be equitable for e^1 , feasible though not equitable for e^2 then s^2 , the choice for e^2 , should not have been feasible in e^1.

A5) thus guarantees logic consistency of what is socially deemed to be equitable for different environments. If one interprets F^e as the choice set of some fictious planner then A5) assumes rational choice behaviour in very much the same sense as known from the revealed preference approach to consumer's behaviour. Cf. Richter (1981a) if more information is required.

A5) entails that the mapping $e \mapsto F^e$ "factorizes". I.e. $S^{e^1} = S^{e^2}$ implies $F^{e^1} = F^{e^2}$. Hence we may sloppily write $F^e = F(S^e)$ so that F defines a choice correspondence: $F(S^e) \subseteq S^e$.

1.4 Minimizing aggregate sacrifice

Adhering to the standard welfare approach one might proceed as follows: Fix some arbitrary welfare function $W : \mathbb{R}^n \to \mathbb{R}$ with (at least) non-negative first partial derivatives and solve

$$W\ (u_1(y_1-t_1),\ldots,u_n(y_n-t_n)) \to \max \quad \text{for} \quad t \in T(e). \qquad (4)$$

From today's point of view one might argue that the truely important contribution of the ability-to-pay approach in Mill's tradition was to develop an alternative maxim. Besides *maximization of aggregate welfare minimization of aggregate sacrifice* was established as independent principle:

$$W\ (\ldots,u_i(y_i)-u_i(y_i-t_i),\ldots) \to \min \quad \text{for} \quad t \in T(e). \qquad (5)$$

Although the verbal literature addresses both principles their exact relationship remains clouded. Musgrave (1959, p.110), e.g., refers to "writers who wished to allocate taxes ... so as to minimize aggregate sacrifice and maximize welfare". The conjunction "and" makes one infer that there are no conflicting aims. This is true under the utilitarian summation rule (which is the only one Musgrave had in mind). However, with respect to the same non-linear $W : \mathbb{R}^n \to \mathbb{R}$ maximization of aggregate welfare (4) and minimization of aggregate sacrifice (5) will lead to conflicting solutions. Just consider the case when $W(s) := \max s_i$. Then (4) would define a most disputable maxim whereas (5) would be equivalent (in the optimum for $C = N$) to (AS).

The formally orientated literature on optimum income taxation seems to have a clear preference for maximizing welfare as against minimizing sacrifice. Revealing examples are Atkinson (1973), Atkinson and Stiglitz (1980, lect. 13), and Sadka (1976).

Theorem 1 now tells us to minimize aggregate sacrifice. More precisely, under A1) - A5) every T^* equals some T^W where $T^W(e)$ denotes the solutions of (5). Theorem 1 makes additional statements about the functional form of admissible welfare functions W. The proof is relegated to the appendix (AI).

Theorem 1: Every T^* holding A1) - A5) is *norm-induced*. I.e., there is some norm $\|\cdot\| = \|\cdot\|^{T^*}$ in $\mathbb{R}^n$ s.t. $T^* = T^{\|\cdot\|}$ where $t^e \in T^{\|\cdot\|}(e)$ by definition iff (6)

$$\|(u_i(y_i)-u_i(y_i-t_i^e))_{i\in N}\| = \min_{t\in T(e)} \|(u_i(y_i)-u_i(y_i-t_i))_{i\in N}\|$$

$\|\cdot\|$ is monotonic, i.e.

$0 \leq s^1 \leq s^2$ implies $\|s^1\| \leq \|s^2\|$.

A *norm* is a mapping $W : \mathbb{R}^n \to \mathbb{R}$ holding:

N1) $s = 0$ if $W(s) = 0$,

N2) $W(as) = |a|\, W(s)$ and

N3) $W(s^1+s^2) \leq W(s^1) + W(s^2)$ for all $a \in \mathbb{R}$, $s, s^i \in \mathbb{R}^n$.

Hence any norm is a convex function. Differentiability is not ensured. Monotonicity is the appropriate substitute for non-negative first partial derivatives.

Consider the most prominent examples of norms, called *p-norms* $\|\cdot\|_p$, $p \in [1,\infty]$. For $s \in \mathbb{R}^n_+$ - the case $s \notin \mathbb{R}^n_+$ is of no interest here - they are defined by:

$$\|s\|_p := \begin{cases} \sqrt[p]{\sum_{i \in N} s_i^p} & (p \in [1,\infty)) \\ \max_{i \in N} s_i & (p = \infty) \ . \end{cases}$$

To see how the classical sacrifice concepts (MS)/(AS) are included in theorem 1 evaluate (6) for $\|\cdot\| = \|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively. (MS) and (AS) are in fact first-order optimality conditions of the corresponding minimizations - subject to $C = N$, $t^e \in \mathbb{R}^n_{++}$. Under the stated assumptions these first-order conditions are necessary as well as sufficient (see also below). (MS) and (AS) thus turn up as special cases in a uniform axiomatically justified theory. Note that the nature of theorem 1 does not allow to discriminate against any p-norm. There is no indication that the classical concepts (AS) and (MS) define preferable standards of equity and distributive justice. It is merely accidental that the 1-norm (as minimization) and the ∞-norm (as first-order condition) allow more suggestive interpretations.

