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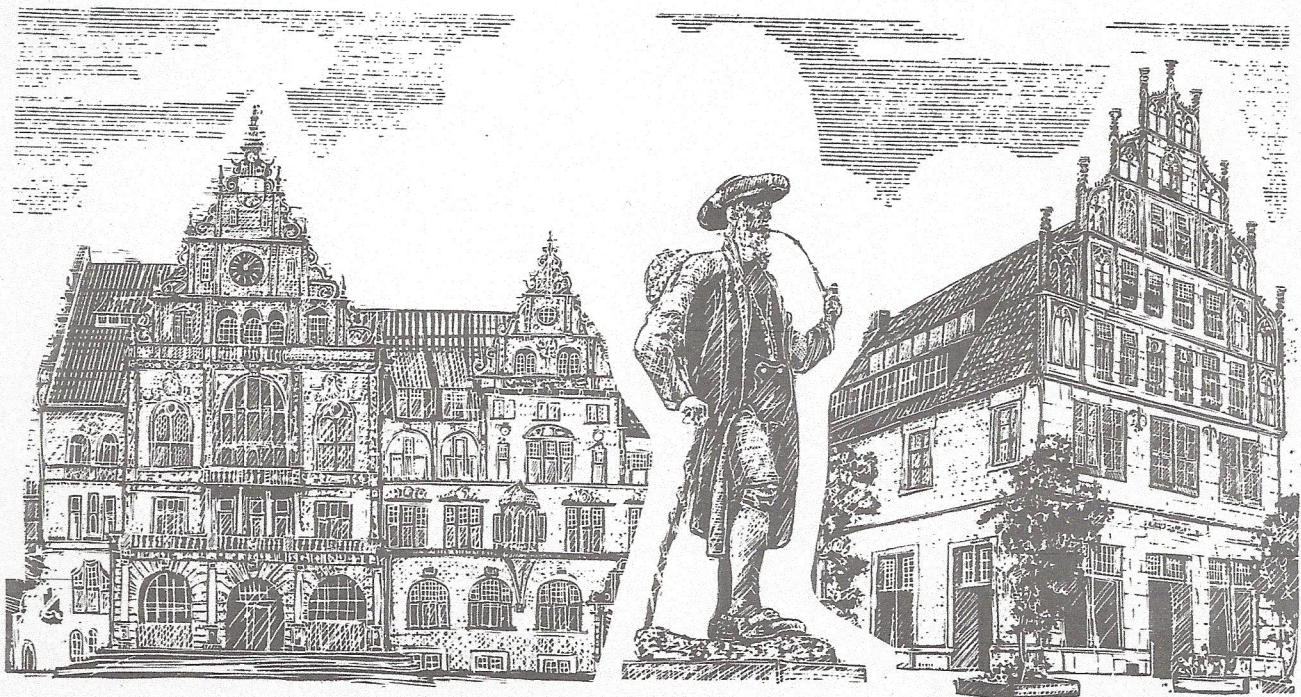## Institute of Mathematical Economics

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R.W. Scholz

Sequential Two Person-Zero-Sum-Games
with Incomplete Information and Inci-
dental Sequence of Moves

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H. G. Bergenthal

R.W. Scholz*

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Abstract:

Value and optimal strategies for infinitely repeated zero-sum-games with incidental sequence of moves and general information matrices fulfilling a condition of consistence are determined in this paper.

As special cases one gets a) a generalization of the results of PONSSARD & ZAMIR [1973] about games in which the informed player always moves first, b) the case of simultaneous moves discussed in KOHLBERG's [1975] paper on which several notes are made and c) the case in which the informed player always is the second to move.

Each variety of sequence of moves brings up a different value and different constructions of optimal strategies for the un-informed player.

Contents

1. Introduction

Sequential games with incomplete information have been discussed by several authors. Due to HARSANYI [1967, 1968], who showed the generality of models for which incomplete information has to be stated only to the payoff functions, a mathematical analysis of games with incomplete information became possible. AUMANN & MASCHLER [1968] determined the value of infinitely two-person-zero-sum-games with incomplete information for one player and simultaneous moves. PONSSARD & ZAMIR [1973] discussed the case where the uninformed player is always the first to move. It was noticed that both varieties of sequence of moves lead to different values and especially the uninformed player's constructions of optimal strategy are different. Among others, KOHLBERG [1976] generalized the special case, where both players are told the pure strategies of the other one after every repetition by general information matrices. So it is possible to model that an uninformed player is able to distinguish differently well between two different strategies at various 'types'.

To analyse games with infinite repetitions and order of sequence determined by chance, to find the value and optimal strategies for both players is the central theme of this paper. The discussion will be restricted to finite strategy sets and it will be assumed that (only) one player has incomplete information

about the payoff matrices; this means, one player (Player II)
knows only a probability distribution p by which one type
(matrix) will be selected out of a finite set of types
respectively. In the class of games considered here it is
supposed that Player II receives an information about the
choice of strategies by Player I by a fairly arbitrary
information pattern (information matrix) during each
repetition. At first we analyse the special cases in which
Player I moves first or second or both players move
simultaneously with probability 1, and then we analyse
the more general model with incidental sequence of moves
which is closer to real life situations. Also from a
methodological point of view it is interesting whether--even
for the considered general case--the mathematical principle
of concavication becomes applicable as it is for the special
case of simultaneous moves and the case where the informed
player is always the first to move.

In section 3 the results of PONSSARD & ZAMIR about games in
which Player II always is the second to move can be generalized
by general information matrices.In section 4 we give a corrected
proof for the optimality of certain strategies for the class of
games discussed in KOHLBERG's paper [1975]. In section 5
the relatively worse situation for Player II, where he has
to draw first and being uninformed, will be discussed. We
shall see that also in this case the idea used in KOHLBERG's
[1975] paper leads to success. The uninformed player has to apply

stationary strategies for blocks and derive the stationary stra-
tegy in the next block from the observed frequence of information
he had received before.

Furthermore, a sufficient condition will be given under which
the veiling of information of the informed player in the three
cases of sequence of move will not handicap each other. The
author prefers to give mathematical strength to proof and
definitions which will not be found and are not present in
most manuscripts in game-theory. This mathematical strength will
hopefully help us to avoid mistakes which creep-in easily other-
wise. A price paid for this formalization is that there is more
work in definitions, this being especially the case due to the
model's generality.

2. <u>Description of Sequential Games with Incidental Sequence of
   Moves; Definitions and Preliminaries</u>

Description of the games $\Gamma_n(p,q)$ and $\Gamma_\infty(p,q)$:

1.) By a random mechanism a matrix G^α is selected out of a
finite set of known r x s payoff-matrices G^1, ..., G^m with pro-
bability $p^1, ..., p^m$. Player I will be informed of the choice of
$\alpha \in \{1, ... , m\}$, Player II knows but the probabilities p^1, ...,
p^m, which are also known to Player I. - Each matrix G^α is allied
with an information matrix H^α.

2.) By chance it is determined and told both players which sequence-modality of moves will happen in the following k-th repetition of the game, whereas $0 \leq k < n$ in the game $\Gamma_n(p,q)$ and $0 \leq k \leq \infty$ in $\Gamma_\infty(p,q)$.

2a) With probability q^1 Player I moves first; this means, Player I chooses a row i; Player II is informed about the i-th row of H^α, thereafter Player II chooses a column j. The choice j is told to Player I.

2b) With probability q^2 Player II is the first to move; this means, Player II chooses a column j, Player I - knowing j - chooses a row i. Player II is told the information $h^\alpha_{i,j}$.

2c) With probability q^3 both players draw simultaneously; this means Player I chooses a row i, (afterwards) Player II a column j. Player I is informed of the choice j, Player II by $h^\alpha_{i,j}$.

3.) After each repetition Player II 'pays' $g^\alpha_{i,j}$ Player I, whereas the value which is to be payed is only told to Player I.

4.) If $k < n-1$, steps 2.) and 3.) are repeated. If $k = n-1$, the game $\Gamma_n(p,q)$ is finished. In the game $\Gamma_\infty(p,q)$ steps 2.) and 3.) are repeated in any case. The payoff-function in the game $\Gamma_n(p,q)$ is the arithmetical mean of the payoffs of the repetitions. The definitions of the value and the payoff-function of the game $\Gamma_\infty(p,q)$ is given in Definition 2.

Games which fit into this description may be mathematically defined as follows:

<u>Definition</u> 2.1: A two-person-zero-sum-game $\Gamma_n(p,q)$ (resp. $\Gamma_\infty(p,q)$) with information pattern for repetitions and sequence of moves dependent on chance is given by the ordered set $\{R,S, \{G^1,\ldots,G^m\}, \{H^1,\ldots,H^m\}, p,q\}$ and a number n-1, $n \in N$ (resp. ∞) of repetitions, whereas are

(i) $R := \{1,\ldots,r\}$ the set of pure strategies of Player I,

       $S := \{1,\ldots,s\}$ the set of pure strategies of Player II in each repetition.

(ii) $G^1,\ldots,G^m$ r x s payoff-matrices,

       $G^\alpha := (g_{i,j}^\alpha)_{i\in R, j\in S}$, $1 \le \alpha \le m$, $g_{i,j}^\alpha \in \mathbb{R}$.

(iii) $H^1,\ldots,H^m$ r x s information-matrices, the information pattern for Player I; $H^\alpha := (h_{i,j}^\alpha)_{i\in R, j\in S}$ with $1 \le \alpha \le m$, whereby is presupposed, that:

       with j, $j' \in S$, $j \ne j'$ implies $h_{i,j}^\alpha \ne h_{i,j'}^\alpha$.

$H := \bigcup_{\substack{i\in R, j\in S \\ \alpha\in\{1,\ldots,m\}}} \{h_{i,j}^\alpha\} := \{\mu_k / \mu_k := (0,\ldots,0,\underset{\underset{\text{k-th position}}{\uparrow}}{1},0,\ldots,0) \in \mathbb{R}^{|H|}, 0 \le k \le |H|\}$

(iv) p probability distribution on $M := \{1,\ldots,m\}$

(v) $q := (q^1,q^2,q^3)$ probability distribution on $Z := \{1,2,3\}$, the set of possible sequence of moves.

As the condition in Definition 1, (iii) implies, that $\Gamma_n(p,q)$ and $\Gamma_\infty(p,q)$ are games with perfect recall, we do not define mixed strategies but without loss of generality behaviour strategies

$\sigma := (\sigma_k)_{k\in\mathbb{N}}$ and $\tau := (\tau_k)_{k\in\mathbb{N}}$, whereby σ_k (resp. τ_k) is the behaviour of Player I (resp. II) in the k-1-th repetition.

If $F := \{(f^1,\ldots,f^r) \,/\, f^i \geq 0 \; \forall \, i \in R, \; \sum\limits_{i=1}^{r} f^i = 1\}$,

$\sigma_1 : M \times (\{1,3\} \cup \{2\} \times S) \to F$

$\sigma_k : M \times \prod\limits_{i=1}^{k-1} (\{1,3\} \times R \times S \cup \{2\} \times S \times R) \times (\{1,3\} \cup \{2\} \times S) \to F$ for $k \geq 2$;

$\Sigma := \{\sigma = (\sigma_k)_{k \in \mathbb{N}}\}$ assigns the set of all strategies of Player I in the game $\Gamma_\infty(p,q)$.

Thus $Y := \{(y^1,\ldots,y^s) \,/\, y^j \geq 0 \; \forall \, j \in S, \; \sum\limits_{j=1}^{s} y^j = 1\}$

and hence $\tau_1 : ((\{1\} \times \bigcup\limits_{\substack{i \in R \\ \alpha \in M}} \{h^\alpha(i)\}) \cup \{2,3\}) \to Y$ whereby here

$h^\alpha(i) := (h^\alpha_{i,1},\ldots,h^\alpha_{i,s})$; $h^\alpha(i)$ denotes the i-th row of the information matrix H^α and $T := \{\tau = (\tau_k)_{k \in \mathbb{N}}\}$ the set of strategies of Player II in the game $\Gamma_\infty(p,q)$.

Let Ω be defined as the set of all infinite games:

$\Omega := M \times \prod\limits_{n=1}^{\infty} (\{1,3\} \times R \times S \cup \{2\} \times S \times R)$

so by a fixed pair of strategies σ, τ each given finite game sequence $(\alpha, a_1, b_1, c_1; \ldots; a_n, b_n, c_n)$ is assigned to a "natural realisation probability".

