Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 103

L.P.-Games with Sufficiently Many Players

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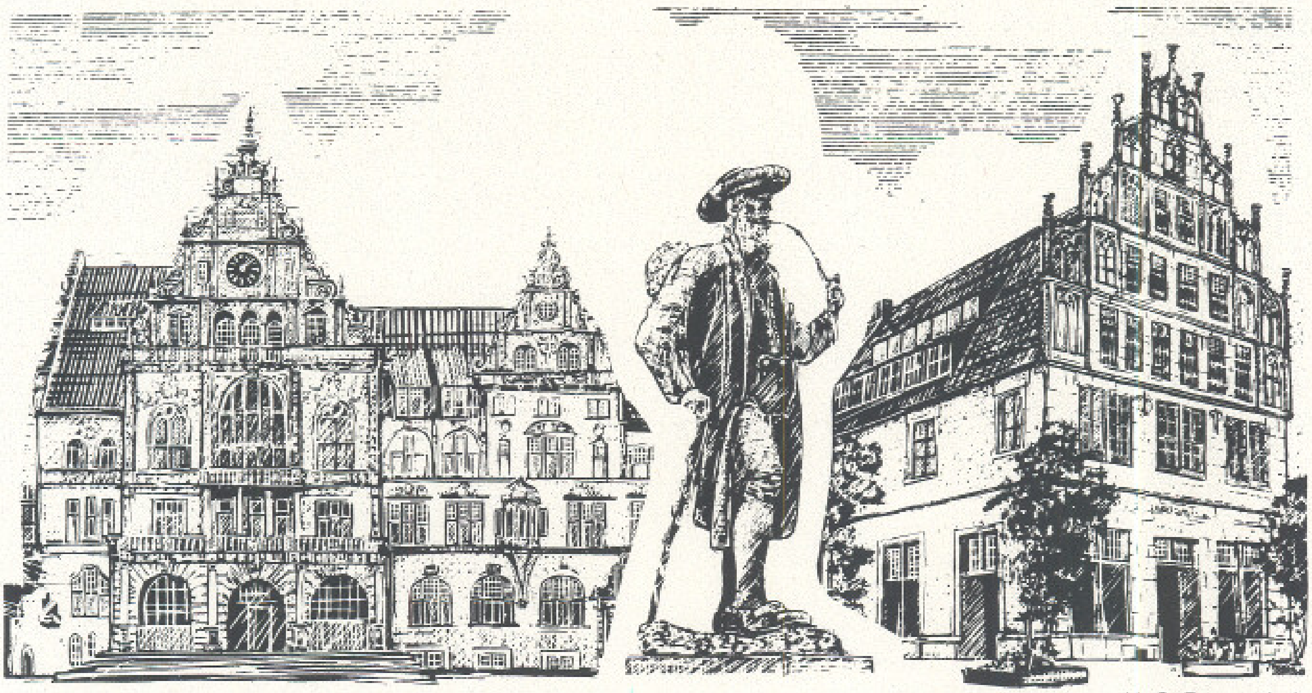
Dezember 1980



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L.P.-games are n-person cooperative games arising from a linear program as studied by OWEN and BILLERA-RAANAN. Employing a version of nondegeneracy of games developed by H.-G. WEIDNER and the author, we show that methods from the Geometry of Numbers are the suitable tool in order to obtain lower bounds for the number of players of each type that ensure that the core and the competitive euqilibrium coincide.

Introduction:

The term L.P.-Game stands for linear production game or linear program game. These games were introduced by G. OWEN [3], who showed that the core of these games converges to the set of competitive equilibrium, that is to those payoffs to the players which are induced by the shadow prices (optimal solution of the dual programme of the grand coalition). OWEN uses the well known technique of introducing "replica markets" in order to state his convergence theorems. On the other hand L.J. BILLERA and J.RAANAN [1] considered the same type of games in the non atomic frame work. These authors showed that OWEN's result holds true in the sense that every measure in the core is induced by an optimal solution to a dual problem. They also considered the (asymptotic) value of L.P.-Games and were concerned with exactness of such games.

All authors were aware that L.P.-Games are a special case of market games and that, therefore, many results may be obtained by applying the well established theory of market games either for the replica case or for the non atomic case. However, many proofs are much easier due to the linear structure imposed on the game and thus the L.P.-Game exhibits certain structures and properties that cannot be found in the general frame work of market games.

This paper is an attempt to link convergence theorems about the core and the competitive equilibrium with a version of nondegeneracy (for additive set function or for games generated by such functions) which was introduced in [5]. As was pointed out in [5] extensively, "nondegeneracy" has something to do with extreme games and their solution concepts. Moreover, it is a result mentioned in [5] that nondegeneracy may be guaranteed if "sufficiently many small players" are participating. The term sufficiently many can be made very precise: Depending on the relative size of the initial allocations alloted to the players there are exact bounds for every type of player in order to guarantee that a game is nondegenerate.

In previous papers this result was applied to studying the question of characterizing extreme games. In this note it is shown that non degeneracy may also serve in order to formulate the equivalence of core and competitive equilibrium in a finite frame work, at least as L.P.-Games are concerned. As it turns out a certain version of non-degeneracy is sufficient to guarantee the coincidence of core and competitive equilibrium. Therefore, in principle it is possible to obtain exact bounds for every type of player, depending on the relative size of the initial resources such that, if these bounds are reached and sufficiently many players of each type available (in particular, sufficiently many "small" players), then core and competitive equilibrium coincide. It should be stressed that this shows that the coincidence does not depend on the replica model: We have a much more precise notion of "a large player set" then is offered by either the replica version or the non atomic version of a market (note that OWEN of course is aware that the convergence is finite in case of a unique dual optimal solution).

Hence the result of this note may roughly be stated as follows. In the frame work of L.P.-Games there are player sets of "medium size" such that the core and the competitive equilibrium coincide. In order to define "medium size" exact lower bounds for every type of player may be specified. The clue to link the equivalence theorem and the exact lower bounds is nondegeneracy theory as developed in [4] [5] [6].

Indeed, nondegeneracy theory leads to problems of the "Geometry of Numbers" (MINKOWSKI's 2^{nd} theorem), more exactly, to the problem of determining certain lattice constants. The determination of these lattice constants yields the desired lower bounds. Hence, by contrast to the replica or the nonatomic version of a game or a market, the study of lattice constants seems to be closely related to the study of games with "medium sized" player sets.

§ 1 L.P.-games and nondegeneracy

The following introductory definitions are due to OWEN [3], see also BILLERA and RAANAN [1]. The clue of this section is the definition of the system of "weak additivity sets" $\underline{Q}$ of an L.P.-Game; see Definition 1.5., Theorem 1.6., and Corollary 1.7. OWEN and BILLERA - RAANAN were implicitly working with this set. The first author, because he is working "along the diagonal" - and diagonal sets are elements of Q, and the latter authors because they are working "close to the diagonal ".

Let $\Omega = \{1,\ldots,n\}$ denote the set of players. For $j = 1,\ldots,m$ let $b^j \in \mathbb{R}^n_+$, b^j is interpreted as an additive set function (over Ω) by the convention

$$b^j (S) = \sum_{i\in S} b^j_i$$

and describes the distribution of resources j over Ω . Hence, $b = (b^1,\ldots,b^m)$ is a "vector-valued measure", we write

$$b (S) = (b^1(S),\ldots,b^m(S)).$$

Next, let $A = (a_{jk})_{\substack{j = 1,\ldots,m \\ k = 1,\ldots,l}} \geq 0$ be an $m \times l$ matrix ("input-output-matrix") and let $c \in \mathbb{R}^l_+$. Given such a tripel

$$\mathfrak{A} = (A,b,c)$$

we consider, for $S \subseteq \Omega$

$$v(S) = v^{\mathfrak{A}}(S) = \max \{c\, x \mid x \in \mathbb{R}^l_+,\ A\, x \leq b(S)\}$$

which defines a function $v = v^{\mathfrak{A}} : \underline{P} \to \mathbb{R}_+$, where $\underline{P}$ is the power set of Ω (the coalitions).

