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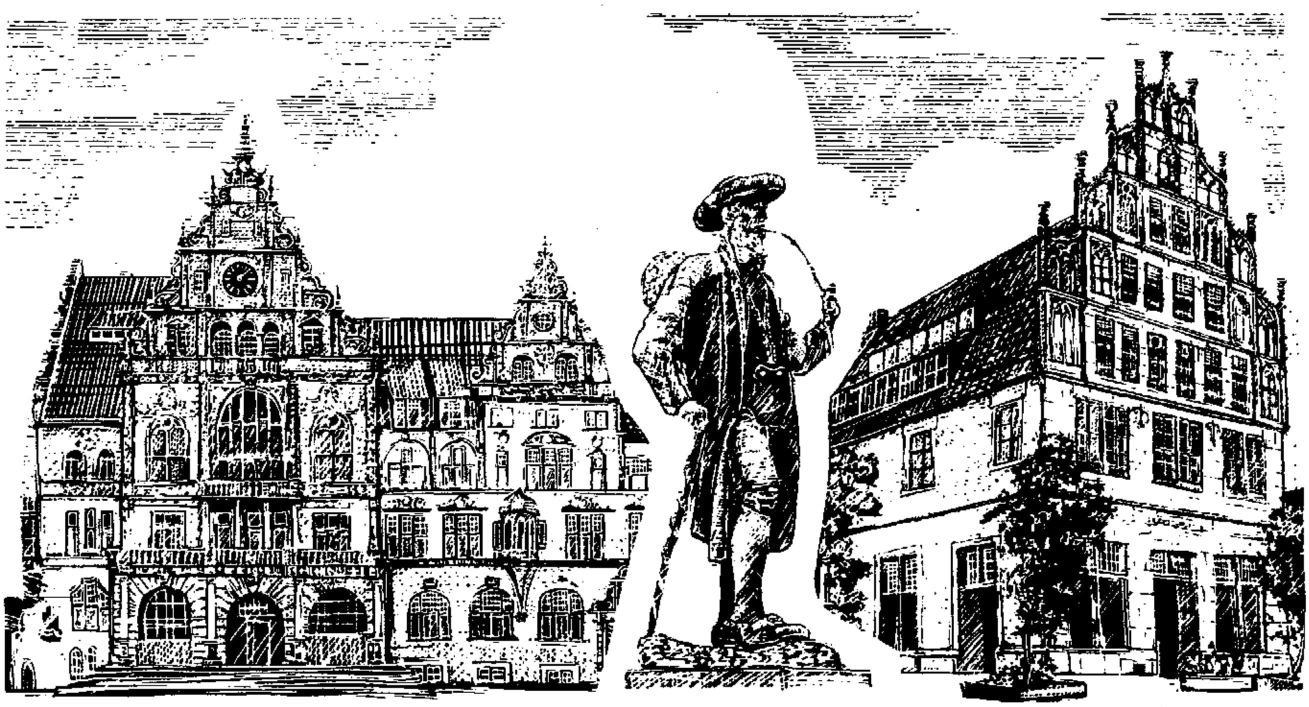

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ABSTRACT

A class of games is investigated, where each player has to decide whether to enter a market or not. A player's payoff is the difference between market profits and entry costs. The game is symmetric with respect to market profits, but asymmetric with respect to entry costs. The general solution concept developed by J.C. Harsanyi and R. Selten is applied to the situation. The solution is the pure strategy equilibrium point, where each player inside the market has lower entry costs than every player outside the market. Examples will be considered, which are special cases of the class under consideration.

Equilibrium Point Selection in a Class of Market Entry Games

by

Reinhard Selten and Werner Güth

Introduction

It is the purpose of this paper to investigate a class of games, where each player has to decide whether to enter a market or not. A player's payoff is the difference of two components, market profits and entry costs. The game is symmetric with respect to market profits, but entry costs are different for each player. Since only the entry decisions are modelled explicitly each player has two pure strategies.

It is assumed that the incentive to enter is a non-increasing function of the number of players who enter the market. It will be shown that under this assumption the game always has equilibrium points in pure strategies, moreover, the number m of players in the market is the same for all pure strategy equilibrium points. With the help of the solution concept developed by J.C. Harsanyi and R. Selten one of these equilibrium points will be selected as the solution of the game. John C. Harsanyi and Reinhard Selten recently agreed to change their solution concept in several aspects. Our treatment will be based on the new version and not on that outlined in earlier publications (J.C. Harsanyi, 1975, 1977, W. Güth, 1978). The solution of the game is that equilibrium point where the firms with the lowest entry costs enter the market.

Finally, examples of oligopoly situations will be discussed which are special cases of the class under consideration.

1. The Game Structure

The game we want to investigate is an n-person game in normal form where each player i has two pure strategies 0 and 1; here 0 stands for the decision not to enter and 1 indicates the decision to enter the market.