$\text{prob} \overbrace{((\alpha; a_1, b_1, c_1; \ldots; a_n, b_n, c_n)}^{:= S_n} \times \overbrace{\prod\limits_{k=n+1}^{\infty} (\{1,3\} \times R \times S \cup \{2\} \times S \times R))}$

$:= p^\alpha \prod\limits_{k=1}^{n} q^{a_k} \cdot \sigma_k(\alpha, A_k)^{B_k} \, \tau_k(A'_k)^{B'_k}$ for $n \in \mathbb{N}$

The definitions of a_k, b_k, c_k, A_k etc. appear clearly from the definition of Ω, Definition 1 and the definition of the strategies, so i.e. $b_k \in R$ resp. $b_k \in S$ according to $a_k \in \{1,3\}$ or $a_k \in \{2\}$, that means which sequence of moves will occur on the k-th run.

It is easy to show [compare SCHOLZ, 1976] that there exists an unique probability-measure $pr_{p,q,\sigma,\tau}$ on $\bar{R} := {}^B\{S_n \ / \ S_n :=$

$= (\alpha; a_1, b_1, c_1, \ldots, c_n) \times \prod_{k=n+1}^{\infty} (\{1,3\} \times R \times S \cup \{2\} \times S \times R)$,

$n \in \mathbb{N} \ \{0\} \ \}$ in Ω with: $pr_{p,q,\sigma,\tau}(S_n) = prob \ (S_n)$ for S_n arbitrary, $n \in \mathbb{N}$.

This is proved by applying the Consistence Theorem of KOLMOGOROFF [see TAYLOR, 1966, p.153], whereby it is best to do to apply a generalization of it, where the spaces can be Polish (that means separate and complete) [see MAMMITZSCH, 1975, p.9].

On the probability space k we define the random variables $\tilde{\alpha}$, $\tilde{a}_k$, $\tilde{i}_k$, $\tilde{j}_k$ by $\tilde{\alpha}(\omega) = \alpha$, $\forall \omega \in \Omega$, analogous for $\tilde{a}_k$

$\tilde{i}_k(\omega) := \begin{cases} b_k & \text{if } a_k \in \{1,3\} \\ c_k & \text{if } a_k = 2 \end{cases} \quad \forall \ \omega \in \Omega$, analogous for $\tilde{j}_k$.

and furthermore: $\tilde{g}_k := \tilde{g}^{\tilde{\alpha}}_{\tilde{i}_k, \tilde{j}_k}$ and $\tilde{\tilde{g}}_k := \frac{1}{k} \sum_{i=1}^{k} \tilde{g}_i$.

Remark: Without intending to identify function and function-
value, sometimes when obvious is abbreviately written:
$\tilde{\alpha} = \tilde{\alpha}(\omega)$.

Definition 2.2: $v(p)$ is the value of $\Gamma_\infty(p,q)$ if for all $\varepsilon > 0$
there are $\sigma_\varepsilon \in \Sigma$, $\tau_\varepsilon \in T$, $N(\varepsilon) \in \mathbb{N}$, so that for all $n > N(\varepsilon)$

$$E_{p,q,\sigma_\varepsilon,\tau} \tilde{\tilde{g}}_n := E_{pr_{p,q,\sigma_\varepsilon,\tau}} (\tilde{\tilde{g}}_n) \geq v - \varepsilon \ \forall \ \tau \in T$$

$$E_{p,q,\sigma,\tau_\varepsilon} \tilde{\tilde{g}}_n := E_{pr_{p,q,\sigma,\tau_\varepsilon}} (\tilde{\tilde{g}}_n) \leq v + \varepsilon \ \forall \ \sigma \in \Sigma$$

σ_ε (resp. τ_ε) is called an ε-good strategie; a strategie σ_0
(resp. τ_0) is called an optimal strategie, if it is ε-good for
all $\varepsilon > 0$.

The fundamental theorem which will be applicated is a Minimax-
Theorem [comp. VOGEL 1970, p.124].

Theorem 2.1: Let K_1, K_2 be convex sets in $\mathbb{R}^n$ resp. $\mathbb{R}^k$, $n,k \in \mathbb{N}$.
Be $f(x,y)$ concave on K_1 for each fixed $y_0 \in K_2$ and convex on K_2
for each fixed $x_0 \in K_1$. Furthermore K_1 compact and for each $y_0 \in K_2$
$f(x,y_0)$ semi-upper-continuous on K_1, then follows $\sup_{K_1} \inf_{K_2} f(x,y)$

$\inf_{K_2} \max_{K_1} f(x,y)$.

If $f : K \to \mathbb{R}$, $K \subset \mathbb{R}^n$ convex; then cav $f : K \to \mathbb{R}$ is defined by cav $f(x) := \sup\{\eta \ / \ (x,\eta) \in \mathcal{h}\{(x,\eta) \ / \ \eta \leq f(x)\}\}$ $\forall x \in K$ cav f is the smallest concave function which majors f ($\mathcal{h}(x)$ denotes the convex hull of x).

A corollar from the Theorem of CARATHEODORY see [EGGLESTON, 1963, p.35] brings up an approximation of the concavication of a function f by convex-combinations of values of f (ROCKAFELLA, Theorem 10.2. [1970]).

Theorem 2.2: $f : K \to \mathbb{R}$, $K \subset \mathbb{R}^n$ convex, then:

$$\text{cav } f(x) = \sup\{\sum_{l=1}^{n+1} \lambda_l f(x_l) \ / \ \sum_{l=1}^{n+1} \lambda_l x_l = x, \ x_l \in K, \ \lambda_l \geq 0, \ \sum_{l=1}^{n+1} \lambda_l = 1\}.$$

3. The Class of Games where the Informed Player Makes the First Move

In this section the informed player moves first with probability 1, so we introduce some simplifications to clarify the constructions.

A strategy σ of Player I is here sufficiently defined by

$$\sigma_1 : M \to F; \quad \sigma_k : M \times \prod_{l=1}^{k-1} R \times S \to F \quad \text{for } k \geq 2;$$

a strategy of Player II by

$$\tau_1 := \bigcup_{\substack{i \in R \\ \alpha \in M}} \{h^\alpha(i)\} \to Y \quad \text{and for } k \geq 2 \text{ by}$$

$$\tau_k := (\prod_{n=1}^{k-1} \bigcup_{\substack{i \in R \\ \alpha \in M}} (\{h^\alpha(i)\} \times S)) \times \bigcup_{\substack{i \in R \\ \alpha \in M}} \{h^\alpha(i)\} \to Y$$

Furthermore the elements $h^\alpha(i)$ themselves will be identified with the unit vektors $\mu_k \in \mathbb{R}^\rho$, $k = 1, \ldots, \rho := \mid \bigcup_{\substack{i \in R \\ \alpha \in M}} \{h^\alpha(i)\} \mid$

and for the probability distribution $(1,0,0)$ in this section we frequently write abbreviating a "1".

It is clear, that the value of $\Gamma_1(p,1)$, the game without repetitions, is only dependent on σ_1 and τ_1, so we define:

$$\Sigma_1 := \{\sigma_1 := (x(1), \ldots, x(m)) \, / \, x(\alpha) \in F, \alpha \in M\} .$$

$$T_1 := \{\tau_1 := y(\mu_1), \ldots, y(\mu_\rho) \, / \, y(\mu_k) \in Y, k = 1, \ldots, \rho\} .$$

For the value $v_1^1(p)$ of the game $\Gamma_1(p,1)$ we will show

<u>Theorem 3.1</u>: a) $v_1^1(p)$ exists for all $p \in P := \{p \in \mathbb{R}^m /$

$$\sum_{\alpha=1}^{m} p^{\alpha} = 1, \quad p^{\alpha} \geq 0 \ \forall \ \alpha \in M\}$$

b) $v_1^1 : P \to \mathbb{R}$ is concave.

<u>Proof</u>: a) $\Sigma_1 \cdot \mathbb{CR}^{r \cdot m}$, $T_1 \subset \mathbb{R}^{r \cdot s}$ are convex, $E_{p,1,\sigma,\tau} \ \tilde{g}_1$ linear in σ_1 resp. τ_1. Hence the preliminaries of Theorem 2.1. are fulfilled; as Σ_1, T_1 compact even follows

$$v_1^1(p) = \min_{\tau_1 \in T_1} \ \max_{\sigma_1 \in \Sigma_1} \ E_{p,1,\sigma_1,\tau_1} \ \tilde{g}_1$$

$$= \max_{\sigma_1 \in \Sigma_1} \ \min_{\tau_1 \in T_1} \ E_{p,1,\sigma_1,\tau_1} \ \tilde{g}_1$$

b) $$v_1^1(p) = \min_{\tau_1 \in T_1} \ \max_{\sigma_1 \in \Sigma_1} \ \sum_{\alpha=1}^{m} p^{\alpha} \sum_{i=1}^{r} x^i(\alpha) \sum_{j=1}^{s} g_{i,j}^{\alpha} \cdot y^j(h^{\alpha}(i))$$

$$= \min_{\tau_1 \in T_1} \ \sum_{\alpha=1}^{m} p^{\alpha} \ (\max_{x(\alpha) \in F} \ \sum_{i=1}^{r} x^i(\alpha) \sum_{j=1}^{s} g_{i,j}^{\alpha} \cdot y^j(h^{\alpha}(i)))$$

$$= \min_{\tau_1 \in T_1} \ \sum_{\alpha=1}^{m} p^{\alpha} \cdot \max_{i \in R} \ (\sum_{j=1}^{s} g_{i,j}^{\alpha} \cdot y^j(h^{\alpha}(i)) \ )$$

v_1^1 is the minimum of linear functions, hence concave.

<u>Remarks</u>: The always existing optimal strategies of Player II in the game $\Gamma_1(p,1)$ are denoted by $\hat{y} := (\hat{y}(\mu_1), \ldots, \hat{y}(\mu_\rho))$.

An additional requirement to the information matrices in this section is:

(+) there exists $(x(1), \ldots, x(m))$ with $x(\alpha) \in F$ for all $\alpha \in M$, so that for all $\alpha, \beta \in M$: $\quad x(\alpha) \cdot (h^{\alpha}(1), \ldots, h^{\alpha}(r))' = x(\beta) \cdot (h^{\beta}(1), \ldots, h^{\beta}(r))$

The class of information patterns from Definition 1 fulfilling (+) is never empty.

Example 3.1:

$$H^1 = \ldots = H^m = \begin{pmatrix} \mu_1 & \ldots & \mu_s \\ \vdots & & \vdots \\ \mu_1 & \ldots & \mu_s \end{pmatrix}, \mu_k = (0,\ldots,0,\overset{\overset{\text{k-th position}}{\downarrow}}{1},0,\ldots,0) \in \mathbb{R}^s$$

For this system of information matrices each $x=(x(1),\ldots,x(m))$ with $x(\alpha) \in F$ for all $\alpha \in M$, fulfills (+). If we have this system, Player II gets no essential information, so the choice of α cannot be concluded.

Example 3.2: If $\{H^1,\ldots,H^m\}$ in such a manner, that for all $\alpha \in M$ there is an $i(\alpha) \in R$ so that $(h^\alpha_{i(\alpha),1},\ldots,h^\alpha_{i(\alpha),s}) = (\mu_{j_1},\ldots,\mu_{j_s})$, $\mu_{j_1},\ldots,\mu_{j_s}$ unit vektors in $\mathbb{R}^{|H|}$ and $h^\alpha_{i,j}$ arbitrary for $i \neq i(\alpha)$ for all $\alpha \in M$, then x with $x(\alpha) = (0,\ldots,\overset{\overset{\text{i}(\alpha)\text{-th position}}{\uparrow}}{1},\ldots,0)$ fulfills (+)

and there is i.g. $H^\alpha \neq H^\beta$ for $\alpha \neq \beta$.

Remarks: The illustrative significance and interpretation of strategies, which fulfill (+) will become clear and is given in front of Theorem 3.2. Presently is only stated, that these strategies are called "nonseperating strategies".

To determine the value of $\Gamma_\infty(p,1)$ the auxiliary game $\Delta (p,(1,0,0))$ has to be discussed. For $p \in P$, $H^1,\ldots,H^m$ as in Definition 1 $N^1_s(p)$ assigns the set of nonseperating strategies in $\Gamma_1(p,1)$.

$Ns^1(p) := \{(x(1),\ldots,x(m))/x(\alpha)\cdot(h^\alpha(1),\ldots,h^\alpha(r))' =$

$\quad = x(\beta)\cdot(h^\beta(1),\ldots,h^\beta(r))' \ \forall\alpha,\beta \text{ mit } p^\alpha,p^\beta>0, x(\alpha), x(\beta)\in F\}$

For $\sigma_1\in N_s^1(p) \neq \emptyset$, $\tau_1\in T_1$

$E_p^1(\sigma_1,\tau_1) := \sum_{\alpha=1}^{m} \sum_{i=1}^{r} \sum_{j=1}^{s} p^\alpha x^i(\alpha) \ g_{i,j}^\alpha \ y^j(h^\alpha(i))$, the game

$\Delta(p,(1,0,0)) := \{Ns^1(p), T_1, E_p^1(\sigma_1,\tau_1)\}$ is given in strategic

or normal form.