What disturbs one most about theorem 1 is the fact that a fairly large class of welfare functions should be eligible under taxation according to ability to pay. As a matter of fact, those could feel supported who always held the ability-to-pay approach indeterminate altogether. (A most lively and extensive debate about this determinacy question was carried out around 1970 in the German speaking literature. Cf. Haller (1972/73) for further references.) For the proponents of ability to pay any argument is however welcome that allows to further restrict the class of admissible norms. In particular, the search for a set of axioms seems to be called for that allows to single out (at least) all p-norms (so that (AS) and (MS) are both included.) Below, we shall show a way how to obtain all p-norms with $p < \infty$. (The argument is rather standard. It exploits the fact that for $p < \infty$ $\|\cdot\|_p$ is a monotone transformation of an additively separable welfare function.) It is clearly non-satisfactory that just the norm should be excluded that would sustain (AS). However, related theorems, say Nagumo (1930), Debreu (1960), and Roberts (1980, theorem 8) - to name just a few - indicate that a tough mathematical problem is at stake. The convincing axiomatization of all p-norms (including $\|\cdot\|_\infty$) needs still to be found.

In more conventional phraseology (6) could be addressed as $\|\cdot\|$-*least sacrifice prescription*. Note, however, that the interpretative contents of "least" have been generalized vis-à-vis the traditional understanding. Here, "least" is contingent on the welfare function $\|\cdot\|$ which may very well differ from utilitarian summation ($\|\cdot\|_1$).

Let us now consider some arbitrary norm which is twice continuously differentiable in $\mathbb{R}^n_{++}$. (By that seemingly innocuous assumption $\|\cdot\|_\infty$ was excluded, already).

Introducing Lagrangean multipliers we obtain ($\partial_i = \frac{\partial}{\partial s_i}$)

$$\partial_i \|(u_j(y_j) - u_j(y_j - t^e_j))_{j\in N}\| \, u'_i(y_i - t^e_i) = \text{const } (i\in N) \quad (7)$$

as first-order condition to (6) - subject to $C = N$ and $t^e \in \mathbb{R}^n_{++}$. As u_i is concave and $\|\cdot\|$ monotonic $\|(u_j(y_j) - u_j(y_j - \cdot))_{j\in N}\|$ is convex in $\mathbb{R}^n_{++}$. From consumption theory it is known that the first-order condition is sufficient if some convex (concave) function is minimized (maximized) with respect to the "budget constraint" $\Sigma t_i = g$.

The denomination of the classical equal sacrifice concepts might suggest to call (7) $\|\cdot\|$-*sacrifice concept*. Such term would generalize the concepts of equal absolute and equal marginal sacrifice in a most natural way. It would uniformly address the first-order optimality condition for $\|\cdot\|$-least aggregate sacrifice. However, as mentioned in the introduction such a phraseology might be misleading.

Fix two arbitrary sacrifice profiles s^1, s^2 that differ only with respect to their k-th component. Let t^{s^1}, t^{s^2} be the allocations of tax shares that are related to s^1, s^2 by axiom A4). Is it normatively justified to require that the Lorenz curves of $(t_i^{s^j})_{i\in N\setminus\{k\}}$ $(j=1,2)$ should be identical? There may be some doubt. However, if we accept the position, i.e. A6), we are able to single out all welfare functions $\|\cdot\|_p$ with $p < \infty$.

A6) *Independence*: For all $k\in N$, $s\in \mathbb{R}^n_{++}$ let the Lorenz curve of $(t_i^s)_{i\in N\setminus\{k\}}$ be independent of s_k.

Two non-zero, non-negative vectors induce identical Lorenz curves iff they are scalar multiples of each other. From here it is easy to conclude that A6) is equivalent to:

$$\frac{\partial}{\partial s_k}(t_j^s \,/\, t_i^s) = o \quad \text{for all} \quad s\in \mathbb{R}^n_{++}$$

and all i, j, k that are different from one another.

<u>Corollary:</u> Assume A6) and $T^* = T^{\|\cdot\|}$ for some in $\mathbb{R}^n_{++}$ continuously differentiable $\|\cdot\|$. Then $\|\cdot\| = \|\cdot\|_p$ with $p \in [1,\infty)$.

