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Equlibrium Point Selection in a Bar-
gaining Situation with Opportunity

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Equilibrium Point Selection in a Bargaining Situation with
Opportunity Costs.

By Reinhard Selten and Ulrike Leopold

It is the purpose of this paper to analyze a simple bargaining
situation with the help of a noncooperative solution theory de-
veloped by John C. Harsanyi and Reinhard Selten (Harsanyi 1976,
Harsanyi-Selten 1980). This theory selects one equilibrium point
in every noncooperative game.

The bargaining problem considered here is a very simple one. Two
bargainers, player 1 and player 2, can propose one of two agree-
ments U or V, wehre U is more favorable for player 1 and V is more
favorable for player 2. An agreement is reached if it is proposed
by both players. In addition to U and V, each of both players has
a third alternative, namely not to bargain at all. Thereby he
avoids opportunity costs of bargaining which arise whether an
agreement is reached or not. One may think of an illegal deal
where bargaining involves a risk of being punished.

The model has the form of a 3x3-bimatrix game. The game has three
pure strategy equilibrium points. Two of these equilibrium points
correspond to the agreements U and V and the third one is the stra-
tegy pair where both players do not bargain at all. It is shown
that each of these three equilibrium points can be the solution of
the game. With the exception of degenerate cases a complete over-
view over all possible cases is obtained. The interpretation of
the result shows that the behavior of the solution as a function of
the parameters is not unreasonable.

A similar problem will be investigated in a chapter of the book
manuscript on the solution concept (Harsany - Selten 1980). There,
arbitrarily many different agreements on the division on one money
unit are considerd but only one of the players has the option not
to bargain at all. One may hope that finally it will be possible
to extend the analysis of this paper in a similar way to a range of
possible agreements.

A short version of this paper will be published in German (Selten -
Leopold 1980).

## I. 2-person-games in normal form

A 2-person-game $G = (\Phi_1, \Phi_2, H)$ in normal form consists of two finite non-empty sets of pure strategies $\Phi_1$ and $\Phi_2$ of the players 1 and 2 and of a payoff function H which assigns a payoff vector $H(\varphi) = (H_1(\varphi), H_2(\varphi))$ with real components to every strategy combination

$$\varphi = (\varphi_1, \varphi_2) \in \Phi = \Phi_1 \times \Phi_2 .$$

A mixed strategy $q_i$ of player i is a probability distribution over player i's pure strategy set $\Phi_i$. The probability assigned to $\varphi_i$ is denoted by $q_i(\varphi_i)$. Moreover $Q = Q_1 \times Q_2$ is the set of combinations of mixed strategies $q = (q_1, q_2)$. The definition of the payoff function H is extended from $\Phi$ to Q in the usual way.

The combination $q \in Q$ which contains $q_i$ and $q_j$ with $i \neq j$ is sometimes denoted by $q_i q_j$. No distinction will be made between a pure strategy $\varphi_i$ and that mixed strategy which assigns probability 1 to this pure strategy.

A mixed strategy $r_i \in Q_i$ is called best reply to $q_j \in Q_j$ with $j \neq i$, if we have:

$$H_i(r_i \, q_j) = \max_{q_i \in Q_i} H_i(q_i \, q_j)$$

We say that a combination $r = (r_1, r_2)$ is a best reply to $q = (q_1, q_2)$ if $r_1$ is a best reply to $q_2$ and $r_2$ is a best reply to $q_1$.

An equilibrium point is a strategy combination $r \in Q$ which is a best reply to itself. An equilibrium point $r = (r_1, r_2)$ is called strong, if r is the only best reply to r. A strong equilibrium point is always an equilibrium point in pure strategies.

## II. The bargaining problem

The bargaining problem considered here is a 2-person-game in normal form $G = (\Phi_1, \Phi_2, H)$ with $\Phi_1 = \{U_1, V_1, W_1\}$ and $\Phi_2 = \{U_2, V_2, W_2\}$ whose payoff function is given by the following matrix :

|  | $U_2$ | $V_2$ | $W_2$ |
|---|---|---|---|
| $U_1$ | u      1 | o      o | o      b |
| $V_1$ | o      o | 1      v | o      b |
| $W_1$ | a      o | a      o | a      b |

$$u > v > 1$$
$$o < a < 1$$
$$o < b < 1$$

Figure 1: The bargaining problem. The quares contain player 1's payoff in the upper left corner and player 2's payoff in the lower right corner. It is assumed that the parameters a, b, u and v satisfy the above inequalities.

The game can be interpreted as a bargaining problem with opportunity costs. There are two possible contracts U and V. The strategies $U_i$ and $V_i$ consist in naming the respective contract by player i.

An agreement results if and only if both players name the same contract; otherwise bargaining results in conflict. Each of the two players i has a third possibility $W_i$, which stands for not bargaining at all.

It is assumed that opportunity costs of size a and b respectively can be avoided by choosing $W_i$.

In order to have something definite in mind, the opportunity costs can be thought of as the risk of cooperation in an illegal deal.

One has to imagine that the risk of punishment is totally independent of whether a contract is concluded or not. The payoffs for

$$U = (U_1, U_2) \qquad \text{and } V = (V_1, V_2)$$

are net-payments, whose computation already takes into account the punishment-risk. The units of utility are normed in such a way, that for both players the contract unfavorable to him yields payoff 1 for him. This can be done without loss of generality.

If the strategies $W_1$ and $W_2$ were not available the bargaining problem would have the form of a 2x2-game with two strong equilibrium points,

in which the payoffs outside the equilibrium-points are zero. The theory of John C. Harsanyi and Reinhard Selten selects in such games the strong equilibrium point with the greater payoff product (Harsanyi-Selten, 1980). It is not necessary here to explain this point in more detail. In view of $u > v$, in our case $U = (U_1, U_2)$ is the solution of the 2x2-game, which results by crossing out the strategies $W_1$ and $W_2$. Therefore one can expect that also in the 3x3-game $U = (U_1, U_2)$ often will be the solution. It is interesting to see where this is the case and where the opportunity costs yield other results. As we shall see later each of the three strong equilibrium points $U = (U_1, U_2)$ , $V = (V_1, V_2)$ and $W = (W_1, W_2)$ can be the solution of the game.

## III. The Harsanyi - Selten - theory

It is not necessary to give a complete description of the theory of Harsanyi and Selten; we shall only outline those parts of the theory which are needed in the analysis of our bargaining problem.

The theory achieves perfectness of the selected equilibrium point by looking at uniformly perturbed games of the original game and then going to the limit with the perturbance parameter $\varepsilon$ (Harsanyi-Selten, 1980). Instead of doing this we shall apply the procedures for equilibrium selection directly to the unperturbed game. Since we are not interested in border cases where the perturbances really matter, it can be expected that the results are not influenced by the ommission of perturbances. The analysis, however, is greatly simplified in this way.

The process of solving a game begins with a procedure of reduction and decomposition. In our case this procedure does not change the game. Therefore the procedure of reduction and decomposition will not be explained in detail. Games which cannot be further decomposed or reduced like our game here, are called irreducible.

Irreducible games may still have structural properties, which must be taken into account. In this respect certain substructures called formations are of importance.

Consider a game $G' = (\Psi_1, \Psi_2, H')$ which is derived from the game $G = (\Phi_1, \Phi_2, H)$ in the following way:
The pure strategy-sets $\Psi_1$ and $\Psi_2$ are subsets of the respective sets $\Phi_1$ and $\Phi_2$. The new payoff function $H'$ is the restriction of $H$ to

$\Psi = \Psi_1 \times \Psi_2$. Such a substructure $G' = (\Psi_1, \Psi_2, H')$ is a
formation of G if for i = 1,2 each pure best reply $\varphi_i$ in G to a
mixed strategy $q_j$ with $q_j(\varphi_j) = 0$ for $\varphi_j \notin \Psi_j$ always belongs to $\Psi_i$.

This property can be interpreted in the following way: best replies
to strategies within the formation are within the formation. A formation
is a closed substructure with respect to best replies.
The above definition of the formation can not be directly generalized
to n-person-games (n > 2); in these games one also considers best replies
to so-called joint mixtures, which are not essential for 2-person-games.
Since we are regarding only 2-person-games, we can ommit the general
definition.
A primitive formation is a formation which has no proper subformation.
Primitive formations are of special importance for the theory. Consider
a strong equilibrium-point $\pi = (\pi_1, \pi_2)$ of $G = (\Phi_1, \Phi_2, H)$ then
$G' = (\Psi_1, \Psi_2, H')$ with $\Psi_1 = \{\pi_1\}$, $\Psi_2 = \{\pi_2\}$ and $H'(\pi) = H(\pi)$ is
a primitive formation of G. Usually not all primitive formations do
arise in this way by strong equilibrium points; however in our bargaining
problem this is actually the case.
It is easy to see that the game has three primitive formations namely
those generated by $U = (U_1, U_2)$ , $V = (V_1, V_2)$ and $W = (W_1, W_2)$.
The first step in the solution of an irreducible game is the determination
of its primitive formations and their solutions. The solutions of the
primitive formations are considered to be natural candidates for a
solution of the game. The set of all solutions of primitive formations
is called the first candidate set $\Omega_1$.