The tripel

$$\Delta^{\mathfrak{A}} = (\Omega, \underline{P}, v^{\mathfrak{A}})$$

is called and <u>L.P.-game</u>. Because of

(2) $v(S) = v^{\mathfrak{A}}(S) = \min \{y\, b(S) \mid y \in \mathbb{R}^m_+,\ y\, A \geq c\}$

a core element of this game is easily obtained as follows. Pick an <u>optimal</u> solution $\bar{y}$ for the "dual Ω-program ,i.e., $\bar{y} \in \mathbb{R}^m_+$ such that

(3) $v(\Omega) = \bar{y}\, b\,(\Omega) = \min \{y\, b\,(\Omega) \mid y \in \mathbb{R}^m_+,\ yA \geq c\}$,

then $\bar{y}$ is <u>feasible</u> for (2) (as the constraints do not depend on S) and hence

(4) $\bar{y}\, b(S) \geq v(S)$.

But (3) and (4) show that $\bar{y}\, b\,(\cdot) = \bar{y}_1\, b^1(\cdot)+\ldots+\bar{y}_m\, b^m(\cdot)$ is an element of the core of v, we write

(5) $\bar{y}\, b \in \mathcal{C}\,(v)$.

We shall henceforth assume that <u>b is normalized</u>, i.e.,

(6) $b\,(\Omega) = e = (1,\ldots,1) \in \mathbb{R}^m_+$.

For any $z \in \mathbb{R}^m_+$ let

(7) $f\,(z) = f^{A,c}(z) := \max \{cx \mid x \in \mathbb{R}^l_+,\ A\,x \leq z\}$

denote the value of the "linear program " (A,z,c) such that $v = v^{\mathfrak{A}} = v^{(A,b,c)} = f^{A,c} \circ b = f \circ b$.

Finally, for $\bar{y} \in \mathbb{R}^m_+$ denote

$$Q_0 = Q_0^{A,c} = Q_{0\,\bar{y}}^{A,c}$$

$= \{z \in \mathbb{R}^m_+ \mid \min \{y\, z \mid y \in \mathbb{R}^m_+,\ y\, A \geq c\} = \bar{y}\, z\}$

(8) $= \{z \in \mathbb{R}^m_+ \mid \bar{y}$ is a dual optimal solution for $(A,z,c)\}$

$= \{z \in \mathbb{R}^m_+ \mid y \in \mathbb{R}^m_+,\ y\, A \geq c$ implies $y\, z \geq \bar{y}\, z\}$

$= \{z \in \mathbb{R}^m_+ \mid f\,(z) = \bar{y}\, z\}$

Remark 1.1.: 1. If $\bar{y}$ is optimal for the dual Ω - program (c f. (3)), then clearly

$$b\,(\Omega) = e \in Q_o \ .$$

2. Suppose that, in addition, $\bar{y}$ is the unique optimal solution for the dual Ω-program Then, it is not hard to see that there is $\varepsilon > o$ such that for $|z-e|<\varepsilon$ it follows that $z \in Q_o$. That is, an ε-neighborhood of $b\,(\Omega) = e$ is contained in Q_o. Hence, $\bar{y}$ is the (unique) optimal (dual) solution for (A, z, c) $(|z-e|<\varepsilon)$ or

$$f\,(z) = \bar{y}\, z \quad (|z-e|<\varepsilon) \ .$$

The feasible set of (2) and (3), i.e.,

$$\{y \in \mathbb{R}^m_+ \mid y\, A \geq c\}$$

is a convex polytope. Let λ be sufficiently large such that

(9) $Y := \{y \in \mathbb{R}^m_+ \mid y\, A \geq c\ ,\ y\, e \leq \lambda\}$

contains a (relative) neighborhood of all extreme points of this polytope. Then we have

<u>Lemma</u> 1.2. Given A and c, let $\bar{y} \in \mathbb{R}^m_+$ be an extreme optimal solution for the dual Ω-program, i.e., $\bar{y}$ satisfies (3) and is extreme in Y. Suppose that

$$\hat{y}^p \quad p = 1,\ldots, q \leq m$$

are those vertices of Y that are adjacent to $\bar{y}$. Then

$$Q_o = \{z \in \mathbb{R}^m_+ \mid (\hat{y}^p - \bar{y})\, z \geq o \ (p=1,\ldots,q)\}$$

If the dual Ω-program is non degenerate (in the sense of Linear Programming), then $q = m$.

No proof shall be offered as this is standard procedure in Linear Programming and Convex Analysis.

<u>Lemma</u> 1.3. Given A and c, let $\bar{y}$ be an extreme optimal solution for the dual Ω-program, i.e., $\bar{y}$ satisfies (3) and is extreme in Y.

Then there is a matrix $\Lambda = (\lambda^p_j)_{\substack{p=1,\ldots,q \\ j=1,\ldots,m}}$ such that

$$Q_o = \{z \in \mathbb{R}^m_+ \mid \Lambda z \geq o\} ,$$

the rows of which are the vectors $\hat{y}^p - \bar{y}$ given by Lemma 1.2.

If the dual Ω-program is non degenerate (in the L.P.-sense), then $q = m$, i.e., Λ is an $m \times m$ matrix which is obtained as follows.

Introduce slack variables $s_1,\ldots,s_{l+1}$; i.e., define a mapping $S : \mathbb{R}^m_+ \to \mathbb{R}^{m+l+1}$ by

$$y \longrightarrow (y, s_1,\ldots,s_{l+1}) =: y'$$

via

$$s_k = y A_{\cdot k} - c_k , \quad k = 1,\ldots,l ,$$

$$s_{l+1} = \lambda - y e ,$$

i.e.

$$y \longrightarrow (y, Ay - c, \lambda - y e)$$

(where $A_{.k}$ is the k'th row of A). Also let

$$A' := \begin{pmatrix} a_{11} & \dots & a_{1l} & 1 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_{m1} & \dots & a_{ml} & 1 \\ -1 & & 0 & 0 \\ & & & \cdot \\ & & & \cdot \\ & & & \cdot \\ 0 & & -1 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} A & e \\ -I & 0 \\ 0 & 1 \end{pmatrix}$$

$$c' := (c,\lambda) \quad , \quad Y' := \{y' \in \mathbb{R}_{+}^{m+l+1} \mid y'A' = c'\} \ ,$$

such that $S : Y \longrightarrow Y'$ is 1 - 1 .

Now, if $\bar{y}' = S(\bar{y})$ corresponds to $\bar{y}$, then $\bar{y}'$ is a vertex of Y'. Define $J' \subseteq \{1,\dots,m+l+1\}$ by

$$J' = \{j \mid \bar{y}'_j > 0\} \quad ,$$

then $|J'| = l+1$ (nondegeneracy). Now

$$(\lambda_j^k)_{\substack{j \in J' \\ k \notin J'}} \qquad (j,\ k \in \{1,\dots,m+l+1\})$$

is the "simplex tableau" corresponding to $\bar{y}'$, i.e., defined by

$$A'_{k\cdot} = \sum_{j \in J'} \lambda_j^k \ A'_{j\cdot} \qquad (k \notin J')$$

(the $A'_{j\cdot}$ $(j \in J')$ being linear independent). Write

(1)

$$\Lambda' = \; k \begin{pmatrix} \overset{j}{\vdots} & \begin{matrix} 1 & & 0 \end{matrix} \\ \cdots\cdots \; -\lambda_j^k \; \cdots\cdots & \\ \vdots & \begin{matrix} 0 & & 1 \end{matrix} \end{pmatrix} \begin{matrix} J'^c \end{matrix} \quad m$$

$$\underbrace{\qquad\qquad}_{\substack{J' \\ l+1}} \; \underbrace{\qquad\qquad}_{\substack{J'^c \\ m}}$$

Then Λ is obtained by cancelling all rows from Λ' that correspond to slack variables, i.e.,

$$\Lambda = \; k \begin{pmatrix} \overset{j}{\vdots} & \begin{matrix} 1 & & 0 \end{matrix} \\ \cdots\cdots \; \lambda_j^k \; \cdots\cdots & \\ \vdots & \begin{matrix} 0 & & 1 \end{matrix} \end{pmatrix} \begin{matrix} J'^c \end{matrix} \quad m$$

$$\{j \mid \bar{y}_j > o\} \qquad\qquad \{j \mid \bar{y}_j = o\}$$

$$\{1,\ldots,m\}$$

$$m$$

Again, no proof is offered; the technique employed is standard in L.P.-framework in order to find the adjacent vertices $\hat{y}^p$ $(p=1,\ldots,m)$ of $\bar{y}$ in the non degenerate case. The additional restriction

"$y\, e \leq \lambda$" is introduced in order to ensure compactues of Y - otherwise some edges touching $\bar{y}$ might be unbounded and would not allow for neighbouring vertices.

