

Universität Bielefeld/IMW

**Working Papers
Institute of Mathematical Economics**

**Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung**

Nr. 59

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September 1977



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an der

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COMPLEXITY AND SOCIAL DECISION
RULES

by

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Abstract

This paper attempts to show how a particular concept and measure of complexity, as derived from automata theory, can be meaningfully interpreted in a program of 'limited rationality' regarding individual or social choices. The complexity measure appears to be a natural consequence of looking at a decision rule as a finite-state machine that computes preferences bounded by computational constraints. By factoring the social decision process into component processes it is demonstrated how searching for improvement depends on 'structural' and 'computational' limitations.

COMPLEXITY AND SOCIAL DECISION RULES

HANS W. GOTTINGER

1. INTRODUCTION

The notion of 'global rationality' underlying the construction of 'economic man' that is generally accepted at least in normative economics has come increasingly under attack by those who care for more fruitful behavioral assumptions in economic reasoning. This notion is intrinsically related to various optimization programs that have been implemented in economics but that have been found only of limited use in realistic, complex situations. H.A. Simon [8] deserves credit having observed the limitation of global rationality and suggesting a modification of this program by introducing his concept of 'limited rationality.' To a great extent these ideas were carried forward in studying human thought processes where it was found that decision-makers, for purposes of problem-solving, go through several stages of goal formation, a hierarchical representation of goals, super- and subgoals, where at every stage goal attainment rather than optimization is called for. Such programs are motivated by the complexity of problem-solving tasks that are treated successfully by decomposing problem-solving in a sequential way and by associating to every stage of the process the attainment of a subgoal. Goal-oriented behavior, therefore, is non-optimizing behavior

and only improvement-related with respect to the attainment of the next goal in a sequence. (G.W. Ernst and A. Newell [2]).

Simon [11] relates a need for revision of the 'economic man' to the limitation of access of information and computational capacities being available to human decision-makers. The computational dimension is probably the most important aspect of characterizing 'limited rationality', in fact, this point has been brought up in a similar connection by H. Leibenstein [6] where he interprets 'rationality' in terms of 'calculatedness' (computability) and tightness or looseness of calculatedness is supposed to cover the whole spectrum between rationality and limited rationality.

The computational dimension of limited rationality as applied to the social choice process is analyzed here in a more rigorous fashion than has been done before. It turns out that complexity is an essential tool for analyzing constraints on the decision process. Moreover, any axiomatic system of 'limited rationality', yet to be defined, must contain complexity as a primitive notion.

The paper attempts to show how a social decision function can be constructed, by unconventional tools, such that it is compatible with individual decision functions. Complexity enters the construction as the basic limiting factor.

2. CHOICE PROCESSES AND COMPLEXITY

On the level of individual or social choice problems complexity relates to the ability or inability of human beings

to make effective choices in a consistent or rational way. In this regard complexity exhibits some kind of uncertainty that cannot be treated properly in terms of probabilities.

One clear indication when complexity enters individual decision-making is given by not being able to prove that a utility function representing preferences or choices does exist. If this proves to be a legitimate question on the level of individual decision-making, it is even more so on a social choice level.¹⁾ F.S. Roberts [9,p.127] proposes two ways out of this dilemma:

' ... one approach to the decision-making is to describe a procedure whereby we can modify or redefine or make explicit our preferences in the course of decision-making in order to become more "rational" (i.e., that such a utility function will exist).'

A second approach, somewhat less demanding, is to settle for a utility assignment which best approximates the utility function.

It is doubtful that the first approach leads to a satisfactory solution. Since even if it is possible to teach individuals how to act more rationally than they used to behave, they will never be 'perfect computers' and there is a threshold of complexity beyond which they cannot effectively handle situations, for instance, making choices among many alternatives. Put in a different way, you can try to teach subjects how to make optimal decisions in a simple course of actions, as J. Marschak [7] suggests on the basis of psychological studies on that matter. But still teaching optimality does not cope with the problem that people simply make mistakes because of complexity or 'embarras de richesse' in selecting among many alternatives -- in the same way as people may understand simple

arithmetical rules but cannot solve complicated arithmetical problems in the large because of time, resource and computational constraints.

The alternative then is that people adopt reasonable behavior strategies (in the sense of being within their 'computational budget') which cope with the intrinsic complexity of (social) choices, e.g. those rules exhibiting non-optimizing behavior.

Regarding the second approach, much of the contribution by measurement theory has been in the direction of weakening preference requirements (for example, Luce's semi-order theory, avoiding indifference, but admitting thresholds).

The weaker assumptions aim at reducing the computational burden of decision-makers, yet they fail to make explicit the complexity bounds in forming decision rules.