<u>Proof:</u> Fix $s \in \mathbb{R}^n_{++}$. By A4) $(e^s, t^s) \in E^{lin} x T^{\|\cdot\|}(e^s)$ exists s.t. $s_i = \alpha_i^s \, t_i^s$. Hence $t^s >> o$. (7) yields $\partial_j \|s\| / \partial_i \|s\| = \alpha_i^s / \alpha_j^s = (s_i t_j^s)/(s_j t_i^s)$.
By A6) $\partial_k \, (\partial_j \|s\| / \partial_i \|s\|) = o$. The assertion now follows by, say, Hicks (1965, p. 335).

A final comment on the reversal of theorem 1 which is "the trivial direction": One can construct monotonic norms $\|\cdot\|$ s.t. $T^{\|\cdot\|}$ does not pass A1). However, all such norms meet A2)-A5). Furthermore $T^{\|\cdot\|_p}$ meets A1) for $p \in [1,\infty]$ and A6) for $p \in [1,\infty)$. All this is mentioned without proof which is straightforward but tedious.

1.5 <u>On progressivity</u>

For $\|\cdot\| = \|\cdot\|_p$, $p \in [1,\infty)$, (7) reduces to

$$(u_i(y_i) - u_i(y_i - t_i^e))^{p-1} \, u'(y_i - t_i^e) = \text{const} \quad (i \in N). \qquad (8_p)$$

If we add (AS), i.e. $p = \infty$,

$$u_i(y_i) - u_i(y_i - t_i^e) = \text{const} \quad (i \in N) \qquad (8_\infty)$$

we obtain a one-dimensional family of (generalized) sacrifice concepts stated for the case that $C = N$ and $t^e \in \mathbb{R}^n_{++}$.

They are all normatively justified by the preceeding axiomatization. Let us now turn to the question whether for *any* $p \in [1,\infty]$ $t^e = t^{\|\cdot\|_p}(e)$ implies moderately progressive taxation - irrespective of the utility functions under consideration. Two qualifications should be made before stating the (negative) answer.

First, we have to bear in mind that we are arguing in a utilitarian frame-work. We should not expect progressivity to pertain to individuals with different utility functions. So let us only make comparisons among individuals with identical utility functions. W.l.o.g. $u_i = U$ for all $i \in N$.

Secondly, the progressivity requirement needs some comment. We would like to know whether (8) ensures a (strictly) rising average tax rate:

$$y_i < y_j \quad \text{implying} \quad t_i^e/y_i < t_j^e/y_j \ . \qquad (9)$$

The answer given below relies on an application of the implicit function theorem to (8) - where constancy with respect to $i \in N$ is replaced by constancy with respect to y_j out of some neighbourhood of y_i. This means that (9) is only checked for "infinitesimally close" y_i and y_j. (Note that a mathematical problem, a "passage to the limit" $|N| \to \infty$, is behind the stage. Progressivity is analyzed in a setting with continuously many taxpayers. The sacrifice concepts (8) are however axiomatized for finite N, only.)

Fix $i \in N$, put $u_i = U$ and define tax function $T_U = T_U^{\|\cdot\|_p}$ with $T_U(y_i) = t_i^e$, locally: I.e., solve

$$(U(Y) - U(Y-T_U(Y))^{p-1} U'(Y-T_U(Y)) = \text{const} \qquad (10_p)$$

or

$$U(Y) - U(Y-T_U(Y)) = \text{const}, \qquad (10_\infty)$$

respectively, for $Y \in \mathcal{O}(y_i)$. (10) is the "continuous version" of (8). (Note that the usage of capital T deviates from the proceeding sections.) Instead of (9) $\frac{d}{dY} \frac{T(Y)}{Y} > 0$ is now being checked at $Y = y_i$.

As Samuelson (1947, p. 227) recognized the key to progressivity is the *elasticity of marginal utility*

$$\eta_U(Y) := - Y \frac{U''(Y)}{U'(Y)}$$

Note the following elementary properties:

$$\eta_U(Y) \geq 0 \quad \text{whenever} \quad Y \geq 0 \ .$$

$$\eta_U = \eta_{\alpha U + \beta} \quad \text{for all} \quad \alpha, \beta, \in \mathbb{R} ;$$

$$\eta_{\log} = 1, \ \eta_{p_{\sqrt{\cdot}}} = 1 - \frac{1}{p} \quad \text{for} \quad p \in (0, \infty);$$

$$\eta_{U^1}(Y) > \eta_{U^2}(Y) \quad \text{iff some } \bar{U} \text{ exists with } \bar{U}' > 0,$$

$$\bar{U}'' < 0 \quad \text{such that} \quad U^1(Y) = \bar{U}(U^2(Y)).$$

(Cf. Pratt, 1964, p. 128.)