However, we want to stress that the matrix Λ may be obtained from A and c by standard procedures of L.P.-theory.

Remark 1.4. Clearly $b\,(\Omega) = e > 0$. Therefore, if $\bar{y}$ is the unique optimal solution for the dual Ω-program for some A, c (cf. (3)), it follows from Remark 1.1. that $\Lambda\, e > 0$. (note that Λ has now row vector which is zero).

Definition 1.5. Let $\mathfrak{A} = (A, b, c)$ and let $\bar{y}$ be an optimal solution of the dual Ω-program (3). Let $Q_o = Q_{o\bar{y}}^{A,c}$. Define

$$Q = Q_{\bar{y}}^{\mathfrak{A}} = \{S \in P \mid b(S) \in Q_o,\ b(S^c) \in Q_o\}\ .$$

Theorem 1.6. The following are properties of $\underline{\underline{Q}}$:

1. $\underline{\underline{Q}} = \{S \in \underline{\underline{P}} \mid \bar{y}$ is S dual optimal as well as S^c dual optimal$\}$

2. $\underline{\underline{Q}}$ is closed under the formation of complements.

3. $\bar{y}\, b\,(S) = v\,(S) = v^{\mathfrak{A}}(S)$ if and only if $S \in \underline{\underline{Q}}$.

4. $\underline{\underline{Q}}$ is the system of "weak additivity sets" of $v = v^{\mathfrak{A}}$, i.e.

$$\underline{\underline{Q}} = \{S \in \underline{\underline{P}} \mid v\,(S) + v\,(S^c) = v\,(\Omega)\}\ .$$

Proof 1. follows by inspection of Q_o,
2. is trivial.
3. is an immediate consequence of 1..
4. is easy:

If $S \in \underline{\underline{Q}}$, then $S^c \in \underline{\underline{Q}}$ and $(v = v^{\mathfrak{A}})$

$$v(S) = \bar{y}\, b(S) \ , \ v(S^c) = \bar{y}\, b(S^c) \ ;$$

hence

$$v(S) + v(S^c) = \bar{y}\,(b(S) + b(S^c)) = \bar{y}\, b(\Omega) = v(\Omega) \ .$$

On the other hand, suppose that

$$v(S) + v(S^c) = v(\Omega)$$

for some $S \in \underline{\underline{P}}$. Then, as $\bar{y}$ is feasible for the dual S and S^c program

$$\begin{aligned} \bar{y}\, b(S) &\geq v(S) = v(\Omega) - v(S^c) \\ &= \bar{y}\, b(\Omega) - v(S^c) \geq \bar{y}\, b(\Omega) - \bar{y}\, b(S^c) \\ &= \bar{y}\, b(S) \ . \end{aligned}$$

Obviously, "$=$" prevails and we have

$$\bar{y}\, b(S) = v(S) \ , \ \bar{y}\, b(S^c) = v(S^c) \ ,$$

indicating that $S, S^c \in \underline{\underline{Q}}$, q.e.d.

<u>Corollary</u> 1.7. Given $\mathfrak{A}$ and $\bar{y}$ as in Definition 1.5., we have

$$\underline{\underline{Q}} = \{S \in \underline{\underline{P}} \mid 0 \leq \Lambda\, b(S) \leq \Lambda\, e\} \ .$$

Recall that Λ (depending on A, c, $\bar{y}$) is specified by Lemma 1.3., also $\Lambda e > 0$ if $\bar{y}$ is unique. The Corollary follows from Lemma 1.3. and Definition 1.5. as

$$\begin{aligned} \underline{\underline{Q}} &= \{S \in \underline{\underline{P}} \mid \Lambda\, b(S) \geq 0 \ , \ \Lambda\, b(S^c) \geq 0\} \\ &= \{S \in \underline{\underline{P}} \mid \Lambda\, b(S) \geq 0 \ , \ \Lambda\, e \geq \Lambda\, b(S)\} \ . \end{aligned}$$

Let us now recall a definition of nondegeneracy that has been used extensively in [4] [5] [6] .

Definition 1.8. An additive, normalized ($m(\Omega) = 1$), set function m on $\underline{\underline{P}}$ (i.e., an n-vector) is said to be nondegenerate with respect to a subsystem $\underline{\underline{P}}_o \subseteq \underline{\underline{P}}$ if $m = (m_1,\dots,m_n)$ is the unique solution of the linear system of equations in variables $y_1,\dots,y_n$ given by

$$\sum_{i\in S} y_i = m(S) \quad (S\in\underline{\underline{P}}_o)$$

$$\sum_{i\in\Omega} y_i = 1$$

(we write "m n.d. $\underline{\underline{P}}_o$" in this case).

Corollary 1.9. Let $\mathfrak{A}$ and $\bar{y}$ be given as in Definition 1.5. and let $\underline{\underline{Q}} = \underline{\underline{Q}}^{\mathfrak{A}}_{\bar{y}}$. Define $\bar{m} := \bar{y}\, b$. If $\bar{m}$ n.d. $\underline{\underline{Q}}$, then

$$\mathcal{C}(v) = \mathcal{C}(v^0) = \{\bar{m}\} = \{\bar{y}\, b\} .$$

Proof Let $\mu \in \mathcal{C}(v)$. For $S \in \underline{\underline{Q}}$ we have

$$\mu(\Omega) = \mu(S) + \mu(S^c)$$

$$\geq v(S) + v(S^c) = v(\Omega) = \mu(\Omega)$$

hence

$$\mu(S) = v(S) = \bar{y}\, b(S) = \bar{m}(S) \quad (S\in\underline{\underline{Q}}) .$$

As $\bar{m}$ n.d. $\underline{\underline{Q}}$, it follows that $\mu = \bar{m}$, q.e.d.

The following observation is useful . Given $\mathfrak{A}$ and $\bar{y}$, consider

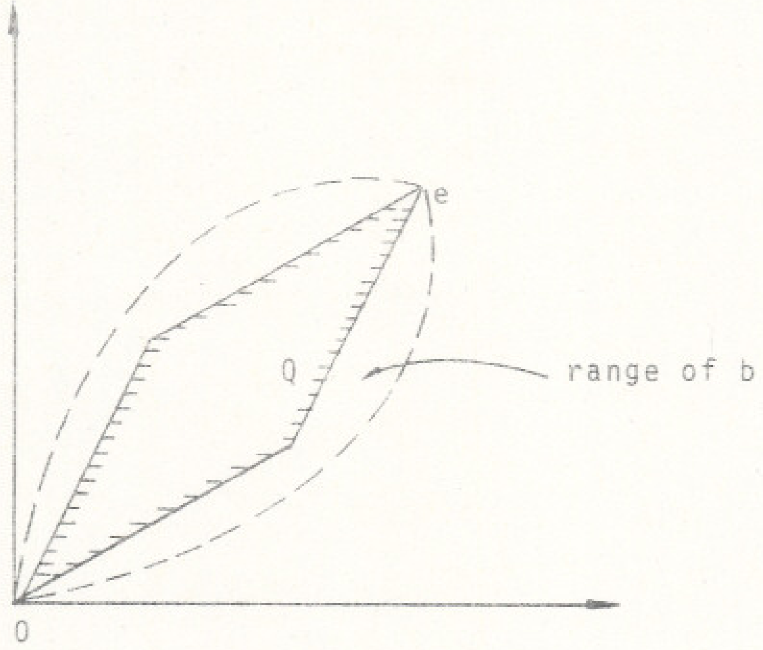
$$Q := \{z \in \mathbb{R}^m_+ \mid o \leq \Lambda z \leq \Lambda e\}$$

such that

$$\underline{\underline{Q}} = b^{-1}(Q).$$

Q is an intersection of stripes

$$Q = \{z \in \mathbb{R}^m_+ \mid 0 \leq \Lambda_{p.}\, z \leq \Lambda_{p.}\, e,\ p = 1,\dots,q.\}$$



and the question is whether b throws sufficiently many sets S into Q such that $\overline{m} = \overline{y}\, b$ is uniquely defined by it's values on these sets, i.e., whether the range of b is "sufficiently dense in Q".