Each player i has entry costs C_i. The entry costs are different for different players. The players are numbered in such a way that we have:

(1) $$C_1 < C_2 < \ldots < C_n$$

Let m be the number of players who enter the market. Those who enter the market receive R_m and those who do not enter the market receive S_m as market profits. Let $\pi = (\pi_1, \ldots, \pi_n)$ be a combination of pure strategies where π_i is one of both pure strategies of player i and let $m(\pi)$ be the number of players with $\pi_i = 1$. Then the payoff function H_i of player i is as follows:

(2) $$H_i(\pi) = \begin{cases} R_{m(\pi)} - C_i & \text{for } \pi_i = 1 \\ S_{m(\pi)} & \text{for } \pi_i = 0 \end{cases}$$

The expression

(3) $$A_m = R_m - S_{m-1}$$

will be called <u>incentive to enter</u>. We shall assume that A_m is a non-increasing function of m:

(4) $$A_1 \geq A_2 \geq \ldots \geq A_n$$

Assumption (4) will be referred to as incentive monotonicity. In order to exclude border cases, which would complicate the analysis without adding much to the economic significance of the results, we require non-degeneracy in the following sense:

(5) $C_i \neq A_m$ for i, m = 1,...,n

(6) $C_i + C_j \neq C_k + C_l$ for i, j, k, l = 1,...,n pairwise different

For the same reason equality has been excluded in (1).

Obviously, the game is fully specified by 3n real numbers, namely $C_1,\ldots,C_n$, $R_1,\ldots,R_n$, $S_0,\ldots,S_{n-1}$, satisfying (1),(4), (5) and (6). In the following a game of this kind will be called a market entry game.

Note that we do not assume anything on the sign of C_i, R_m and S_m. Even if for the application, which we have in mind, the C_i generally will be positive numbers.

2. Pure Strategy Equilibrium Points

Let $\bar{m}$ be that integer which satisfies

(7) $C_{\bar{m}} < A_{\bar{m}}$ or $\bar{m} = 0$

and

(8) $C_{\bar{m}+1} > A_{\bar{m}+1}$ or $\bar{m} = n$

Since C_m is monotonically increasing and A_m is montonically non-increasing in m and since equalities are excluded by (5), it is clear that exactly one $\bar{m}$ exists which satisfies (7) and (8). As we shall see, $\bar{m}$ is the equilibrium number of players who enter the market.

Let $\pi = (\pi_1,\ldots,\pi_n)$ be a pure strategy combination and let π_i' be a strategy of player i, then π/π_i' denotes that strategy combination which results from π by substituting π_i' for π_i and leaving all other components of π unchanged:

(9) $\pi/\pi_i' = (\pi_1,\ldots,\pi_{i-1}, \pi_i', \pi_{i+1},\ldots,\pi_n)$

With the help of this notation an equilibrium point in pure strategies can be defined as a pure strategy combination

$\pi^* = (\pi_1^*, \ldots, \pi_n^*)$ with

$$(10) \quad H_i(\pi^*) = \max_{\pi_i \in \Pi_i} H_i(\pi^*/\pi_i)$$

for $i = 1,\ldots,n$ where Π_i is player i's set of pure strategies, in our case the set $\{0,1\}$.

Theorem 1: Let $\bar{\pi} = (\bar{\pi}_1,\ldots,\bar{\pi}_n)$ be the following pure strategy combination

$$(11) \quad \bar{\pi}_i = \begin{cases} 1 & \text{for } i = 1,\ldots,\bar{m} \\ 0 & \text{for } i = \bar{m}+1,\ldots,n \end{cases}$$

where $\bar{m}$ is defined by (7) and (8). The strategy combination $\bar{\pi}$ is an equilibrium point in pure strategies.

Proof: Obviously, the payoffs for $\bar{\pi}$ are as follows:

$$(12) \quad H_i(\bar{\pi}) = \begin{cases} R_{\bar{m}} - C_i & \text{for } i = 1,\ldots,\bar{m} \\ S_{\bar{m}} & \text{for } i = \bar{m}+1,\ldots,n \end{cases}$$

Consider a player i with $i = 1,\ldots,\bar{m}$. If he deviates to $\pi_i = 0$ he receives

$$(13) \quad H_i(\bar{\pi}/\pi_i) = S_{\bar{m}-1}$$

In view of (3) and (7) this is smaller than $H_i(\bar{\pi})$. Now, consider a player $i = \bar{m}+1,\ldots,n$. If he deviates to $\pi_i = 1$ he receives

$$(14) \quad H_i(\bar{\pi}/\pi_i) = R_{\bar{m}+1} - C_i$$

In view of (3) and (8) this is smaller than $H_i(\bar{\pi})$.

Theorem 2: The pure strategy combination $\pi^*=(\pi_1^*,\ldots,\pi_n^*)$ is an equilibrium point if and only if the following conditions are satisfied:

$$(15) \quad m(\pi^*) = \bar{m}$$

(16) $C_i < A_{\bar{m}}$ for every i with $\pi_i^* = 1$

(17) $C_i > A_{\bar{m}+1}$ for every i with $\pi_i^* = 0$

Proof: In the same way as in the proof of theorem 1 it can be seen immediately that each player looses by deviating from a strategy combination π^* satisfying (15), (16), and (17). This proves the if-part of the theorem.

Now suppose that π^* is an equilibrium point. We must show that (15), (16), and (17) are satisfied. Define $m^* = m(\pi^*)$. Inequalities (16) and (17) must be satisfied with m^* instead of $\bar{m}$, since otherwise in view of (3) a deviation would be profitable for at least one player.

It remains to show $m^* = \bar{m}$. Suppose that $m^* < \bar{m}$. Then at least one of the players $i = 1,\ldots,\bar{m}$ employs the strategy $\pi_i^* = 0$. For this player i we have $C_i < A_{m^*+1}$ since $A_{m^*+1} \geq A_{\bar{m}}$. This player i could gain by deviating to $\pi_i = 1$.