Many choice processes in the real world, in contrast to theoretical constructs used by choice theorists, represent essentially ill-structured problems to the extent that solutions of these problems are not readily available and they involve an excessive amount of computational power. In general, a problem is considered to be well-structured if it satisfies a number of criteria, the most important of which relate to the existence of at least one problem space that provides for solvability with the help of a practicable (reasonable) amount of computation or search. Apparently well-structured problems such as theorem-proving and chess playing in artificial intelligence turn out in many instances to be ill-

structured, given the problem-solving power of problem-solving methods. There seems to be an intrinsic relationship between well- or ill-structuredness of a problem and the threshold of complexity (in von Neumann's sense) below which a system shows a regular, stable and predictable behavior but beyond which often quite different, sometimes counterintuitive modes of behavior can occur. A problem can be well-structured in the small, but ill-structured in the large.. According to H. Simon [13] 'the difficulty stems from the immense gap between computability in principle and practical computability in problem spaces as large as those of games like chess.' This generally applies to complicated choice processes.

Therefore, the problem of complexity is similar to the problem a chess player faces when searching for a 'satisfactory' strategy in chess. The social choice problem resembles the choice of strategies in chess-playing to the extent that the decision-maker is involved in a choice problem of combinatorial dimension. To search for all game-theoretically possible alternatives goes far beyond the computational ability of the human being.

One conclusion, therefore appears to be obvious: we have to depart from behavioral hypotheses involving optimizing behavior, as convenient as it might be in mathematical terms, since it does not come to grips with non-trivial choice problems in complex situations. We do not have to leave the grounds of rationality, a rule-of-thumb method may be rational in a restrictive sense, thus we have to view it in terms of

'limited rationality.' Rule-of-thumb methods may be applied for various reasons: either because the individual faces expected costs of computation to be far beyond expected utility of further searches in choice-theoretic behavior or he (she) is faced with an immense mass of alternatives to the effect that he (she) is psychologically outstripped by the ensuing 'complexity of computation.' Chess players tend to choose simpler decision rules, they do not consider all possible strategies and pick up the best, but generate and examine a rather small number, making a choice as soon as they discover one that they regard as satisfactory. According to H.Simon [12], 'limits of rationality in chess involve (a) uncertainty about the consequences that would follow from each alternative, (b) incomplete information about the set of alternatives, and (c) complexity preventing the necessary computations from being carried out.'

All three properties may be subsumed under a more general concept of complexity in choice-theoretic situations. For example, uncertainty and lack of information may here assume different aspects to what is widely known in statistical decision theory and the economics of uncertainty, e.g. uncertainty resulting from computational incapability when faced with a large number of choice alternatives. These are essentially non-probabilistic situations. Thus, complexity is an important tool for evaluating decision rules, in fact, it may prove instrumental for an axiomatic analysis of 'bounded rationality' which is still lacking.

3. SOME FORMAL PREREQUISITES

We present here some formal definitions toward developing a more general theory of complexity for social choice situations that may prove useful to understand the concepts to be used throughout the following section. In this particular context, such a general theory of complexity has been introduced earlier by C. Futia [3], more generally see Gottinger [5].

(1) If A is a non-empty set of symbols, then let A^* represent the set of all strings whose members are elements of A , i.e. $A^* = \{(a_1, \dots, a_n) : n \geq 1 \text{ and } a_j \in A\}$. Then we define a sequential machine as a function $f: A^* \rightarrow B$ where A is the basic input set, B is the output set and $f(a_1, \dots, a_n) = b_n$ is the output at time n if a_j is the input at time j ($1 \leq j \leq n$). This is the external description of a sequential machine by specifying a function $f: A^* \rightarrow B$.

The internal description involves a circuit $(A, B, Z, \lambda, \delta)$, where A and B are defined as above, Z is the (nonempty) set of internal states, $\delta: Z \times A \rightarrow B$ is the output function, $\lambda: Z \times A \rightarrow Z$ is the next-state function. The step from the external to the internal description of a system is referred to as identification. It is a problem to show that given f we may find a C and a $z \in Z$ such that C 'realizes' f with $f = C_z$.

For example, let $C_z: A^* \rightarrow B$ be the system given by starting $C = (A, B, Z, \lambda, \delta)$ in state $z \in Z$, then C_z is defined inductively in a straight-forward way:

$$C_z(a_1) = \delta(z, a_1)$$

$$C_z(a_1, \dots, a_n) = C_{\lambda(z, a_1)}(a_2, \dots, a_n) \text{ for } n > 2.$$

(2) Let $f: A^* \rightarrow B$ a machine. Then f^S , the semigroup of f , is given by the congruence \equiv_f on A^* where for $t, r, \in A^*$, $t \equiv_f r$ if and only if $f(\alpha t \beta) = f(\alpha r \beta)$ for all $\alpha, \beta \in A^* \cup \{1\}$. Then, if $[t]_f$ denotes the equivalence class of the equivalence relation \equiv_f containing t , we have $f^S = \{[t]_f : t \in A^*\}$ and $[t]_f \cdot [r]_f = [tr]_f$ (where tr denotes the product in A^* and \cdot denotes the product in f^S). $\{1\}$ is the empty string.

(3) A semigroup S is combinatorial if and only if each subgroup of S is of order 1.