Intuitively, it may suffice to have "many small players", i.e., many small values of b^j_i ($i \in \Omega$, $j = 1,\dots,m$). It is OWEN's [3] result that replication of b is a procedure which ensures "small players". On the other hand, BILLERA-RAANAN [1] show that a non-atomic b is a possible version. Of course in these cases it seems reasonable to assume that the range of b is "sufficiently dense in Q".

However, we feel that the study of nondegeneracy yields more precise insight into the meaning of "many small players". E.g., the reader may want to turn to a result represented by Theorem 3.5. of [5] , see also page 38 in [4]. In the context of that paper, non-degeneracy of an additive set function m is equivalent to non-degeneracy of an integer valued M. The set $\underline{\underline{Q}} = \underline{\underline{Q}}$ is a "contoûr line" or "constancy set": $\underline{\underline{Q}}_\lambda = \{S \mid M(S) = \lambda\}$ (λ natural).

According to the different weights M is capable of attaining as values, M is split via

$$M_i = \sum_{\rho=1}^{r} g_\rho 1_{K_\rho}(i) \qquad (i \in \Omega)$$

i.e.

$$M(S) = \sum_{\rho=1}^{r} g_\rho \, | S \cap K_\rho |$$

where $o < g_1 < \ldots < g_r$ are integers and $\Omega = K_1 + \ldots + K_r$ is a decomposition. Roughly, Theorem 3.5. of [ ] states the following: There are precise bounds L_ρ ($\rho = 1 \ldots r$) such that given some number theoretical properties of λ ($\lambda \in$ ideal spanned by $g_1, \ldots, g_r$)

$$| K_\rho | \geq L_\rho \qquad \rho = 1 \ldots r$$

is a sufficient condition for non degeneracy of M with respect to $\underline{\underline{Q}}$. The bounds depend on $g_1, \ldots, g_r$ and close inspection shows (see e.g. formula (26) (27) of [5] that bounds are large for small ρ, in other words we must have many small players in order to ensure nondegeneracy.

This is the main goal of our presentation: To show that although the nondegeneracy system $\underline{\underline{Q}}_\lambda$, as used in [5], is not the same as the one we use in our present context (see Corollary 1.9.), the results are roughly comparable: It is in principle possible to define lower bounds for the players such that if there are sufficiently many players (and in particular sufficiently many small players) then $\overline{m}$ is nondegenerate with respect to $\underline{\underline{Q}}$ and hence, by Corollary 1.9. the core and the "competitive equilibrium" coincide. This will be explained in Section 3. Section 2 is dealing with two examples.

§ 2 Examples

Example 2.1. Let $l = 1$ such that x ranges in $\mathbb{R}^1_+$, $c \in \mathbb{R}^1_+$ and

$$A = (a_{j1})_{j=1,\dots,m} =: (a_j)_{j=1\dots m} .$$

Then $v = v^{\mathfrak{OL}}$ is given by

$$\begin{aligned} v(S) &= \max\{cx \mid x \in \mathbb{R}^1_+ ,\ a_j x \leq b^j(S)\ (j=1,\dots,m)\} \\ (1) \qquad &= c \min_{\substack{j\in\{1,\dots,m\} \\ a_j > 0}} \frac{b^j(S)}{a_j} , \end{aligned}$$

assuming of course that $a \neq 0$. The S-dual problem is given by

$$(2) \qquad v(S) = \min\{\sum_{j=1}^{m} y_j\, b^j(S) \mid y \in \mathbb{R}^m_+,\ y_1 a_1 + \dots + y_m a_m \geq c\}$$

Consider this for $s = \Omega$. Then

$$(3) \qquad v(\Omega) = \min\{\sum_{j=1}^{m} y_j \mid y \in \mathbb{R}^m_+,\ y_, a_1 + \dots y_m a_m \geq c\} .$$

Let $\mathrm{J} := \{j \mid \frac{c}{a_j} = v(\Omega)\} = \{j \mid a_j = \max_{j'} a_{j'}\}$,

then, clearly the extreme solutions of (3) are given by

$$\frac{c}{a_j}\ e^j \ (j \in \mathrm{J}).$$

Therefore we pick $j_o \in \mathrm{J}$ and let $\bar{y} := \frac{c}{a_{j_o}}\ e^{j_o}$.

The neighbouring extremes of Y (given λ sufficiently large) are

$$\frac{c}{a_j}\ e^j \ (j \neq j_o) \quad \text{and} \quad e^{j_o}$$

such that

$$Q_o = \{z \in \mathbb{R}^m_+ \mid (\bar{y}^p - \bar{y})\, z \geq 0\ (p=1,\dots,m)\}$$

$$= \{z \in \mathbb{R}^m_+ \mid \frac{c}{a_j} z_j \geq \frac{c}{a_{j_o}} z_{j_o} \ (j \neq j_o),\ \lambda z_{j_o} \geq \frac{c}{a_{j_o}} z_{j_o}\}$$

$$= \{z \in \mathbb{R}^m_+ \mid \frac{z_j}{a_j} \geq \frac{z_{j_o}}{a_{j_o}} \ (j \neq j_o)\} \ .$$

Next, given b , we find

$$\underline{\underline{Q}} = \{S \in \underline{\underline{P}} \mid \frac{b^j(S)}{a_j} \geq \frac{b^{j_o}(S)}{a_{j_o}} \ , \ \frac{1-b^j(S)}{a_j} \geq \frac{1-b^{j_o}(S)}{a_{j_o}} \ (j \neq j_o)\}$$

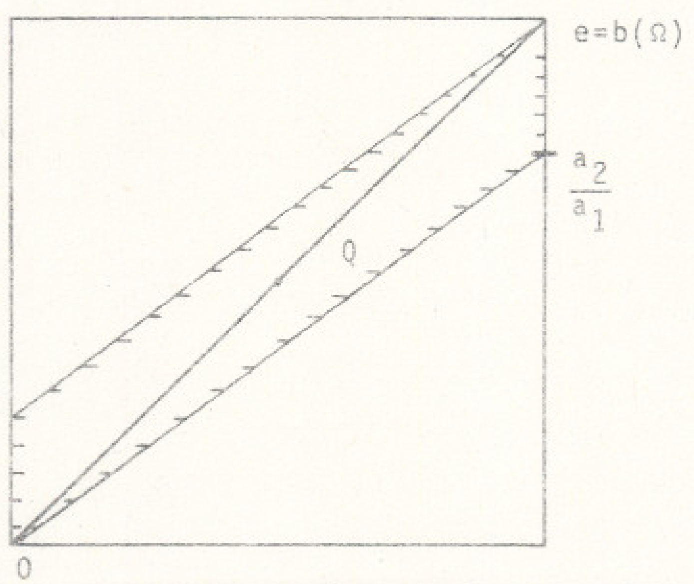
(4) $$= \{S \in \underline{\underline{P}} \mid 0 \leq \frac{b^j(S)}{a_j} - \frac{b^{j_o}(S)}{a_{j_o}} \leq \frac{1}{a_j} - \frac{1}{a_{j_o}} \ (j \neq j_o)\} \ .$$

Note that $\bar{y}\, b = \bar{m} = \frac{b^{j_o}}{a_{j_o}}$.