It is not necessary here to define the solution of a primitive formation
in general. In our case each of the three primitive formations consists
of only one equilibrium point, which therefore must be the solution of
the primitive formation;
therefore we have
$$\Omega_1 = \{U , V , W\}$$
If $\Omega_1$ consists of only one element, then this only first candidate
is the solution. If $\Omega_1$ contains more than one element, a second
candidate set $\Omega_2$ must be constructed. For this purpose we need the
concept of payoff dominance.

We say that an equilibrium point r <u>payoff dominates</u> an equilibrium
point q, if we have

$$H_i(r) > H_i(q) \qquad \text{for } i = 1,2$$

The <u>second candidate set</u> $\Omega_2$ is the set of all elements of $\Omega_p$ which
are not payoff dominated by other candidates in $\Omega_1$ .

In our case we observe that W is payoff dominated by U and V, but between
U and V there is no payoff dominance. Therefore we have

$$\Omega_2 = \{ U , V \} \qquad .$$

In cases where $\Omega_2$ consists only of one element, this candidate is the
solution of the game. In our case there are two candidates in $\Omega_2$.
Therefore we shall try to select one of both candidates in $\Omega_2$ with the
help of the risk dominance concept which will be explained next.

<u>Risk dominance</u> is a relation between two equilibrium points U and V of
the same game with the interpretation that the risk dominant equilibrium
is in some sense the less riskyone, if only these two equilibrium points
are believed to be possible solutions of the game. The following three
cases can arise in the risk dominance comparison between two equilibrium
points U and V:

U $\succ$ V     U risk dominates V

V $\succ$ U     V risk dominates U

U | V     there is no risk dominance relationship between U and V

The definition of risk dominance is based on a hypothetical process of
expectation formation which is modelled with the help of the tracing
procedure (Harsanyi 1976). The <u>tracing procedure</u> can be thought of as a
mathematical description of a reasoning process which starts with an
arbitrary strategy combination $p = (p_1, p_2)$ of a game $G = (\Phi_1, \Phi_2, H)$
which is gradually transformed to an equilibrium point $r = T(G, p)$ of
the game G. (We restrict our explanations to the 2-person-case). The
original combination $p = (p_1, p_2)$ is called the prior combination. It
has the interpretation of a naive theory of behavior, which is the
starting point for considerations, which finally yield the equilibrium
point T(G, p). Later we shall look at the tracing procedure in more
detail. In the following we shall first introduce the special "bicentric"
prior combination on which the definition of risk dominance is based.
The naive theory of behavior which underlies the <u>bicentric prior</u>
<u>distribution</u>, proceeds from the assumption that player i expects that

the other player plays $U_j$ or $V_j$ . The subjective probability for player i
is z for $U_j$ and 1-z for $V_j$. The expectation of player i is described
by the mixed strategy $z U_j + (1-z)V_j$ of player j. It is assumed that
player i selects a mixed best reply which assigns equal positive
probabilities to all pure best replies, this <u>central</u> <u>best</u> <u>reply</u> is called
$r_i^z$ . Assume that z is a random variable uniformly distributed over the
interval $[0,1]$ . The <u>bicentric</u> <u>prior</u> strategy $p_i$ is the probability
distribution over the strategies $\varphi_i \in \Phi_i$ which arises in this way:

$$p_i(\varphi_i) = \int_0^1 r_i^z(\varphi_i)\, dz \qquad \text{for all } \varphi_i \in \Phi_i$$

$p = (p_1, p_2)$ is the <u>bicentric</u> <u>prior</u> <u>strategy</u> <u>combination.</u> U risk
dominates V in cases where the application of the tracing procedure
yields $T(G,p) = U$ and V risk dominates U for $V = T(G,p)$. It may also
happen that there is no risk dominance relationship between U and V.
We now describe the linear tracing procedure.

In some extreme cases an unequivocal result can only be received
with help of the more complicated logarithmic tracing procedure which
is not considered here any further.

We define a family of games $G^t = (\Phi, H^t)$ with the tracing parameter
$t \in [0,1]$ . The number 1-t can be interpreted as the degree of
confidence placed in the prior combination. The payoff function $H^t$
results from $G = (\Phi, H)$ in the following way :

$$H_i^t(\varphi_i \ \varphi_j) = tH_i(\varphi_i \ \varphi_j) + (1-t)\, H_i(\varphi_i \ p_j)$$

for $i = 1,2$ with $j \neq i$ .

For t=0 the strategy $\varphi_j$ chosen by the opponent does not be of any
importance for the payoff of player i. For t = 1 the influence of the
prior strategy combination disappears totally. $G^1$ is the original game
G.

Let us call $\mathcal{R}$ the set of all pairs (q, t) with $q \in Q$ and $t \in [0,1]$ .
Further, let E be the set of all $(r, t) \in \mathcal{R}$ with the property that r is an
equilibriumpoint of $G^t$. We call E the graph of the equilibrium points.
The best reply $r^0$ to p in G is almost always determined in an unambiguous
way; if this is the case, then $(r^0, 0)$ is the only point of the form
(q, 0) in E. It can be shown, that almost always there is exactly one
continuous path, which leads in E from $(r^0, 0)$ to a uniquely determined
$(r^1, 1)$. The equilibrium point $r^1$ is the <u>result</u> T(G, p) <u>of the tracing</u>

procedure applied to p.

The way in which the solution is determined with help of risk dominance is shown here for the case that $\Omega_2$ consists of two elements. If one of the two equilibrium points of $\Omega_2$ risk dominates the other, then this one is the solution.

If there is no risk dominance between the two candidates, the solution will also be determined with the help of the tracing procedure, however, together with another prior strategy combination. Consider $\Omega_2 = \{U, V\}$. The centroid of $\Omega_2$ is the strategy combination $c = (c_1, c_2)$ with $c_i = \frac{1}{2} U_i + \frac{1}{2} V_i$ for i = 1,2 ; this means that each of both players uses his equilibrium strategies in U and V with the same probability. For U | V , i.e. the case in which there is no risk dominance between U and V, T(G, c) is the solution of the game.

In the analysis of our bargaining problem, we shall neglect border cases which pose difficulties in the application of the tracing procedure. These border cases form a set of lower dimension in the parameter space. Wherever we shall describe a region where the tracing procedure applied to the bicentric prior or the centroid yields one of the three strong equilibrium points U, V, and W as the final result, we shall really mean, that this is true with the exception of border cases. It would be tedious to point this out in every single case; border cases may also arise along lines inside the regions, if the analysis requires case distinctions based on strong inequalities . We shall make no attempt to keep track of the excluded border cases in detail. One can expect that an extension of the analysis to the border cases would not add much to the interpretation of the overall result.

## IV . Computation of the bicentric prior strategy combination

The bicentric prior strategies can easily be determined with the help of the graphical representations in figure 2.

The drawings show the payoff for each player i reached by $U_i$, $V_i$ and $W_i$ if his opponent plays $zU_j + (1-z)V_j$.
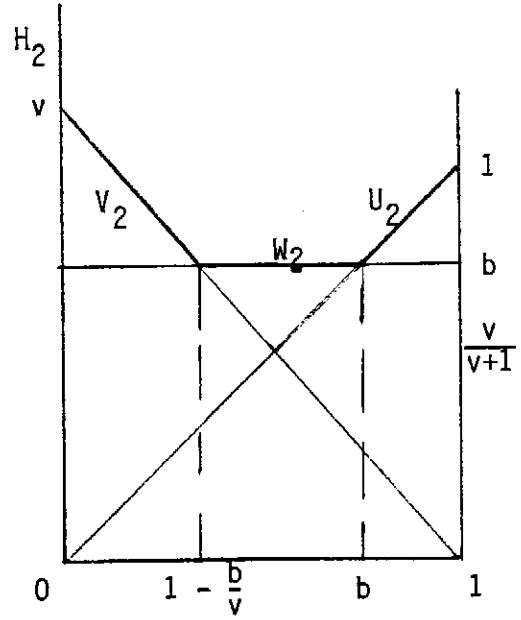
For the computation of the prior strategy of player i it is important to distinguish several cases.
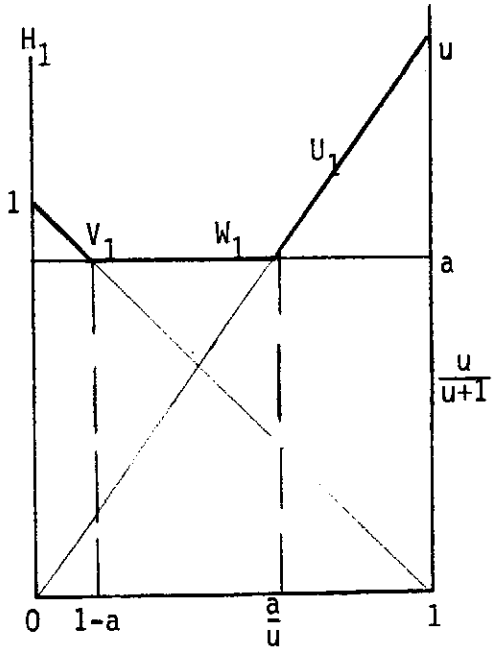
$$p_1(U_1) = 1 - \frac{a}{u}$$
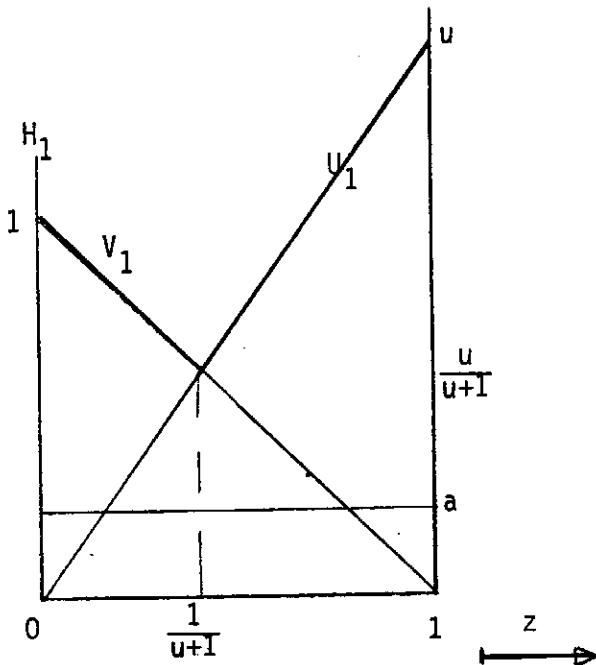
$$p_1(V_1) = 1 - a$$