(4) A right mapping semigroup or right transformation semigroup is a pair (X, S) , where X is a nonempty set, and S is a subsemigroup of $F_R(X)$ the semigroup of all mappings of X into X under the multiplication $(f \cdot g)(x) = g(f(x))$. For each $x \in X$, $s \in S$, let $xs = (x)s$. Then the following conditions are satisfied:

$$(1) \quad x(s_1 s_2) = (x s_1) s_2.$$

$$(2) \quad s_1, s_2 \in S \text{ and } s_1 \neq s_2 \text{ imply } x s_1 \neq x s_2 \text{ for some } x \in X.$$

(5) (Wreath Product) Let (X_j, S_j) be right mapping semigroups for $j = 1, \dots, n$. Let $X = X_n \times \dots \times X_1$. Let S be the semigroup of $F_R(X)$ consisting of all functions $\psi : X \rightarrow X$ satisfying the two following conditions:

(i) (triangular action) If $p_k : X \rightarrow X_k$ denotes the k th projection map, then for each $k = 1, \dots, n$ there exists $f_k : X_k \times \dots \times X_1 \rightarrow X_k$ such that

$$p_k \psi(t_n, \dots, t_{k+1}, t_k, \dots, t_1) = f_k(t_k, \dots, t_1)$$

for all $t_i \in X_i$, $i = 1, \dots, n$.

(ii) (kth component action lies in S_k) We require $f_1 \in S_1$, and, for all $k = 2, \dots, n$ and all $\alpha = (t_{k-1}, \dots, t_1) \in X_{k-1} \times \dots \times X_1$, the function $g_\alpha \in F_R(X_k)$ given by $g_\alpha(y_k) = f_k(y_k, t_{k-1}, \dots, t_1)$ is an element of S_k .

Then $(X_n, S_n) \} \dots \} (X_1, S_1) = (X, S)$ is the wreath product of $(X_n, S_n), \dots, (X_1, S_1)$, and $(X_n, S_n)w \dots w(X_1, S_1)$ is the abstract semigroup determined by (X, S) .

(6) Let (X, S) and (Y, T) be right mapping semigroups. Then we write $(X, S) \mid (Y, T)$, read (X, S) divides (Y, T) , if and only if (1) there exists a subset Y' of Y and a subsemigroup T' of T such that Y' is invariant under the action of T' (i.e., $Y'T' \subseteq Y'$); and (2) there exists a map $\theta : Y' \twoheadrightarrow X$ (\twoheadrightarrow means onto) and an epimorphism $\phi : T' \twoheadrightarrow S$ such that $\theta(yt) = \theta(y)\phi(t)$ for all $y \in Y'$, $t \in T'$.

(7) (Krohn-Rhodes Decomposition [8]) Let (X, S) be a right mapping semigroup. Then the (group complexity $\#_G(X, S) = \#_G(S)$) is defined to be the smallest non-negative integer n such that

$$S \mid (Y_n, C_n)w(X_n, G_n)w \dots w(Y_1, C_1)w(X_1, G_1)w(Y_0, C_0)$$

holds with G_1, \dots, G_n being finite groups and C_0, \dots, C_n finite combinatorial semigroups (flip-flops), i.e. the minimal number of alternations of blocks of simple groups and blocks of combinatorial semigroups necessary to obtain (X, S) . Hence by making full use of decomposition results on sequential machines one could redefine complexity in terms of the phase space decomposition.

Therefore, complexity finds its group-theoretic roots in the fact that the transformation semigroup can be simulated (realized) by the wreath product of all pairs of component machines whose semigroups are simple groups and those machines whose semigroups are finite combinatorial semigroups (= flip-flop machines). Intuitively speaking, a combinatorial semigroup corresponds to a machine that virtually does no computation but rather switches inputs and outputs among various input-output configurations. This property reminds us of information theory when selecting events which have information measure zero. These types of machines generate regular patterns to be expected, they do not yield any surprise. Therefore, their behavior does not produce information. Since everybody understands it, it cannot be complex. This result has some immediate impact on possible applications. It suggests that if we are able to detect subsystems that behave like flip-flops we could erase these subsystems without changing the structural complexity associated to other subsystems but, nevertheless, decreasing the computational complexity in terms of length of computations.

On the other hand, simple groups conform to machines that perform simple arithmetic operations (such as addition, multiplication,...). Many examples of that sort have been given by John Rhodes [8]. A simple group constitutes the basic (irreducible) complexity element which increases the complexity of the machine by just one unit. Hence punching out groups of that kind in the decomposition lowers complexity at most by one. Now what is the significance of the Kroh-Rhodes theory? It shows

us to which extent we can decompose a machine into components that are primitive, irreducible and that the solution depends on the structure of components and on the length of computation. Hence complexity does not depend only on how long a chain of components there are, but also on how complicated each component is. Therefore, complexity takes account of the total number of computations in a chain (the computational aspect) but also of the inherent complexity of the subsemigroups (submachines) hooked together via the wreath product (the structural aspect). The structural aspect can heuristically be represented by the amount of 'looping' in a computer program that computes S on X . This has been proposed by C. Futia [3] for computing sequential decision or search rules.²⁾ These are the key features of an algebraic theory of complexity.