Suppose that $m^* > \bar{m}$. Then at least one of the players $i=\bar{m}+1,\ldots,n$ employs the strategy $\pi_i^* = 1$. For this player i we have $C_i > A_{m^*}$, since $A_{\bar{m}} \geq A_{m^*}$. This player i could gain by deviating to $\pi_i = 0$.

This shows that π^* cannot be an equilibrium point unless we have (15).

Remark: The proof of theorem 2 shows that all the pure strategy equilibrium points are strong in the sense that every player looses by deviating to another strategy.

3. Elimination of Dominated Strategies

It can be seen easily that a player i has a dominated strategy if and only if either

(18) $C_i > A_1$

or

(19) $C_i < A_n$

holds. In case (18) the strategy $\pi_i = 1$ and in the case (19) the strategy $\pi_i = 0$ is dominated. In both cases we have strict dominance, weak dominance is excluded by (5).

It is intuitively clear that the game situation remains essentially unchanged if the players with dominated strategies are fixed at their undominated strategies. In this way, the original game can be mapped into another game where the set of players is reduced by the elimination of those who have dominated strategies in the original game. It can be seen without difficulties that after an appropriate renumbering of the remaining players the new game is in the general class of games considered here. Therefore, we do not suffer a real loss of generality if we restrict our attention to games where no player has a dominated strategy. Accordingly, we shall assume

(20) $C_i < A_1$ for $i = 1,\dots,n$

(21) $C_i > A_n$ for $i = 1,\dots,n$

It may, of course, happen that new dominated strategies occur after we have gone through the process of eliminating strategies once, but after a finite number of repetitions we shall receive a game without dominated strategies.

The solution concept developed by John C. Harsanyi and Reinhard Selten specifies a reduction procedure which among other things eliminates dominated strategies (Harsanyi 1975, 1977; Güth 1978). It could be formally shown that the application of the solution concept leads to the result that in our case instead of a game with dominated strategies that game must be solved which is received by iterated elimination of dominated strategies. In order to avoid lengthy details we shall not do this here. We shall restrict our attention to games satisfying (20) and (21). Such games will be called <u>regular market entry games</u>.

4. Formations

The solution concept requires us to look at certain substructures of the game called formations. Consider a game which results from the original one by restricting the pure strategy sets of the players to nonempty subsets of the original pure strategy sets; the new payoff function is the old one restricted to the new set of strategy combinations. Games which arise in this way are called formations if for every player i the following is true for every joint mixture of pure strategy combinations of the other players in the new game: Player i's best replies to this joint mixture are the same in the original and the new game.

A primitive formation is a formation which has no proper subformation. Consider a pure strategy equilibrium point $\pi = (\pi_1,\dots,\pi_n)$ which is strong in the sense that for $i = 1,\dots,n$ player i's equilibrium strategy π_i is the only best reply to π. Clearly, we receive a primitive formation if we restrict every player i's pure strategy set to the single strategy π_i. It may happen that a game has primitive formations which do not arise from strong pure strategy equilibrium points. As we shall see, this is not the case in the games considered here.

In the solution concept developed by John C. Harsanyi and Reinhard Selten the primitive formations are of special significance. In a game which cannot be further decomposed or reduced the solutions of the primitive foramtions are considered to be natural solution candidates which must be compared with each other in order to eliminate all but one if this is possible.

Theorem 3: In a market entry game the primitive formations are exactly those substructures which arise from a pure strategy equilibrium point $\pi = (\pi_1,\dots,\pi_n)$ by restricting every player i's strategy set to his equilibrium strategy π_i.

Proof: As we have seen at the end of section 2, all pure strategy equilibrium points are strong. This shows that they yield primitive formations. It remains to show that no primitive formation can have more than one strategy for any player. Let C be the set of players which have two strategies in their restricted pure strategy set. Since the other players are fixed at their formation strategies the players in C find themselves in a situation which after an appropriate renumbering of the players is equivalent to a game in our class. Therefore, by theorem 1 the formation has a strong pure strategy equilibrium point. It follows by the definition of a formation that this equilibrium point is a strong equilibrium point of the whole game, too. This shows that the formation is not primitive.

5. Payoff Dominance

As we have explained before, the solutions of primitive formations are regarded as natural candidates for the solution of the game. They are called initial candidates. The set of all initial candidates is referred to as the first candidate set since in the process of finding the solution a sequence of a finite number of candidate sets may have to be considered.

In our case the initial candidate set is the set of all strong equilibrium points.

We say that an equilibrium point φ payoff dominates an equilibrium Ψ if we have

(22) $H_i(\varphi)>H_i(\Psi)$ for $i = 1,\dots,n$

The next step after the determination of the first candidate set consists in the elimination of all initial candidates which are payoff dominated by other initial candidates. In this way one receives the second candidate set.

In order to see how in our case the second candidate set differs from the first one we investigate the circumstances under which one strong equilibrium point $\varphi = (\varphi_1,\dots,\varphi_n)$ payoff dominates another strong equilibrium point $\Psi = (\Psi_1,\dots,\Psi_n)$.