$Ns^1(p)$ is compact as $Ns^1(p) \subset \prod_{\alpha=1}^{m} F$, $\prod_{\alpha=1}^{m} F$ bounded and $Ns^1(p)$

closed. Furthermore $Ns^1(p)$ convex, T_1 compact and convex and

$E_p^1(\sigma_1,\tau_1)$ linear in σ_1 and τ_1, so the preliminaries of Theorem

3.1 are fulfilled and we get:

The value $u^1(p)$ of $\Delta(p,(1,0,0))$ exists.

$$u^1(p) := \max_{\sigma_1\in Ns^1(p)} \ \min_{\tau_1\in T_1} \ E_p^1(\sigma_1,\tau_1) = \min_{\tau_1\in T_1} \ \max_{\sigma_1\in Ns^1(p)} \ E_p^1(\sigma_1,\tau_1)$$

In the following derivation of optimal strategies of Player I

in $\Gamma_\infty(p,1)$ we will use stationary strategies. The principle

is also available for the other varieties of sequence of moves.

We call a strategy $ST_1(x)\in\Sigma$ of Player I stationary in the game

$\Gamma_\infty(p,1)$, if the choice of $i\in R$ in each repetition depends

merely on $\alpha\in M$, more exact:

$\sigma := ST_1(x)\in\Sigma$ with $x := (x(1),\ldots,x(m)), x(\alpha)\in F \ \forall\alpha\in M$

if $\sigma_k(\alpha;i_1,j_1,\ldots,i_k,j_k) = \sigma_1(\alpha)\in F \ \forall k\geq 2$,

$\alpha \in M$, $(i_1,j_1,\ldots,i_k,j_k) \in \prod_{i=1}^{k} R \times S$.

$\tau_k^{j_k}(\cdot)$ shall stand for the j_k-th component of $\tau_k(\cdot)$ and define

$$h_k := h^{\alpha_k}(i_{k_0}), j_{k_0} \quad \text{for all } \alpha_k \in M, i_{k_0} \in R, j_{k_0} \in S, k \in \mathbb{N}$$

and analogously: $\tilde{h}_k := h^{\tilde{\alpha}}(\tilde{i}_k), \tilde{j}_k$.

For $x \in Ns^1(p)$ we get of course for α, β with $p^\alpha, p^\beta > 0$

i) $$\sum_{i:h^\alpha(i)=h^{\alpha_0}(i_0)} x^i(\alpha) = \sum_{i:h^\beta(i)=h^{\alpha_0}(i_0)} x^i(\beta)$$

hence

<u>Corollar</u>: For strategies $ST_1(x)$ holds: x fulfills the condition (+)
if and only if $\tilde{\alpha}$ and $h^{\tilde{\alpha}}(\tilde{i}_1)$ are stochastic independent. As for
$x \in Ns^1(p)$, $\tau \in T$ with $p^\beta, p^{\beta'} > 0$ and S_k defined according to page 6

$$pr_{p,1,ST_1(x),\tau}(h^{\tilde{\alpha}}(\tilde{i}_1)=h^{\alpha_0}(i_0) / \tilde{\alpha}=\beta)$$

$$= \frac{1}{p^\beta} \cdot pr_{p,1,ST_1(x),\tau}(\tilde{\alpha}=\beta, h^{\tilde{\alpha}}(\tilde{i}_1)=h^{\alpha_0}(i_0))$$

$$= \frac{1}{p^\beta} \cdot \sum_{S_k:\alpha=\beta, h^\beta(i_1)=h^{\alpha_0}(i_0)} \text{prob } S_k$$

$$= \frac{1}{p^\beta} \cdot \sum_{i_1:h^\beta(i_1)=h^{\alpha_0}(i_0)} p^\beta \cdot x^{i_1}(\beta)$$

i) $$= \frac{1}{p^{\beta'}} \cdot \sum_{i:h^{\beta'}(i)=h^{\alpha_0}(i_0)} p^{\beta'} \cdot x^i(\beta')$$

$$= pr_{p,1,ST_1(x),\tau}(h^{\tilde{\alpha}}(\tilde{i}_1)=h^{\alpha_0}(i_0)/\tilde{\alpha}=\beta')$$

The above considerations show furthermore that the stochastic independence implies i) and even $x \in Ns^1(p)$.

But even for stationary strategies $ST_1(x)$ with $x \in Ns^1(p)$ can be verified that the choice of α cannot be concluded after any repetitions, consequently

Theorem 3.2: For $ST_1(x), p \in P, x \in Ns^1(p), \tau \in T$ with

$$pr_{p,1,ST_1(x),\tau} (\tilde{h}_1 = h_1, \ldots, \tilde{h}_k = h_k) > 0 \quad \text{we get:}$$

$$pr_{p,1,ST_1(x),\tau} (\tilde{\alpha} = \alpha / \tilde{h}_1 = h_1, \ldots, \tilde{h}_k = h_k) = p^\alpha \; \forall \alpha \in M, \; k \in \mathbb{N}$$

Proof: Induction on k; the proof is conducted in order to demonstrate the core of technique here but also a tribute of mathematical exactness.

$k = 1$ follows at once from Corollar, the theorem be proved for $k-1>1$, if $pr := pr_{p,1,ST_1(x),\tau}$ and

$$B_k^\alpha := \{(\alpha,1,i_1,j_1,\ldots,1,i_k,j_k)/h^\alpha(i_n) = h^{\alpha_n}(i_{n_0}), j_n = j_{n_0}, n \le k\} \forall \alpha \in M, \; k \in \mathbb{N}$$

one can conclude using the equation $\text{prob}(S_k) = pr(S_k)$ and applying the definition of $\text{prob}(S_k)$ (compare p. 6)

$$pr(\tilde{\alpha} = \alpha / \tilde{h}_1 = h_1, \ldots, \tilde{h}_k = h_k)$$

$$= \frac{pr(\tilde{\alpha} = \alpha, \tilde{h}_1 = h_1, \ldots, \tilde{h}_k = h_k)}{\sum_{\beta \in M} pr(\tilde{\alpha} = \beta, \tilde{h}_1 = h_1, \ldots, \tilde{h}_k = h_k)}$$

$$= \frac{\displaystyle\sum_{S_k:(\alpha;i_1,j_1,\ldots j_k)\in B_k^\alpha} \text{prob } S_k}{\displaystyle\sum_{\beta\in M}\;\sum_{S_k:(\beta,i_1,\ldots,j_k)\in B_k^\beta} \text{prob } S_k}$$

$$= \frac{\displaystyle\sum_{(\alpha;i_1,j_1,\ldots,j_k)\in B_k^\alpha} p^\alpha \cdot \prod_{n=1}^{k-1} x^{i_n}(\alpha)\cdot\tau_n(h_1,\ldots,}{\displaystyle\sum_{\beta\in M}\;\sum_{(\beta,i_1,j_1,\ldots,j_k)\in B_k^\beta} p^\beta \cdot \prod_{n-1}^{k-1} x^{i_n}(\beta)\cdot\tau_n(h_1,\ldots,}$$

$$\frac{h_{n-1},h^\alpha(i_n))^{j_n}\cdot(x^{i_k}(\alpha)\cdot\tau_k(h_1,\ldots,h_{k-1},h^\alpha(i_k))^{j_k})}{h_{n-1},h^\beta(i_n))^{j_n}\cdot(x^{i_k}(\beta)\cdot\tau_k(h_1,\ldots,h_{k-1},h^\beta(i_k))^{j_k})}$$

i)
$$= \frac{\tau_k(h_1,\ldots,h_{k-1},h^{\alpha_k}(i_{k_0}))^{j_{k_0}}\cdot\displaystyle\sum_{i_k:h^\alpha(i_k)=h^{\alpha_k}(i_{k_0})} x^{i_k}(\alpha)}{\tau_k(h_1,\ldots,h_{k-1},h^k(i_{k_0}))^{j_{k_0}}\displaystyle\sum_{i_k:h^\alpha(i_k)=h^{\alpha_k}(i_{k_0})} x^{i_k}(\alpha)}$$

$$= \frac{\displaystyle\sum_{(\alpha,i_1,\ldots,j_{k-1})\in B_{k-1}^\alpha} p^\alpha \cdot \prod_{n=1}^{k-1} x^{i_n}(\alpha)\cdot\tau_n(\cdot)^{j_n}}{\displaystyle\sum_{\beta\in M}\;\sum_{(\beta,i_1,\ldots,j_{k-1})\in B_{k-1}^\beta} p^\beta\cdot \prod_{n=1}^{k-1} x^{i_n}(\beta)\cdot\tau_n(\cdot)^{j_n}}$$

$$= pr(\tilde\alpha=\alpha \;/\; h_1=\tilde h_1,\ldots,h_{k-1}=\tilde h_{k-1})$$

$$= \quad p^\alpha$$

Theorem 3.3: For $\sigma = ST_1(x)$; $x := (x(1),\ldots,x(m)) \in Ns^1(p)$

optimal in $\Delta(p,(1,0,0))$, $\tau \in T$ arbitrary follows:

$$E_{p,1,ST(x),\tau} \; \overline{\widetilde{g}}_n \geq u^1(p) \quad \forall \, n \in \mathbb{N}$$

Proof: If $pr_{p,1,ST_1(x),\tau} \; (\widetilde{h}_1 = h_1,\ldots,\widetilde{h}_k = h_k) > 0$ one can conclude:

$$E_{p,1,ST_1(x),\tau} \; (\widetilde{g}_{k+1} \, / \, \widetilde{h}_1 = h_1,\ldots,\widetilde{h}_k = h_k)$$

$$= \sum_{\alpha=1}^{m} \sum_{i=1}^{r} \sum_{j=1}^{s} pr_{p,1,ST_1(x),\tau} \; (\widetilde{\alpha} = \alpha / \widetilde{h}_1 = h_1,\ldots,\widetilde{h}_k = h_k) \cdot$$

$$\cdot \, x^i(\alpha) \cdot g^\alpha_{i,j} \cdot \tau_{k+1}(h_1,\ldots,h_k,h^\alpha(i))^j$$

(Theorem 3.2)

$$= \sum_{\alpha=1}^{m} \sum_{i=1}^{r} \sum_{j=1}^{s} p^\alpha \cdot x^i(\alpha) \cdot g^\alpha_{i,j} \cdot \tau_{k+1}(h_1,\ldots,h_k,h^\alpha(i))^j$$

$$\geq \min_{\tau \in T_1} \sum_{\alpha=1}^{m} \sum_{i=1}^{r} \sum_{j=1}^{s} p^\alpha \cdot x^i(\alpha) g^\alpha_{i,j} \cdot y^j(h^\alpha(i))$$

$$= \max_{\alpha_1 Ns^1(p)} \; \min_{\tau_1 \in T_1} \; E^1_p(\sigma_1,\tau_1) = u^1(p)$$

and so Theorem 3.3 follows immediately.

Theorem 3.4: For $p \in P$ there are $p_\beta \in P$; $\lambda_\beta \geq 0, \beta = 1,\ldots,m+1$ so that:

$$\sum_{\beta=1}^{m+1} \lambda_\beta p_\beta = p \quad \text{and} \quad \sum_{\beta=1}^{m+1} \lambda_\beta u^1(p_\beta) = cav \; u^1(p) \; .$$

<u>Proof:</u> For $p \in P$

$$P(p) := \{(p_1,\ldots,p_{m+1};\lambda_1,\ldots,\lambda_{m+1}) \ / \ p_\beta \in P, \ \sum_{i=1}^{m+1} \lambda_\beta \cdot p_\beta = p, \lambda_\beta \geq 0,$$

$$\sum_{\beta=1}^{m+1} \lambda_\beta = 1\}$$

is bounded and closed. If $\pi_\beta : P(p) \rightarrow \mathbb{R}$ is defined by

$$\pi_\beta(p_1,\ldots,p_{m+1},\lambda_1,\ldots,\lambda_{m+1}) := \lambda_\beta \cdot u^1(p_\beta) \quad \forall \ \beta \in \{1,\ldots,m+1\}$$

then with $P(p) \ni (p_1^{(n)},\ldots,p_{m+1}^{(n)},\lambda_1^{(n)},\ldots,\lambda_{m+1}^{(n)}) \rightarrow$

$\rightarrow (p_1,\ldots,p_{m+1},\lambda_1,\ldots,\lambda_{m+1}) \in P(p)$ and reasoned by $p_\beta^{(n)} \rightarrow p_\beta$

for big n implies $Ns^1(p_\beta^{(n)}) \subset Ns^1(p_\beta)$ follows that:

$$\lim_{n \rightarrow \infty} \sup \ u^1(p_\beta^{(n)}) \leq u^1(p_\beta), \quad \text{hence also}$$

$$\lim_{n \rightarrow \infty} \sup \ \lambda_\beta^{(n)} u^1(p_\beta^{(n)}) \leq \lambda_\beta \cdot u^1(p_\beta), \quad \text{that means } \pi_\beta \text{ is upper-}$$

semi-continuous.