4. AN EXAMPLE OF A DECISION OR SEARCH RULE

The subsequent example has been adapted as an illustration from a similar search problem presented by Futia [3]. An individual, as a member of society (or voter) is subsequently confronted in a 'large' market of public goods to choose among different kinds of commodities or services (nuclear energy, missiles, health care, etc.) offered to him for sale by different government agencies at different prices (i.e. tax rates). In order to receive a tax rate quotation (or possibly some other relevant information) from any given agency, the voter must incur some (not necessarily monetary) cost constituting his marginal search cost. The voter's goal is: given a certain bundle of public goods that satisfies his aspiration he wants to search for low tax rates such that his final

taxes (plus total search costs) will be kept as low as possible.

This problem can be formalized as follows:

Let t_i denote the tax rate quotation of agency i . Let $t = (t_1, \dots, t_n)$ be the tax structure and suppose $t_i \in [0, 1] = I$. Denote by I^n the n -dimensional cartesian product of I , and define a probability density F on I^n representing the voter's initial belief about which tax rates the agencies are likely to quote. The order of quotations presented to the voter is considered to be irrelevant, thus, for simplicity, it is assumed that F is symmetric, i.e. if p is a permutation of $(1, 2, \dots, n)$ and if $t^p = (t_{p(1)}, \dots, t_{p(n)})$, then $F(t) = F(t^p)$.

The set-up of this problem enables to construct a decision rule which prescribes to the voter, for each i , whether to stop searching after receiving i quotations or whether to continue searching on the basis of the i quotations he has received. A decision rule is assumed to be a mapping from a set of observations into a set of actions. In this problem, for each i , let the set of actions be $A = \{\text{'accept'}, \text{'reject'}\}$, and the set of observations be $O_i = I^i$. Then a decision rule is a sequence of functions $D = (D_1, \dots, D_{n-1})$, where $D_i: O_i \rightarrow A$ if $(t_1, \dots, t_i) \in O_i$, then $D_i(t_1, \dots, t_i)$ records the voter's decision to either accept the tax rates that have been quoted to him and choose (by vote) the given bundle of public goods presented to him, or to continue searching and reject tax rates t_1, \dots, t_i .

Now it is perfectly legitimate to ask for this kind of problem what is the voter's optimal decision rule? This question could be answered by the machinery provided in statistical decision theory to find optimal solutions for search problems

(see Gottinger [5a]).

Instead, here we are interested in the basic ill-structuredness of the problem given by the complexity of the decision rule. To this end, on the basis of the previous section, we proceed to associate with every decision rule D a (computer) program f_D which computes D . This permits us to define the complexity of the program by the amount of 'looping' between subprograms (computational complexity) and the intrinsic complexity of the subprograms (structural complexity). Hence, a sequential machine is used as a metaphor for determining complexity of sequential decision rules. This can be further illustrated by elaborating on the problem above by using the sequential machine framework.

Let A = set of observable tax rates = finite subset of $[0,1]$.
Let $B = \{\text{'stop'}, \text{continue to } i+1, i=1,2,\dots,n\}$. Then the machine f_D is defined inductively on the length of the input sequence by

$$\begin{aligned} &D_1(t_1) \text{ if } m = 1, \text{ or if} \\ &f_D(t_1, \dots, t_{m-1}) = \text{'stop'} \text{ or } = D_i(t_{m-i}, \dots, t_m) \\ &\qquad\qquad\qquad \text{if } f_D(t_1, \dots, t_{m-1}) = \\ &\qquad\qquad\qquad \text{'continue to } i+1 \text{' } \end{aligned}$$

The computational length and the structural complexity of subsystems that are needed to compute f_D reflects a measure of complexity for f_D (equivalently for the decision rule D). Obviously, optimal is a rule that is generally more complex and more expensive but which may very well be beyond the computational power and sophistication of the voter. Hence the

voter, facing an ill-structured problem wants to make it well-structured by seeking a decision rule which matches his computational ability and sophistication.

5. COMPLEXITY OF DECISION RULES

We suppose that the decision-maker identifies alternatives in his choice space and does express preferences between at least two alternatives by simply computing, else he finds alternatives 'undecidable' or 'incomparable' that cannot be computed. Preference statements are therefore translated into computing devices, indifference statements are kept out because of possible vagueness. The decision-maker represented as a simple finite state machine, can be decomposed according to performing these tasks. ³⁾ In the first case the job to be done, e.g. computing preferences, is achieved by a simple group machine (that is a decision machine acting as a simple group in the mathematical sense), in the second case the activity consists of a combinatorial machine, acting as a 'flip-flop' which does not compute anything. ⁴⁾ Realizing a decision-rule therefore means a decomposition of the decision process according to the decomposition of machines into component machines that 'hooked' together (via the wreath product) realize the overall machine. Of course, the complexity of decision rules may vary; a 'sophisticated' decision-maker may activate more simple groups, less flip-flops, or groups that compute faster, more accurately and more reliably. This type of decision-maker will carry more structural complexity in the sense given in the previous section.