We first observe that the following equations hold for players who use the same strategy in φ and Ψ:

(23) $H_i(\varphi) = H_i(\Psi) = S_{\bar{m}}$ for $\varphi_i = \Psi_i = 0$

(24) $H_i(\varphi) = H_i(\Psi) = R_{\bar{m}} - C_i$ for $\varphi_i = \Psi_i = 1$

Since payoff dominance is defined by strong inequality we must have

(25) $\varphi_i \neq \Psi_i$ for $i = 1,\ldots,n$

if φ payoff dominates Ψ. Therefore a payoff dominance relationship requires

(26) $n = 2\bar{m}$

since those players who are in the market in one equilibrium point must be outside the market in the other. If φ payoff dominates Ψ, thus the following conditions must be satisfied:

(27) $R_{\bar{m}} - C_i > S_{\bar{m}}$ for i with $\varphi_i = 1$

(28) $S_{\bar{m}} > R_{\bar{m}} - C_i$ for i with $\varphi_i = 0$

On the left hand side of the inequalities (27) and (28) we find the payoffs in φ and on the right hand side those in Ψ. This shows that the players i with $\varphi_i = 1$ and $\Psi_i = 0$ must be the players $1,\ldots,\bar{m}$. Consequently, φ cannot payoff dominate Ψ unless we have

(29) $\varphi = (1,\ldots,1,0,\ldots,0)$

(30) $\Psi = (0,\ldots,0,1,\ldots,1)$

This shows that the second candidate set either agrees with the first one or has exactly one candidate less, namely $(0,\ldots,0,1,\ldots,1)$.

6. Risk Dominance

In order to narrow down the second candidate set a notion of risk dominance is employed. For any two equilibrium points it can be determined whether one risk dominates the other.

It may also happen that there is no risk dominance relationship between two equilibrium points. The definition of risk dominance is based on the idea that for the purpose of comparing two equilibrium points φ and Ψ one has to look at a hypothetical situation where one player i is convinced that either φ or Ψ is the solution and that all other players know which one of both equilibrium points is the solution.

It may happen that $\varphi_i = \Psi_i$. In this case player i does not have to know whether φ or Ψ is the solution. He just can play $\varphi_i = \Psi_i$.

Consider a player i with $\varphi_i \neq \Psi_i$; he must have a subjective probability z for φ being the solution. With this probability z his expected payoff for playing a pure strategy π_i is equal to

$$(31) \qquad E_i(\pi_i,z) = zH_i(\varphi/\pi_i) + (1-z)H_i(\Psi/\pi_i)$$

We say that π_i is a best reply to z if π_i is a pure strategy which maximizes $E_i(\pi_i,z)$. Let r_i^z be that mixed strategy of player i which assigns equal positive probability to all best replies to z and zero probability to all other pure strategies. It is reasonable to suppose that player i will use the strategy r_i^z if his payoff expectation is given by (31). He has no reason to prefer one of the pure best replies to z.

For each of the players i with $\varphi_i \neq \Psi_i$ we define a strategy p_i which can be thought of as a preliminary theory on his behavior. Since this strategy plays a similar role as a prior probability distribution it is called player i's prior strategy. Assume that z is a random variable uniformly distributed over the interval [0,1]. If this is the case the probability, with which any pure strategy π_i is used by player i, is given by the following integral:

$$(32) \qquad p_i(\pi_i) = \int_0^1 r_i^z (\pi_i)dz$$

Equation (32) defines player i's prior strategy.

We shall first define risk dominance for the case that we have $\varphi_i \neq \Psi_i$ for $i = 1,\ldots,n$. The definition is based on the application of the tracing procedure to the prior strategy combination $p = (p_1,\ldots,p_n)$. The tracing procedure which will not be described here in detail (see Harsanyi 1975,1977) can be thought of as a mathematical model of a reasoning process which starts with a prior strategy combination and ends with an equilibrium point. The initial non-equilibrium strategy combination is gradually changed into an equilibrium point.

In the case $\varphi_i \neq \Psi_i$ for $i = 1,\ldots,n$ we say that φ risk dominates Ψ if φ is the equilibrium point which results if the tracing procedure is applied to p.

If for some players i we have $\varphi_i = \Psi_i$ the definition of risk dominance requires that the tracing procedure is not performed in the original game but in a restricted game; the restricted game is derived from the smallest formation which contains both φ and Ψ by fixing the players i with $\varphi_i = \Psi_i$ at these strategies. Obviously, in our case this formation contains both strategies for all players i with $\varphi_i \neq \Psi_i$. The restricted game differs from the original one by the fact that it has fewer players, namely those with $\varphi_i \neq \Psi_i$. The payoffs in the restricted game are derived from the original ones by fixing the players with $\varphi_i = \Psi_i$ at the strategies.

φ risk dominates Ψ if the application of the tracing procedure to the prior strategy combination in the restricted game yields φ as the final equilibrium point.

Since it may happen that neither φ nor Ψ results from the application of the tracing procedure it is not excluded that neither one of both equilibrium points risk dominates the other.

In the analysis of the market entry games investigated here it will not be necessary to apply the tracing procedure. It is known that the result of the tracing procedure is the best reply to the prior combination if this best reply is a strong equilibrium point. This is the situation which we find in the risk dominance computations arising in the analysis of our model.

7. Strategic Distance and Maximal Stability

Not all risk dominance comparisons between elements of the second candidate set are regarded as equally important. Comparison between equilibrium points,which are in a certain sense near to each other,are given priority.

Ideally, one would want to select an equilibrium point which risk dominates each other equilibrium point in the second candidate set. Unfortunately, this is not always possible. In view of this fact it seems to be reasonable to look for equilibrium points which are undominated by other equilibrium points within a large neighbourhood. One would like to make this neighbourhood as large as possible.

In order to make these ideas more precise it is necessary to introduce a measure of strategic distance. The measure adopted in the theory of Harsanyi and Selten is closely connected to the way in which the prior strategies are computed. For the sake of simplicity we shall explain the distance measure for the case $\varphi_i \neq \Psi_i$ for i = 1,...,n. If we have $\varphi_i = \Psi_i$ for some players the same definitions apply to the restricted game.

