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Coalition probabilities in a non-cooperative model of three-person quota game bargaining

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Experiments on three-person characteristic function games show that different coalitions are formed with different frequencies. Typically in asymmetric three person quota games without the grand coalition and with zero payoffs for single players the coalition of the two stronger players is more frequent than the other two-person coalitions [5] [6]. Recently Levinsohn and Rapoport have found a very interesting empirical regularity. The relative frequency of the coalition of the two stronger players seems to be linearly related to a structural index which measures the inequality between the quotas [6].

In view of these empirical findings one may ask the question whether it is possible to give a game theoretical explanation for the phenomen of different frequencies for different coalitions. Generally, human decision behavior cannot be expected to conform to the logic of game-theoretical reasoning [12]. Nevertheless, it is necessary to explore the possibilities of rational explanation. Game theoretical analysis can provide important insights even if the results are of limited descriptive validity.

The theory of the bargaining set developed by R. Aumann and M. Maschler suggests that in a three person quota game without the grand coalition a two person coalition will form where the players in the coalition receive their quotas as payoffs. No predictions on coalition frequencies are made [2]. Several other theories which can be applied to the situation come to the same conclusion. This is true for the von Neumann-Morgenstern solution (applied to payoff configurations instead of imputations) and for the John Cross-solution and its variants [1],[3], [15]. Unlike these theories, equal share analysis excludes the coalition of the two weaker players and sometimes does not permit any other coalition than that of the two stronger players [10]. At least under some experimental conditions, those coalitions which are excluded by equal share analysis do occur with non-negligible relative frequencies [5],[7]. Obviously, a more adequate theory should lead to positive probabilities for these coalitions, too.

It is the purpose of this paper to present a non-cooperative bargaining model which permits the derivation of coalition probabilities. Predicted frequencies are not blatantly dissimilar to observed frequencies even if there are significant deviations.

1. Three-person quota games

Let v be the characteristic function of a three person game without the grand coalition and with zero payoffs for single players:

(1) $v(1) = v(2) = v(3) = 0$

(2) $v(12) = a \quad v(13) = b \quad v(23) = c$

We assume that a, b and c are positive. Without loss of generality the numbers of the players can be chosen in such a way that we have

(3) $a \geq b \geq c > 0$

Moreover we assume

(4) $b + c > a$

in order to secure that the quotas q_1, q_2, q_3 are positive. These quotas are defined as follows:

(5) $q_1 = \frac{a+b-c}{2}$

(6) $q_2 = \frac{a+c-b}{2}$

(7) $q_3 = \frac{b+c-a}{2}$

The grand coalition 123 cannot be formed; the function v is not defined for this coalition. For the sake of simplicity we exclude the limiting cases c = 0 and b+c = a.

2. The bargaining model

The bargaining model has the form of an infinite extensive game. The game is played over infinitely many periods t = 1,2,3, ...in the sense that the rules permit plays of infinite length but plays of finite length are also possible and in fact are more advantageous for the players. Every period has the same structure; a play which does not end before period t leads to a subgame at the beginning of period t which is homeomorphic to the whole game. Therefore the game is completely described by the rules for an arbitrary period t and the payoff specification for infinite plays.

2.1 Rules for period t: For t = 1,2,3,..., period t, if it is reached, is played as follows. At the beginning of the period the three players have full information on all previous choices including those of the random player. The period is subdivided into four steps. The first three steps determine a tentative coalition C_t.

Step 1: A random choice selects one of the players 1,2,3. Each of the three players is chosen with the same probability 1/3. The result of the random choice is made known to all players.

Step 2: Let i be the player selected in step 1. Player i has to propose either coalition ij or ik, where j and k are the other two players. Player i's proposal is made known to all players.

Step 3: Let ij be the coalition proposed by player i. Player j can either accept ij or propose jk. The tentative coalition C_t is ij if he accepts ij and jk if he proposes jk. The tentative coalition C_t is made known to all players.

Step 4: Let players g and h be the members of C_t and let X_{gh} be the set of all vectors $x = (x_1, x_2, x_3)$ with

(9) $x_g + x_h = v(gh)$, $x_g \geq 0$, $x_h \geq 0$

and

(10) $x_m = 0$

where m is the player who is not in C_t. Players g and h simultaneously and independently select proposals $x^g \varepsilon X_{gh}$ and $x^h \varepsilon X_{gh}$, respectively. If both proposals agree then the game ends and the players 1,2,3 receive the payoffs x_1, x_2, x_3 specified by the common proposal $x = x^g = x^h$. In the conflict case $x^g \neq x^h$ the game proceeds to period t+1.