The following theorem generalizes Samuelson's (1947, p. 227) result for $p = \infty$ to arbitrary $p \in [1,\infty]$. For a proof see appendix (AII).

<u>Theorem 2</u>: With the above notation, notably

$$T_U = T_U^{\|\cdot\|_p} \ , \quad p \in [1,\infty] \ , \quad \mathcal{O} := (y_i - T_U(y_i), y_i) \ ,$$

we obtain

a) $\eta_U(Y) > 1$ for all $Y \in \mathcal{O}$ implies progressivity of T_U at y_i;

b) $\eta_U(Y) < 1 - \frac{1}{p}$ for all $Y \in \mathcal{O}$ implies regressivity of T_U at y_i.

The important - though deceiving - message of theorem 2 is that for *all* $p \in (1,\infty]$ iso-elastic utility functions exist (e.g. $U(Y) = Y^{1/(p-1)}$ for $p < \infty$) such that regressive taxation follows under the corresponding generalized sacrifice concept (10_p). (AS) is an extremal case ($p = \infty$) which admits the logically strongest assertions when compared to other $p \in (1,\infty)$. All utility functions U that imply regressivity under any $p \in (1,\infty)$ imply the same under (AS). This uniformity of results might be striking.

As $\eta_U \geq 0$ for all concave U the implication b) is void for $p = 1$. This is well in line with the known characteristic of equal marginal sacrifice that it never leads to regressivity but to immoderate progressivity by equalizing incomes after tax.

2. A sacrifice concept implying moderate progressivity

I shall propose a non-classical sacrifice concept that does not fit into the axiomatic frame-work of section 1 which, however, upon application implies *moderate progressivity for all utility functions* U :

$$1 > T'_U(Y) > T_U(Y)/Y.$$

The defect is that I do not know of any truely convincing normative justification. Actually, to be precise, this new sacrifice concept admits an axiomatization on some *meta-level* as a "compromise" on Nash's and Kalai-Smorodinsky's solutions for n-person bargaining problems (Richter, 1981b). This axiomatization will convince the game theorist though probably not the theorist in public finance. The latter might be expecting axioms that have normative strength in themselves and do not recur to different solution concepts of general bargaining theory.

The further discussion will therefore be confined to some heuristic motivation. We start from equal proportional sacrifice (PS) and note that the induced choice correspondence T^* is invariant under linear transformations of utilities:

$$T^*(u,y,g,N) = T^*((\alpha_i u_i)_{i\in N},Y,g,N)$$

for all $\alpha_i \neq 0$. One might wonder that T^* is not invariant under affine transformations. The more transformation invariance is given the less informational requirements have to be met. Such reasoning suggests to redefine (PS):

$$\frac{u_i(y_i) - u_i(y_i - t_i^e)}{u_i(y_i) - u_i(o)} = c \quad (i\in N) \tag{11}$$

(where y_i is assumed to be positive. The solution concept corresponding to (11) is known in game theory as Kalai-Smorodinsky's monotone solution. The authors (1975) provided an axiomatization - which was later extended from 2 to n persons by Huttel and Richter (1980). This axiomatization is however difficult to justify when restricted to the present context of taxation.) The reformulation of (PS) would mean no real change if the assumption where correct that the classics tacitly used to visualize utility functions as passing through the origin $(u_i(o) = o)$. If T^* is defined by (11) we obtain invariance under affine transformations:

$$T^*(u,y,g,N) = T^*((\alpha_i u_i + \beta_i)_{i\in N},Y,g,N)$$

for all $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i \neq o$. If $t_i^e < y_i$ condition (11) is equivalent to

$$\frac{u_i(y_i) - u_i(y_i - t_i^e)}{u_i(y_i - t_i^e) - u_i(o)} = \frac{c}{1-c} \quad (i\in N) . \tag{12}$$

If we interpret (12) as first-order optimality condition we are led to consider the following minimizations:

$$\left\| \left(\frac{u_i(y_i) - u_i(y_i-t_i)}{u_i(y_i-t_i) - u_i(o)} \right)_{i \in N} \right\| \rightarrow \min \quad , \quad \sum_N t_i = g \ . \qquad (13)$$

((12) is first-order condition when $\|\cdot\| = \|\cdot\|_\infty$.)
Denote by $T^{\|/\|}(e)$ the set of all t^e solving (13). The obvious parallel structure of $T^{\|/\|}$, here, and of $T^{\|\cdot\|}$, above in section 1, suggests to interpret (13) as follows: The individual taxpayer's sacrifice is here defined by $u_i(y_i) - u_i(y_i-t_i)/(u_i(y_i-t_i) - u_i(o))$. It deviates from the absolute sacrifice notion underlying section 1. Here, the absolute sacrifice is relativized by the distance to zero income, measured again on a subjective basis. Zero income is a natural candidate for subsistence level. One could thus argue that the difference between (6) and (13) comes from the underlying sacrifice notion whereas the same set of aggregation rules seems to be admissible.