Let r^z be the following strategy combination

(33) $r^z = (r_1^z,...,r_n^z)$

where the r_i^z are defined as in section 6. We call r^z the average best reply to z. The interval $0 \leq z \leq 1$ is divided into a finite number of subintervals where different strategy combinations are average best replies to z. Some of these subintervals are single points. Those strategy combinations,which are average best replies on a subinterval of positive length, are called essential for the comparison between φ and Ψ.

The strategic distance $e(\varphi,\Psi)$ is defined as the number of average best replies which are essential for the comparison between φ and Ψ.

If two strong equilibrium points φ and Ψ are compared the distance $e(\varphi,\Psi)$ is at least 2 since φ and Ψ are always essential average best replies.

The distance measure $e(\varphi,\Psi)$ has the following interpretation: It counts the number of critical subjective probabilities z where at least one player shifts from one best reply to another. The greater this number of critical probabilities is the greater is the confusion which arises in the risk dominance comparison between φ and Ψ. In this sense $e(\varphi,\Psi)$ measures the intensity of confusion. We may say that a preference for more clear cut comparisons is involved in giving priority to comparisons at small strategic distances.

The diameter $\bar{e}$ of the second candidate set is defined as the greatest distance $e(\varphi,\Psi)$ between two equilibrium points φ and Ψ in the second candidate set. For a given equilibrium point φ in the second candidate set let $\sigma(\varphi)$be the greatest number k among the integers $1,\ldots,\bar{e}$ such that the second candidate set contains no Ψ with $e(\varphi,\Psi) \leq k$ which risk dominates φ. This number $\sigma(\varphi)$ is called the stability radius of φ. Let $\bar{\sigma}$ be the maximal stability radius of equilibrium points in the second candidate set. The elements φ of the second candidate set with $\sigma(\varphi) = \bar{\sigma}$ are called maximally stable.

If the second candidate set contains only one maximally stable element then this equilibrium point is the solution of the game. The determination of the solution is more complicated if there are several maximally stable equilibrium points. Fortunately, this more difficult case need not concern us here since the analysis of our model will exhibit a unique maximally stable equilibrium point.

8. Strategic Distances in Market Entry Games.

If a regular market entry game has only one pure strategy equilibrium point then this must be the equilibrium point $\bar{\pi}$ described in theorem 1. Obviously, $\bar{\pi}$ is the solution if the game has no other equilibrium point. We can restrict our attention to the case that there are at least two strong equilibrium points.

Consider two different strong equilibrium points $\varphi = (\varphi_1,\ldots,\varphi_n)$ and $\Psi = (\Psi_1,\ldots,\Psi_n)$. Let M be set of all players i with $\varphi_i \neq \Psi_i$. The set M is the player set of the restricted game for the risk dominance comparison between φ and Ψ. A player i in M must have an incentive not to enter if there are $\bar{m}$ other players in the market and he must have an incentive to enter if there are $\bar{m}$-1 other players inside the market. Otherwise he could not have different strategies in both equilibrium points. Hence

$$(34) \quad A_{\bar{m}+1} < C_i < A_{\bar{m}} \quad \text{for all } i \in M$$

In order to compute the prior strategies p_i and the strategic distance $e(\varphi,\Psi)$ we must determine the best replies to z. Consider a player i with $\varphi_i = 1$ and $\Psi_i = 0$. For $\pi_i = 1$ and $\pi_i = 0$ equation (31) assumes the following form:

$$(35) \quad E_i(1,z) = z(R_{\bar{m}} - C_i) + (1-z)(R_{\bar{m}+1} - C_i)$$

$$(36) \quad E_i(0,z) = z\, S_{\bar{m}-1} + (1-z)S_{\bar{m}}$$

Let $Z(\varphi_i)$ and $Z(\Psi_i)$, respectively, be the subintervals of $0 \leq z \leq 1$ where φ_i and Ψ_i are best replies to z. It follows by (35) and (36) that we have

$$(37) \quad Z(\varphi_i) = \left[\frac{C_i - A_{\bar{m}+1}}{A_{\bar{m}} - A_{\bar{m}+1}}, 1\right] \quad \text{for } \varphi_i = 1$$

$$(38) \quad Z(\Psi_i) = \left[0, \frac{C_i - A_{\bar{m}+1}}{A_{\bar{m}} - A_{\bar{m}+1}}\right] \quad \text{for } \varphi_i = 1$$

Now consider a player $i \in M$ with $\varphi_i = 0$ and $\Psi_i = 1$. Here we have

$$(39) \quad E_i(1,z) = z(R_{\bar{m}+1} - C_i) + (1-z)(R_{\bar{m}} - C_i)$$

$$(40) \quad E_i(0,z) = z\, S_{\bar{m}} + (1-z)S_{\bar{m}-1}$$

This yields

$$(41) \quad Z(\varphi_i) = \left[\frac{A_{\bar{m}} - C_i}{A_{\bar{m}} - A_{\bar{m}+1}}, 1\right] \quad \text{for } \varphi_i = 0$$

$$(42) \quad Z(\Psi_i) = \left[0, \frac{A_{\bar{m}} - C_i}{A_{\bar{m}} - A_{\bar{m}+1}}\right] \quad \text{for } \varphi_i = 0$$