2.2 Infinite plays: The payoffs attached to infinite plays are zero for each player.

2.3 Interpretation: We may call steps 1,2 and 3 the coalition formation phase and step 4 the payoff bargaining phase. The coalition formation phase may be thought of as a game of perfect information whose outcome is a tentative coalition. The bargaining phase is a very simple model of payoff bargaining within a coalition.

The perfect information character of the coalition formation phase facilitates the analysis. Overall symmetry is achieved by the random choice at step 1. The remaining two steps 2 and 3 of the coalition formation phase are just sufficient in order to permit the formation of any of the three two-person coalitions, regardless of the outcome of the random choice.

2.4 Behavior strategies: A behavior strategy s_i of player i is a function which assigns a probability distribution s_{iu} over the choices at u to every information set of player i; in this paper a behavior strategy will always be a finite behavior strategy in the sense that s_{iu} assigns positive probabilities to a finite number of choices only. Since more general behavior strategies are not considered the word "finite" will be omitted. With respect to the bargaining model, the only restriction imposed by the finiteness condition concerns step 4.

Let $s = (s_1,s_2,s_3)$ be a combination of behavior strategies for the bargaining model, i.e. a triple of behavior strategies, one for each personal player. Since the bargaining model is an infinite game, it is necessary to explain what is meant by player i's expected payoff $H_i(s)$ at s. Fortunately, in the case of the bargaining model no difficulties arise. Let $H_{it}(s)$ player i's expected payoff up to period t (payoffs arising later are neglected). Since s permits only a finite number of possibilities for the course of the game up to period t, the expected payoff $H_{it}(s)$ is well defined. Moreover, the sequence $H_{i1}(s)$, $H_{i2}(s)$,... is non-decreasing and bounded from the above by a, since payoffs can be obtained only at the end of the game by agreement at step 4. Therefore, the sequence converges to a limit. Player i's expected payoff $H_i(s)$ is defined as this limit. Since infinite plays yield zero payoffs, it is natural to define expected payoffs for combinations of behavior strategies in this way.

3. Nature of the proposed solution

A reasonable game theoretical solution of a non-cooperative game cannot be anything else than an equilibrium point or a set of equilibrium points. Theories which prescribe non-equilibrium behavior are self-destroying prophecies since at least one player has an incentive to deviate if he believes that the other players obey the prescriptions.

Unfortunately, one cannot expect that human decision making is perfectly rational. This is especially true for game situations. Learning processes rather than abstract thinking determine the behavior of experienced players. Mathematical models of game learning like the Brown-Robinson process converge to equilibrium points if they converge at all, but this is by no means always the case [8], [9], [14]. Nevertheless, equilibrium point theory may still be a useful tool for the prediction of experienced behavior where learning processes do not converge. Numerical examples convey the impression that in such cases one can expect a tendency towards cyclical fluctuations around an equilibrium point in mixed strategies. These fluctuations may produce average results similar to those at the mixed equilibrium point.

Admittedly, the hope for empirical relevance of mixed strategy equilibrium points as approximations of experienced average behavior is based on weak evidence. Nevertheless, it seems to be worthwhile to pursue this approach.

The bargaining model introduced in section 2 has many equilibrium points. Reasonable criteria will be used to select a unique one. In view of the descriptive purpose of the theory to be presented it is not advisable to apply the method of selection developed by John C. Harsanyi and the author [4]. Instead of this the analysis will be based on requirements which are specifically addressed to the problem at hand without any ambition to provide a solution concept for a large class of games.

In view of the limited scope of the theory to be presented it seems to be adequate to avoid lengthy formal definitions of technical terms whose usual meaning is sufficiently clear.

The unique equilibrium point to be selected will be an equilibrium point in behavior strategies. We shall refer to this equilibrium point as "the solution" of the bargaining game. In the following the requirements characterizing the solution will be introduced together with the necessary terminological explanations.

<u>3.1 Subgame consistency</u>: Subgame consistency of an equilibrium point in behavior strategies requires that the strategies of the players should be invariant with respect to homeomorphisms between subgames. A more formal definition of subgame consistency will not be given here. Instead of this it will be explained what the requirement means for the bargaining game introduced in section 2.