Even if we accept this hypothesis crucial questions remain open.

The sacrifice notion revealed by (13) is less "natural", less trivial, than the notion of absolute sacrifice. That easily explains why the classics never considered it. However, it has definite appeal, not least by the fact that it requires less informational assumptions than absolute sacrifice. Hence, an axiomatic justification seems to be called for - even more than in section 1. This is an open problem.

The axiomatization of $\{T^{\|/\|}\}$ - or of any subsets - is open altogether. Theorem 1 cannot be adapted in an abvious manner. Furthermore it is not clear, exactly which $T^{\|/\|}$ guarantees all the distributional characteristics we like them to have. Since Cohen-Stuart (1889) $T^{\|/\|_\infty}$ is known to generate regressive tax schedules for selected utility functions. Only $T^{\|/\|_1}$ is easily seen to meet all classical expectations. For this purpose solve

$$\sum_{i \in N} \frac{u_i(y_i) - u_i(y_i - t_i)}{u_i(y_i - t_i)} \to \min, \quad \sum_{i=1}^{N} t_i = g \qquad (14)$$

(where $u_i(\cdot)$ has been transformed to satisfy $u_i(o) = o$). Computing the first-order condition and passing to the "continuous version" yields

$$\frac{U(Y)\ U'(Y-T_U(Y))}{[U(Y-T_U(Y))]^2} = \text{const} \quad (Y \in \mathcal{O}(y_i)) \qquad (15)$$

where $U = u_i$, $U(o) = o$, $T_U(y_i) = t_i^e$, and t^c optimal for (14).

We might address (15) as *utilitarian proportional sacrifice*: *Utilitarian* summation, when applied to the *proportional sacrifice* notion $(u_i(y_i) - u_i(y_i - t_i))/(u_i(y_i - t_i) - u_i(o))$, implies an optimality condition the continuous version of which is (15).

<u>Theorem 3</u>: Let T_U satisfy (15). Then

$$1 > T_U'(Y) > T_U(Y)/Y \qquad (\text{for all } Y \in \mathcal{O}(y_i))$$

where the right-hand side is subject to the condition $0 < T_U(Y)/Y \leq 1/2$.

<u>Proof:</u> A III.

Utilitarian proportional sacrifice thus leads to moderate progressivity whenever the average tax rate is not greater than one half. This assertion holds for all utility functions under consideration. Inserting isoelastic utility functions $U_\varepsilon(Y) = Y^\varepsilon/_\varepsilon$ into (15) yields a tax function

$$T_\varepsilon(Y) = Y - bY^{\frac{\varepsilon}{1+\varepsilon}}$$

with *constant residual progression* $\frac{\varepsilon}{1+\varepsilon}$.

Such tax functions have much been discussed in public finance starting with Edgeworth (1919) and Vickrey (1972, p. 458). They are characterizable by various interesting properties. Cf., say, Jakobsson (1976), Genser (1980), and Richter and Hampe (1981). Note, however, that the residual

progression $\frac{\varepsilon}{1+\varepsilon}$ has to belong to the interval $(0,1/2]$ if T_ε is to be justified by (15)! This follows as we assumed $U(o) = o$ $(\varepsilon > o)$ and $U'' \leq o$ $(\varepsilon \leq 1)$.
A residual progression of one half or less is rarely - if ever - obtained by effective tax schedules. Cf. Richter and Hampe (1981).

Still, there are good reasons to further study utilitarian proportional sacrifice. The interested reader is referred to Richter (1981b).

Appendix

A I.

The "real-world" conflict of determining equitable tax shares $T^*(e) \subseteq T(e)$ is analyzed in the utility space. I.e. we determine T^* via the characterization of equitable utility profiles $F(S^e) \subseteq S^e$. S^e is interpreted as "bliss-point problem" in the sense of Richter (1981a). Zero sacrifice is the natural candidate to figure as bliss-point. We then focus on the correspondence $F : \{S^e \mid e \in E\} \rightarrow \mathbb{R}^n_+$ and show - by reference to theorem 2, Richter (1981a) - that some monotonic $\|\cdot\|$ exists s.t.

$$s^e \in F(s^e) \quad \text{iff} \quad \|s^e\| = \min_{s \in S^e} \|s\| \quad . \qquad (16)$$

From here, $T^* = T^{\|\cdot\|}$ follows immediately.