Equations (37),(38), (41) and (42) show that player i shifts from his best reply Ψ_i to his best reply φ_i at the following critical probability z_i:

$$(43) \quad z_i = \frac{C_i - A_{\bar{m}+1}}{A_{\bar{m}} - A_{\bar{m}+1}} \quad \text{for } \varphi_i = 1$$

$$(44) \quad z_i = \frac{A_{\bar{m}} - C_i}{A_{\bar{m}} - A_{\bar{m}+1}} \quad \text{for } \varphi_i = 0$$

In view of (34) it is clear that $0 < z_i < 1$ holds for $i = 1,\dots,n$. Suppose that any two z_i are different from each other. It can be seen immediately that in this case there are $|M| + 1$ different essential average best replies r^z where $|M|$ is number of players in M; the $|M|$ critical probabilities z_i subdivide the interval $0 \le z \le 1$ into $|M| + 1$ subintervals. This yields $e(\varphi,\Psi) = |M| + 1$

The distance may be smaller than $|M| + 1$ if several of the z_i are equal to each other. In view of (1) two z_i of the form (43) for different players cannot be equal to each other; the same is true for two z_i of the form (44). Therefore for $i \neq j$ we cannot have $z_i = z_j$ unless both players use different strategies in φ. Suppose that for $\varphi_i = 0$ and $\varphi_j = 1$ the critical probabilities z_i and z_j are equal. Then it follows by (43) and (44) that we must have

$$(45) \quad C_i + C_j = A_{\bar{m}} + A_{\bar{m}+1}$$

In the case $\varphi_i = 1$ and $\varphi_j = 0$ the same condition must be satisfied. In view of the non-degeneracy assumption (6) there can be only one pair i,j with $i \neq j$ such that (45) holds. Suppose that i,j is a pair of this kind and that both players are in M. Then $e(\varphi,\Psi)$ is equal to $|M|$.

Since the number of players in the market is $\bar{m}$ in both equilibrium points, $|M|$ must be an even number. For $|M|=2$ the distance is either $e(\varphi,\Psi) = 2$ or $e(\varphi,\Psi) = 3$. For $|M|=4$ the distance $e(\varphi,\Psi)$ is either 4 or 5. Obviously for $|M| > 2$ the distance $e(\varphi,\Psi)$ is always greater than for $|M| = 2$.

As we shall see, a regular market entry game has one and only one equilibrium point, namely $\bar{\pi}$ as defined by (11) which is not risk dominated by any other equilibrium point at a distance of at most 3. From what we have said, it is clear that nothing more has to be shown in order to prove that $\bar{\pi}$ is the solution.

9. Risk Dominance in Market Entry Games

For the reasons mentioned above it will be sufficient to look at risk dominance comparisons between equilibrium points φ and Ψ with only two players using different strategies in φ and Ψ or, in other words, with $|M| = 2$. Let i and j be the two players in M and assume

(46) $\varphi_i = 1$ and $\varphi_j = 0$

It follows by the definition of p_i in (32) and by (37), (38), (41) and (42) that we have

$$(47) \quad p_i(\varphi_i) = \frac{A_{\bar{m}} - C_i}{A_{\bar{m}} - A_{\bar{m}+1}}$$

$$(48) \quad p_i(\Psi_i) = \frac{C_i - A_{\bar{m}+1}}{A_{\bar{m}} - A_{\bar{m}+1}}$$

$$(49) \quad p_j(\varphi_j) = \frac{C_j - A_{\bar{m}+1}}{A_{\bar{m}} - A_{\bar{m}+1}}$$

$$(50) \quad p_j(\Psi_j) = \frac{A_{\bar{m}} - C_j}{A_{\bar{m}} - A_{\bar{m}+1}}$$

Player i's strategy $\pi_i = 1$ is his only best reply to p_j in the restricted game if

$$(51) \quad p_j(\varphi_j)(R_{\bar{m}} - C_i) + p_j(\Psi_j)(R_{\bar{m}+1} - C_i) > p_j(\varphi_j)\, S_{\bar{m}-1} + p_j(\Psi_j)\, S_{\bar{m}}$$

This inequality compares player i's expected payoff for his strategy $\pi_i = 1$ with the expected payoff for $\pi_i = 0$. (51) can be written as follows:

$$(52) \quad p_j(\varphi_j)(A_{\bar{m}} - C_i) + p_j(\Psi_j)(A_{\bar{m}+1} - C_i) > 0$$

With the help of (49) and (50) it can be seen that this is equivalent to the following condition:

$$(53) \quad (C_j - A_{\bar{m}+1})(A_{\bar{m}} - C_i) > (A_{\bar{m}} - C_j)(C_i - A_{\bar{m}+1})$$

Here we have used the fact that in view of (34) we have $A_{\bar{m}} > A_{\bar{m}+1}$. The case $A_{\bar{m}} = A_{\bar{m}+1}$ is not excluded by the model but by the assumption that there are at least two different strong equilibrium points. Suppose that we have $C_i < C_j$. It can be seen with the help of (34) that all factors in (53) are positive. The first factor of the left-hand side of (53) is greater than the second factor on the right-hand side, and the second factor on the left-hand side is greater than the first one on the right-hand side. In this way we can see that (53) holdsif and only if C_i is smaller than C_j, i.e. for $i < j$.