$$p_1(W_1) = \frac{a}{u} - (1-a)$$

$$p_2(U_2) = 1 - b$$

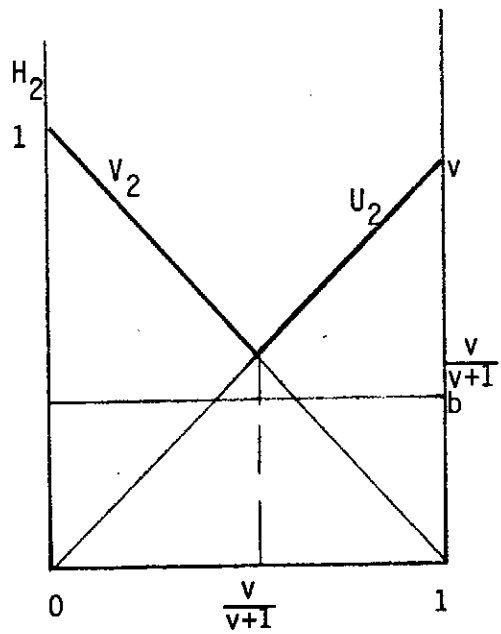$$p_2(V_2) = \frac{v-b}{v}$$

$$p_2(W_2) = b - \frac{v-b}{v}$$

$$p_1(U_1) = \frac{u}{u+1}$$

$$p_1(V_1) = \frac{1}{u+1}$$

$$p_1(W_1) = 0$$

$$p_2(U_2) = \frac{1}{v+1}$$

$$p_2(V_2) = \frac{v}{v+1}$$

$$p_2(W_2) = 0$$

Figure 2: Determination of the bicentric prior strategies.

At the intersection point of the lines for $U_i$ and $V_i$ players 1 and 2 obtain the payoffs $\frac{u}{u+1}$ respectively $\frac{v}{v+1}$ , if they play one of these strategies. In the upper half of the figure the opportunity costs a and b are greater than the intersection point payoffs. Therefore $W_1$ and $W_2$ can be best replies to $zU_j + (1-z)V_j$ in these cases. In the lower part of the figure the opportunity costs are below the intersection point payoffs.

The probabilities prescribed by the prior strategies can be determined easily with the help of elementary geometry. The results are shown below the respective graphical representations.

It is necessary to distinguish four cases with respect to the bicentric prior strategy combination. The cases will be labeled as shown in figure 3.

|  | $b < \frac{v}{v+1}$ | $b > \frac{v}{v+1}$ |
|---|---|---|
| $a < \frac{u}{u+1}$ | case (1) | case (2) |
| $a > \frac{u}{u+1}$ | case (3) | case (4) |

Figure 3: Case distinction with respect to the bicentric prior.

## V . Investigation of case (1)

In this case the opportunity costs a and b are relatively small. It can be shown that in case (1) the equilibrium point U is always the solution. The result agrees with the analysis of the 2x2-game obtained by erasing the strategies $W_i$. We can say that relatively small opportunity costs do not have any influence on the solution.

The following payoffs must be computed to receive the best reply of player i to the prior strategy $p_j$ of the other players:

$$H_1(U_1\ p_2) = p_2(U_2)H_1(U_1\ U_2) + p_2(V_2)\ H_1(U_1\ V_2) = \frac{u}{v+1}$$

$$H_1(V_1\ p_2) = p_2(U_2)H_1(V_1\ U_2) + p_2(V_2)\ H_1(V_1\ V_2) = \frac{v}{v+1}$$

$$H_1(W_1\ p_2) = a$$

$$H_2(p_1\ U_2) = p_1(U_1)H_2(U_1\ U_2) + p_1(V_1)\ H_2(V_1\ U_2) = \frac{u}{u+1}$$

$$H_2(p_1\ V_2) = p_1(U_1)H_2(U_1\ V_2) + p_1(V_1)\ H_2(V_1\ V_2) = \frac{v}{u+1}$$

$$H_2(p_1\ W_2) = b$$

since we have u > v, the following is valid:

$$H_1(U_1\ p_2) > H_1(V_1\ p_2)$$

$$H_2(p_1\ U_2) > H_2(p_1\ V_2)$$

Moreover, the restrictions on the opportunity costs yield:

$$H_1(U_1\ p_2) > H_1(W_1\ p_2)$$

$$H_2(p_1\ U_2) > H_1(p_1\ W_2)$$

It follows from the four inequalities, that $U = (U_1, U_2)$ is the only best reply to $p = (p_1,\ p_2)$. If a strong equilibrium point is the only best reply to the prior-strategy combination, then this strong equilibrium point is the result of the tracing procedure. Therefore we have

T(G,p) = U, or in other words, U risk dominates V. Consequently <u>the equilibrium point U is the solution of the game in case (1)</u>.

The equations for $H_i(U_i\ p_j)$ and $H_i(V_i\ p_j)$ also hold for the 2x2 game, which results by erasing the strategies $W_i$. Also this game has the solution U.

## VI. <u>Best replies to the prior in case (2)</u>

The payoffs of the pure strategies against the prior strategy of the opponent are shown by the following equations:

$$H_1(U_1\ p_2) = p_2(U_2)\ H_1(U_1\ U_2) + p_2(V_2)\ H_1(U_1\ V_2) + p_2(W_2)\ H_1(U_1\ W_2) = u(1-b)$$

$$H_1(V_1\ p_2) = p_2(U_2)\ H_1(V_1\ U_2) + p_2(V_2)\ H_1(V_1\ V_2) + p_2(W_2)\ H_1(V_1\ W_2) = \frac{v-b}{v}$$

$$H_1(W_1\ p_2) = a$$

$$H_2(p_1\ U_2) = p_1(U_1)\ H_2(U_1\ U_2) + p_1(V_1)\ H_2(V_1\ U_2) + p_1(W_1)\ H_2(W_1\ U_2) = \frac{u}{u+1}$$

$$H_2(p_1\ V_2) = p_1(U_1)\ H_2(U_1\ V_2) + p_1(V_1)\ H_2(V_1\ V_2) + p_1(W_1)\ H_2(W_1\ V_2) = \frac{v}{u+1}$$

$$H_2(p_1\ W_2) = b$$

These equations and the assumptions on the parameters yield the following results on the conditions under which various payoff inequalities hold:

$H_1(U_1 \ p_2) > H_1(V_1 \ p_2)$   if   $\dfrac{uv-v}{uv-1} > b$

$H_1(U_1 \ p_2) > H_1(W_1 \ p_2)$   if   $u(1-b) > a$

$H_1(V_1 \ p_2) > H_1(W_1 \ p_2)$   if   $v(1-a) > b$ .

$H_2(p_1 \ U_2) > H_2(p_1 \ V_2)$   always

$H_2(p_1 \ U_2) > H_2(p_1 \ W_2)$   if   $\dfrac{u}{u+1} > b$

$H_2(p_1 \ V_2) > H_2(p_1 \ W_2)$   never

The conditions on the right are necessary and sufficient for the inequalities on the left. All six inequalities remain valid if ">" is replaced by "<" and "always" is replaced by "never".