We have to check all assumptions met for theorem 2 in Richter (1981a). These assumptions are labelled by A1), A2),... but are here addressed by $\bar{A}1)$, $\bar{A}2)$,..., to avoid confusion from double naming. Note that, e.g. $\bar{A}1)$ is A5) of section 1, above. So the mapping $\bar{A}\cdot) \to A\cdot)$ is no "identity".

The trivial checkings are the following: S^e is convex from below for all $e\in E$ since u_i is concave. $\bar{A}1)$ is A5) as mentioned, already. $\bar{A}3)$, namely $aF(S^e) = F(aS^e)$ for all $a\in\mathbb{R}_{++}$, $e\in E$, follows from A2. $\bar{A}2)$ (Pareto efficiency) follows by definition of $F(S^e)$ and by making use of A1a).

$\bar{A}4)$ translates into the requirement:

$$\forall h\in\mathbb{R}^n_+ \setminus \{o\} \quad \forall c\in\mathbb{R}_+ \quad \exists\, e\in E:$$

$$S^e = S(h,c) := \{s \geq o \mid h\cdot s \geq c\} .$$

<u>Proof:</u> Fix h,c and define $e = (u,y,g,C)\in E^{lin}$ as follows: $g := c$, $u_i(Y) := \alpha_i(Y-y_i)$ where $\alpha_i > o$ is chosen to hold $\alpha_i = h_i^{-1}$ in case of $h_i > o$. y may be arbitrary. Put $C := \emptyset$ if $c = o$ and $C := \{i\in N \mid h_i > o\}$, else. Show $S^e = S(h,c)$. Assume $C \neq \emptyset$ or $c > o$, equivalently. (The case $C = \emptyset$ is straightforward.)

Fix $s\in S^e$. Hence $s_i \geq \alpha_i t_i$ for some $t \geq o$ with $\Sigma_C t_i = g$.

$h \cdot s = \Sigma_C h_i s_i = \Sigma_C s_i/\alpha_i \geq \Sigma_C t_i = g = c$. I.e. $s \in S(h,c)$. For the reversal, fix $s \in S(h,c)$ and put $t_i := c(h \cdot s)^{-1} h_i s_i$ if $i \in C$ and $t_i := o$, else. (Note that $h \cdot s \geq c > o$.) Then $t \geq o$ and $\Sigma_C t_i = c = g$, i.e. $t \in T(e)$. Show, finally, that $s_i \geq u_i(y_i) - u_i(y_i - t_i) = \alpha_i t_i$. This is trivial for $i \notin C$ and else follows from: $\alpha_i t_i = c(h \cdot s)^{-1} s_i \leq s_i$.

The last axiom we have to verify is

$\bar{A}5$) $F \cdot S(\cdot,\cdot) : (\mathbb{R}^n_+ \setminus \{o\}) \times \mathbb{R}_+ \to \mathbb{R}_+$ is

a) surjective,

b) closed-valued,

c) zero-continuous.

b) is entailed by A3) and by continuity of utility functions.

a) Fix $s \in \mathbb{R}^n_+$. We have to construct $h \in \mathbb{R}^n_+$, $\neq o$ and $c \in \mathbb{R}_+$ s.t. $s \in F(S(h,c))$. Choose $e := e^s$ according to A4). Put $c := g$; $h_i := \alpha_i^{-1}$ for $i \in C$ and $h_i := o$, else. Check $S^e = S(h,c)$. By A4,5) $s \in F(S^{e^s}) = F(S(h,c))$.

c) As indicated in Richter (1981a) zero-continuity follows from $\bar{s} \in F(S(h,c))$, $\bar{s}_i > \bar{s}_k$ implying $h_j \geq h_k$. (17)

To prove (17) fix $S(h,c)$ and define $e = e^{h,c} \in E^{lin}$ with $S(h,c) = S^e$ as before. Because of A5) we may substitute S^e for $S(h,c)$ in (17). If $C = \emptyset$ then $\bar{s} = o$ by A1a) and nothing is to prove. Else, assume $C \neq \emptyset$, $\bar{s}_j > \bar{s}_k$.

First case: $k \notin C$. By definition of C $h_k = o \leq h_j$.
Second case: $k \in C$. A1a) implies $j \in C$. The assertion now follows by definition of $e = e^{h,c} \in E^{lin}$ and by A1b):
$o < h_j^{-1} = u_j' \leq u_k' = h_k^{-1}$.

A II.