Player j's strategy $\pi_j = 0$ is his only best reply to p_i in the restricted game if

$$(54) \quad p_i(\varphi_i)S_{\bar{m}} + p_i(\Psi_i)S_{\bar{m}-1} > p_i(\varphi_i)(R_{\bar{m}+1} - C_j) + p_i(\Psi_i)(R_{\bar{m}} - C_j)$$

With the help of (47) and (48) it can be seen in the same way as above that (54) is equivalent to (53). Consequently, player j's best reply to p_i is φ_j if and only if $C_i < C_j$ holds.

It is now clear that both φ_i and φ_j are the only best replies to the prior strategies in the restricted game if and only if we have $C_i < C_j$. (Note that the limiting case $C_i = C_j$ is excluded by (1)). If the tracing procedure is applied in the restricted game for the comparison of φ and Ψ the final result will be φ in the case $C_i < C_j$ and Ψ in the case $C_j < C_i$. If (46) holds for a comparison with $|M| = 2$ then φ risk dominates Ψ if and only if $C_i < C_j$ holds.

10. The Solution of Regular Market Entry Games

It now can be shown without much difficulty that $\bar{\pi}$ as given by (11) is the solution. For this purpose we shall prove two assertions (a) and (b):

(a): Let Ψ be a strong equilibrium point with $e(\bar{\pi},\Psi) \leq 3$ and $\Psi \neq \bar{\pi}$. Then $\bar{\pi}$ risk dominates Ψ.

(b): Let Ψ be a strong equilibrium point with $\Psi \neq \bar{\pi}$. Then there exists a strong equilibrium point φ with $e(\varphi,\Psi) \leq 3$ such that φ risk dominates Ψ.

It is clear that (a) and (b) have the consequence that $\bar{\pi}$ is the only maximally stable strong equilibrium point in the second candidate set. (We have shown in section 5 that $\bar{\pi}$ is not payoff dominated). It follows by (a) that the stability radius of $\bar{\pi}$ is at least 3 and it follows by (b) that the stability radius of any other strong equilibrium point is smaller than 3.

We now proceed to show (a). Let i and j be the two players who use different strategies in $\bar{\pi}$ and Ψ and assume $\bar{\pi}_i = 1$ and $\bar{\pi}_j = 0$. It follows by definition (11) of $\bar{\pi}$ that we must have $C_i < C_j$. In view of the result of section 9 this shows that (a) is true.

It remains to show (b). The equilibrium point Ψ prescribes $\Psi_i = 0$ to at least one of the players $1,\ldots,\bar{m}$ and $\Psi_j = 1$ to at least one of the players $\bar{m}+1,\ldots,n$. Let i and j be two such players. Define $\varphi = (\varphi_1,\ldots,\varphi_n)$ as follows: $\varphi_i = 1$, $\varphi_j = 0$ and $\varphi_k = \Psi_k$ for every k different from i and j. Since in view of (1) we have $C_i < C_j$ it follows by the result of section 9 that φ risk dominates Ψ.

As the result of our analysis we can state the following theorem:

Theorem 4: The solution of a regular market entry game is the equilibrium point $\bar{\pi}$ described in theorem 1.

11. The Linear Cournot-Market Entry Game

The most obvious example is based on the linear Cournot-model. The model is embedded in a two stage-game which can be described as follows:

Stage 1 (entry stage): n potential suppliers 1,...,n simultaneously choose $\pi_i = 0$ (no entry) or $\pi_i = 1$ (entry). The choices are made without knowing the decisions of the others. At the end of stage 1 the vector $\pi=(\pi_1,\ldots,\pi_n)$ is made known to all players 1,...,n.

Stage 2 (supply stage): All players with $\pi_i = 1$ simultaneously choose a supply $x_i \geq 0$. The choices are made without knowing the decisions of the others. For $\pi_i = 0$ we define $x_i = 0$.

Payoffs: The payoff functions H_i are defined as follows:

$$(55) \qquad H_i = \begin{cases} x_i p - kx_i - C_i & \text{for } \pi_i = 1 \\ 0 & \text{for } \pi_i = 0 \end{cases}$$

where the price p is given by

$$(56) \qquad p = \min\,[b - ax,\ 0]$$

and x is total supply:

$$(57) \qquad x = x_1 + \ldots + x_n$$

a, b and k are positive constants with $b>k$. The entry costs $C_1,\ldots,C_n$ are positive and satisfy inequality (1).

The two stage-game model is a game in extensive form. Each situation,which can arise at the beginning of stage 2, corresponds to a subgame. We call these subgames supply decision subgames. For every vector $\pi = (\pi_1,\ldots,\pi_n)$ with $\pi \neq (0,\ldots,0)$ the game has one supply decision subgame Γ_π. There are no other proper subgames.

A natural solution concept which can be applied to the extensive game is that of subgame perfect equilibrium point. An equilibrium point is called subgame perfect if it induces an equilibrium point on every subgame (Selten 1973 and 1975). In order to find the subgame perfect equilibrium points of the two stage-model we have to look at the equilibrium points of the supply decision subgames.