E.g., for $m = 2$, $a_1 > a_2$, $j_o = 1$; we have the following sketch:

$$Q_o = \{z \in \mathbb{R}^2_+ \mid \frac{z_2}{a_2} \geq \frac{z_1}{a_1}\}$$

$$Q = \{z \in \mathbb{R}^2_+ \mid \frac{z_1}{a_1} \leq \frac{z_2}{a_2} \leq \frac{z_1}{a_1} + \frac{1}{a_2} - \frac{1}{a_1}\} \ .$$



If $\bar{J} = \{j_o\}$ and $\bar{y}$ is the <u>unique</u> solution then $a_{j_o} > a_j$ $(j \neq j_o)$ and we have the trivial

Lemma 2.2. In the situation of Example 2.1., let $\delta_j : \frac{1}{a_j} - \frac{1}{a_{j_o}} > 0$ $(j \neq j_o)$ and assume that b^{j_o} is orthogonal to b^j $(j \neq j_o)$. Then, if

(5) $$b_i^j \leq a_j \delta_j \quad (j \neq j_o)$$

it follows that

$$\mathcal{C}(v) = \{\bar{m}\} = \{\frac{b^{j_o}}{a_{j_o}}\} .$$

Proof Let $m \in \mathcal{C}(v)$.

Let i be an element of the carrier of b^j $(j \neq j_o)$. Then inspection of (4) shows $i \in \underline{Q}$. From this it follows that $m_i = \bar{m}_i = 0$. On the other hand, if i is in the carrier of b^{j_o} then

$$v(\Omega) - m_i = m(\Omega) - m_i = m(\Omega - i)$$
$$\geq v(\Omega - i) = \frac{b^{j_o}(\Omega - i)}{a_{j_o}} = v(\Omega) - \bar{m}_i ,$$

i.e. $\bar{m} \geq m$. As $m, \bar{m} \in \mathcal{C}(v)$, we have $m = \bar{m}$.

Note that we did not show that $\bar{m}$ n.d. $\underline{Q}$. As the next example shows, this would similarly work with bounds (5) cut by approximately half their size. However, as this procedure is too coarse anyway (as shall be treeted in section 3) we do not elaborate on this topic.

In any case (5) tells us that players commanding resources $j \neq j_o$ should be small (and as $b^j(\Omega) = 1$, this means that there should be many of them).

Example 2.3. ("replicated L.P.-games").

Let ${}^k\Omega_i : = \{i, i+n, \ldots, i + (k-1)n\}$,

$$^k\Omega := \sum_{i\in\Omega} \Omega_i^k ,$$

and let $^kb \in \mathbb{R}_+^{kn}$ be defined by $^kb_j^\imath := \frac{1}{k} b_j^i$ $(\imath \in {}^k\Omega_i)$.

Then $^k\mathfrak{a} := (A, {}^kb, c)$ is the <u>k-fold replication of</u> $\mathfrak{a} = (A,b,c)$ and $^kv := v^{{}^k\mathfrak{a}}$ is the <u>k-fold replication of</u> $v^{\mathfrak{a}}$. We prefer to consider this version (division of resources by $\frac{1}{k}$) since

$$^kb(\Omega) = b(\Omega) = e$$

$$^kv(\Omega) = v(\Omega)$$

follows at once.

This in turn implies that the dual problem is the same for Ω and $^k\Omega$ since

$$\min\{y\,{}^kb(\Omega) \mid y \in \mathbb{R}_+^m , yA \geq c\}$$

$$= \min\{y\, b(\Omega) \mid y \in \mathbb{R}_+^m , yA \geq c\} .$$

Hence, $\bar{y} \in \mathbb{R}_+^m$ is an optimal solution for the dual $^k\Omega$ - problem if and only if it is an optimal solution for the dual Ω-problem.

<u>Theorem</u> 2.4. Let $\mathfrak{a} = (A,b,c)$ and let $\bar{y}$ be an extreme solution of the dual Ω- (and $^k\Omega$ -) problem. Also, let Λ be the corresponding tableau of the simplex algorithm as given by Lemma 1.3.. Assume $\Lambda e > 0$. If

(6) $$k \geq 1 + 2 \max_{\substack{p=1\ldots q\\ i\in\Omega}} \frac{|\Lambda_{p.} b_i|}{\Lambda_{p.} e} ,$$

then

$$\mathfrak{e}({}^kv) = \{\bar{y}\,{}^kb\} .$$

<u>Proof</u> Note that while $Q_o = Q_o \bar{y}^{Ac}$ is the same for $\mathfrak{a}$ and ${}^k\mathfrak{a}$,

$$\underline{Q} = \underline{Q}\,(b) \quad \text{and} \quad {}^k\underline{Q} : = \underline{Q}\,({}^k b)$$

are very well different. We are going to show that ${}^k\bar{m} : = \bar{y}\,{}^k b$ n.d. ${}^k\underline{Q}$ if (6) is satisfied.

To this end, (6) is at once rewritten to imply

(7) $$\frac{k-1}{k}\ \Lambda_{p.}\ \frac{e}{2} + \frac{1}{k}\,\Lambda_{p.}\ b_i \geq 0 \ (p=1,\ldots,Q)$$

and

(8) $$\Lambda_{p.}\ \frac{e}{2} + \frac{1}{k}\ \Lambda_{p.}\ b_i \leq \Lambda_{p.} e \ (p=1,\ldots,q)\ .$$

Now, according to whether k is even or odd, we may find a coalition $S \subseteq {}^k\Omega$, which has $\frac{k}{2}$ or $\frac{k-1}{2}$ players ι of each type i. Clearly

$${}^k b\ (S) = \frac{k}{2}\ \frac{1}{k}\, b\ (\Omega) = \frac{e}{2}$$

or

$${}^k b\ (S) = \frac{k-1}{k}\ \frac{e}{2}$$

respectively. Thus, $S \in {}^k\underline{Q}$. Pick $\iota \in \Omega_i$, $\iota \notin S$. As ${}^k b_\iota = \frac{1}{k}\, b_i$, we find

(9) $$\Lambda_{p.}\ {}^k b\ (S+\iota) = \Lambda_{p.}\ {}^k b\ (S) + \Lambda_{p.}\ {}^k b_\iota$$

$$\geq \frac{k-1}{k}\ \Lambda_{p.}\ \frac{e}{2} + \frac{1}{k}\,\Lambda_{p.}\ b_i \geq 0$$

(the inequality uses (7)). Also

(10) $$\Lambda_{p.}\ {}^k b\ (S+\iota) \leq \Lambda_{p.}\ \frac{e}{2} + \frac{1}{k}\,\Lambda_{p.}\ b_i \leq \Lambda_{p.}\ e$$

(using (8)), $(p=1,\ldots,Q)$. Now, (9) and (10) show that $S+\iota \in {}^k\underline{Q}$ as well.

Now, in order to show nondegeneracy let μ be a solution of the "defining system" of equations as introduced by Definition 1.8. As S and $S+\iota \in {}^{k}\underline{\underline{Q}}$, we have

$$ {}^{k}\bar{m}\ (S) = \mu\ (S) $$

$$ {}^{k}\bar{m}\ (S+\iota) = \mu\ (S+\iota)\ , $$

and hence, ${}^{k}\bar{m}_{\iota} = \mu_{\iota}$. As ι may be chosen arbitrarily and S accordingly, we have $\mu = {}^{k}\bar{m}$, q.e.d.

Of course, the above theorem is a reformulation of OWEN's [3] result about the finite convergence of core and competitive equilibrium. However, we are somewhat more precise as bounds for the coincidence of both concepts are specified. Nevertheless these bounds are the first off hand numbers one can obtain by studying nondegeneracy theory and as will turn out they are far too coarse.

In fact, the result of the theorem does not depend on the particular replicated form of the market. All we have used is the fact that an element in the range of b may be constructed, which is close to $\frac{e}{2}$. Then, if players are sufficiently small we may add singletons to the set which generates this element, thus generating sufficiently many sets in ${}^{k}\underline{\underline{Q}}$ in order to ensure nondegeneracy of $\bar{y}\,{}^{k}b$ with respect to this system.