A (social) decision rule is a sequential decision rule and as such is considered to be a finite state machine (associated to a finite semigroup), and according to complexity theory it has a finite decomposition. In this regard the results of Krohn-Rhodes complexity theory apply. The idea involved here is to factor a social choice process into parts (components) where the global process is modelled as a transformation semigroup associated to a social decision rule, and the local parts are represented by transformation subsemigroups. The new tools originate from decomposition results in automata theory.

Consider a choice set of finitely many alternatives $X = \{a, b, \dots, x, y, z\}$ and let $D_i = 1$ iff i prefers x to y , $D_i = 0$ iff i is 'undecided' about x and y , $D_i = -1$ iff i prefers y to x . Let D be a nonempty set of decision rules D_i , X a nonempty collection of subsets of X , a social decision function (SDF) then is a function $F: X \times D \rightarrow P(X)$, $P(X)$ being the power set. A SDF for individual i is given by $F(\{x, y\}, D_i)$, $x, y \in X$.

Social decision functions are in fact decision machines in the sense that they decide on propositions about accepting or rejecting social states, computing them by discrimination, (preference, non-preference). By doing this, they generate as outputs decision rules and induce next states representing changes in preference profiles or configurations. There is good reason to argue that we should leave out indifference statements since they cannot clearly be distinguished from the phenomenon of 'undecidability'. Intransitive in-

difference arises in situations where a chain of indifferences, each of which seems reasonable, adds up to a sufficiently large difference to yield a definite preference between the first and the last items in the chain. We would like to avoid intransitive indifference, therefore we require the decision machine only to accept preference rather than indifference statements.

In order to construct such a decision machine let us state the following

Problem: Let $X^n = X_1 x \dots x X_n$ be the social choice set when the DM is confronted with a sequence of finitely many social alternatives. Let $A_0 \subseteq A_1 \subseteq \dots \subseteq A_n$ be those sets of alternatives in which the DM can actually find comparisons (in the sense that he prefers alternatives in these sets and finds himself in a position to compute preferences). Let A be a nonempty collection of all A_0, A_1, \dots, A_n . Then he constructs selection functions $\rho_0, \rho_1, \dots, \rho_n, \rho_i : X^n \rightarrow A$ such that for all $x_i \in X_i$, $\rho(x_i) \in A_i$. In a way, ρ_i constitutes a reduction mechanism by reducing all possible alternatives with which the DM is confronted to those which are computed as actual choices.⁵⁾ It is said that the DM accepts the decision rule $D_i(x_0, \dots, x_i)$ if $\rho(x_0, \dots, x_i) \in A_i$, more explicitly, accept $D_0(x_0)$ if $\rho(x_0) \in A_0$, accept $D_1(x_0, x_1)$ if $\rho(x_0, x_1) \in A_1$, etc.

There is an upper bound, representing the complexity bound of the DM, beyond which he is unable to compute his preferences. The upper bound somewhat restricts him in selecting decision rules which are 'beyond his complexity.' Therefore, let $k(D)$ be the largest integer satisfying the

bound such that $A_{k(D)-1} \not\subseteq A_{k(D)}$. How is the bound to be determined?

In a different context, regarding the complexity of (dynamic) finite-state systems, I distinguish between design and control complexity.

To recall (cf. Gottinger [5]), under design complexity I understand that complexity (number) associated to the transformation semigroup in which full use of the system potential is made. Under control complexity I understand that specific complexity (number) that results from computations which keep the entire system or at least part of it under complete control. A qualitatively stable decision rule would be a rule for which design and control complexity coincide. However, in most practical cases design complexity will exceed control complexity. Since one cannot assume that the control complexity of an average (unsophisticated) DM can be increased by teaching him how to behave in a rational manner one should pick up designs of decisions rules for which there is a reasonable understanding and control. 6)

Example. In a game of chess the number of all possible strategies to achieve a chess-mate corresponds to the design complexity of a chess-playing program. The number of all actual strategies chosen by a particular chess player to achieve success corresponds to his control complexity. Given two chess players both initially endowed with the same knowledge of how to play chess, then if in a sufficiently long sequence of repetitive plays one does better than the other, he exhibits a better understanding of the game, e.g. a higher control complexity.

In a certain way both concepts are naturally associated to 'programs of optimization' and 'programs of satisficing or bounded rationality), respectively. That is to say, design complexity pertains to that decision rule (which is best in some appropriate sense), in general an optimization principle is involved, which, however, cannot be realized given the limited computational resources of the DM (control complexity).

To which extent this bound can effectively be determined by experiments appears to be a problem in experimental psychology. However, it is possible, at least in principle, to give a set of criteria under which it can be determined whether a DM chooses decision rules violating his bound of complexity. ⁷⁾ Whenever individuals violate in experiments a set of consistency postulates (such as transitivity), namely those which they have accepted at the very beginning, they will realize that they have committed computational errors. Thus commitment of errors or violating consistency postulates seem to be suitable criteria for determining complexity bounds of computation. In experimental situations, subjects then have to be confronted with various decision rules of a different complicated character and the class of decision rules in which no errors or almost no errors occur constitute those which satisfy the control complexity of the DM.