As sums of upper-semi-continuous functions are upper-semi-con-

tinuous follows $\pi : P(p) \rightarrow \mathbb{R}, \ \pi := \sum_{\beta=1}^{m+1} \pi_\beta$ is upper-semi-con-

tinuous. So the statement of Theorem 3.4 can be obtained from

Theorem 2.2 as real upper-semi-continuous functions do have a

real maximum on compact sets.

We will now assume, that Player I, when he determines his

strategies, uses a random mechanism;--more exact: by chance-

after the choice of α, with certain probabilities--one strategy is selected out of a finite set of strategies, which is defined before the game starts.

<u>Theorem 3.5</u>: Thus in $\Gamma_\infty(p,1)$ for $p^\alpha > 0$ $\forall \alpha \in M$, p_β, λ_β, $\beta = 1, \ldots, m+1$ as determined in Theorem 3.4:

If Player I is consulting a random mechanism which chooses, if $\alpha \in M$ is selected, the strategy $ST_1(x_\beta)$ with probability

$\dfrac{\lambda_\beta \cdot p_\beta^\alpha}{p^\alpha}$, whereby x_β is optimal in $\Delta(p_\beta,(1,0,0))$; that means

$$p(\sigma_0 = ST^1(x_\beta) \ / \ \tilde{\alpha} = \alpha) = \frac{\lambda_\beta \cdot p_\beta^\alpha}{p^\alpha}, \ \beta = 1, \ldots, m+1$$

then for a strategy σ_0 defined on that way even:

$$E_{p,1,\sigma_0,\tau} \ \tilde{\overline{g}}_k \geq \text{cav } u^1(p) \quad \forall \ k \in \mathbb{N}, \ \tau \in T.$$

<u>Proof</u>: Thus, for $\beta = 1, \ldots, m+1$, $\alpha \in M$

1.) $pr(\sigma_0 = ST_1(x_\beta)) = \sum\limits_{\alpha \in M} pr(\sigma_0 = ST^1(x_\beta), \ \tilde{\alpha} = \alpha)$

$= \sum\limits_{\alpha \in M} pr(\sigma_0 = ST^1(x_\beta) \ / \ \tilde{\alpha} = \alpha) \cdot pr(\tilde{\alpha} = \alpha) = \sum\limits_{\alpha \in M} \dfrac{\lambda_\beta \cdot p_\beta^\alpha}{p^\alpha} \cdot p^\alpha = \lambda_\beta$

2.) $pr(\tilde{\alpha} = \alpha \ / \ \sigma_0 = ST^1(x_\beta))$

$= \dfrac{pr(\sigma_0 = ST_1(x_\beta) \ / \ \tilde{\alpha} = \alpha) \cdot pr(\tilde{\alpha} = \alpha)}{pr(\sigma_0 = ST_1(x_\beta))} = \dfrac{\dfrac{\lambda_\beta \cdot p_\beta^\alpha}{p^\alpha} \cdot p^\alpha}{\lambda_\beta} = p_\beta^\alpha$

1.) and 2.) imply for $k \in \mathbb{N}$, $\tau \in T$ arbitrary

$$E_{p,1,\sigma_0,\tau} \; \widetilde{\overline{g}}_k \;\; = \sum_{\beta=1}^{m+1} \; pr(\sigma_0=ST_1(x_\beta))\cdot E_{p,1,\sigma_0,\tau} \; (\widetilde{\overline{g}}_k/\sigma_0=ST_1(x_\beta))$$

$$= \sum_{\beta=1}^{m+1} \lambda_\beta\cdot E_{p_\beta,1,ST_1(x_\beta),\tau} \; \widetilde{\overline{g}}_k \overset{(3,3)}{\geq} \sum_{\beta=1}^{m+1} \lambda_\beta\cdot u^1(p_\beta) \overset{(3.4)}{=} cav\; u^1(p)$$

therefrom Theorem 3.5 follows immediately.

Theorem 3.6: $v_1^1(p) = cav\; u^1(p)$ $\forall$ $p \in P$ with $p^\alpha > 0$ $\forall \alpha \in M$

Proof: $Ns^1(p)\subset\Sigma_1$, therefore $u^1(p) \leq v_1^1(p)$, further since

Theorem 3.1b, v_1^1 concave even: $cav\; u^1(p) \leq v_1^1(p)$.

To prove the inverse relation we define for $p \in P$, $\sigma := \sigma_1 =$

$=(x(1),\ldots,x(m))$, $\tau := \tau_1 \in T$ and $\mu := h^{\alpha_0}(i_0) \in \bigcup_{\substack{\alpha\in M \\ i\in R}} \{h^\alpha(i)\}$

$p(\mu) := (p^1(\mu),\ldots,p^m(\mu)) \in P$ by

$p^\alpha(\mu) := pr_{p,1,\sigma,\tau} \; (\widetilde{\alpha}=\alpha \; / \; h^{\widetilde{\alpha}}(\widetilde{i}_1) = \mu)$

$= \dfrac{pr_{p,1,\sigma,\tau} \; (\widetilde{\alpha}=\alpha, h^{\widetilde{\alpha}}(\widetilde{i}_1)=\mu)}{pr_{p,1,\sigma,\tau} \; (h^{\widetilde{\alpha}}(\widetilde{i}_1)=\mu)}$

$= \begin{cases} \dfrac{p^\alpha \cdot \displaystyle\sum_{i:h^\alpha(i)=\mu} x^i(\alpha)}{\displaystyle\sum_{\beta\in M,} \; \displaystyle\sum_{i:h^\beta(i)=\mu} p^\alpha\cdot x^i(\alpha)} & \text{if denominator positive} \\[20pt] 0 & \text{else} \end{cases}$

and

$$x(\mu) := (x_\mu(1),\ldots,x_\mu(m)) \in \Sigma_1 \qquad \text{by}$$

$$x^i_\mu(\alpha) := pr_{p,1,\sigma,\tau}(\tilde{i}_1 = i \,/\, h^{\tilde{\alpha}}(\tilde{i}_1) = \mu,\; \tilde{\alpha} = \alpha)$$

$$= \begin{cases} \dfrac{x^i(\alpha)}{\displaystyle\sum_{i:h^\alpha(i)=\mu} x^i(\alpha)} & \text{if } \mu \in \displaystyle\bigcup_{i\in R} \{h^\alpha(i)\} \\[2em] \delta_{1,i} & \text{else} \end{cases}$$

for $\alpha \in M$, $i \in R$ and if $\sigma_{i,k}$ is the KRONECKER-symbol.

Thus for $\sigma := \sigma_1 = (x(1),\ldots,x(m))$

$$E_{p,1,\sigma,\tau}\,(\tilde{g}_1 \,/\, h^{\tilde{\alpha}}(\tilde{i}_1) = \mu) = E_{p(\mu),1,x(\mu),\tau}\,\tilde{g}_1$$

for $x(\mu) \in Ns^1(p(\mu))$ one can conclude for $\tau^* \in T$ arbitrary:

$$v^1_1(p) = \max_{\sigma\in\Sigma_1} \min_{\tau\in T_1} E_{p,1,\sigma\,\tau}\,\tilde{g}_1$$

$$= \max_{\sigma\in\Sigma_1} \min_{\tau\in T_1} \sum_{k=1}^{\rho} pr_{p,1,\sigma,\tau}(h^{\tilde{\alpha}}(\tilde{i}_1)=\mu_k)\cdot E_{p,1,\sigma,\tau}\,(\tilde{g}_1/h^{\tilde{\alpha}}(\tilde{i}_1)=\mu_k)$$

$$= \max_{\sigma\in\Sigma_1} \sum_{k=1}^{\rho} pr_{p,1,\sigma,\tau^*}(h^{\tilde{\alpha}}(\tilde{i}_1)=\mu_k)\cdot \min_{\tau\in T_1} E_{p,1,\sigma,\tau}(\tilde{g}_1/h^{\tilde{\alpha}}(\tilde{i}_1)=\mu_k)$$

$$= \max_{\sigma\in\Sigma_1} \sum_{k=1}^{\rho} pr_{p,1,\sigma,\tau^*}(h^{\tilde{\alpha}}(\tilde{i}_1)=\mu_k)\cdot \min_{\tau\in T_1} E_{p(\mu_k),1,x(\mu_k),\tau}\,\tilde{g}_1$$

$$\leq \max_{\sigma \in \Sigma} \sum_{k=1}^{\rho} pr_{p,1,\sigma,\tau*}(h^{\widetilde{\alpha}}(\widetilde{\imath}_1)=\mu_k)$$

$$\cdot \max_{x \in Ns^1(p(\mu_k))} \min_{\tau \in T_1} E_{p(\mu_k),1,x(\mu_k),\tau} \widetilde{g}_1$$

$$= \max_{\sigma \in \Sigma} \sum_{k=1}^{\rho} pr_{p,1,\sigma,\tau*}(h^{\widetilde{\alpha}}(\widetilde{\imath}_1)=\mu_k) \cdot u^1(p(\mu_k))$$

but even for all $\sigma \in \Sigma_1$ follows

$$\sum_{k=1}^{\rho} pr_{p,1,\sigma,\tau*}(h^{\widetilde{\alpha}}(\widetilde{\imath}_1)=\mu_k) \cdot p(\mu_k)$$

$$= \sum_{k=1}^{\rho} (\sum_{\beta \in M, i:h^{\beta}(i)=\mu_k} p^{\beta} \cdot x^i(\beta)) \cdot (p^1(\mu_k),\ldots,p^m(\mu_k))$$

$$= (\sum_{k=1}^{\rho} p^1 \cdot \sum_{i:h^1(i)=\mu_k} x^i(1),\ldots, \sum_{k=1}^{\rho} p^m \sum_{i:h^m(i)=\mu_k} x^i(m)) = p$$

hence JENSENS inequation for concave functions implies

$$v_1^1(p) \leq \max_{\sigma \in \Sigma} \sum_{k=1}^{\rho} pr_{p,1,\sigma,\tau*}(h^{\widetilde{\alpha}}(\widetilde{\imath}_1)=\mu_k) \cdot u^1(p(\mu_k))$$

$$\leq \sum_{k=1}^{\rho} pr_{p,1,\sigma*,\tau*}(h^{\widetilde{\alpha}}(\widetilde{\imath}_1)=\mu_k) \cdot cav\, u^1(p(\mu_k))$$

$$\leq cav\, u^1(\sum_{k=1}^{\rho} pr_{p,1,\sigma*,\tau_1^*}(h^{\widetilde{\alpha}}(\widetilde{\imath}_1)=\mu_k) \cdot p(\mu_k)) = cav\, u^1(p).$$

<u>Theorem 3.7</u>: If $\hat{y} := (\hat{y}(\mu_i),\ldots,\hat{y}(\mu_s)) \in T_1$ optimal strategy

in $\Gamma_1(p,1)$, then for the strategy $\tau_0 =: ST_1(\hat{y})$ with $\tau_{0_k}(h^\alpha(i_1),$

$j_1;\dots;h^\alpha(i_k)) = \hat{y}(h^\alpha(i_k)) \; \forall \; k \in \mathbb{N}$ and $h^\alpha(i_1),j_1,\dots,h^\alpha(i_k)$

arbitrary follows: $E_{p,1,\sigma,\tau_0} \; \overline{\tilde{g}}_n \leq cav \; u^1(p) \; \forall \; \sigma \in \Sigma, \; n \in \mathbb{N}$.

The proof is easily conducted showing, that for each sequence,

$\alpha,h^\alpha(i_1),j_1,\dots,h^\alpha(i_k)$ which for any fixed $\sigma,\tau = ST^1(\hat{y})$ may

appear with positive probability the expectation of the payoff

is $\leq v_1^1(p) = cav \; u^1(p)$.

Summarizing Theorem 3.2 - 3.7 we get

<u>Theorem 3.8</u>: The value $v^1(p)$ of $\Gamma_\infty(p,1)$ exists and $v^1(p) =$

cav $u^1(p)$. The strategy σ_0 in Theorem 3.5 is an optimal stra-

tegy for Player I, the strategy τ_0 in Theorem 3.7 is an optimal

strategy for Player II.

<u>Remarks</u>: 1) Also for the value $v_n^1(p)$ of $\Gamma_n(p,1)$ holds, as

can be taken from the proofs: $v_n^1(p) = v^1(p)$

2) If the information matrices additionally (compare Example 3.2)
fulfill

  i) $H^\alpha = H^\beta \; \forall \; \alpha, \; \beta \in M$

ii) $h^\alpha(i) \neq h^\alpha(i')$ for $i,i' \in R, i \neq i'$

then, as a special case we get the results of PONSSARD and

ZAMIR [1973].