As we shall see each of the Γ_π has a uniquely determined equilibrium point which can be found as follows: For $\pi_i = 1$ we have

$$(58) \qquad H_i = x_i(b-k-ax) - C_i$$

It follows that for $x_i > 0$ the condition

$$(59) \qquad \frac{\partial H_i}{\partial x_i} = b - k - ax - ax_i = 0$$

must be satisfied at the equilibrium point. Let m be the number of players with $x_i > 0$ at the equilibrium point. Summing the necessary conditions (59) for all i with $x_i > 0$ yields

$$(60) \qquad x = \frac{m}{m+1} \; \frac{b-k}{a}$$

This together with (59) leads to the following conclusions:

$$(61) \qquad x_i = \frac{1}{m+1} \; \frac{b-k}{a} \qquad \text{if } x_i > 0$$

$$(62) \qquad p = k + \frac{b-k}{m+1}$$

Since p is greater than k a player i with $\pi_i = 1$ cannot have the equilibrium supply $x_i = 0$; a sufficiently small $x_i > 0$ would still yield a price greater than k and thereby raise player i's payoff above $-C_i$. Consequently, m is nothing else than the number of players with $\pi_i = 1$ and (61) holds for all these players. It can be seen immediately that the marginal conditions are not only necessary but also sufficient for equilibrium. Consequently, we have found an equilibrium point which is uniquely determined.

The equilibrium profits for Γ_π are functions of π; player i's equilibrium payoff will be denoted by $H_i(\pi)$. Equations (55), (61) and (62) yield

$$(63) \quad H_i(\pi) = \begin{cases} \frac{1}{a}\left(\frac{b-k}{m(\pi)+1}\right)^2 - C_i & \text{for } \pi_i = 1 \\ 0 & \text{for } \pi_i = 0 \end{cases}$$

where $m(\pi)$ is the number of players with $\pi_i = 1$ in π.

The subgame perfect equilibrium points of the two stage-model can be found with the help of the truncated game which results from the original extensive form if every subgame Γ_π is replaced by the corresponding payoff vector $H(\pi) = (H_1(\pi),\dots,H_n(\pi))$. This truncated game will be denoted by $\bar{\Gamma}$.

As we shall see, the truncated game $\bar{\Gamma}$ is a market entry game if the non-degeneracy conditions (5) and (6) are satisfied. The theory of Harsanyi and Selten can be used in order to select one of the equilibrium point of $\bar{\Gamma}$. For this purpose we can make use of the results obtained above. Together with the uniquely determined subgame equilibria the solution of $\bar{\Gamma}$ determines a subgame perfect equilibrium point for the two stage-model. It is natural to think of this equilibrium point as the solution of the two-stage model even if the theory of Harsanyi and Selten has been developed for finite games only and, therefore, cannot be applied directly to a game with infinitely many pure strategies.

In order to see that $\bar{\Gamma}$ is a market entry game if the non-degeneracy conditions (5) and (6) are given it remains to show that (4) is satisfied.

In our case we have

$$(64) \quad R_m = \frac{1}{a}\left(\frac{b-k}{m+1}\right)^2$$

$$(65) \quad S_m = 0$$

This yields

$$(66) \qquad A_m = \frac{1}{a}\left(\frac{b-k}{m+1}\right)^2$$

Equation (66) shows that A_m is a decreasing function of m. Assumption (4) is satisfied. If,in addition to this, (5) and (6) are satisfied the solution of $\bar{\Gamma}$ is given by theorem 4. Those $\bar{m}$ players who have the lowest entry costs enter the market while the other players stay outside. This is the result one would intuitively expect. Nevertheless, it is important to see that it can be obtained by a general theory without making use of ad hoc-arguments related to specific features of market entry games.

12. Further Possible Applications of Market Entry Games

In the following we do not want to describe examples of market entry games in detail. Instead of this we shall give verbal descriptions of economic situations which could be modelled as market entry games under appropriate assumptions on functional form and parameters.

In the last section,we have opened up the linear Cournot-oligopoly model by adding a market entry stage which precedes the supply decisions. The same embedding procedure can be applied to many other oligopoly models, e.g. to models with differentiated products where prices, advertising expenditures or quality parameters are the decision variables.

It is also possible to look at two stage-models where the first stage is not an entry stage but an innovation stage. Suppose that each of n suppliers can adopt a method of production which saves labour but requires investment costs which are different for different firms in view of variation of technical experience. The second stage subgame equilibria will depend on the number m of suppliers who have adopted the new production method at the innovation stage.

In a paper by W. Güth and U. Meyer (1979) a multi-stage-oligopoly model is investigated which describes a situation where a successful production method already used by one oligopolist can be imitated by his competitors. One may think of a patent whose time of protection is running out. It has been shown in the paper that the analysis of this model leads to a market entry game.

T.C. Schelling (1973) discusses several examples of game situations where each player i has just two pure strategies $\pi_i = 0$ and $\pi_i = 1$. Some of these games typically will satisfy all structural relationships of market entry games. Applying our results, which are based on the equilibrium selection theory of Harsanyi and Selten, these games now can be more thoroughly investigated.

There are many examples in the area of public choice which, in principle, permit a description in terms of market entry games even if the application of a game theoretical theory of equilibrium point selection does not seem to be really adequate. Suppose, for example, that vaccination against an infectious disease, say polio, is available on a voluntary basis. Since the danger of infection is a decreasing function of the number of people vaccinated the incentive to obtain vaccination also decreases with this number. Those who obtain vaccination receive a utility R derived from security against disease and from which an individually different cost component C_i must be subtracted which stands for the inconvenience of vaccination (it is assumed that utilities are additive in these components; the assumption that R is equal to everybody can be secured by an appropriate scaling of utility units). Whereas $R_m = R$ is constant the utility S_m obtained by the vaccination of m other individuals increases with m. Therefore A_m is a decreasing function of m. Clearly, theorems 1 and 2 can be applied to this model but it is debatable whether it makes sense to apply the selection theory of Harsanyi and Selten. The game model assumes that every player knows the costs C_i of every other player. If information is incomplete in this respect a different and more difficult game is played.

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