Now we turn our attention to the question, under which conditions a combination of pure strategies is the only best reply to the bicentric prior strategy combination. The result is shown in figure 4. As we shall see five of the nine possible strategy combinations are excluded by the assumptions for the parameters.

|  | $U_2$ | $V_2$ | $W_2$ |
|---|---|---|---|
| $U_1$ | $a < u(1-b)$<br>$b < \dfrac{u}{u+1}$<br>$b < \dfrac{uv-v}{uv-1}$ | excluded | $a < u(1-b)$<br>$\dfrac{u}{u+1} < b < \dfrac{uv-v}{uv-1}$ |
| $V_1$ | excluded | excluded | $\dfrac{uv-v}{uv-1} < b < v(1-a)$<br>$\dfrac{u}{u+1} < b$ |
| $W_1$ | excluded | excluded | $u(1-b) < a$<br>$v(1-a) < b$<br>$\dfrac{u}{u+1} < b$ |

Figure 4: Only best replies to the bicentric prior in case (2).

We first show that $V_2$ cannot be a best reply to $p_1$. If $V_2$ were the best reply to p, the following inequality would have to be satisfied:

$$\frac{v}{u+1} > \frac{u}{u+1}$$

This is impossible, in view of $u > v$. Moreover

$$\frac{v}{u+1} > b$$

contradicts $\frac{u}{u+1} < b$ . It can be seen that $(V_1, U_2)$ can never be a best reply to p, since this would imply:

$$\frac{u}{u+1} > b > \frac{uv-v}{uv-1}$$

and therefore

$$u^2v - u > u^2v - uv + uv - v$$

which is a contradiction to $u > v$.

Moreover, a best reply $(W_1, U_2)$ would require the following condition:

$$u(1-b) < a < \frac{u}{u+1}$$
$$\frac{u-a}{u} < b < \frac{u}{u+1}$$
$$u^2 - ua + u - a < u^2$$
$$\frac{u}{u+1} < a$$

This is excluded by case (2). Therefore $(W_1, U_2)$ is never a best reply to p. The conditions for the remaining subcases are nothing else than the conditions for the relevant payoff inequalities.

In cases where a strong equilibrium point of the game is the only best reply to the prior strategy combination, this is also the final result of the tracing procedure.

If U is the only best reply to p, then U risk dominates V and U is the solution of the game. If, however, W is the only best reply to p, then there is no risk dominance relationship between U and V and it becomes necessary to apply the tracing procedure to the centroid prior distribution. In the two remaining subcases of figure 4, where $(U_1, W_2)$ and $(V_1, W_2)$ are the best replies to p it is necessary to look more closely at the path of the tracing procedure in order to determine the final result $T(G,p)$. This will be done in the next section.

## VII. Application of the tracing procedure to the bicentric prior distribution in case (2)

If $\varphi = (\varphi_1, \varphi_2)$ is the only best reply to the bicentric prior combination p, then we can find a greatest intervall of the form $0 \leq t < t'$ where $\varphi$ is a strong equilibrium point of $G^t$. With the exception of degenerate cases the situation at the upper border t' will be as follows: There is one player i such that his strategy $\varphi_i$ will remain his only best reply to $\varphi_j$ in $G^t$ for $t=t'+\varepsilon$ for sufficiently small $\varepsilon$, but for the other player j another strategy $\psi_j$ becomes his best reply to $\varphi_i$. The number t' is called <u>switch point</u> from $\varphi = (\varphi_1, \varphi_2)$ to $\Psi = (\varphi_1, \psi_2)$.

The <u>switch point</u> t' from $\varphi = (\varphi_1, \varphi_2)$ to $\Psi = (\varphi_1, \psi_2)$ can be defined independently of whether $\varphi$ is an equilibrium point in $G^0$ or not. t' is the solution of the following linear equation:

$$H_2^{t'}(\varphi_1 \varphi_2) = H_2^{t'}(\varphi_1 \psi_2)$$

or more detailed:

$$t' H_2(\varphi_1 \varphi_2) + (1-t') H_2(c_1 \varphi_2) = t' H_2(\varphi_1 \psi_2) + (1-t') H_2(c_1 \psi_2)$$

Switch points from $(\varphi_1, \varphi_2)$ to $(\psi_1, \varphi_2)$ are defined in an analogous manner. Of course, we cannot talk about switch points, unless the respective equation has a uniquely determined solution. Values outside of the interval $0 \leq t \leq 1$ do not have any meaning.

Let $\varphi = (\varphi_1, \varphi_2)$ be the only best reply to the prior strategy combination. The border t' of the greatest interval $0 \leq t \leq t'$ with the property that $\varphi$ is equilibrium point of $G^t$ is called the <u>first switch point</u>. With the exception of degenerate cases this is a switch point from $\varphi$ to a combination of the form $(\psi_1, \varphi_2)$ or of the form $(\varphi_1, \psi_2)$. In the first case player 1, in the second case player 2 is called the <u>first switcher</u>. If the first switch point is already a switch point to a strong equilibrium point of G, then this equilibrium point is already the result of the tracing procedure (except for border cases). The search on the path has come to an end by the first switch. This situation prevails in the subcase of figure 4 in which $(U_1, W_2)$ is the only best reply. In order to show this we compute the payoffs of each player for the game $G^t$:

$H_1^t (U_1 \ W_2) = (1-t) \ u \ (1-b)$

$H_1^t (V_1 \ W_2) = (1-t) \ \frac{v-b}{v}$

$H_1^t (W_1 \ W_2) = a$

$H_1^t (U_1 \ W_2)$ is always greater than $H_1^t (V_1 \ W_2)$ and therefore the switch can only go to $(W_1, W_2)$ if player 1 is the first switcher. If, however, player 2 is the first switcher, then the switch can only go to $(U_1, U_2)$ for the computations yield the following results:

$H_2^t(U_1 \ U_2) = \frac{u+t}{u+1}$

$H_2^t(U_1 \ V_2) = (1-t) \ \frac{v}{u+1}$

$H_2^t(U_1 \ W_2) = b$

The payoff for the combination $(U_1, U_2)$ is the only one which increases with t. The payoffs in the other cases are decreasing, respectively constant. The first switch goes to U or to W. ( border cases in which both switching points are equal are not considered.)
The switching points are the following ones:

from $(U_1, W_2)$ to $(W_1, W_2)$ : $t_1 = 1 - \frac{a}{u(1-b)}$

from $(U_1, W_2)$ to $(U_1, U_2)$ : $t_2 = bu + b - u$

The smaller switching point decides in which direction the first switch leads: For $a < (1-b)^2 u(u+1)$ the final result of the tracing procedure is U; for $a > (1-b)^2 u(u+1)$ the final result of the tracing procedure is W.


We now apply the same kind of reasoning to the case where $(V_1, W_2)$ is the best reply to p. In this subcase the switch can go to $(W_1, W_2)$ if player 1 is the first switcher and to $(V_1, V_2)$ if player 2 is the first switcher. A switch to $U_1$ or to $U_2$ can be excluded, since player 1's payoff for $(U_1, W_2)$ and player 2's payoff for $(V_1, U_2)$ are decreasing in t, wheras the payoffs for $(W_1, W_2)$ are constant. The switch points are as follows:

from $(V_1, W_2)$ to $(W_1, W_2)$ : $t_3 = 1 - \frac{av}{v-b}$

from $(V_1, W_2)$ to $(V_1, V_2)$ : $t_4 = \frac{bu+b-v}{uv}$

We have

$$t_3 < t_4 \quad \text{for} \quad a > \frac{(v-b)^2(u+1)}{uv^2}$$

$$t_4 < t_3 \quad \text{for} \quad a < \frac{(v-b)^2(u+1)}{uv^2}$$

W is the result of the tracing procedure in the first case and V is the result in the second case. If the result is V, then V risk dominates U and V is the solution of the game.

In order to obtain a better overview over the results for case (2) we show that

$$a < \frac{(v-b)^2(u+1)}{uv^2} \qquad \text{implies} \quad b < v(1-a)$$

and

$$b > v(1-a) \qquad \text{implies} \quad a > \frac{(v-b)^2(u+1)}{uv^2}$$

In order to prove the first implication we rearrange the first inequality with a on the left hand side as follows

$$(v-b)^2 > \frac{uv^2 a}{u+1}$$

In view of $v > b$ an equivalent transformation yields:

$$b < v - v \sqrt{\frac{u}{u+1}\, a}$$

Since $a < u/(u+1)$ holds we can conclude:

$$b < v - v\sqrt{a^2} = v(1-a)$$

The second implication follows by the fact that $b > v(1-a)$ implies

$$b > v - v\sqrt{\frac{u}{u+1}\, a}$$

which yields

$$(v-b)^2 < \frac{v^2 u}{u+1}\, a$$

With the help of the two implications the results obtained up to now for case(2) can be summarized as follows :

$$L(G) = U \begin{cases} \text{for } a < u(1-b) \text{ and } b < \frac{uv-v}{uv-1} \text{ and } b > \frac{u}{u+1} \text{ and } a < (1-b)^2 u(u+1) \\ \text{for } a < u(1-b) \text{ and } b < \frac{uv-v}{uv-1} \text{ and } b < \frac{u}{u+1} \end{cases}$$

$$L(G) = V \quad \text{for } a < \frac{(v-b)^2(u+1)}{uv^2} \text{ and } \frac{uv-v}{uv-1} < b < v(1-a) \text{ and } \frac{u}{u+1} < b$$

$$T(G,p)=W \begin{cases} \text{for } \frac{(v-b)^2(u+1)}{uv^2} < a \text{ and } \frac{uv-v}{uv-1} < b < v(1-a) \text{ and } \frac{u}{u+1} < b \\ \text{for } a < u(1-b) \text{ and } b < \frac{uv-v}{uv-1} \text{ and } b > \frac{u}{u+1} \text{ and } a > (1-b)^2 u(u+1) \end{cases}$$