4. The Class of Games where the Uninformed Player is Always the Second to Move

In the games considered in this section the uninformed player moves first with probability 1, so we arrange some abbreviations.

A strategy σ of player I is sufficiently defined here by

$$\sigma_1 : M \times S \to F$$

$$\sigma_k : M \times (\prod_{i=1}^{k-1} S \times R) \times S \to F \quad \text{for } k \geq 2;$$

a strategy τ of Player II by

$$\tau_1 : \{2\} \to Y$$

$$\tau_k : \{2\} \times \prod_{i=1}^{k-1} H \to Y \quad \text{for } k \geq 2.$$

τ_1 simply allies an y to the sequence of moves "2".

The probability distribution $(0,1,0)$ is abbreviated by a "2".

For games, in which the uninformed player is the second to move we will additionaly request, that
(*) there exists $(x(1,1),\ldots,x(\alpha,j),\ldots,x(m,s))$, $x(\alpha,j) \in F$ for all $\alpha \in M$, $j \in S$ so that:

$$x(\alpha,j) \cdot H^{\alpha}(j) = x(\beta,j) \cdot H^{\beta}(j) \quad \forall \alpha,\beta \in M, \ j \in S$$

whereby $H^\alpha(j) := (h^\alpha_{1,j},\ldots,h^\alpha_{r,j})'$ $\forall \alpha \in M$, $j \in S$.

Remarks:

1) The system of information matrices in example 3.1. fulfills (*).

2) Requirement (+) is stronger than (*).

3) The interpretation of (*) is equivalent to those of (+).

According to the case in which the informed player always moves first we here consider an auxiliary game $\Delta_2(p,(0,1,0)) := \{Ns^2(p), T_1, E^2_p(\sigma_1,\tau_1)\}$, whereby for $p \in P$, $Ns^2(p)$ contends all tuples $\sigma_1 := ((x(1,1),..,x(m,s))$ which for $\alpha,\beta \in M$; $p^\alpha,p^\beta > 0$ fulfill (*) and here also T_1 can be identified with Y (resp. τ_1 with $y \in Y$). By making use of the Minimax-Theorem it can be demonstrated once more that the value $u^2(p)$ of $\Delta_2(p,(0,1,0))$ exists: $u^2(p) = \max\limits_{\sigma_1 \in Ns^2(p)} \min\limits_{\tau_1 \in T_1} E^2_p(\sigma_1,\tau_1) =$

$$\min\limits_{\tau_1 \in T_1} \max\limits_{\sigma_1 \in Ns^2(p)} E^2_p(\sigma_1,\tau_1).$$

Although it can be shown, that with

$\Sigma_1 := \{\sigma_1 := (x(1,1),..,x(m,s)) \;/\; x(\alpha,j) \in F \; \forall \alpha \in M,\; j \in S\}$

the value v^2_1 of $\Gamma_n(p,2)$ exists:

$$\dot{v}_1^2(p) = \min_{\tau_1 \in T_1} \quad \max_{\sigma_1 \in \Sigma_1} \quad \sum_{\alpha=1}^{m} \sum_{i=1}^{r} \sum_{j=1}^{s} p^\alpha \cdot x^i(\alpha,j) g_{i,j}^\alpha \cdot y^j$$

$$= \min_{j \in S} \sum_{\alpha=1}^{m} p^\alpha \max_{i \in R} g_{i,j}^\alpha$$

$v_1^2(p)$ is the minimum of linear functions and hence concave on P, further you can infer as in section 3

$u^2(p) \leq v_1^2(p) \quad \forall p \in P$, so cav $u^2(p) \leq v_1^2(p)$.

As Example 4.1 demonstrates "<" can occur;

Example 4.1: $R = S := \{1,2\}$

$$G^1 := \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad G^2 := \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$H^1 = H^2 := \begin{pmatrix} \mu_1 & \mu_3 \\ \mu_2 & \mu_4 \end{pmatrix} \quad ; \quad \mu_j := (\delta_{j,1}; \ldots; \delta_{j,4}) \quad j=1,\ldots,4.$$

$p_0 := (1/2, 1/2)$; $q := (0,1,0)$ then for the thereby defined game $\Gamma_1(p_0,2) = \{R,S, \{G^1,G^2\}, \{H^1,H^2\}, p,q\}$ follows

$v_1^2(p_0) = \min \left(\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 1; \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2 \right) = 1.5$; but cav $u(p_0) < v_1^2(p_0)$

as the special information matrices imply here

$$u^2(p) = \min_{i \in S} \quad \max_{i \in R} \quad \sum_{\alpha=1}^{2} p^\alpha \cdot g_{i,j}^\alpha, \text{ therefore}$$

$$u^2(p) = \begin{cases} \min\ (1;(1-p^1)\cdot 2)=1 & \text{for } p^1 \in [0,\tfrac{1}{2}] \\[2ex] \min\ (2p^1;1) = 1 & \text{for } p^1 \in [\tfrac{1}{2},1] \end{cases}$$

hence $\quad \operatorname{cav} u^2(p_0) = 1 < 1.5 = v_1(p_0)$

If we want to determine the value of the game $\Gamma_\infty(p,2)$ and to construct optimal strategies, partially we have to conduct alternative proofs as will be pointed out when discussing the "behaviour" of Player II.

At first, we will define here too stationary strategies:

$\sigma =: ST_2(x) \in \Sigma$ with $x := (x\,(1,1),\ldots,x(m,j))$, $x(\alpha,j)) \in F$

$\forall\ \alpha \in M,\ j \in S\ $ by $(k \geq 2)$

$$\sigma_k(\alpha,j_1,i_1,\ldots,j_k) := \sigma_1(\alpha,j_k) = x(\alpha,j_k)$$

for all $\alpha \in M$; $j_k \in S$; $(j_1,i_1,\ldots,i_k) \in \displaystyle\prod_{i=1}^{k-1} S \times R.$

For stationary strategies $(ST_2(x)$ with $x \in Ns^2(p)$ follows:

i) $\displaystyle\sum_{\substack{i:h^\alpha_{i,j} = \mu}} x^i(\alpha,j) = \sum_{\substack{i:h^\beta_{i,j} = \mu}} x^i(\beta,j)$

    for all $\alpha \in M$, $j \in S$ and all unit vectors $\mu \in \mathbb{R}^{|H|}$, hence

    similar to section 4.4:

ii) for strategies $ST_1(x)$ we obtain: x fulfills (*) if and only if $\tilde{\alpha}$ and $h^{\tilde{\alpha}}_{\tilde{i}_1,\tilde{j}_1}$ are independent.

Completely analogous to section 3 we can show the existence of $p_\beta^2 \in P$, $\lambda_\beta^2 > 0$ thus

$$\sum_{\beta=1}^{m+1} \lambda_\beta^2 \, p_\beta^2 = p \quad \text{and} \quad \sum_{\beta=1}^{m+1} \lambda_\beta^2 \, u^2(p_\beta^2) = \text{cav } u^2(p), \quad \text{so:}$$

<u>Theorem 4.1</u>: If p_β^2, λ_β^2 is determined as above, then for games $\Gamma_\infty(p,2)$ with $p^\alpha > 0$ $\forall \alpha \in M$ state:

If Player I is consulting a random mechanism which chooses, - if $\alpha \in M$ is selected, - the strategy $ST_2(x_\beta^2)$ with probability

$$\frac{\lambda_\beta^2 \cdot p_\beta^{\alpha,2}}{p^\alpha} \, , \quad \text{whereby } x_\beta^2 \text{ is optimal in } \Delta \, (p_\beta^2,(0,1,0)), \text{ i.e.}$$

$$p(\sigma_0 = ST_2(x_\beta)/\widetilde{\alpha} = \alpha) = \frac{\lambda_\beta^2 \cdot p_\beta^{\alpha,2}}{p^\alpha} \quad \beta = 1,\ldots,m+1,$$

then for a strategy defined in this way:

$$E_{p,2,\sigma_0,\tau} \, \widetilde{\widetilde{g}}_k \geq \text{cav } u^2(p) \quad \forall k \in \mathbb{N}, \ \tau \in T:$$

<u>Remarks</u>: To distinguish the λ's and p_β's above from those of section 3 they are indicated with a "2".

Without loss of generality we will request now, that

$\left| g_{i,j}^\alpha \right| \leq 1$ $\forall \alpha \in M$, $i \in R$, $j \in S$. Since for $p_0 \in P$,

$p_0^\alpha > 0$ $\forall \alpha \in M$, there exists $\varphi \in \mathbb{R}^m$ with:

$$\varphi \cdot p_0 = \text{cav } u^2 (p_0),$$

$$\varphi \cdot p \geq u^2 (p) \quad \forall \ p \in P$$

[compare BONNESEN & FENCHEL, 1971], so you get for a game $\Gamma'_\infty (p_0, 2)$ which originates from $\Gamma_\infty (p_0, 2)$ replacing G^α by the payoff-matrices $G'^\alpha := (g'^\alpha_{i,j}) := (g^\alpha_{i,j} - \varphi^\alpha)$: a strategy of Player II, which secures ε, secures $\text{cav } u^2(p_0) + \varepsilon$ in the game $\Gamma_\infty(p_0, 2)$.

Remarks: For a game $\Delta'(p,(0,1,0))$ corresponding to $\Gamma'_\infty (p,2)$ follows:

$$\text{cav } u'^2 (p_0) = 0 \text{ and}$$

$$\text{cav } u'^2 (p) \leq 0 \quad \forall \ p \in P.$$

In the following we need a corollar proved by KOHLBERG [1976b, p.13]

Corollar: Let K be a closed set in $\mathbb{R}^m$. For $\xi \notin K$, we denote by $c(\xi)$ a closest point in S to ξ. Suppose (ξ_n) is a sequence of points in $\mathbb{R}^m$ that satisfies:

i) if $\overline{\xi}_n := \frac{1}{n} \sum_{i=1}^{n} \xi_i \in K$, then the hyperplane through

$c(\xi_n)$, perpendicular to the line segment

$[\xi_n, c(\xi_n)]$ seperates $\overline{\xi}_n$ and ξ_{n+1}

ii) $|\xi_n| \leq 1 \ \forall \ n \in N$

then: $d_n \leq \frac{c}{\sqrt{n}}$ where $d_n := d(\xi_n, K)$ and $c := \max(2, d_1)$.

We now want to look for a strategy of Player I, which secures $\varepsilon = \text{cav } u'^2(p_0) + \varepsilon$ in $\Gamma'_\infty(p_0, 2)$. As shown in Example 4.1, i.g. $v_2'^1(p_0) > \text{cav } u'^2(p_0)$; Player II has to consider the information he gets and therefrom he has to deduce the strategy which he will choose later on, if he wants to come down to cav $u'^2(p_0)$.

To realize and to formulate this idea, there is need of some more definitions. Let

$$Q := \prod_{l=1}^{s} \left(\{\mathcal{O}\} \cup \{x / x = \sum_{k=1}^{|H|} \lambda_k \mu_k, \; \lambda_k \geq 0, k=1,\ldots,|H|; \; \sum_{k=1}^{|H|} \lambda_k = 1\} \right),$$

and with $\hat{q} := (q_1, \ldots, q_s)' \in Q \subset \mathbb{R}^{s \cdot |H|}$ for $\alpha \in M, j \in S$

$$F(\alpha, j, \hat{q}) := \{f \in F \cup \{\mathcal{O}\} \; / |f \cdot H^\alpha(j) - q_j| = 0\}$$

$$\xi^\alpha(\hat{q}, j) := \begin{cases} \max_{f \in F(\alpha, j, \hat{q})} f \cdot G^\alpha \cdot \mathcal{f}_j' & \text{if } F(\alpha, j, \hat{q}) \neq \emptyset \\ -\infty & \text{else} \end{cases}$$

whereby $\mathcal{f}_j' := (\delta_{1,j}, \ldots, \delta_{s,j})$ and δ_{ij} the KRONECKER symbol.

Remarks:

1.) Of course for $F(\alpha, j, \hat{q}) \neq \emptyset$ there exists $f_{j,\alpha} \in F$ thus:

$$f_{j,\alpha} \cdot G^\alpha \cdot \mathcal{f}_j' = \max_{f \in F(\alpha, j, \hat{q})} f \cdot G^\alpha \cdot \mathcal{f}_j' \text{, as } F(\alpha, j, \hat{q}) \text{ closed and}$$

bounded and

$\varphi : F(\alpha, j, \hat{q}) \to \mathbb{R}$ defined by $\varphi(f) := f \cdot G^\alpha \cdot \mathcal{f}_j'$ is continuous for all $\alpha \in M, j \in S$.