§ 3 Lower bounds for the number of types

As in previous applications nondegeneracy leads to the reformulation of our problem by means of sum combinatorial or number theoretical device. As it turns out the number of players sufficient in order to ensure that the core and the competitive equilibrium coincide depends on certain lattice constants. This is done by introducing different types of players which are in our present context characterized completely by their initial outfit. However, the number of players of each type may vary arbitrarily above of certain bounds.

Suppose there are $\rho = 1,\dots,r$ different types of players involved in an L.P.-game. Let $g_\rho \in \mathbb{R}^m_+$, $\rho = 1,\dots,r$ denote type ρ's initial allocation. Thus, the g_ρ are an enumeration of the different ones of the b_i. Put

(1) $$K_\rho := \{ i \mid b_i = g_\rho \} \;,\; k_\rho := | K_\rho | .$$

Then

(2) $$b(\cdot) = \sum_{\rho=1}^{r} g_\rho \, | K_\rho \cap \cdot | \;,$$

and accordingly

(3) $$\bar{m} = \bar{y}\, b(\cdot) = \sum_{\rho=1}^{r} \bar{y}\, g_\rho \, | K_\rho \cap \cdot |$$

$$=: \sum_{\rho=1}^{r} \bar{g}_\rho \, | \;_\rho \cap \cdot | \;,$$

("equal treatment" when $\bar{m} \in \mathcal{C}(v)$ is generated by an optimal dual solution $\bar{y}$).

Let us fix $\mathfrak{O}\!\mathfrak{l}$ and $\bar{y}$ for the moment.

<u>Definition</u> 3.1. $\underline{\underline{Q}}$ separates $i, j \in \Omega$ if there is $S \in \underline{\underline{Q}}$ such that

$$i \in S, j \notin S \quad \text{or} \quad j \in S, i \notin S .$$

<u>Lemma</u> 3.2. If $\mu \in \mathcal{C}(v)$ and $\underline{Q}$ separates $i,j, \in K_\rho$, then

$$\mu(\cdot) = \sum_{\rho=1}^{r} \bar{\mu}_\rho \mid K_\rho \cap \cdot \mid$$

for appropriate $\bar{\mu}_\rho \in \mathbb{R}_+$ $(\rho = 1,\dots,r)$.

<u>Proof</u> Let $i,j \in K_\rho$ for some ρ.

Pick S as given by Definition 3.1. such that $i \in S$, $j \notin S$. Note that $b_i = b_j = g_\rho$ and hence $b(S-i+j) = b(S)$. From this, it follows that $\Lambda b(S-i+j) = \Lambda b(S)$ and hence

$$S-i+j \in \underline{Q} \quad .$$

Therefore, as $\mu \in \mathcal{C}(v)$, we have

$$\mu(S) = \bar{m}(S)$$

$$\mu(S-i+j) = \bar{m}(S-i+j),$$

which implies

$$\mu_j - \mu_i = \bar{m}_j - \bar{m}_i = 0.$$

Therefore, $\mu_i = \mu_j = \bar{\mu}_\rho$ $(i,j \in K_\rho)$, q.e.d.

Of course, Lemma 3.2. is a trivial version of equal treatment in the core. The separation condition indicated by Definition 3.1. is indeed not a very strong one: In most applications treated so far it turned out that the separation condition is automatically satisfied if the sufficient condition for "admissibility" (see the following definition) is satisfied.

<u>Definition</u> 3.3. Let $G = (g_1,\dots,g_\rho) \in \mathbb{R}_+^{m \times r}$ and $k = (k_1,\dots,k_r) \in \mathbb{N}^r$. Then (G,k) is said to be <u>admissible</u>

("for $\underline{\underline{Q}}$ " or "for $(\mathfrak{a}, \bar{y})$") if there is a matrix

$$A = (\alpha_\rho^\sigma)_{\sigma,\rho = 1,\ldots,r} \tag{3}$$

with <u>integer elements</u> α_ρ^σ such that

(4)
1. $0 \leq \alpha_\rho^\sigma \leq k_\rho \quad (\sigma,\rho = 1,\ldots,r)$,
2. A is non singular
3. $0 \leq \Lambda \sum_{\rho=1}^{n} \alpha_\rho^\sigma g_\rho \leq \Lambda e$.

We shall use the term <u>strongly admissible</u> if, in addition, for every ρ there is σ such that $0 < \alpha_\rho^\sigma < k_\rho$ is satisfied.

The reader is obliged to compare this definition of admissibility with the one given in [5] [6] .

<u>Theorem</u> 3.4. Given $\mathfrak{a}$ and $\bar{y}$, let $\bar{m} : = \bar{y} b$ as usual and suppose that G and k are specified by (1) and (2). If (G,k) is strongly admissible, then $\mathcal{C}(v) = \{\bar{m}\}$, provided $\Lambda e > 0$.

<u>Proof:</u> Suppose $\mu \in \mathcal{C}(v)$ and let $A = (\alpha_\rho^\sigma)_{\sigma,\rho=1,\ldots,r}$ be defined by (3).

Step 1: Let $i,j \in K_\rho$ for some ρ. Pick $\bar{\sigma}$ such that

$$0 < \alpha_\rho^{\bar{\sigma}} < k_\rho \quad .$$

Then, there is $S \in \underline{\underline{P}}$ such that $|S \cap K_\rho| = \alpha_\rho^{\bar{\sigma}}$. Obviously, S can be chosen such that $i \in S$, $j \notin S$ and hence $\underline{\underline{Q}}$ separates i,j. By Lemma 3.2., it follows that

$$\mu = \sum_{\rho=1}^{r} \bar{\mu}_\rho |K_\rho \cap \cdot| \tag{5}$$

with appropriate $\bar{\mu}_\rho$ $(\rho=1,\ldots,r)$. Next, this procedure may be repeated for <u>any</u> σ, i.e., there is S^σ sucht that

$$| S^\sigma \cap K_\rho | = \alpha_\rho^\sigma \quad .$$

By condition 3. of (3) it follows that

$$0 \leq \Lambda b\,(S^\sigma) = \Lambda \sum_{\rho=1}^{r} g_\rho \, | S^\sigma \cap K_\rho |$$

$$= \Lambda \sum_{\rho=1}^{r} \alpha_\rho^\sigma \, g_\rho \leq \Lambda \, e$$

and hence $S^\sigma \in \underline{\underline{Q}}$. Therefore

$$\sum_{\rho=1}^{r} \bar{\mu}_\rho \, \alpha_\rho^\sigma = \sum_{\rho=1}^{r} \bar{\mu}_\rho \, | S^\sigma \cap K_\rho | = \mu\,(S^\sigma)$$

$$= \bar{m}\,(S^\sigma) = \sum_{\rho=1}^{r} \bar{g}_\rho \, | S^\sigma \cap K_\rho | = \sum_{\rho=1}^{r} \bar{g}_\rho \, \alpha_\rho^\sigma \; (\rho=1,\ldots,r),$$