We only prove for $p \in (o,1)$. Put $T_U(y_1) = t_1^e = t_1$.
Implicit differentiation of (10_p) yields:

$$\frac{d}{dY} T_U(y_1) =$$

$$\frac{(p-1)\ U'(y_1-t_1)[U'(y_1-t_1)-U'(y_1)]-[U(y_1)-U(y_1-t_1)]U''(y_1-t_1)}{(p-1)(U'(y_1-t_1))^2-[U(y_1)-U(y_1-t_1)]U''(y_1-t_1)}$$

Hence $\quad \frac{d}{dY} T_U \gtreqless \frac{T_U}{Y} \quad \Longleftrightarrow$

$$(p-1)\ U'(Y-T_U)[(Y-T_U)U'(Y-T_U)-YU'(Y)]$$

$$\gtreqless \quad (Y-T_U)[U(Y)-U(Y-T_U)]U''(Y-T_U) \quad \Longleftrightarrow$$

$$(p-1)[YU'(Y)-(Y-T_U)U'(Y-T_U)]$$

$$\lesseqgtr \quad \eta_U(Y-T_U)[U(Y)-U(Y-T_U)] \qquad (18)$$

The latter relation has to be evaluated at $Y = y_1$.

a) $\eta_U(Y) > 1 \quad \Longleftrightarrow \quad -\frac{YU''(Y)}{U'(Y)} > 1$

$\Longleftrightarrow \quad 0 > U'(Y)+YU''(Y) = \frac{d}{dY} \ YU'(Y)$.

As $\eta_U(Y) > 1$ for all $Y \in \mathscr{O}$ the left-hand side of (18) is negative whereas the right-hand one is non-negative. Hence progressivity is obtained.

b) By assumption $\eta_U(y_1-t_1) \leq 1 - \frac{1}{p}$.

To prove regressivity it is enough by (18) to show

$$y_1U'(y_1)-(y_1-t_1)U'(y_1-t_1) > \frac{1}{p}\,[U(y_1)-U(y_1-t_1)]$$

or

$$\int_{y_1-t_1}^{t_1} \frac{d}{dY}\, YU'(Y)\, dY > \frac{1}{p} \int_{y_1-t_1}^{t_1} U'(Y)\, dY \,.$$

The latter inequality holds if the same inequality is true for the integrands:

$$U'(Y) + YU''(Y) > \frac{1}{p}\, U'(Y) \qquad \text{for all } Y \in \mathscr{O}$$

$$\Longleftrightarrow \qquad 1 - \eta_U(Y) > \frac{1}{p}$$

which is true by assumption.

A III.

Application of the implicit function theorem to (15) yields

$$\frac{dT_U}{dY} = \frac{U'(Y)U'(Y-T_U)[U(Y-T_U)]^2}{U(Y)U''(Y-T_U)[U(Y-T_U)]^2 - 2U(Y)[U'(Y-T_U)]^2U(Y-T_U)} + 1 \,.$$

The denominator is negative which implies a marginal tax rate of less than one. For progressivity we have to show

$$\frac{dT_U}{dY} > \frac{T_U}{Y} \quad \Longleftrightarrow$$

$$YU'(Y)U'(Y-T_U)[U(Y-T_U)]^2$$

$$< \quad (Y-T_U)[-U(Y)U''(Y-T_U)[U(Y-T_U)]^2+2U(Y)[U'(Y-T_U)]^2U(Y-T_U)]$$

$$\Longleftrightarrow \quad \frac{YU'(Y)}{U(Y)} + \frac{(Y-T_U)U''(Y-T_U)}{U'(Y-T_u)} < \frac{2(Y-T_U)U'(Y-T_U)}{U(Y-T_U)} \quad .$$

As $\eta_U \geq 0$, $U' > 0$, $U'' \leq 0$ it suffices to verify

$$Y \leq 2(Y-T_U) \quad .$$

This condition is to hold by assumption.

This paper is a totally rewritten version of an earlier draft which circulated at the World Congress of the Econometric Society, Aix-en-Provence, 1980, under the title "Taxation According to Ability to Pay". I am grateful to D. Bös, W. Rohde, J. Rosenmüller, and R. Selten for stimulating and helpful comments.

References

Atkinson, A.B., 1973, How progressive should income tax be? in: M. Parkin et al., eds., Essays in modern economics (Longman, London) 90-109.

Atkinson, A.B. and J.E. Stiglitz, 1980, Lectures on public economics (McGraw-Hill, Maidenhead).

Cohen-Stuart, A.J., 1889, Bijdrage tot de theorie der progressieve inkomstenbelasting, Engl.translation of extraxts in: R.A. Musgrave and A.T. Peacock, Classics in the theory of public finance (Macmillan, London, 1967).

Debreu, G., 1960, Topological methods in cardinal utility theory, in: K.J. Arrow et al., eds., Mathematical methods in the social sciences (Stanford University Press, Stanford) 16-26.