2.) In the construction of optimal strategies in $\Gamma_\infty(p,2)$ developed in this paper $\xi^\alpha(\hat{q},y)$ play the role of (upper) estimations for the payoffs if $\hat{q}$ is an observed frequence (vector of frequencies) of informational elements if hypothetic, that G^α is the payoff matrix. If for a certain j $F(\alpha,j,\hat{q}) = \emptyset$, the observed frequencies of information are inconsistent with H^α, so one can be sure, that α was not chosen; so the definition $\xi^\alpha(\hat{q},y) = -\infty$ in this case is rational.

Further, we will define the random variables $\widetilde{\widetilde{g}}_k^{\widetilde{\alpha}}(j)$, $\widetilde{q}_{k,j}$; $j \in S$, $k \in \mathbb{N}$ on the probability-field $(\Omega, \mathfrak{R}, pr_{p,2,\sigma,\tau})$ by

$$\widetilde{\widetilde{g}}_k^{\widetilde{\alpha}}(j) := \begin{cases} \dfrac{1}{|\{n \le k / \widetilde{J}_n = j\}|} \sum_{n \le k : \widetilde{J}_n = j} g_{\widetilde{i}_n, j}^{\widetilde{\alpha}} & \text{if the denominator positive} \\ \qquad\qquad 0 & \text{else.} \end{cases}$$

Consequently $\widetilde{\widetilde{g}}_k^{\widetilde{\alpha}}(j)(\omega)$ assigns the median payoff for a sequence ω of those plays of the first k-1 repetitions in which Player II chose column j.

If at least $\widetilde{q}_{k,j} := (\widetilde{q}_{k,j}^{\mu_1}, \ldots, \widetilde{q}_{k,j}^{\mu_{|H|}})$ for μ_k, $1 \le k \le |H|$ (see Definition 2.1) and

$$\widetilde{q}_{k,j}^{\mu_k} := \begin{cases} \dfrac{|\{n \le k / \widetilde{J}_n = j, h_{\widetilde{i}_n, j}^{\widetilde{\alpha}} = \mu_k\}|}{|\{n \le k / \widetilde{J}_n = j\}|} & \text{if denominator positive} \\ \qquad\qquad 0 & \text{else,} \end{cases}$$

$\tilde{q}_k := (\tilde{q}_{k,1}, \ldots, \tilde{q}_{k,s})$; then by this defined random variable

$\xi^\alpha(\tilde{q}_{k,j})$: $\quad \xi^{\tilde{\alpha}}(\tilde{q}_{k,j}) \geq \tilde{\tilde{g}}_k^{\tilde{\alpha}}(j) \quad \forall k \in \mathbb{N}, \; j \in S.$

This can be shown (neglecting $\mathrm{pr}_{p,2,\sigma,\tau}$ nullsets) as follows;

if for $\quad \omega := (\alpha; 2,j_1,i_1; \ldots; 2,j_k,i_k,.)$ arbitrary

$$\tilde{f}_{k,j}^i(\omega) := \frac{|\{n \leq k \;/\; \tilde{i}_n=i, \tilde{j}_n=j\}|}{k} \quad i \in R, \; j \in S, \; k \in \mathbb{N}$$

$$\tilde{\tilde{g}}_k^{\tilde{\alpha}}(j)(\omega) = \sum_{i=1}^r \tilde{f}_{k,j}^i(\omega) \cdot g_{i,j}^\alpha, \text{ we get}$$

$$\xi^{\tilde{\alpha}(\omega)}(\tilde{q}_k(\omega),j) = \max_{f \in F(\tilde{\alpha}(\omega),j,\tilde{q}_{k,j}(\omega))} \sum_{i=1}^r f^i \cdot g_{i,j}^\alpha \geq \tilde{\tilde{g}}_k^{\tilde{\alpha}}(j)(\omega).$$

As $(\tilde{f}_{k,j}^1(\omega), \ldots, \tilde{f}_{k,j}^r(\omega)) \in F(\tilde{\alpha}(\omega),j,\tilde{q}_{k,j}(\omega)) \quad \forall j \in S$

$$(\tilde{f}_{k,j}^1(\omega), \ldots, \tilde{f}_{k,j}^r(\omega)) \cdot H^\alpha(j) = \tilde{q}_{k,j}(\omega).$$

If further for $\alpha \in M, \; \hat{q} \in Q, \; y \in Y$

$$\xi^\alpha(\hat{q},y) := \sum_{j=1}^s \xi^\alpha(\hat{q},j) \cdot y^j \text{ and } \quad \xi(\hat{q},y) := (\xi^1(q,y), \ldots, \xi^m(q,y))$$

we can prove the following corollar

Theorem 4.2: For every $\varepsilon > 0, p \in P$ there exists $y \in Y$, so that

for the game $\Gamma_\infty'(p,2)$:

$$p \cdot \xi(\hat{q},y)' \leq \varepsilon \; \forall \; \hat{q} \in Q$$

Proof: i) E_p^2 viewed as a function of (σ_1, τ_1) is continuous on $Ns^2(p) \times T_1$; for every $\varepsilon > 0$ there exists $\tau_1^* =: y$, $y^j > 0 \ \forall j$, so that $\max\limits_{\sigma_1 \in Ns^2(p)} E_p(\sigma_1, \tau_1^*) < \varepsilon$ (compare Remarks on p. 30)

ii) For $\varepsilon > 0$, $p \in P$ choose y so that $y^j > 0$ and i) is fulfilled, the definitions yield:

$p \cdot \xi(\hat{q}, y)'$

$$= \begin{cases} \sum\limits_{\alpha=1}^{m} p^\alpha \cdot \sum\limits_{j=1}^{s} y^j \cdot \max\limits_{f \in F(\alpha, j, \hat{q})} \sum\limits_{i=1}^{r} f_j^i \cdot g_{i,j}'^{\alpha}, & \text{if } F(\alpha, j, \hat{q}) \neq \emptyset \\ & \text{for all } \alpha \in M \text{ with } p^\alpha > 0, j \in S. \\ -\infty & \text{else} \end{cases}$$

as $y^j > 0 \ \forall j \in S$.

If then $p \cdot \xi(q, y)' \neq -\infty$ choose

$f_{j,\alpha} := (f_{j,\alpha}^1, \ldots, f_{j,\alpha}^r) \in F \quad j = 1, \ldots, s \quad$ so that

$$\sum\limits_{j=1}^{s} \sum\limits_{i=1}^{r} f_{j,\alpha}^i \cdot g_{i,j}^\alpha \cdot y^j$$

$$= \sum\limits_{j=1}^{s} \max\limits_{f \in F(\alpha, j, \hat{q})} \sum\limits_{i=1}^{r} f^i \cdot g_{i,j}'^\alpha \cdot y^j \quad \text{for } F(\alpha, j, \hat{q}) \neq \emptyset \\ \text{and } p^\alpha > 0$$

which is obviously possible.

With $\hat{x}(\alpha, j) := f_{j,\alpha}$ if $F(\alpha, j, \hat{q}) \neq \emptyset$ and $p^\alpha > 0$,

$\hat{x}(\alpha, j) \in F$ arbitrary else, one gets:

$(\hat{x}(1,1),\ldots,\hat{x}(\alpha,j),\ldots,x(m,s)) \in Ns^2(p)$, hence also for this case

$$p \cdot \xi(\hat{q},y)' = \sum_{\alpha \in M} \sum_{j=1}^{s} \sum_{i=1}^{r} \hat{x}^i(\alpha,j) \cdot g_{i,j}^{!\alpha} \cdot y^j \cdot p^\alpha$$

$$\leq \max_{x \in Ns^2(p)} \sum_{\alpha=1}^{m} \sum_{j=1}^{s} \sum_{i=1}^{r} x^i(\alpha,j) \cdot g_{i,j}^{!\alpha} \cdot y^j \cdot p^\alpha$$

$$= \max_{\sigma_1 \in Ns^2(p)} E_p(\sigma_1, \tau_1^*) < \varepsilon$$

In accordance to ideas of KOHLBERG's construction [1976 b] we can prove the following Theorems 4.3 and 4.4 which are the core of the construction.

<u>Theorem 4.3</u>: For all $\varepsilon > 0$ there exists $N := N(\varepsilon)$, so that for any sequence $(\hat{q}_n)$, $\hat{q}_n \in Q$ you may construct a sequence (y_n), $- y_1 \in Y$ arbitrary, $y_n := y(\hat{q}_1,\ldots,\hat{q}_{n-1}) \in Y$ for $n \geq 2$ -, so that for all $\forall n \geq \mathbb{N} : \tilde{\xi}_n^\alpha \leq \varepsilon$;

whereby $\xi_n^\alpha := \xi^\alpha(\hat{q}_n,y_n)$ and $\bar{\xi}_n^\alpha := \frac{1}{n} \sum_{k=1}^{n} \xi_k^\alpha$.

<u>Proof:</u> Let $K := \{(x^1,\ldots,x^m) \in \mathbb{R}^m / x^\beta \leq \varepsilon \; \forall \beta \in M\}$.

We will offer the following procedure:

if it is assumed that ξ_k^α is computed for $k=1,\ldots, n$ (for $n=1$ $\xi_1^\alpha := \xi^\alpha(\hat{q}_1,y_1)$, therefore computable), then take $p_{n+1} \in P$ so that

i) if $\bar{\xi}_n \notin K$: $p_{n+1} \in P$ parallel with $\bar{\xi}_n' - c(\xi_n')$

whereby $\quad \bar{\xi}_n'^{\alpha} = \begin{cases} 0 \text{ if } \bar{\xi}_n^{\alpha} = -\infty \\ \\ \bar{\xi}_n^{\alpha} \text{ else} \end{cases}$

and $c(\bar{\xi}_n')$ defined as above.

ii) if $\bar{\xi}_n \in K \quad p_{n+1} \in P$ arbitrary, only fulfilling $p_{n+1}^{\alpha} = 0$ if $\bar{\xi}_n^{\alpha} = -\infty$. (If $\xi_n^{\alpha} = -\infty \ \forall \alpha \in M \quad p_{n+1} \in P$ can be chosen completely arbitrary.)

If then $\bar{\xi}_n \notin K$ and $c(\bar{\xi}_n) := c(\xi_n')$, then

$c(\xi_n^{\alpha}) = \epsilon \forall \alpha$ with $p_{n+1}^{\alpha} > 0$. Thus

$p_{n+1} \cdot c(\bar{\xi}_n^{\alpha})' = \epsilon \leq p_{n+1} \cdot \bar{\xi}_n'$.

Now choose $y_{n+1} := y_{n+1}(\hat{q}_1, \ldots, \hat{q}_n)$ with positive components ϵ-good in $\Delta'(p_{n+1}, (0,1,0))$ then with Theorem 4.2 for $\hat{q}_{n+1} \in Q$ arbitrary.

$p_{n+1} \cdot \xi(\hat{q}_{n+1}, y_{n+1})' \leq \epsilon \quad$ therefore

$p_{n+1} \cdot \xi(\hat{q}_{n+1}, y_{n+1})' \leq p_{n+1} \cdot c(\bar{\xi}_n)' \leq p_{n+1} \bar{\xi}_n'$. As further

$c(\bar{\xi}_n) \in C := \{x \in \mathbb{R}^m \ / \ p_{n+1} \cdot x = \epsilon \}$ and p_{n+1} parallel to

$\bar{\xi}_n - c(\bar{\xi}_n)$, the hyperplane C sticks perpendicular on $[c(\bar{\xi}_n), \bar{\xi}_n]$ (HESSE's Theorem).

2. If $\xi_n^\alpha > -\infty$ $\forall \alpha \in M$, $n \in \mathbb{N}$ then the preliminaries of the corollar are fulfilled and we get the theorem.

3. If $\xi_n^\alpha = -\infty$ for some $\alpha \in M$, $n \in \mathbb{N}$ we define:

$M' := \{\alpha \in M \ / \ \exists \ \hat{q}_n, j_n \in S \text{ with } F(\alpha, j, \hat{q}_n) = \emptyset\} \subset M$

For $\alpha' \in M'$ there exists a minimal $n_{\alpha'}$ for which $F(\alpha', j, \hat{q}_{n_{\alpha'}}) = \emptyset$

If now

$$(\xi_n^*)^\alpha := \begin{cases} \xi_n^\alpha & \text{for } n \le n_{\alpha'} \\ \\ 0 & \text{for } n \ge n_{\alpha'} \end{cases} \qquad \text{for } \alpha' \in M'$$

$$(\xi_n^*) := \xi_n^\alpha \qquad \text{for } \alpha \in M - M' \ ,$$

follows $(\xi_n^*)^\alpha > -\infty$ for $n > \hat{n}$, $\hat{n}$ for example the maximum of the $n_{\alpha'}$'s.