and, as A is non-singular, it follows that

$$(\bar{\mu}_1,\ldots,\bar{\mu}_r) = (\bar{g}_1,\ldots,\bar{g}_r) \, .$$

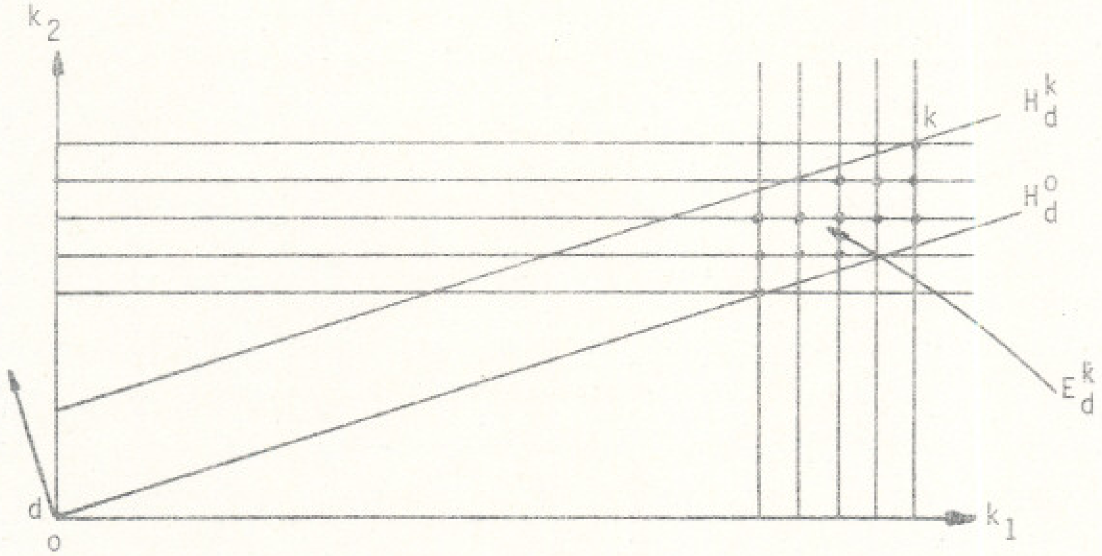
Thus, $\mu = \bar{m}$, q.e.d.

We now have reformulated nondegeneracy in terms of admissibility. The question of finding bounds for the number of players of each type may therefore be answered once it is possible to solve the problems specified by (4). In other words, given the "weights" $G = (g_1 \ldots, g_r)$ and the "box size numbers" $k = (k_1, \ldots, k_r)$, under what condition is it possible to find the matrix A such that (4) is satisfied?

<u>Definition</u> 3.5. Let $k \in \mathbb{N}^r$. For any vector $d \in \mathbb{R}^r$, consider the hyperplanes with normal d passing through 0 (say H_d^0) and passing through k (say H_d^k) and let E_d^k be the intersection of the "stripe between" H_d^0 and H_d^k and the rectangle generated

by k (in $\mathbb{N}^r$), i.e.,

$$E_d^k = \{x \in \mathbb{N}^r \mid o \leq dx \leq dk,\ o \leq x \leq k\}.$$



We shall say that k is <u>sufficiently large</u> w.r.t. a matrix $D \in \mathbb{R}^{q \times r}$, if, taking the rows $D_{1.},\dots,D_{q.}$ of D, we find r linearly independent $\mathbb{N}^r$-vectors in

$$E_D^k := \bigcap_{p=1}^{q} E_{D_{p.}}^k ,$$

non of which equals k.

Theorem 3.6. Let $A \in \mathbb{R}_+^{m \times l}$ and $c \in \mathbb{R}_+^l$. Suppose, $\bar{y} \in \mathbb{R}_+^m$ is an extreme solution of

$$\min \{y\, e \mid y \in \mathbb{R}_+^m,\ y\, A \geq c\}.$$

Let $\Lambda \in \mathbb{R}^{q \times m}$ be specified by Lemma 1.3. assume $\Lambda\, e > o$. Given $G \in \mathbb{R}_+^{m \times r}$ and $k \in \mathbb{N}^r$ such that $\sum_{\rho=1}^{r} k_\rho g_\rho = e$, b (and hence σ) is defined on $\Omega = \{1,\dots,n\}$ up to permutations of players of the same type by taking $n = \sum_{\rho=1}^{r} k_\rho$.

If k is sufficiently large w.r.t. $D: = \Lambda\, G$, then

$$\mathcal{C}\,(v^{\alpha}) = \{\bar{y}\, b\} = \{\bar{m}\}\ .$$

Proof. By definition,

$$E_D^k = \bigcap_{p=1}^{q} E_{D_{p.}}^k$$

contains r linearly independent $\mathbb{N}^r$ vectors, say $\alpha^1,\dots,\alpha^r$. That is, we have

$$\begin{aligned} & o \le D_{p.}\ \alpha^\sigma \le D_{p.}\ k \\ & o \le \alpha^\sigma \quad\ \le k \qquad (\sigma=1,\dots,r) \\ & \qquad\qquad\qquad\quad (p=1,\dots,q), \end{aligned}$$

or

$$\begin{aligned} & o \le D\, \alpha^\sigma \le D\, k \\ (6) \qquad & \\ & o \le \alpha^\sigma \quad \le k \qquad (\sigma=1,\dots,r). \end{aligned}$$

Observing

$$G\, k = \sum_{\rho=1}^{r} G_{.\rho}\, k_\rho = \sum_{\rho=1}^{r} k_\rho\, g_\rho = e$$

$$G\, \alpha^\sigma = \sum_{\rho=1}^{r} G_{.\rho}\, \alpha_\rho^\sigma = \sum_{\rho=1}^{r} \alpha_\rho^\sigma\, g_\rho \quad (\sigma=1,\dots,r)$$

and $\Lambda\, G = D$, we conclude that (6) is rewritten as

$$\begin{aligned} & o \le \Lambda \sum_{\rho=1}^{r} \alpha_\rho^\sigma\, g_\rho \le \Lambda \sum_{\rho=1}^{r} k_\rho\, g_\rho = \Lambda e \\ & o \le \quad \alpha^\sigma \qquad\quad \le k\ . \end{aligned}$$

Hence, $(G\ ,k)$ is admissible for $(\alpha, \bar{y})$.
In fact, as none of the vectors α^σ equals k it is seen at once that $(G\ ,k)$ is strongly admissible, thus we may apply Theorem 3.4.,

q.e.d.

Theorem 3.6. may be seen as an exact formulation of the requirement that "sufficiently many" players must be available in order to have the core and the c.e. coincide: k has to be sufficiently large in the sense of Definition 3.5.. The test for this property amounts to checking wether a certain convex compact polyhedron in $\mathbb{R}^r$ allows for r linearly independent $\mathbb{N}^r$-vectors. The determination of the "lattice constants" involved in this test is a problem in the Geometry of Numbers; indeed, MINKOWSKI was already concerned with this question.

We shall give one application, assuming for the sake of symmetry that k is even.

<u>Theorem</u> 3.7. Let $A \in \mathbb{R}_+^{m \times l}$, $c \in \mathbb{R}_+^l$ and suppose that $\bar{y}$ is an extreme solution of

$$\min \{yA \mid y \in \mathbb{R}_+^m,\ yA \geq c\} .$$

Let $\Lambda \in \mathbb{R}^{q \times m}$ be specified by Lemma 1.3. and assume $\Lambda e > 0$. Given $G \in \mathbb{R}_+^{m \times r}$ and $k \in \mathbb{N}^r$ sucht that $\sum_{\rho=1}^{r} k_\rho g_\rho = e$, b (and hence $\mathfrak{a}$) is defined on Ω up to permutations of players of the same type. Assume w.l.o.g. $k_1 \leq k_2 \leq \ldots \leq k_r$ and k_ρ even $(\rho = 1, \ldots, r)$. Let $D = \Lambda C$ and let V_D^k be the volume of the convex polyhedron $\{x \in \mathbb{R}^r \mid o \leq x \leq k,\ o \leq Dx \leq Dk\}$. If

(7) $$V_D^k \geq \frac{2\, k_2 \cdot \ldots \cdot k_{r-1}}{\min\limits_{R \subseteq \{1,\ldots,r\}}\ \max\limits_{\substack{\rho \in R \\ p=1,\ldots,q}} \left\{ \frac{1}{k_\rho},\ \frac{|D_{p\cdot}(R)|}{|D_{p\cdot}\, k|} \right\}}$$

then

$$\mathcal{C}(v^{\mathfrak{a}}) = \{\bar{y}b\} = \{\bar{m}\} .$$

<u>Proof</u> Given a convex polyhedron $B \subseteq \mathbb{R}^r$, consider the "successive minima"

(8) $\lambda_\rho := \lambda_\rho(B) := \min\{\lambda \mid \text{there are } \rho \text{ independent } \mathbb{N}^r\text{-vectors in } \lambda B\}$.