Those decision rules are called qualitatively stable. Only qualitatively stable decision rules guarantee that social, economic and political processes can be controlled in any effective way by social choice, otherwise the amount of error, mis-

representation of preferences, etc. could easily lead to a destabilization of the social system, and some degree of rationality can no longer be maintained.

5. A CONSTRUCTION OF COMPATIBLE SOCIAL DECISION RULES

Let P_1, P_2, \dots be sets of computable preference profiles for $i = 1, 2, \dots$ individuals of the social group achieving a common social decision rule D (matching the preference profile of the social group). Let there be D_1, D_2, \dots decision rules acting as sequential machines such that D_k computes the preference profile P_k . Then we define the complexity of the social decision rule $D, \theta(D)$, to be equal to $\min \{ \theta(D_j) : j = 1, 2, \dots \}$.

In short, the complexity of a social decision rule is bounded by the minimum complexity of any individual decision rule D_j which is able to generate any individual preference profile matching the preference profile of the social decision rule. We proceed to associate a social decision rule (SDR) D for the social choice problem with a finite semigroup $S(D)$. We could envisage the social choice process as a transformation semigroup $(X, S(D))$ where X is the set of social choice alternatives each individual (in the social group) is searching for, while elements of $S(D)$ will be finite sequences of preference quotations generating the preference profile.

We could define $X = \{ * \} \cup A_0 \cup A_1 \cup \dots \cup A_{n-1}$ with U disjoint. This is the set of the DM's choice histories. Then $\rho(x_1, \dots, x_i) \in A_i$ represents the history of the DM's preference statements who has completed i searches and has made choices over x_1, \dots, x_i .

A DM will stop searching if further searching will violate reasonable consistency criteria. The stop rule of searching is imposed by the complexity bound of the social decision rule. By construction, the complexity of D , $\theta(D)$, is equal to the complexity of $S(D)$, $\theta(S(D))$. Again the complexity of the SDR D is bounded by the minimum complexity of the individual decision rules D_j (finite state machines) which by interacting realize a compatible social decision rule.

The procedure how to generate a computable SDR when all members of the society set up their own individual decision rules can be described as a sequential game among the members. If the game has a von Neumann value we agree to say that a compatible SDR has been realized.

For simplicity, let us assume that there are only two members of the society which after having computed their individual decision functions want to find a compatible SDR (which satisfies both).

Assume that the game starts in C_{z_0} with strategy ρ constituting the selection rule of the first member of society, then the circuit $C = (A, B, Z, \lambda, \delta)$ is the preference profile with $\lambda : A \times Z \rightarrow Z$ and $\delta : Z \times A \rightarrow B$. Let A be the set of social choices that have been made by Player I (and the configuration is revealed to Player II). Then B is the set of resulting social choices of Player II that adjust to the preference profile of Player I. Z is the set of adjusted social choice configurations of the game as they appear to Player I. $\lambda(z, a)$ and $\delta(z, a)$ are interpreted as follows: if z is the adjusted social choice con-

figuration, as it appears to Player I, and a is the choice which enters as input to Player II, let $z \cdot a$ be the social choice configuration after the choice a is made on the configuration z . Let $D(z \cdot a) = b$ be the decision rule generated by C when the position presented to C is $z \cdot a$. Define $\delta(z, a) = b = D(z \cdot a)$ and furthermore define $\lambda(z, a) = (z \cdot a) \cdot b \equiv (z \cdot a) \cdot D(z \cdot a)$, where z_0 is the initial position. Suppose our SDR can be put in binary form, whenever the 'compute preferences' key is followed we assign 1, otherwise 0.

The latter case will be interpreted as meaning that no consistent preference statement can be made since the number of choices involved is too large and therefore we have to eliminate redundant choice alternatives. Then under these circumstances, we could consider for at least two players the construction of a compatible SDR to be equivalent to a game tree with binary outcomes.

Example. In this game each player plays zero or one successively -- corresponding to the construction of the decision rule. Let us assume the circuit C is a player who responds to the action of the first player, and the circuit C' . W denotes a win for the player, L denotes a loss for the player. The payoff is +1 for W , and -1 for L . Clearly, the von Neumann value for this game is +1 for the player who goes second. Assuming C goes second the strategies achieving the von Neumann value +1 can be listed as follows (and read out of the game tree):