The corollar is applicable if we consider $(\xi_n^*)_{n \ge \hat{n}+1}$ and if we choose $y_{\hat{n}+1}$ arbitrary, $y_n = y(q_{\hat{n}+1}, \ldots, q_{n-1})$ for $n > \hat{n}+1$.

So for $\varepsilon > 0$ there exists $N(\varepsilon)$, so that for all $n > N(\varepsilon)$, $\alpha \in M - M'$

$$\frac{1}{n - \hat{n} + 1} \sum_{k=\hat{n}+1}^{n} \xi_k^\alpha = \frac{1}{n - \hat{n} + 1} \sum_{k=\hat{n}+1}^{n} \xi_k^{*\alpha} < \varepsilon$$

4. As $|\xi_n^\alpha| \leq 1 \ \forall \alpha = 1,\ldots,\hat{n},\ldots$ the statement is proved by 2. and 3.

Remarks: In Theorem 4.2 and 4.3 y and y_n are chosen with positive components.

Analogous to section 3 we define stationary strategies $\tau =: ST_2(y)$ with $y \in Y$ for Player II by

$$\tau_k(2, h_{i_1,j_1}^\alpha, \ldots, h_{i_{k-1},j_{k-1}}^\alpha) = \tau_1(2) = y$$

$$\forall \ (h_{i_1,j_1}^\alpha, \ldots, h_{i_{k-1},j_{k-1}}^\alpha) \in \prod_{n=1}^{k-1} H$$

Theorem 4.4: For all $\eta > 0$ there exists $K := K(\eta)$, so that for $k \geq K$:

$$pr_{p,2,\sigma,ST_2(y)} (\tilde{g}_k > \xi^{\tilde{\alpha}}(\tilde{q}_k, y) + \eta) \leq \eta \quad \forall \sigma \in \Sigma, ST_2(y) \in T \text{ with } y \in Y.$$

Proof: If $pr_{p,2,\sigma,ST_2(y)}$ is abbreviated by pr we get as

$$\bar{\tilde{g}}_k = \sum_{j=1}^{s} \frac{|\{n \leq k / \tilde{j}_n = j\}|}{k} \cdot \tilde{g}_k(j) \leq \sum_{j=1}^{s} \frac{|\{n \leq k / \tilde{j}_n = j\}|}{k} \xi^{\tilde{\alpha}}(\tilde{q}_k, j)$$

$$pr(\bar{\tilde{g}}_k > \xi^{\tilde{\alpha}}(\tilde{q}_k, y) + \eta) \leq pr(\sum_{j=1}^{s} \left| \frac{|\{n \leq k / \tilde{j}_n = j\}|}{k} - y^j \right| > \eta)$$

$$\leq \sum_{j=1}^{s} pr(\left| \frac{|\{n \leq k / \tilde{j}_n = j\}|}{k} - y^j \right| > \eta) \leq \sum_{j=1}^{s} \frac{y^j - (1 - y^j)}{k \cdot \eta^2}$$

$$\leq \frac{1}{k} \cdot \sum_{j=1}^{s} \frac{1}{4 \cdot \eta^2} \leq \eta .$$

for k big as inferred from the Theorem of BERNOULLI; further-more as can be seen, k can be determined independently on σ and $ST_2(y)$ resp. y.

Description of Principle of Construction
for ε-Strategies of Player II

τ_ε is called an ε-strategy of Player II if $E_{p,2,\sigma,\tau_\varepsilon} \tilde{g}_k' \leq \varepsilon$ in $\Gamma_\infty'(p_0,2)$ for all $\sigma \in \Sigma$ if $k > k_0$ and k_0 suitably determined.

To determine an ε-strategy of Player II we will, incorporating what follows and looking for transparence in the ideas of proofs - subdivide the game in "sections", whereby each section itself again is subdivided in N "blocks" of length K.

Hereby for $\varepsilon > 0$ N is determined to Theorem 4.3 and the length of the blocks $K := K(\varepsilon/N)$ according to Theorem 4.4.

Player II has to use a stationary strategy $ST_2(y_{u,w})$ in the w-th block of the u-th section, whereby here a strategy is called stationary if in each repetition in the u-th section (independent on the information inside the w-th block) with unchanged proba-bilities it chooses the pure strategies $j=1,\ldots,s$.

Thereby $y_{u,1} \in Y$ can be choosen arbitrarily but $y_{u,1}^j > 0$ for all $j \in S$, $u \in \mathbb{N}$. $y_{u,w}$ for $2 \leq w \leq N$ is constructed according to Theorem 4.3 whereby:

$y_{u,w} := y(\widetilde{Q}(u,1),\ldots,\widetilde{Q}(u,w-1))$ and thereby

$\widetilde{Q}(u,w) := (\widetilde{Q}_1(u,w),\ldots,\widetilde{Q}_S(u,w))$ and $\widetilde{Q}_j(u,w)$ denotes the relative frequency of information in the w-th block of the u-th section for repetitions in which column j is chosen by Player II.

Interpretation: You can interpret the above described proceeding of Player II as follows (compare proof of Theorem 4.3): Player II has to derive from the observed frequencies of the first w-blocks of a section an estimation p_{n+1} of the choice of $\alpha \in M$. Then he has to take a y_{n+1} ϵ-good in $\Delta'(p_{n+1},(0,1,0))$ and has to play the stationary strategy $ST_2(y_{n+1})$ in the w+1-th block. At the end of a block $\xi^{\widetilde{\alpha}}(\widetilde{Q}(u,w+1),y_{n+1})$ (with probability near 1) can be taken as an upper bound for the median payoff of the w+1-th block. Length of blocks and number of sections can be adjusted in such a manner, that at the end of a section the expectance of the median payoff will become $\leq \epsilon$.

Execution of the Construction

Given $\epsilon > 0$ determine $N := N(\epsilon)$ according to Theorem 4.3; $K := K(\frac{\epsilon}{N})$ according to Theorem 4.4.

On the probability field $(\Omega, \mathbb{R}, \mathrm{pr}_{p_0, 2, \sigma, \tau})$ we define $\widetilde{Q}(u,w)$ for $u \in \mathbb{N}$, $1 \le w \le N$, $j \in S$ and $\mu \in H$

$$
\widetilde{Q}_j^\mu (u,w) := \begin{cases} \dfrac{1}{K} \cdot \dfrac{|\{n/(u-1)KN+(w-1)K<n\le(u-1)KN+wK, h_{\widetilde{i}_n, \widetilde{\jmath}_n}^{\widetilde{\alpha}}=\mu, \widetilde{\jmath}_n=j\}|}{|\{n/(u-1)KN+(w-1)K<n\le(u-1)KN+wK, \widetilde{\jmath}_n=j\}|} & \\[2em] \qquad\qquad\qquad\qquad \text{if denominator} \ne 0 & \\[1em] 0 \qquad\qquad\qquad\qquad \text{else} \end{cases}
$$

Hence

$$\widetilde{Q}_j(u,w) := (\widetilde{Q}_j^{\mu_1}(u,w), \ldots, \widetilde{Q}_j^{\mu_{|H|}}(u,w)) \qquad \text{for } j \in S$$

$$\widetilde{Q}(u,w) := (\widetilde{Q}_1(u,w), \ldots, \widetilde{Q}_S(u,w)); \text{ and}$$

$$\widetilde{G}(u,w) := \frac{1}{K} \sum_{l=(u-1)KN+(w-1)K+1}^{(u-1)KN+wK} \widetilde{g}_l'.$$

Now an ε-strategy τ_ε for $\omega := (\alpha; 2, j_1, i_1, \ldots, 2, j_k, i_k, \ldots)$,

can be defined by: $\tau_{\varepsilon_k}(2, h_{i_1, j_1}^\alpha, \ldots, h_{i_{k-1}, j_{k-1}}^\alpha) := y_{u,1} \in Y, y_{u,1}^j > 0 \forall j$

for $(u-1)KN < k \le (u-1)KN+K$,

$$\tau_{\varepsilon_k}(2, h_{i_1, j_1}^\alpha, \ldots, h_{i_{k-1}, j_{k-1}}^\alpha) := y_{u,w} := y(\widetilde{Q}(u,1), \ldots, \widetilde{Q}(u,w-1),$$

$y_{u,w}$ according to Theorem 4.3 with positive components

for $(u-1)KN+(w-1)K<k\leq(u-1)KN+wK$; $u\in\mathbb{N}$, $2\leq w\leq N$ for all

$$(2,h^{\alpha}_{i_1,j_1},\ldots,h^{\alpha}_{i_{k-1},j_{k-1}})\in\{2\}\times\prod_{n=1}^{k-1}H.\text{ Theorem 4.4 implies then}$$

$$\text{pr}_{p_0,2,\sigma,\tau_\varepsilon}(\tilde{G}(u,w)>\xi^{\tilde{\alpha}}(\tilde{Q}(u,w);y_{u,w})+\varepsilon)<\frac{\varepsilon}{N}\ \forall\sigma\in\Sigma\text{ and with}$$

$$\bar{\tilde{G}}(u):=\frac{1}{N}\sum_{w=1}^{N}\tilde{G}(u,w),$$

$$\bar{\xi}^{\tilde{\alpha}}(u):=\frac{1}{N}\sum_{w=1}^{N}\xi^{\tilde{\alpha}}(\tilde{Q}(u,w),y_{u,w})\text{ then}$$

$$\text{pr}_{p_0,2,\sigma,\tau_\varepsilon}(\bar{\tilde{G}}(u)>\bar{\xi}^{\tilde{\alpha}}(u)+\varepsilon)<\varepsilon\ \forall\sigma\in\Sigma.$$

Since according to the construction for $\omega\in\Omega$, $u\in\mathbb{N}$ arbitrary, $\bar{\xi}^{\tilde{\alpha}}(u)(\omega)<\varepsilon$ for all $\alpha\in M$ we get

$$E_{p_0,2,\sigma,\tau_\varepsilon}\bar{\tilde{G}}(u)\leq\varepsilon\cdot 1+(1-\varepsilon)\cdot 2\varepsilon<3\varepsilon\ \ \forall\sigma\in\Sigma,$$

finally for $k>k_0$, $k_0>u_0\ KN$ with $u_0>\frac{1}{\varepsilon}$ whereby for k there

exists the unique construction $k=u_k\ KN+w_kK+r_k,u_k<0,$

$0\leq w_k<N$, $0\leq r_k<K$:

$$E_{p_0,2,\sigma,\tau_\varepsilon}\tilde{g}'_k=E_{p_0,2,\sigma,\tau_\varepsilon}\frac{1}{k}(\sum_{u=1}^{u_k}NK\cdot\bar{\tilde{G}}(u)+\sum_{l=u_kKN+1}^{k}\tilde{g}'_l)$$

$$\leq \frac{1}{k} (u_k \cdot NK3\varepsilon + NK1) \leq \frac{1}{u_k \cdot NK} (u_k NK \cdot 3\varepsilon + NK) < 4\varepsilon \quad \forall \sigma \in \Sigma ,$$

thus τ_ε is an ε-strategy.

In conformity with remarks on p. 28, this strategy even secures cav $u(p_0) + \varepsilon$ in $\Gamma_\infty (p_0, 2)$ for Player II. On the other hand Theorem 4.1 gives a strategy which guarantees cav $u(p_0)$ for Player I, thus

<u>Theorem 4.5</u>: The value of $v^2(p)$ of the game $\Gamma_\infty(p, 2)$ exists: $v^2(p) = $ cav $u^2(p) \quad \forall p \in P$.

Further the strategy τ_ε given above is an ε-good strategy in $\Gamma_\infty(p, 2)$.

<u>Remarks</u>: As can be seen now, for the value v_n^2 of the game $\Gamma_\infty(p, 2)$: $v_n^2(p) \geq v_\infty^2(p) = $ cav $u^2(p) \quad \forall p \in P$.

As following example shows $\neq$ can occur.

<u>Example 4.2</u>: If $p_0 = (\frac{1}{2}, \frac{1}{2})$ then for the game defined in Example 4.1: cav $u^2(p_0) = 1 \cdot u^2(p_0)$, hence:

for x optimal in $\Delta(p(0,1,0))$ $E_{p_0, 2, ST(x), \tau} \ \widetilde{\overline{g}}_k \geq u^2(p_0) = 1$

and $\text{pr}_{p_0, 2, ST(x), \tau} (\widetilde{\alpha} = \alpha / h_{\widetilde{i}_1, \widetilde{j}_1}^{\widetilde{\alpha}} = h_{i_k, j_k}^{\alpha}, \ldots, h_{\widetilde{i}_k, \widetilde{j}_k}^{\widetilde{\alpha}} = h_{i_k, j_k}^{\alpha}) = p^\alpha$

$\forall \alpha \in M, k \in \mathbb{N}, \tau \in T$ and $h_{i_1, j_1}^\alpha, \ldots, h_{i_k, j_k}^\alpha$ with

$\text{pr}_{p_0, 2, ST(x), \tau} (h_{\widetilde{i}_1, \widetilde{j}_1}^{\widetilde{\alpha}} = h_{i_1, j_1}^{\alpha}, \ldots, h_{\widetilde{i}_k, \widetilde{j}_k}^{\widetilde{\alpha}} = h_{i_k, j_k}^{\alpha}) > 0$.