## VIII. The application of the tracing procedure to the centroid prior combination in case (2).

Wherever W is the final result $T(G, p)$ of the tracing procedure applied to p, there is no risk dominance between U and V. Here we receive the solution as the result $T(G, c)$ of the tracing procedure applied to the centroid $c = (c_1, c_2)$ of $\Omega = \{U, V\}$.

| | $U_2$ | $V_2$ | $W_2$ |
|---|---|---|---|
| $U_1$ | $\frac{1+t}{2} u$ $\qquad \frac{1+t}{2}$ | $\frac{1-t}{2} u$ $\qquad \frac{1-t}{2} v$ | $\frac{1-t}{2} u$ $\qquad b$ |
| $V_1$ | $\frac{1-t}{2}$ $\qquad \frac{1-t}{2}$ | $\frac{1+t}{2}$ $\qquad \frac{1+t}{2} v$ | $\frac{1-t}{2}$ $\qquad b$ |
| $W_1$ | $a$ $\qquad \frac{1-t}{2}$ | $a$ $\qquad \frac{1-t}{2} v$ | $a$ $\qquad b$ |

Figure 5: The games $G^t$ which appear in the application of the tracing procedure to the centroid $c = (c_1, c_2)$ of $\{U, V\}$.

Figure 5 shows the games $G^t$ which appear in the application of the tracing procedure to the centroid $c = (c_1, c_2)$ of $\{U, V\}$. We can see immediately that for $t = 0$ the strategies $V_1$ and $U_2$ cannot be best replies since u and v are greater than 1.

Since in case (2) we have

$$a < \frac{u}{u+1} < \frac{u}{2}$$

$W_1$ cannot be a best reply for $t=0$ either. $W_2$ is a best reply for $t=0$ if we have $b > v/2$. Our conclusions with respect to the only best replies to c are summarized in figure 6.

|  | $U_2$ | $V_2$ | $W_2$ |
|---|---|---|---|
| $U_1$ | excluded | $b < \frac{v}{2}$ | $\frac{v}{2} < b$ |
| $V_1$ | excluded | excluded | excluded |
| $W_1$ | excluded | excluded | excluded |

Figure 6: Only best replies to the centroid prior combination c
in case (2).

We have to determine the path of the tracing procedure applied to c in
order to find T(G, c). We first look at the subcase $(U_1,V_2)$ of figure 6.
In order to see where the first switch occurs we compute the following
switch points:

from $(U_1,V_2)$ to $(V_1,V_2)$ : $t_5 = \frac{u-1}{u+1}$

from $(U_1,V_2)$ to $(W_1,V_2)$ : $t_6 = \frac{u-2a}{u}$

from $(U_1,V_2)$ to $(U_1,U_2)$ : $t_7 = \frac{v-1}{v+1}$

from $(U_1,V_2)$ to $(U_1,W_2)$ : $t_8 = \frac{v-2b}{v}$

$t_5 < t_6$ holds for $a < \frac{u}{u+1}$ which is satisfied in case (2). The inequality
$t_7 < t_5$ is equivalent to $u > v$. Moreover $t_8 < t_7$ holds for $b > \frac{v}{v+1}$ which
is satisfied in case (2). This shows that $t_8$ is the smallest of the four
switch points. Player 2 is the first switcher and the switch goes to
$(U_1,W_2)$. We must be aware of the possibility that not all combinations
$(U_1, q_2)$ with $q_2 = (1-\alpha)V_2 + \alpha W_2$ are equilibrium points in $G^{t_8}$. Actually
this problem does not arise here, since player 1's payoff is the same
for $(U_1,V_2)$ and $(U_1, W_2)$ and a deviation to $V_1$ need not be considered in
view of the fact that $V_1$ yields less payoff then $W_1$ against $W_2$. Therefore

the path of the tracing procedure continues at $(U_1, W_2)$ after $t = t_8$ until the next switch occurs.

Since $(U_1, W_2)$ is not yet an equilibrium point of G, at least one further switch must occur. We can exclude the possibility that player 1 mooves over to $V_1$; figure 5 shows that player 1's payoff for $V_1$ is always less than that for $U_1$ if player 2 plays $W_2$. This leaves only two possibilities, a switch to $(W_1, W_2)$ or to $(U_1, U_2)$. The switch points are as follows:

from $(U_1, W_2)$ to $(W_1, W_2)$ : $t_6 = \dfrac{u-2a}{u}$

from $(U_1, W_2)$ to $(U_1, U_2)$ : $t_9 = 2b-1$

$t_6 < t_9$ holds if and only if we have
$$a > u(1-b)$$

In order to see that the smaller one of the both numbers $t_6$ and $t_9$ is the next switch point, it is also necessary to convince ourselves that both are greater than the first switch point $t_8$. We know already that $t_8 < t_6$ holds and it can be seen easily that $t_8 < t_9$ holds for $b > v/(v+1)$, which is satisfied in case (2).

In the subcase $(W_1, W_2)$ at figure 4 the inequality $a > u(1-b)$ holds. It follows that under the conditions which hold there, the second switch goes to W and W is the result of the tracing procedure and the solution of the game. However, it is also necessary to look at the possibility that $T(G, p) = W$ arises in the subcase $(V_1, W_2)$ of figure 4. We receive $T(G, c) = U$ if the following conditions are satisfied

$$\frac{uv-v}{uv-1} < b < v(1-a)$$
$$\frac{(v-b)^2(u+1)}{uv^2} < a < u(1-b)$$
$$b < \frac{v}{2}$$

We receive $T(G,c) = W$ in subcase $(V_1, W_2)$ of figure 4, if $a > u(1-b)$ holds.

We now look at the subcase $(U_1, W_2)$ of figure 6. The situation is the same as in the previous subcase after the first switch to $(U_1, W_2)$. The first switch from $(U_1, W_2)$ must either go to $(W_1, W_2)$ or to $(U_1, U_2)$. The switch points $t_6$ and $t_9$ have been computed above. The switch goes to W. for $a > u(1-b)$ and to U for $a < u(1-b)$. In the former case W is the

solution of the game and in the latter case U is the solution.

The results obtained in case (2) can be summarized as follows :

$$
L(G) = \begin{cases}
U & \text{for } a < u(1-b) \text{ and } b < \dfrac{uv-v}{uv-1} \\[2mm]
& \text{for } \dfrac{(v-b)^2(u+1)}{uv^2} < a < u(1-b) \\[2mm]
& \text{and } \dfrac{uv-v}{uv-1} < b \\[2mm]
V & \text{for } a < \dfrac{(v-b)^2(u+1)}{uv^2} \text{ and } \dfrac{uv-v}{uv-1} < b \\[2mm]
W & \text{for } \dfrac{(v-b)^2(u+1)}{uv^2} < a \text{ and } u(1-b) < a
\end{cases}
$$

## IX. Application of the tracing procedure to the bicentric prior in case (3).

The payoffs for pure strategies against the opponent's bicentric prior strategies are as follows:

$$H_1(U_1\ p_2) = \frac{u}{v+1}$$

$$H_1(V_1\ p_2) = \frac{v}{v+1}$$

$$H_1(W_1\ p_2) = a$$

$$H_2(p_1\ U_2) = \frac{u-a}{u}$$

$$H_2(p_1\ V_2) = v(1-a)$$

$$H_2(p_1\ W_2) = b$$

In view of $u > v$ player 1's payoff for $(U_1, p_2)$ is always greater than that for $(V_1, p_2)$. Therefore $V_1$ cannot be a best reply to $p_2$. We now show that $V_2$ cannot be a best reply to $p_1$. For this purpose it is sufficient to prove the following inequality

$$\frac{u-a}{u} > v\ (1-a)$$

An equivalent transformation yields

$$a > \frac{uv-u}{uv-1}$$

It can be seen easily that in view of u > v we have

$$\frac{uv-u}{uv-1} < \frac{u}{u+1}$$

In case (2) we always have $u/(u+1)$ < a. Consequently $V_2$ cannot be player 2's best reply to $p_1$.

We can exclude the possibility that $(U_1, W_2)$ is the only best reply to p; in order to do this we observe that this would imply:

$$b > \frac{u-a}{u} > \frac{u - \frac{u}{v+1}}{u} = \frac{v}{v+1}$$

which contradicts b < $v/(v+1)$. The condition $(u-a)/u$ < b is equivalent to a > u(1-b). With this fact in mind, we can see that Figure 7 describes the conditions under which the nine pure strategy combinations are only best replies to the bicentric prior p.

|  | $U_2$ | $V_2$ | $W_2$ |
|---|---|---|---|
| $U_1$ | $a < \frac{u}{v+1}$ | excluded | excluded |
| $V_1$ | excluded | excluded | excluded |
| $W_1$ | $\frac{u}{v+1} < a < u(1-b)$ | excluded | $a > u(1-b)$ |

Figure 7: Only best replies to the bicentric prior in case (3)

In the subcase $(U_1, U_2)$ of figure 7 the equilibrium point U risk dominates V and is the solution of the game. In the subcase $(W_1, W_2)$ there is no risk dominance between U and V and we have to apply the tracing procedure to the centroid of $\{U, V\}$ in order to find the solution.