All subgames beginning at step 2 after the random selection of a specific player i are homeomorphic. Subgame consistency requires that at step 2 player i always chooses ij and ik with the same probabilities, independently of t and the prior history of the game. Similarily the probabilities of the decisions of a player at step 3 must be always the same. All subgames beginning with the same tentative coalition C_t at step 4 of some period t are homeomorphic. The members of C_t must always behave in the same way.

Note that subgame consistency is not a part of the rules of the bargaining game. The players are permitted to play any behavior strategy. Only the solution will be required to have the property of subgame consistency.

Subgame consistency should not be confused with subgame perfectness which requires that an equilibrium point should be induced on every subgame [11],[12]. A subgame perfect equilibrium point in behavior strategies may not be subgame consistent and vice versa.

3.2 Positive coalition offer probabilities: We say that a subgame consistent equilibrium point in behavior strategies for the bargaining game has positive coalition offer probabilities if it always prescribes positive choice probabilities to both choices of a player who decides at steps 2 or 3.

As has been pointed out before, empirical observations suggest that all two person coalitions are formed with positive probability. Apparently learning does not extinguish the tendency to form any of these coalitions. Therefore, the requirement of positive coalition offer probabilities seems to be a reasonable condition to be imposed on a solution which tries to represent the behavior of experienced players.

3.3 Independent positive coalition probabilities: We call a tentative coalition where an agreement is reached a final coalition. Let α, β and γ be the probabilities with which 12, 13 and 23 are reached as final coalition by a given equilibrium point of the bargaining game. We say that an equilibrium point of the bargaining game has independent positive coalition probabilities if the conditional probabilities that 12, 13 and 23 are reached as final coalitions after the random choice at step 1 of period 1 are positive and do not depend on the player who has been selected. This means that the conditional probabilities for 12, 13 and 23 given step 1 of period 1 are always the same positive probabilities α β and γ.

3.4 Nash bargaining property: Consider a subgame consistent equilibrium point in behavior strategies of the bargaining game and let w_1, w_2 and w_3 be the expected equilibrium payoff of players 1,2 and 3, respectively. Since a subgame at the beginning of a period t is always homeomorphic to the whole game, w_1, w_2 and w_3 are also the expected payoffs for these subgames.

Suppose that gh is the tentative coalition C_t at some period t. The situation of the players g and h may be described as a bargaining problem concerning the division of v(gh) among g and h with w_g and w_h as conflict payoffs. A profitable agreement can be reached, if we have

(11) $\quad w_g + w_h < v(gh)$

The application of Nash's bargaining theory to this situation specifies an agreement where the surplus above $w_g + w_h$ is split evenly among both players. This agreement corresponds to the following proposal $x = (x_1,x_2,x_3) \varepsilon X_{gh}$:

(12) $\quad x_g = w_g + \frac{1}{2}(v(gh) - w_g - w_h)$

(13) $\quad x_h = w_h + \frac{1}{2}(v(gh) - w_g - w_h)$

(14) $\quad x_m = 0$

where m is the player not in C_t.

We say that a subgame consistent equilibrium point in behavior strategies of the bargaining game has the Nash bargaining property if it satisfies the following condition for any two players g and h for which (11) holds: Whenever g and h find themselves together in a coalition C_t they both choose the proposal $x \varepsilon X_{gh}$ described by (12), (13) and (14). This proposal is called the Nash proposal.

The Nash bargaining property may be interpreted as a condition which secures that in a step 4 situation both players suffer the same loss should they fail to reach the equilibrium solution. In this sense the equilibrium agreements are required to be balanced with respect to the risk of conflict.

4. The solution

As we shall see, the four requirements of subgame consistency , positive coalition offer probabilities, independent positive coalition probabilities and the Nash bargaining property determine a unique equilibrium point in behavior strategies for the bargaining game. As long as uniqueness has not yet been proved we shall call any equilibrium point with these properties "a solution".

Lemma 1: A solution always prescribes the same proposal $x \varepsilon X_{gh}$ to both members g and h of a tentative coalition C_t.

Proof: Subgame consistency requires that the proposals do not depend on prior history. The proposals for the tentative coalition gh are always the same. No agreement at gh can ever be reached unless the proposals of g and h agree. In view of the requirement of independent positive coalition probabilities both proposals **must agree.**

Remark: Only the requirements of subgame consistency and independent positive coalition probabilities have been used in this proof.

Lemma 2: A solution has the property that every equilibrium play ends immediately after step 4 of period 1 with the formation of a two-person coalition.

Proof: Lemma 2 is an immediate consequence of lemma 1.