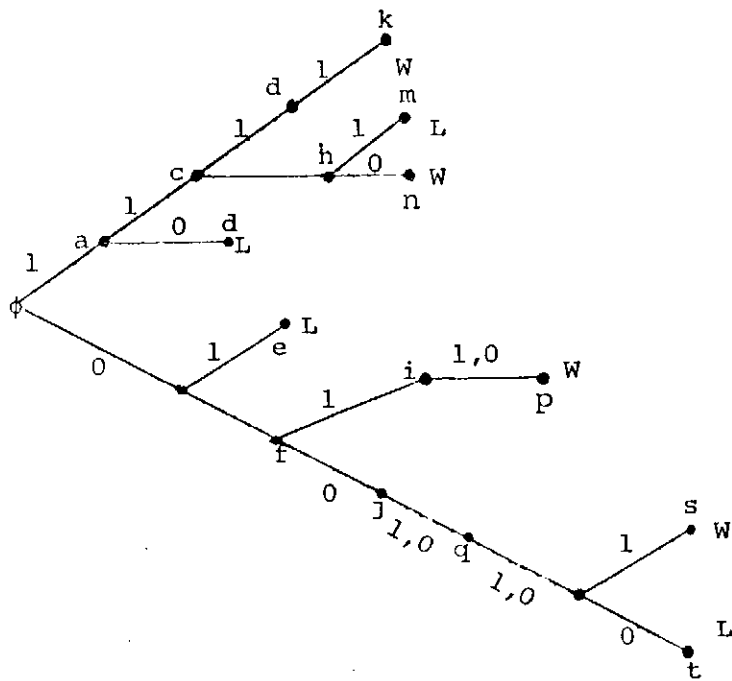


Fig. Game tree with binary outcomes

(winning strategies for C)

- (a) $(\phi, 1), (a, 1), (c, 1), (g, 1), (k, W),$
- (b) $(\phi, 1), (a, 1), (c, 0), (h, 0), (n, W),$
- (c) $(\phi, 0), (b, 0), (f, 1), (i, (1, 0)), (p, W),$
- (d) $(\phi, 0), (b, 0), (f, 0), (j, (1, 0)), (q, (1, 0)), (v, 1), (s, W)$

Let $C = (A, B, Z, \lambda, \delta)$ be defined as follows:

$$A = \{0, 1(O, L), (O, W), (1, L), (1, W)\}, B = A,$$

$$Z = \{\phi, a, b, c, d, e, f, \dots, r, s, t\}.$$

Then $C_{z_0}^* : A \rightarrow B$ induces a sequential social decision rule to which there is associated a complexity, the complexity of the transformation semigroup (X, S) . The problem is to find a minimal complexity of the transformation semigroup that permits a construction of a social decision rule compatible with the choice behavior of individual members of the society. In view of (a)-(d) we succeed in doing this by finding the string of minimal length, i.e. the decision rule with the minimal complexity. The upper bound for the complexity follows from the following result:

Proposition (J. Rhodes): Let S be a semigroup of mappings on the finite set X (sequential choice space). Let r be the maximum range (or fixed points) of any idempotent $e = e^2 \in S$. Then $\#_G(S) \leq r-1$.

Proof. Let I be the ideal generated by the idempotents of S . Then S/I is combinatorial and $I \leq \{f: X \rightarrow X : |f(x)| \leq r\} = I_r$. Further $I_k, k=1, \dots, n$ are the ideals of $F_R(X)$, the semigroup of all mappings of X into X . Then by the results of Rhodes et al. [8] it can be shown that $\#_G(S) = \#_G(I)$ and $\#_G(I_r) \leq r-1$.
q.e.d.

6. SUMMARY AND EXTENSION

We have noticed how choice processes could be factored into component subprocesses and how these are associated to properties of transformation semigroups. A social choice process could be understood as a sequential game, as an interaction between individual choice processes in such a way that the interaction generates a SDR that is compatible with all individual choice processes. To achieve this, we use new tools of 'limited rationality', derived from automata theory, embodied in the system of social decision-making. Complexity as a crucial factor in the choice of decision rules is related to limitations of human decision-making in terms of their capacity to recall, memorize and compute only relatively few items among which consistent choices can be made. In contrast to conventional social choice theory we only consider preference profiles that are in a certain sense 'computable', thus restricting the social choice process to reasonable behavior rules. It is not clear so far to which extent the ideas expressed herein will have an impact on traditional social choice theory, namely relating to Arrow-type impossibility or possibility theorems. In actual human decision-making, alternatives are often examined sequentially, consequently we consider this approach to be basically of sequential type, whereas traditional theory is static, e.g. all alternatives are evaluated before a choice is made. Furthermore, in view of Arrow's assumptions on constructing a social welfare function (SWF) it appears that the assumption of 'unrestricted domain' of the choice set will no longer hold be-

cause of imposing strict computational requirements.

An obvious extension would consist of using complexity of decision rules as a primitive notion for an axiomatization of economic behavior that introduces explicitly behavioral assumptions related to limited computability. The DM is not only limited in his choice behavior by computational requirements, equally important, he is also restricted by acting as a member of a group or social class where in order to achieve some consensus (for example, a common group decision function) he has to adjust his behavior to past choices of other group members. This is illustrated by looking at the adjustment mechanism as a sequential game. The determinants of the game (environmental conditions, previous choice configurations) are themselves determined as outcomes of complicated cognitive processes, bounded by complexity. Complexity of this kind virtually covers two aspects: one is structural, the other computational. Structural complexity here relates to the 'sophistication' of the DM, how he can reason when confronted with difficult tasks, depending on his problem-solving capability (as discussed in the example of the missionaries and the cannibals, see Ernst and Newell [2]).