We now turn our attention to the subcase $(W_1, U_2)$. In order to determine the path of the tracing procedure we look at the following payoffs:

$$H_1^t (W_1 \, U_2) = a$$

$$H_1^t (U_1 \, U_2) = (1-t) \frac{u}{v+1} + tu = \frac{u}{v+1} + t \frac{uv}{v+1}$$

$$H_1^t (V_1 \, U_2) = (1-t) \frac{v}{v+1}$$

$$H_2^t (W_1 \, U_2) = (1-t) \frac{u-a}{u}$$

$$H_2^t (W_1 \, V_2) = (1-t) \, v \, (1-a)$$

$$H_2^t (W_1 \, W_2) = b$$

In view of $u > v$ and $(u-a)/u > v(1-a)$ the first switch cannot go to $V_1$ or $V_2$. Consequently the switch must go either to $(U_1, U_2)$ or to $(W_1, W_2)$. The corresponding switch points are as follows:

from $(W_1, U_2)$ to $(U_1, U_2)$ : $\quad t_{10} = \frac{a(v+1)-u}{uv}$

from $(W_1, U_2)$ to $(W_1, W_2)$ : $\quad t_{11} = \frac{u(1-b)-a}{u-a}$

We have

$$t_{10} < t_{11} \quad \text{for} \quad b < \frac{(u-a)^2(v+1)}{u^2 v}$$

$$t_{11} < t_{10} \quad \text{for} \quad b > \frac{(u-a)^2(v+1)}{u^2 v}$$

U is the result of the tracing procedure for $t_{10} < t_{11}$ and W is the result of the tracing procedure for $t_{11} < t_{10}$. In the former case U risk dominates V and is the solution of the game. In the latter case we have $T(G, p) = W$ .

The results obtained up to now can be summarized as follows:

$$L(G) = U \quad \text{for} \quad a < \frac{u}{v+1}$$

$$\text{for} \quad \frac{u}{v+1} < a < u(1-b) \quad \text{and} \quad b < \frac{(u-a)^2(v+1)}{u^2 v}$$

$$T(G, p) = W \quad \text{for} \quad \frac{u}{v+1} < a < u(1-b) \quad \text{and} \quad \frac{(u-a)^2(v+1)}{u^2 v} < b$$
$$\text{for } a > u(1-b)$$

## X. Applications of the tracing procedure to the centroid prior strategy combination in case (3).

In case (3) we have

$$b < \frac{v}{v+1} < \frac{v}{2}$$

Figure 5 shows that therefore player 2's best reply to player 1's centroid strategy $c_1$ is always $V_2$. Moreover player 1's best reply to $c_2$ is $U_1$ for $a < u/2$ and $W_1$ for $a > u/2$. We can conclude that the conditions for only best replies to c are those which are shown in figure 8.

|  | $U_2$ | $V_2$ | $W_2$ |
|---|---|---|---|
| $U_1$ | excluded | $a < \frac{u}{2}$ | excluded |
| $V_1$ | excluded | excluded | excluded |
| $W_1$ | excluded | $\frac{u}{2} < a$ | excluded |

Figure 8: Only best replies to the centroid of {U, V} for $T(G, p) = W$ in case (3).

We first look at the path of the tracing procedure in subcase $(U_1, V_2)$ of figure 8. We have to look at the same switch points $t_5$, $t_6$, $t_7$ and $t_8$ as in section VIII. As before $t_7 < t_5$ follows by $u > v$. Moreover $t_7 < t_8$ holds for $b > v/(v+1)$, which is satisfied in case (3). It can also be seen easily that $t_6 < t_7$ holds for $a > u/(v+1)$, which is satisfied, whenever $T(G, p) = W$ holds. This shows that $t_6$ is the smallest switch point. Player 1 is the first switcher and the switch goes to $(W_1, V_2)$. Player 2's payoff in $G^{t_6}$ is the same for $(U_1, V_2)$ and $(W_1, V_2)$. Therefore all strategy combinations $(q_1, V_2)$ with $q_1$ of the form $q_1 = (1-\alpha)U_1 + \alpha W_1$ are equilibrium points in $G^{t_6}$. The path of the tracing procedure goes through all these equilibrium points.

Now we must find out, where the next switch goes to. Figure 5 shows that player 1's payoff for $(U_1, V_2)$ is decreasing in t. Therefore a switch from $(W_1, V_2)$ to $(U_1, V_2)$ can be excluded. Player 2's payoff for $(W_1, U_2)$ is always smaller than that for $(W_1, V_2)$. Therefore a switch to $(W_1, U_2)$ can be excluded. The remaining two possibilities are $(V_1, V_2)$ and $(W_1, W_2)$. The corresponding switch points are as follows:

from $(W_1, V_2)$ to $(V_1, V_2)$ : $t_{12} = 2a - 1$

from $(W_1, V_2)$ to $(W_1, W_2)$ : $t_8 = \dfrac{v-2b}{v}$

We know already that $t_6 < t_8$ holds. The inequality $t_6 < t_{12}$ is satisfied for $a > u/(u+1)$, which is true in case (3).

We have

$t_{12} < t_8$     for     $b < v(1-a)$

$t_8 < t_{12}$     for     $b > v(1-a)$

The final result of the tracing procedure is V for $t_{12} < t_8$ and W for $t_8 < t_{12}$. In the former case V is the solution and in the latter case W is the solution.

We now turn our attention to the subcase $(W_1, V_2)$ of figure 8. For the same reasons as above the switch from $(W_1, V_2)$ cannot go anywhere else than to $(V_1, V_2)$ and $(W_1, W_2)$. As before the corresponding switch points are $t_{12}$ and $t_8$. The solution is V for $t_{12} < t_8$ and W for $t_8 < t_{12}$.

In order to get a better overview over the results we prove that in case (3)

$$b < \frac{(u-a)^2(v+1)}{u^2 v} \quad \text{implies} \quad a < u(1-b)$$

and

$$a > u(1-b) \quad \text{implies} \quad b > \frac{(u-a)^2(v+1)}{u^2 v}$$

In order to prove the first implication we rearrange the first inequality with b on the left hand side as follows:

$$(u-a)^2 > \frac{bu^2v}{v+1}$$

In view of u > a an equivalent transformation yields

$$a < u - u\sqrt{\frac{v}{v+1} \, b}$$

Since b < v/(v+1) holds we can conclude:

$$a < u - u\sqrt{b^2} = u(1-b).$$

The second implication follows by the fact that a > u(1-b) implies

$$a > u - u\sqrt{\frac{v}{v+1} \, b}$$

which yields

$$(u-a)^2 < \frac{u^2v}{v+1} \, b$$

The results obtained for case (3) can now be summarized as follows:

$$L(G) = \begin{cases} U & \text{for} \quad a < \frac{u}{v+1} \\[2mm] & \text{for} \quad \frac{u}{v+1} < a \quad \text{and} \quad b < \frac{(u-a)^2(v+1)}{u^2v} \\[2mm] V & \text{for} \quad \frac{u}{v+1} < a \quad \text{and} \quad \frac{(u-a)^2(v+1)}{u^2v} < b < v(1-a) \\[2mm] W & \text{for} \quad \frac{u}{v+1} < a \quad \text{and} \quad v(1-a) < b \\[2mm] & \text{and} \quad \frac{(u-a)^2(v+1)}{u^2v} < b \end{cases}$$

## XI. Application of the tracing procedure to the bicentric prior in case (4)

The payoffs for the pure strategies against the opponents bicentric prior strategies are as follows

$$H_1(U_1 \; p_2) = u(1-b)$$

$$H_1(V_1 \; p_2) = \frac{v-b}{b}$$

$$H_1(W_1 \; p_2) = a$$

$$H_2(p_1 \; U_2) = \frac{u-a}{a}$$

$$H_2(p_1 \; V_2) = v(1-a)$$

$$H_2(p_1 \; W_2) = b$$

These equations yield the following conditions for the relevant payoff inequalities:

$H_1(U_1 \ p_2)$ > $H_1(V_1 \ p_2)$ if $\frac{uv-v}{uv-1}$ > b

$H_1(U_1 \ p_2)$ > $H_1(W_1 \ p_2)$ if $u(1-b)$ > a

$H_1(V_1 \ p_2)$ > $H_1(W_1 \ p_2)$ if $v(1-a)$ > b

$H_2(p_1 \ U_2)$ > $H_2(p_1 \ V_2)$ if $a > \frac{uv-u}{uv-1}$

$H_2(p_1 \ U_2)$ > $H_2(p_1 \ W_2)$ if $u(1-b)$ > a

$H_2(p_1 \ V_2)$ > $H_2(p_1 \ W_2)$ if $v(1-a)$ > b

These conditions remain valid if everywhere "<" is substituted for ">".