Lemma 3: Let $x = (x_1, x_2, x_3)$ be the common proposal of both members g and h of a tentative coalition C_t prescribed by a solution. Then the payoffs specified by $x = (x_1, x_2, x_3)$ are as follows:

(15) $x_g = q_g$

(16) $x_h = q_h$

(17) $x_m = 0$

where m is the player not in C_t and q_g and q_h are the quotas of g and h as defined by (5), (6) and (7).

Proof: Consider a player who has to decide at step 3 of period 1. He has to choose between the two 2-person coalitions where he is a member. The requirement of positive coalition offer probabilities demands positive probabilities for both alternatives. According to lemma 2 whichever coalition he selects will be the final one. Since he selects both of them with positive probabilities and moreover the situation where he has to make his decision occurs with positive probability, he must be indifferent between both choices. This has the consequence that he must receive the agreement payoff in both coalitions where he is a member. The argument can be applied to every player. It is clear that only the quota agreements described by (15),(16), (17) satisfy the condition that every player receives the same agreement payoff in both coalitions where he is a member.

Lemma 4: The probabilities α, β and γ with which the final coalitions 12, 13 and 23, respectively are reached by a solution are as follows:

$$(18) \qquad \alpha = \frac{q_1q_2}{q_1q_2 + q_1q_3 + q_2q_3}$$

$$(19) \qquad \beta = \frac{q_1q_3}{q_1q_2 + q_1q_3 + q_2q_3}$$

$$(20) \qquad \gamma = \frac{q_2q_3}{q_1q_2 + q_1q_3 + q_2q_3}$$

Here q_1, q_2 and q_3 are the quotas defined by (5),(6) and (7).

Proof: It is a consequence of lemma 2 and lemma 3 that the equilibrium payoffs are as follows:

$$(21) \qquad w_1 = (\alpha+\beta)q_1 = (1-\gamma)q_1$$

(22) $w_2 = (\alpha+\gamma)q_2 = (1-\beta)q_2$

(23) $w_3 = (\beta+\gamma)q_3 = (1-\alpha)q_3$

The Nash bargaining property requires that the following conditions are satisfied

(24) $q_1 - w_1 = q_2 - w_2$

(25) $q_1 - w_1 = q_3 - w_3$

(26) $q_2 - w_2 = q_3 - w_3$

In view of (21),(22) and (23) this is equivalent to

(27) $\gamma q_1 = \beta q_2 = \alpha q_3$

This together with

(28) $\alpha + \beta + \gamma = 1$

yields (18), (19) and (20).

Lemma 5: Every solution prescribes the same probabilities to the decisions to be made at steps 2 and 3.

Proof: Assume that at step 1 of period 1 player 1 has been selected by the random choice. The situation is graphically represented by figure 1. At step 3 players 2 and 3 must select coalition 23 with the same probability since otherwise player 1 cannot be indifferent between 12 and 13 as he must be in view of positive probabilities for both of his choices. He receives the same agreement payoff q_1 both in 12 and 13 and would prefer the coalition which is chosen with higher probability by the other player.

In the same way unique choice probabilities for the decisions at steps 2 and 3 can be derived for the si-