These quantities are studied within the framework of the "Geometry of Numbers" (see e.g. [2]) (λ_ρ is not to be mixed up with the elements of Λ!).

According to MINKOWSKI's (second) theorem (Theorem V, CH. VIII.4.3., p. 218 of [2]), we have

(9) $\lambda_1 \cdot \ldots \cdot \lambda_r \, V(B) \leq 2^r$

($V(B)$ denoting the volume of B), provided B is symmetric.

As k is assumed to be even, we may enforce symmetricity when E_D^k ($\subseteq \mathbb{N}^r$) is "replaced" by

$$B_D^k := \{x \in \mathbb{R}^r \mid -\frac{k}{2} \leq x \leq \frac{k}{2} \, , \, -\frac{Dk}{2} \leq Dx \leq \frac{Dk}{2}\}$$

($\subseteq \mathbb{R}^r$) . We are going to show that under the conditions of our Theorem, there are r independent $\mathbb{Z}^r$-vectors in B_D^k and hence in E_D^k .

To this end note that B_D^k is given by means of the distance function $F = F_D^k : \mathbb{R}^r \to \mathbb{R}_+$,

(10) $$F(x) = \max_{\substack{\rho=1,\ldots,r \\ p=1,\ldots,q}} \left\{ \left| \frac{2x_\rho}{k_\rho} \right| , \left| \frac{2D_p \cdot x}{D_p \cdot k} \right| \right\}$$

(see Chapter IV of [2]) in a way that F is positively homogeneous, superadditive, and

$$B_D^k = \{x \in \mathbb{R}^r \mid F(x) \leq 1\} .$$

According to (4) , VIII. 1 of [2] ,

(11) $$\lambda_1 = \min_{\substack{x \in \mathbb{Z}^r \\ x \neq 0}} F(x) .$$

Since the lattice we are dealing with is $\mathbb{Z}^r$ and F is monotone, clearly

$$(12) \qquad \lambda_1 = \min_{R \subseteq \{1,\dots,r\}} F(1_R).$$

Writing $D_{p\cdot}\ 1_R = D_{p\cdot}(R)$ ($D_{p\cdot}$ regarded as an additive set function), and observing that $\left|\frac{2x_\rho}{k_\rho}\right|$ equals zero or $\frac{2}{k_\rho}$ at $x = 1_R$, we obtain by (12) and (10):

$$(13) \qquad \lambda_1 = \min_{R \subseteq \{1,\dots,r\}} \max_{\substack{\rho \in R \\ p=1,\dots,q}} \left\{ \frac{2}{k_\rho}, \left| \frac{2\ D_{p\cdot}(R)}{D_{p\cdot}\ k} \right| \right\}.$$

Next, for $\lambda_2 \cdot \dots \cdot \lambda_{r-1}$, we shall take estimates from the cube

$$B^k := \{x \in \mathbb{R}^r \mid -\tfrac{k}{2} \leq x \leq \tfrac{k}{2}\} .$$

Clearly,

$$\lambda_\rho (B^k) = \frac{2}{k_\rho} \qquad (\rho = 1,\dots,r)$$

as we have assumed $k_1 \leq \dots \leq k_r$. Thus

$$(14) \qquad \lambda_\rho (B^k_D) \geq \frac{2}{k_\rho} \qquad (\rho = 1,\dots,r) ,$$

as $B^k_D \subseteq B^k$. Combining we find (again $\lambda_\rho = \lambda_\rho (B^k_D)$) :

$$\lambda_r \leq \frac{2^r}{V(B^k_D)\ \lambda_1 \cdot \dots \cdot \lambda_{r-1}} \qquad \text{(by (9))}$$

$$\leq \frac{2\ k_2 \cdot \dots \cdot k_{r-1}}{V^k_D \cdot \lambda_1} \qquad \text{(by (14))}$$

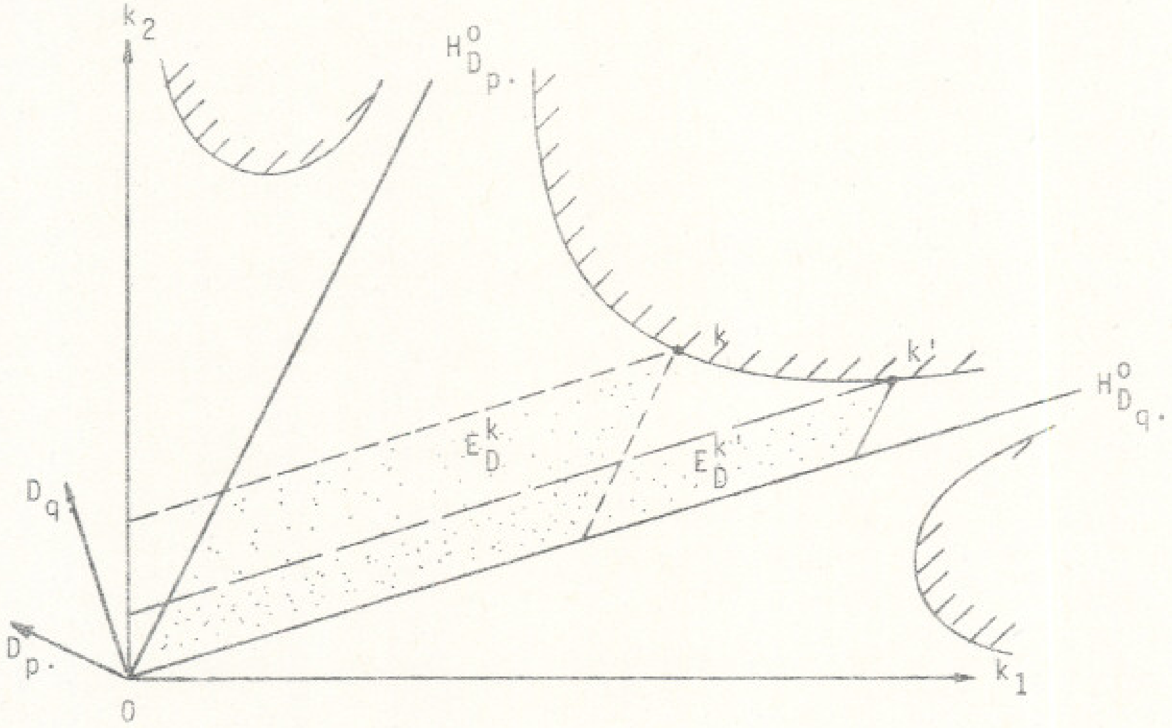
$$\leq 1 \qquad \text{(by (13) and (7))}.$$

This implies that there are r independent $\mathbb{Z}^r$-vectors in B_D^k and hence the same is true for E_D^k. Thus, k is sufficiently large w.r.t. D and, by Theorem 3.6., the proof is completed.

Remark 3.8. 1. Note that the estimate $\lambda_p \geq \frac{2}{k_p}$ is a rather coarse one. Improving this would certainly improve the result of Theorem 3.7.. Indeed, the problem we have at hand is the exact determination of λ_r and hence the Geometry of Numbers enters the field of Game Theory when it comes to give exact estimates for $k = (k_1, \ldots, k_r)$, i.e., for numbers of players of each type.

2. Essentially, MINKOWSKI's theorem tells us that the "volume" of E_d^k in a sense determines wether k is large enough. This is readily understood as follows: the "volume" of E_d^k will be small if $k \in \mathbb{N}^r$ is "close" to one of the hyperplanes $H^o_{D_{p\cdot}}$ (see Definition 3.5.). If k "approaches" $H^o_{D_{p\cdot}}$, then, tracing back Theorem 3.4. to Corollary 1.9., it is seen that $Ae>o$ can no more be guaranteed. This essentially means that the uniqueness of the dual solution eventually is lacking and in this case the "equivalence theorem" of core and c.e. may be wrong. Thus, "close" to the hyperplanes $H^o_{D_{\cdot p}}$, k becomes larger and larger in order to generate a sufficiently large volume and the area of k's

such that the distribution of types indicated by k allows for an equivalence theorem avoides these hyperplanes.



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