We now shall exclude the possibility that $V_1$ is player 1's only best reply to $p_2$. This would require the following inequality

$$v(1-a) > b > \frac{uv-v}{uv-1}$$

In view of $a > u/(u+1)$ we have $1-a < 1/(u+1)$. Therefore the above inequality yields:

$$\frac{v}{u+1} > \frac{uv-v}{uv-1}$$

which is incompatible with $u > v$. We can also exclude the possibility that $V_2$ is the only best reply to $p_1$. This would require:

$$\frac{uv-u}{uv-1} > a > \frac{u}{u+1}$$

which is incompatible with $u > v$ too. The payoff inequalities show that $W_1$ is the only best reply to $p_2$ if and only if $W_2$ is the only best reply to $p_1$. In both cases the same conditions must be satisfied. We can conclude that the conditions under which the nine pure strategy combinations are only best replies to p are those shown in figure 9.

|  | $U_2$ | $V_2$ | $W_2$ |
|---|---|---|---|
| $U_1$ | $a < u(1-b)$ | excluded | excluded |
| $V_1$ | excluded | excluded | excluded |
| $W_1$ | excluded | excluded | $u(1-b) < a$ |

Figure 9: Only best replies to the bicentric prior in case (4).

Since both $(U_1, U_2)$ and $(W_1, W_2)$ are strong equilibrium points we do not really have to apply the tracing procedure to p. For $a < u(1-b)$ the solution is $U$. For $u(1-b) < a$ we have to apply the tracing procedure to the centroid of $\{U, V\}$ .

XII. Application of the tracing procedure to the centroid prior combination in case (4).

With the help of figure 5 it can be seen immediately that the only best replies to the centroid $c = (c_1, c_2)$ of $\{U, V\}$ are as described by figure 10.

|  | $U_2$ | $V_2$ | $W_2$ |
|---|---|---|---|
| $U_1$ | excluded | $a < \dfrac{u}{2}$<br>$b < \dfrac{v}{2}$ | $a < \dfrac{u}{2}$<br>$\dfrac{v}{2} < b$ |
| $V_1$ | excluded | excluded | excluded |
| $W_1$ | excluded | $\dfrac{u}{2} < a$<br>$b < \dfrac{v}{2}$ | $\dfrac{u}{2} < a$<br>$\dfrac{v}{2} < b$ |

Figure 10: Only best replies to the centroid of $\{U,V\}$ for $T\,(G,\ p) = W$ in case (4).

Obviously W is the solution for $a > u/2$ and $b > v/2$. In the other three subcases of figure 8 we have to look at the path of the tracing procudure. We begin with the subcase $(U_1, V_2)$. The switch points $t_5$, $t_6$, $t_7$ and $t_8$ have been computed in section VIII. As before $u > v$ implies $t_7 < t_5$ and $t_8 < t_7$ holds in view of $b > v/(v+1)$. We have

$$t_6 < t_8 \quad \text{for} \quad bu < av$$

$$t_8 < t_6 \quad \text{for} \quad av < bu$$

Therefore the switch goes to $(W_1, V_2)$ for $bu < av$ and to $(U_1, W_2)$ for $av < bu$.

Suppose that we have bu < av and consider the game $G^{t_6}$. All strategy combinations $(q_1, V_2)$ with $q_1 = (1-\alpha)U_1 + \alpha W_1$ are equilibrium points of $G^{t_6}$ since player 2's payoff for $(q_1, U_2)$ is a decreasing function of $\alpha$ and his payoff for $(q_1, U_2)$ is constant. The path of the tracing procedure goes through these equilibrium points. We must look at the possibilities for the next switch. A switch from $(W_1, V_2)$ cannot go to $U_1$ or $U_2$, since player 1's payoff for $(U_1, V_2)$ and player 2's payoff for $(W_1, U_2)$ are decreasing in t whereas the payoffs for $(V_1, V_2)$ and $(W_1, W_2)$ are increasing and constant respectively. The switch must go either to $(V_1, V_2)$ or to $(W_1, W_2)$. The corresponding switch points $t_{12}$ and $t_8$ have been computed in section X. As we have seen there $t_8 < t_{12}$ holds for b > v(1-a), a condition which is always satisfied for T(G, p) = W in case (4). Therefore the second switch goes to $(W_1, W_2)$ and W is the final result of the tracing procedure and the solution of the game.

Suppose that we have av < bu and consider the game $G^{t_8}$. All strategy combinations $(U_1, q_2)$ with $q_2 = (1-\alpha) V_2 + \alpha W_2$ are equilibrium points of $G^{t_8}$, since player 1's payoff for $(V_1, q_2)$ is a decreasing function of $\alpha$ and his payoff for $(W_1, q_2)$ is constant. The path of the tracing procedure goes through these equilibrium points. We must look at the possibilities for the next switch. A switch from $(U_1, W_2)$ cannot go to $V_1$ or $V_2$, since player 1's payoff for $(V_1, W_2)$ and player 2's payoff for $(U_1, V_2)$ are decreasing in t whereas the payoffs for $(U_1, U_2)$ and $(W_1, W_2)$ are increasing in t and constant respectively. The switch must go either to $(U_1, U_2)$ or to $(W_1, W_2)$. The corresponding switch points $t_6$ and $t_9$ have been computed in section VIII. As we have seen there $t_6 < t_9$ holds for a > u(1-b), a condition which is always satisfied for T(G, c) = W in case (4). This shows that here, too, W is the final result of the tracing procedure and the solution of the game.

In the subcases $(U_1, W_2)$ and $(W_1, V_2)$ of figure 10 the situation is essentially the same as in the subcase $(U_1, V_2)$ after one of the two combinations $(U_1, W_2)$ and $(W_1, V_2)$ have been reached by the first switch. For the same reasons as before a switch from $(U_1, W_2)$ or $(W_1, V_2)$ must go to $(W_1, W_2)$. This shows that W is the solution in these subcases, too. We can conclude that W is the solution in case (4) whenever T(G, p) = W holds.

The <u>results</u> <u>obtained</u> <u>for</u> <u>case</u> <u>(4)</u> can be summarized as follows

$$
L(G) = \begin{cases} U & \text{for} \quad a < u(1-b) \\ W & \text{for} \quad u(1-b) < a \end{cases}
$$

## XIII. The solution for all cases

It is important to point out that our results obtained in the cases (1) to (4) hold only with the exception of border cases. Some of these border cases are on lines inside the open regions which appear in the formulas for L(G), since we have neglected cases where two payoffs or two switch points are equal in deriving our results.

In the following we shall combine the results for all four cases, in order to derive an overall picture. In order to do this it will be useful to amalgamate several smaller regions into one bigger one. Of course the results may not hold for the borders of the smaller regions, but this does not matter since we neglect border cases anyhow.

As we have seen in section V the solution in case (1) is always U. The following regions are adjacent to the region of case (1): the first subregion for U in case (2), the first subregion for U in case (3), and the region for U in case (4). These four regions can be amalgamated into one. We obtain:

$$
L(G) = U \quad \text{for} \quad a < \frac{u}{v+1} \quad \text{and} \quad a < u(1-b) \quad \text{and} \quad b < \frac{uv-v}{uv-1}
$$

there are two additional regions with L(G) = U, one for case (2) and one for case (3).

The regions for W in case (2) and case (4) are adjacent . We amalgamate the region for W in case (2) with that part of the region for W in case (4), where a < u/(v+1) holds. Since a = u(1-b) is satisfied for a = u/(v+1) and b = v/(v+1) the inequalities u(1-b) < a and a < v/(v+1) imply b > v/(v+1). Therefore the joint region can be described by the inequalities below.

$$
L(G) = W \quad \text{for} \quad \frac{(v-b)^2(u+1)}{uv^2} < a < \frac{u}{v+1} \quad \text{and} \quad u(1-b) < a
$$

The remaining part of the region for W in case (4) can be amalgamated with the region for W in case (3).

We obtain:

$$L(G) = W \quad \text{for} \quad \frac{u}{v+1} < a \quad \text{and} \quad \frac{(u-a)^2(v+1)}{u^2 v} < b \quad \text{and} \quad v(1-a) < b$$

We have obtained the following result, which holds for all cases with the possible exception of border cases, some of which may be inside the seven subregions listed below

$$L(G) = \begin{cases}
U & \text{for} \quad a < \frac{u}{v+1} \quad \text{and} \quad a < u(1-b) \quad \text{and} \quad b < \frac{uv-v}{uv-1} \\[2mm]
& \text{for} \quad \frac{(v-b)^2(u+1)}{uv^2} < a < u(1-b) \quad \text{and} \quad \frac{uv-v}{uv-1} < b \\[2mm]
& \text{for} \quad \frac{u}{v+1} < a \quad \text{and} \quad b < \frac{(u-a)^2(v+1)}{u^2 v} \\[3mm]
V & \text{for} \quad a < \frac{(v-b)^2(u+1)}{uv^2} \quad \text{and} \quad \frac{uv-v}{uv-1} < b \\[2mm]
& \text{for} \quad \frac{u}{v+1} < a \quad \text{and} \quad \frac{(u-a)^2(v+1)}{u^2 v} < b < v(1-a) \\[3mm]
W & \text{for} \quad \frac{(v-b)^2(u+1)}{uv^2} < a < \frac{u}{v+1} \quad \text{and} \quad u(1-b) < a \\[2mm]
& \text{for} \quad \frac{u}{v+1} < a \quad \text{and} \quad \frac{(u-a)^2(v+1)}{2} < b \quad \text{and} \quad v(1-a) < b
\end{cases}$$

Figures 11 and 12 show graphical representations of the regions for U, V and W in the two special cases u = 1.o5, v = 1.o4 and u = 4.o and v = 3.2 . As one would expect - since u is greater than v - in both cases the region for U is larger than the combined aereas for the region of V. In the second case the region for W is smaller than in the first one. Moreover the second region for V vanishes in the second case. An increase of u or v increases the region of case (1) where always U is the solution and thereby narrows the space left for other solutions.