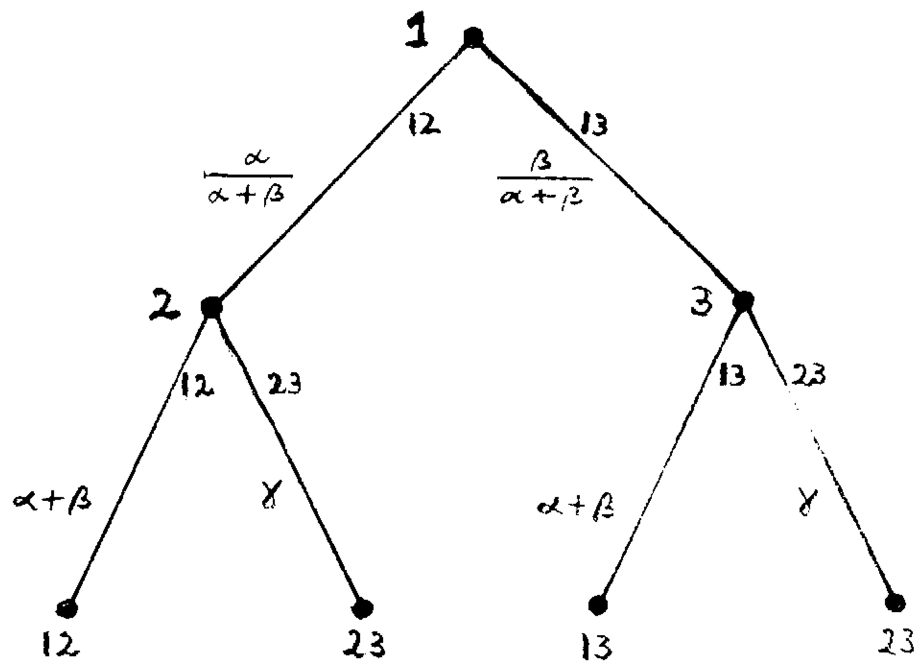

Figure 1: Choice probabilities for steps 2 and 3 after player 1 has been selected at step 1.

tuations which arise if the random choice selects player 2 or 3.

Theorem: The bargaining game has one and only one solution. The probabilities α, β and γ with which 12, 13 and 23, respectively are reached by the solution are given by (18), (19), (20). In the coalition agreements prescribed by the solution the members of the coalition receive their quotas as payoffs.

Proof: It is clear that the behavior strategy combination determined by lemmata 1 to 4 is the only one which can be a solution of the bargaining game. Moreover, it can be seen easily that this strategy combination is in fact an equilibrium point of the bargaining game. A deviation of a player can neither increase his probability of being in the final coalition nor his agreement payoff. Obviously the equilibrium point is subgame consistent. As the proof of lemma 5 shows, the requirements of positive offer probabilities and independent positive coalition probabilities are satisfied. The Nash bargaining property follows from the fact that (24), (25) and (26) hold.

Remarks: As we have pointed out above, subgame consistency does not imply subgame perfectness. The solution is not only subgame consistent but also subgame perfect. This follows immediately by the fact that one subgame in every class of homeomorphic subgames is reached with positive probability. The refined notion of perfectness defined in [13] cannot be applied here, since the game is infinite.

5. Comparison with experimental results

The rules of the bargaining game specified in section 2 could be used as an experimental procedure. Up to now no such experiments have been performed. Nevertheless, it is interesting to compare the theory with experimental results obtained by other formalized bargaining procedures which put the players into the situation of a well defined infinite non-cooperative game in extensive form without any communication possibilities outside the game. Such experiments have been performed by Kahan and Rapoport [5].

The bargaining rules used by Kahan and Rapoport produce a very complicated extensive game. The analysis of this game seems to be quite difficult. It is not clear whether their rules permit an equilibrium point similar to the solution of the bargaining game, but it is not unreasonable to conjecture that this is the case.

Table 1 shows the games used in the experiments by Kahan and Rapoport and table 2 contains the coalition probabilities predicted by the theory for these games. Table 3 shows

Game	a	b	c	q_1	q_2	q_3
I	95	90	65	60	35	30
II	115	90	85	60	55	30
III	95	88	81	51	44	37
IV	106	86	66	63	43	23
V	118	84	50	76	42	8

Table 1: The quota games of Kahan and Rapoport

Game	α	β	γ
I	.424	.364	.212
II	.489	.267	.244
III	.390	.328	.283
IV	.526	.282	.192

Table 2: Coalition probabilities for the game of Kahan and Rapoport

Game	12	13	23	χ^2
I	27 19.94	15 17.09	5 9.97	5.22
II	25 23.47	8 12.80	15 11.73	2.81
III	17 18.70	16 15.73	15 13.57	.31
IV	35 25.26	7 13.51	6 9.22	8.02
V	42 37.04	6 7.06	0 3.90	2.91*

Table 3: Observed and predicted coalition frequencies for the games of Kahan and Rapoport. - Observed values are entered above and predicted values are entered below.

* χ^2-value computed with categories 13 and 23 combined.

that in many cases the predicted values are close to observed values.

For only one of the five games, namely for game IV, the null hypothesis of random deviations from the theory can be rejected at the 5%-level. Nevertheless, there are significant deviations from the theory. This is shown by a Chi-square test for the table as a whole which is significant at the 5%-level (χ^2 = 19.27, nine degrees of freedom).

The theory seems to have a bias towards the underestimation of the probability of the coalition of the two stronger players. One may conjecture that this is the reason for the deviations which cause the rejection of the null hypothesis for the table as a whole. Nevertheless, the agreement of prediction and observation is much better than random. This is shown by a rank correlation between observed and predicted frequencies in table 1. The Spearman rank correlation coefficient is .954 which is significant at the 1%-level.

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