Edgeworth, F.Y., 1919, Graduation of taxes, Economic Journal, repr. in: Papers relating to political economy, vol. II, 1925 (Burt Franklin, New York) repr. in 1970.

Genser, B., 1980, Lorenzgerechte Besteuerung (Verlag der Österreichischen Akademie der Wissenschaften, Wien).

Haller, H., 1972/73, Zur Diskussion über das Leistungsfähigkeitsprinzip, Finanzarchiv N.F. 31, 461-494.

Hicks, J., 1965, Capital and growth (At The Clarendon Press, Oxford).

Huttel, G. and W.F. Richter, 1980, A note on "An impossibility result concerning n-person bargaining games", Disc. Paper, IMW, University of Bielefeld.

Jakobsson, U., 1976, On the measurement of the degree of progression, Journal of Public Economics 5, 161-168.

Kalai, E. and M. Smorodinsky, 1975, Other solutions to Nash's bargaining problem, Econometrica 43, 513-518.

Mill, J.St., 1921, Principles of political economy, ed. W.J. Ashley (Longmans Green & Co., London).

Musgrave, R.A., 1959, The theory of public finance (McGraw-Hill, Tokyo) chap. 5.

Musgrave, R.A. and P.B. Musgrave, 1973, Public finance in theory and practice (McGraw-Hill, New York).

Nagumo, M., 1930, Über eine Klasse der Mittelwerte, Japanese Journal of Math. 7, 71-79.

Pfähler, W., 1978, Normative Theorie der fiskalischen Besteuerung (Peter Lang Verlag, Frankfurt/M.).

Pratt, J.W., 1964, Risk aversion in the small and in the large, Econometrica 32, 122-136.

Richter, W.F., 1981a, Social choice for bliss-point problems, to appear in: Math. Social Sciences.

Richter, W.F., 1981b, A normative justification of progressive taxation: How to compromise on Nash and Kalai-Smorodinsky, to appear in: O. Moeschlin and D. Pallaschke, eds., Game theory and math. economics (North-Holland, Amsterdam).

Richter, W.F. and J.F. Hampe, 1981, On income tax functions, Disc. Paper, IMW, University of Bielefeld, F.R.G.

Roberts, K.W.S., 1980, Interpersonal comparability and social choice theory, Review of Economic Studies 47, 421-439.

Sadka, E., 1976, On progressive income taxation, American Economic Review 66, 931-935.

Samuelson, P.A., 1947, Foundations of economic analysis, (Harvard University Press, Cambridge).

Sen, A.K., 1974, Informational bases of alternative welfare approaches, Journal of Public Economics 3, 374-403.

Sen, A.K., 1977, On weights and measures: informational constraints in social welfare analysis, Econometrica 45, 1539-1573.

Vickrey, W., 1972, Agenda for progressive taxation (Augustus M. Kelley, Clifton).

" WIRTSCHAFTSTHEORETISCHE ENTSCHEIDUNGSFORSCHUNG"

A series of books published by the Institute of Mathematical Economics, University of Bielefeld.

Wolfgang Rohde
Ein spieltheoretisches Modell eines Terminmarktes (A Game Theoretical Model of a Futures Market)
The model takes the form of a multistage game with imperfect information and strategic price formation by a specialist. The analysis throws light on theoretically difficult empirical phenomena.

Vol. 1 176 pages price: DM 24,80

Klaus Binder
Oligopolistische Preisbildung und Markteintritte (Oligopolistic Pricing and Market Entry)
The book investigates special subgame pertect equilibrium points of a three-stage game model of oligopoly with decisions on entry, on expenditures for market potential and on prices.

Vol. 2 132 pages price: DM 22,80

Karin Wagner
Ein Modell der Preisbildung in der Zementindustrie (A Model of Pricing in the Cement Industry)
A location theory model is applied in order to explain observed prices and quantities in the cement industry of the Federal Republic of Germany.

Vol. 3 170 pages price: DM 24,80

Rolf Stoecker
Experimentelle Untersuchung des Entscheidungsverhaltens im Bertrand-Oligopol (Experimental Investigation of Decision-Behavior in Bertrand-Oligopoly Games)
The book contains laboratory experiments on repeated supergames with two, three and five bargainers. Special emphasis is put on the end-effect behavior of experimental subjects and the influence of altruism on cooperation.

Vol. 4 197 pages price: DM 28,60

Angela Klopstech
Eingeschränkt rationale Marktprozesse (Market processes with Bounded Rationality)
The book investigates two stochastic market models with bounded rationality, one model describes an evolutionary competitive market and the other an adaptive oligopoly market with Markovian interaction.

Vol. 5 104 pages price: DM 29,80

Orders should be sent to:

Pfeffersche Buchhandlung, Alter Markt 7, 4800 Bielefeld 1, West Germany.