In order to obtain a non-empty second region for V one has to choose u and v very near to 1 as in the example of figure 11.

W tends to be the solution for relatively high opportunity costs. This is plausible since under these circumstances the risk of bargaining is great: It is not unreasonable to expect coordination of expectations at W.

For relatively low opportunity costs the solution is U. There the situation is similar to the 2x2-game which results by eliminating the strategies $W_1$ and $W_2$.
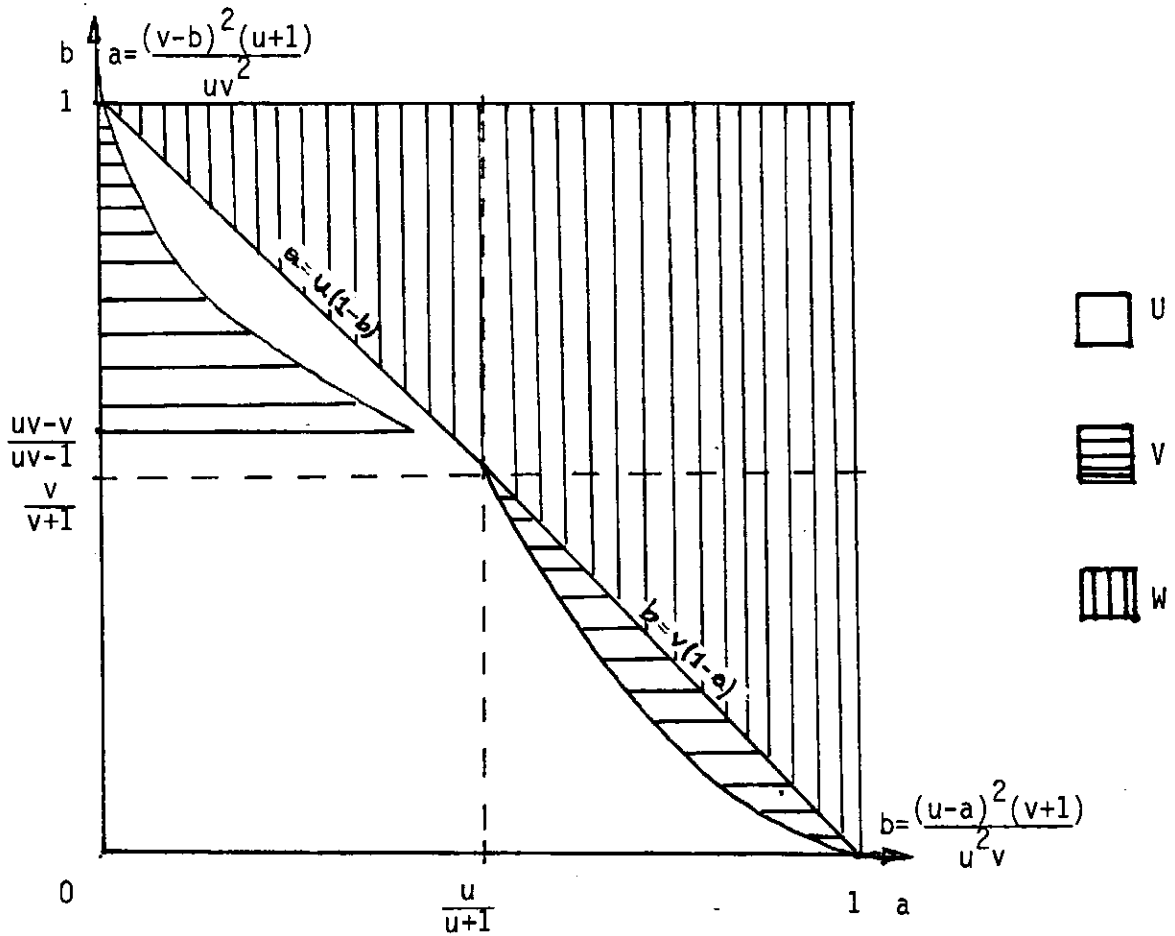
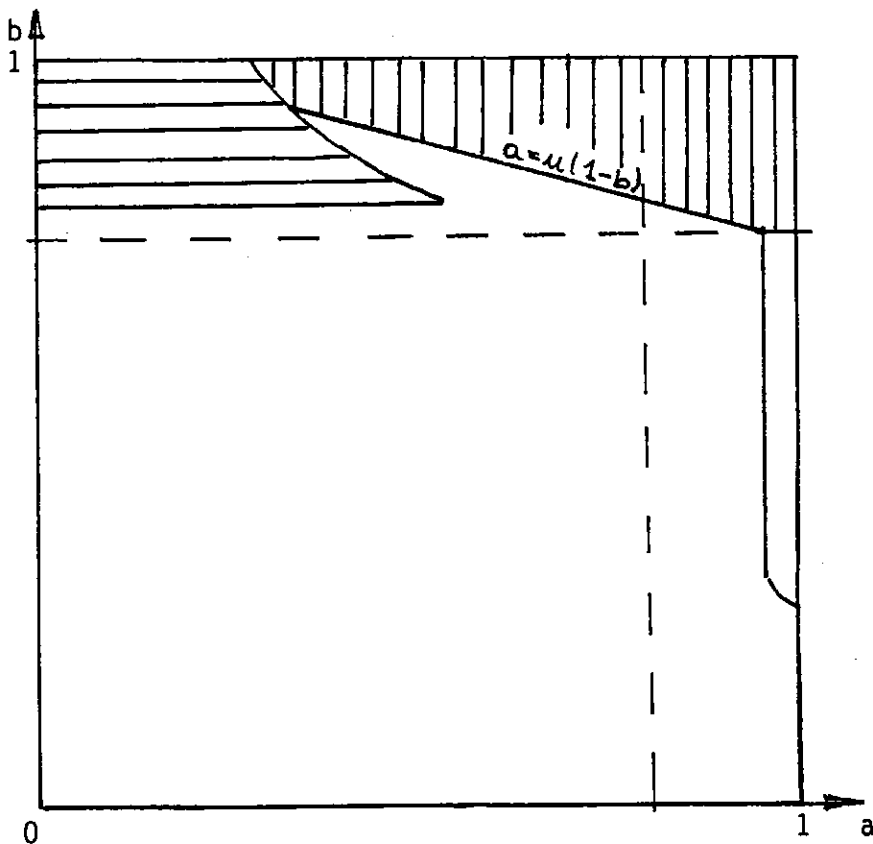Figure 11: risk-dominance-diagramm for u = 1.o5 and v = 1.o4



Figure 12: risk-dominance-diagramm for u = 4.o and v = 3.2

If the opportunity costs are relatively high for one player and relatively low for the other, it can also happen, that V is the solution. In the first region for V player 1 has relatively high opportunity costs and player 2 has low opportunity costs. It is quite understandable that this may improve the bargaining position of player 2 such that his preferred alternative V emerges as the solution.

In the second region for V the same intuitive argument points in the opposite direction. Here player 1 has high opportunity costs and player 2 has low opportunity costs. One would expect that under such circumstances U is the solution. This is in fact the case in the third region for U where the opportunity cost constellation is similar to that of the second region for V. At least at first glance the second region for V looks somewhat strange. However the way in which this region arises in the application of the tracing procedure to the centroid of {U, V} suggests a possible interpretation.

In the second region for V player 2's opportunity costs are not low enough to make W less attractive to him than U. On the other hand V is more attractive to player 2 than W. For player 1, however W is more attractive than U in view of his high opportunity costs. At the end player 2's interest in V turns out to be stronger than player 1's tendency to prefer W, which is not unreasonable since V is more favorable than W for player 1, too.

Whereas the first region for V is due to risk dominance, of V over U, the second region for V is due to tracing the centroid of {U, V} in situations where W is the result of tracing the bicentric prior. Unlike the bicentric prior, the centroid seems to favor player 2's interest, since his best reply to the centroid is $V_2$, if his opportunity costs are sufficiently low, whereas player 1's best reply to the centroid is $W_1$, if his opportunity costs are sufficiently high. Even if it is not quite clear whether this feature of the theory is a desirable one, the above interpretation indicates that it is a defendable one.

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