Computational complexity relates to experience, to the ability to learn doing things, organizing computations. Both factors are likely to be highly correlated, but to a certain degree there will be tradeoffs between both, thus they are comprised in one complexity measure. For a particular decision-making design both factors add up to yielding the control complexity which together with the given design

complexity provides the fundamental evolution complexity relation. This again has a clear interpretation in defining qualitatively stable decision rules.

ACKNOWLEDGEMENT

Research supported in part by Thyssen-Stiftung, Cologne (Köln), W. Germany.

Most work has been done during my visit to Western Management Science Institute, UCLA, in summer of 1976. The help and hospitality of this institution is much appreciated. In particular, I am indebted to Miss Naomi Yano, Los Angeles, and Ms. Goergel, Bielefeld, for typing the manuscript.

FOOTNOTES

- 1) Likewise, a similar problem arises if you want to capture (probabilistic) uncertainty by the representation of finite subjective probability measures. Here it is by no means clear that the representation is unique. P. Suppes [14] reports, in referring to Scott's axioms of finite probability, derived from a qualitative probability structure:
' The more profound difficulty ... is the combinatorial explosion (my italics) that occurs in verifying the axioms when the number of events is large. To check connectedness, for example, we need only consider pairs of events, and to check transitivity, only tuples of events. But, it is fundamental for the kind of axiom schema required to express necessary and sufficient conditions in the finite case that n-tuples of events of arbitrary n must be studied as the number of events increases. As a possible empirical theory of belief, or as a rational one, this seems impractical, and even for fairly small experiments, the effort to determine whether there is a representing unique probability measure requires the use of a moderate size computer facility.'
P. Suppes then sets out to search for simpler axioms, which he terms 'inexact measurement', that attempts to reduce the implicit complexity of finding unique measures of belief.

- 2) In a different context, such a problem-solving machine transforming 'tasks' into 'satisfactory actions' (controls) as a model for an adaptive mechanism has been described by B.R. Gaines [4].

- 3) This decision-making process, organized in this way, is somewhat related to the heuristic conceptualization of the decision-making process as proposed by R. Selten in his 'Chain Store Paradox' [10]. The simple group machine pertains to his level of reasoning which is characterized by a conscious effort to analyse the situation in a rational way on the basis of explicit assumptions whose validity is examined in the light of past experience

and logical thinking. On the other hand, the combinatorial group machine applies to his routine level where 'decisions are made without any conscious effort'. Now it seems evident that the higher level of reasoning brings 'sophistication' in the decision process, increases complexity (structurally) whereas routine decisions do not establish structural complexity by itself. This is not to say, in agreement with Selten, that the higher level always yields the better decision, but this is to say that decision problems of the problem-solving variety require the activation of computational devices with more rather than less structural complexity. But in general, again in agreement with Selten, it depends on the nature of the decision problem.

- 4) According to C. Futia [3], since combinatorial semi-groups ('flip-flops') generate no feedbacks, he argues that feedbacks are only provided by the basic complexity elements, the simple groups, in the Krohn-Rhodes decomposition. Since complexity of his sequential decision rule D , equivalent to the complexity of the associated semi-group $S(D)$, is considered to be proportional to the amount of 'feedback' or 'looping' in a computer program that executes D , it is obvious that he measures only a restrictive notion of complexity, what I call structural complexity. However, he neglects the number of wires or interconnections between all components within the Krohn-Rhodes decomposition, i.e. the length of computations, what I call computational complexity. But only structural plus computational complexity provides a comprehensive measure of complexity for sequential processes. The distinction between both is important, particularly in view of possible tradeoffs between both in the design of decision rules and by comparing decision rules with different designs.

- 5) Now this reduction mechanism induces the choice space to be partitioned into at least two parts, one part which is 'computable', generated by computable preference statements, the other part is 'non-computable', imposed by indecisiveness in choosing among alternatives. Therefore the actual choice space generated by the selection functions is derived from the following equivalence: computable choice space equals given choice space modulo non-computable choice subspace.

- 6) Another way of looking at it utilizes H. Simon's [13] distinction between a well-structured and an ill-structured problem. A stable decision rule is equivalent to a well-structured problem. An unstable decision rule results from the possible 'computational gap' which may occur in the problem-solving process. As Simon [13, p. 186] puts it: '... definiteness of problem structure is largely an illusion when we systematically confound the idealized problem that is presented to an idealized (and unlimitedly powerful) problem-solver with the actual problem that is to be attacked by a problem-solver with limited (even if large) computational capacities'. So, in a way, if the problem-solver's control complexity is below the design complexity of the decision rule, he himself encounters an ill-structured problem, or equivalently, his decision rule is unstable. Then, it is desirable to redesign the decision rule in such a way that his ill-structured problem becomes well-structured to the extent that the new design coincides with the computational power of the problem-solver.

- 7) H. Simon suggests a 'common sense' test based on the introspective knowledge of our own judgmental process.

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