Hence for a strategy σ^* defined by $\sigma_k^*(\cdot)=ST(x)(\cdot), k \leq n$

$$\sigma_k^*(\alpha,j_1,i_1,\ldots,i_{n-1},j) = x(\alpha,j) \text{ with } x=(x(1,1),\ldots,x(m,s)) \text{ optimal}$$

in $\Gamma_1(p_0,2)$;

$$E_{p_0,2,\ \sigma_1^*,\tau}\ \tilde{g}_n \geq \frac{n-1}{n}\cdot u^2(p_0)+ \frac{1}{n}\cdot v_1^2(p_0) = \frac{n-1}{n} + \frac{1.5}{n} > 1 = cav\ u^2(p_0)$$

$$\forall \tau \in T.$$

Description of the Principle of Construction
of Optimal Strategies for Player II

To construct an optimal strategy for Player II in $\Gamma_\infty(p,2)$ we
choose a monotonously decreasing nullsequence.

Player II has to use for a certain number of sequences respec-
tively an ε_t-good strategy, determined in a construction as above.
Thereby each number of sequences playing with ε_t-good strategies
corresponds to a) the total number of sequences in these sections
must be larger than the number of repetitions played before, and
on b) the number has to be "very much larger" than $N_{t+1}\cdot K_{t+1}\cdot$
Hereby $N_{t+1}\cdot K_{t+1}$ assigns the number of games you need to bring
the expectance-value down to $< \varepsilon_{t+1}$ in a construction for an
ε_{t+1}-good strategy.

Construction of an Optimal Strategy

To determine an optimal strategy exactly, we have to do some
epsilontic, which will be conducted here for completeness.

$(\varepsilon_t)_{t \in \mathbb{N}} \searrow 0$. Choose a sequence of integers $(m_t)_{t \in \mathbb{N}}$ fulfilling following conditions

(i) $\quad m_t \ K_t N_t \geq \sum\limits_{l=1}^{t-1} m_l \ K_l N_l$

(ii) $\quad m_t \geq \dfrac{t}{\varepsilon_t} \cdot \dfrac{K_{t+1} \ N_{t+1}}{K_t \cdot N_t}$

whereby K_t, N_t are determined in the construction for an ε_t-good strategy.

As for $k \in \mathbb{N}$ there exists a unique t_k with

$$\sum_{l=1}^{t_k-1} m_l \ K_l \ N_l \leq k < \sum_{l=1}^{t_k} m_l \ K_l \ N_l$$

the divisionalgorithm twice applied implies the unique equation for $k \in \mathbb{N}$

$(+) \quad k = \sum\limits_{l=1}^{t_k-1} m_l K_l N_l + (u_k-1)N_{t_k} K_{t_k} + (w_k-1)K_{t_k} + r_k$

for $\quad t_k \in \mathbb{N}$; $\ 1 \leq u_k \leq m_{t_k}$, $\quad 1 \leq w_k \leq N_k$, $\quad 1 \leq r_k \leq K_{t_k}$.

If now τ_0 is defined for (w.l.o.g.) $\omega = (2, j_1, i_1, 2, \ldots, j_n, i_n, \ldots) \in \Omega$

by: $\tau_{0_k} (2, h^\alpha_{i_1}, j_1, \ldots, h^\alpha_{i_{k-1}}, j_{k-1}) = y_{u_k, w_k}$

$y_{u_k, w_k} := y(\widetilde{Q}(u_k, 1), \ldots, \widetilde{Q}(u_k, w_k-1))$ for $2 \leq w_k \leq N_k$ according to the construction of an ε_{t_k}-good strategy, y_{u_k, w_k} with positive

components, whereby $\widetilde{Q}(u_t, w_t)(\omega)$ assigns the relative frequency

of information in the w_k-th block (of length N_{t_k}) of the

$(\sum\limits_{t=1}^{t_{k-1}} m_t + u_t)$-th section.

Then we get with

$$\widetilde{G}(u(t), w(t)) := \sum_{k=\sum\limits_{l=1}^{t-1} m_l K_l N_l + (u(t)-1)K_t N_t + (w(t)-1)K_t + 1}^{\sum\limits_{l=1}^{t-1} m_l K_l N_l + (u(t)-1)K_t N_t + w(t)K_t} \frac{1}{K_t} \cdot \widetilde{g}'_k$$

for $t \in \mathbb{N}$, $1 \le u(t) \le m_t$; $1 \le w(t) \le N_t$ by construction $\forall \sigma \in \Sigma$, $t \in \mathbb{N}$

$$E_{p,2,\sigma,\tau_0} \widetilde{G}(u(t)) := E_{p,2,\sigma,\tau_0} \frac{1}{N_t} \sum_{w(t)=1}^{N_t} \widetilde{G}(u(t), w(t)) \le \varepsilon_t$$

thus if k according to (+):

$$E_{p,2,\sigma,\tau_0} \widetilde{g}'_k \le E_{p,2,\sigma,\tau_0} \frac{1}{k} (\sum_{\substack{1 \le t \le t_k - 1 \\ 1 \le u(t) \le m_t}} \widetilde{G}(u(t)) N_t K_t +$$

$$+ \sum_{1 \le u(t_k) \le u_{k-1}} N_{t_k} K_{t_k} \widetilde{G}(u(t_k)) + N_{t_k} \cdot K_{t_k} \cdot 1)$$

$$\le \frac{1}{k} (\sum_{t=1}^{t_k - 1} m_t N_t K_t \varepsilon_t + (u_k - 1) \varepsilon_t N_{t_k} K_{t_k} + N_{t_k} K_{t_k})$$

For every $\varepsilon > 0$ there exists $k_1(\varepsilon)$ so that for all $k > k_1(\varepsilon)$

$$\frac{1}{k} \sum_{t=1}^{t_k-1} m_t N_t K_t \varepsilon_t \leq \frac{\sum_{t=1}^{t_k-1} (m_t N_t K_t \ \varepsilon_t}{\sum_{t=1}^{t_k-1} m_t N_t K_t} \leq \frac{\varepsilon}{3} \ .$$

As for $\varepsilon_t \searrow 0$, $n_t := m_t N_t K_t \nearrow \infty$ (which is implied by i))

even $\dfrac{\sum_{t=1}^{k} n_t \cdot \varepsilon_t}{\sum_{t=1}^{k} n_t} \searrow 0$ and as far $k \rightarrow \infty$ also $t_k \rightarrow \infty$, there

is an $k_2(\varepsilon)$, so that for all $k > k_2(\varepsilon)$

$$\frac{1}{k} u_k \varepsilon_{t_k} N_{t_k} K_{t_k} \leq \varepsilon_{t_k} < \frac{\varepsilon}{3} \ ; \text{ there also is } k_3(\varepsilon), \text{ so that for all}$$

$k > k_3(\varepsilon)$

$$\frac{1}{k} N_{t_k} \cdot K_{t_k} \leq \frac{N_{t_k} \cdot K_{t_k}}{m_{t_k-1} N_{t_k-1} K_{t_k-1}} \quad \overset{\text{iii)}}{\leq} \quad \varepsilon_{t_k-1} < \frac{\varepsilon}{3} \ .$$

So for $\varepsilon > 0$ there exists $k(\varepsilon) := \max(k_1(\varepsilon), \ k_2(\varepsilon), \ k_3(\varepsilon))$, so that for all $k > k(\varepsilon)$:

$E_{p,2,\sigma,\tau_0} \ \tilde{g}'_k < \varepsilon \quad \forall \sigma \in \Sigma$. Thus also

$E_{p,2,\sigma,\tau_0} \ \tilde{g}_k < v^2(p) + \varepsilon$ hence τ_0 optimal in $\Gamma_\infty(p,2)$.

5. Notes on KOHLBERG's Paper, the Case of Simultaneous Moves

The class of games now considered in which both players move
simultaneously was discussed by KOHLBERG [1976 b]. Here we will
straight away assume that Player I after each repetition gets
as information the number of the column Player II has chosen,
because [comp. KOHLBERG 1976 b, p.23] the value of the infinitely
repeated game is independent of the (kind of) information
Player I gets and main object of this paper is the analysis
of infinite games.

Similar as above we introduce the following simplifications here.
A strategy of Player II here merely is defined by

$$\tau_1 : \{3\} \to Y \, , \quad \tau_k : \{3\} \times \prod_{n=1}^{k-1} H \quad \text{for } k \geq 2$$

τ_1 allies the sequence of move "3" to an $y \in Y$; for $(0,0,1)$ we
write abbreviated "1".

For the information matrices we will request in this section
(x) that there exists $(x(1),\ldots,x(m))$ with $x(\alpha) \in F$ for all $\alpha \in M$,
so that for all $\alpha, \beta \in M$: $x(\alpha) \cdot H^\alpha = x(\beta) \cdot H^\beta$

Remarks: 1) Requirement (x) is stronger than (*) in section 4
and weaker than (+) in section 3.
2) For interpretation compare remarks on (+) in section 3.

If for $p \in P$ Ns(p) denotes the set of tuples $\sigma_1 = (x(1),\ldots,x(m))$

which fulfill (x) for all $\alpha,\beta \in M$ with $p^\alpha, p^\beta > 0$, then with

$\tau_1 \in T_1 := \{\tau_1 / \tau_1 : \{3\} \to Y\} := Y$ and

$$E_p^3(\sigma_1, \tau_1) := \sum_{\alpha=1}^{m} \sum_{i=1}^{r} \sum_{j=1}^{s} x^i(\alpha) \cdot g_{i,j}^\alpha \cdot y^j \cdot p^\alpha \quad \text{the auxiliary}$$

game $\Delta(p,(0,0,1)$ is given in normal form again. For the value $u^3(p)$ of this game states

$$u^3(p) = \max_{\sigma_1 \in Ns^3(p)} \quad \min_{\tau_1 \in Y} \quad E_p^3(\sigma_1, \tau_1)$$

$$= \min_{\tau_1 \in Y} \quad \max_{\sigma_1 \in Ns^3(p)} \quad E_p^3(\sigma_1, \tau_1).$$

We can show analogously as above for $p \in P$ the existence of $p_1^3, \ldots, p_{m+1}^3 \in P$; $\lambda_1^3, \ldots, \lambda_{m+1}^3 \in \mathbb{R}^+$ with $\sum_{\beta=1}^{m+1} \lambda_\beta^3 = 1$,

$$\sum_{\beta=1}^{m+1} \lambda_\beta^3 p_\beta^3 = p \quad \text{and} \quad \sum_{\beta=1}^{m+1} \lambda_\beta^3 u^3(p_\beta^3) = cav \, u^3(p).$$

As we define stationary strategies $\sigma := ST_3(x) \in \Sigma$ of Player I by $\sigma_k(\alpha, i_1, j_1, \ldots, i_k, j_k) = \sigma_1(\alpha) := x(\alpha) \in F$ $k \geq 2$ for

$i_1 \in R$, $j_1 \in S$, $\alpha \in M$, $0 \leq l \leq k$ we get

Theorem 5.1: For games $\Gamma_\infty(p,3)$ with $p^\alpha > 0$ $\forall \alpha \in M$; p_β^3, λ_β^3

$\beta = 1, \ldots, m+1$ defined as above, follows, if Player I defines his strategy σ_0 by a random mechanism with

$$pr(\sigma_0 = ST^3(x^3_\beta)/\alpha=\tilde{\alpha}) = \frac{\lambda^3_\beta \cdot p^{\alpha,3}_\beta}{p^\alpha} \quad \beta=1,\ldots,m+1 \text{ whereby } x^3_\beta \in NS^3_\beta (p^3_\beta)$$

optimal in $\Delta(p_\beta,(0,0,1))$ then for a strategy defined in this way:

$$E_{p,3,\sigma_0,\tau} \tilde{\tilde{g}}_k \geq cav \ u^3(p_0) \ \forall \tau \in T.$$

For the rest of this section we (analogously to section 4) w.l.o.g. request that $\left|g^\alpha_{i,j}\right| \leq 1 \ \forall \alpha \in M, \ i \in R, \ j \in S$ and $cav \ u^3(p) \leq 0 \ \forall p \in P, \ cav \ u^3(p_0) = 0.$

We define

$$F(\alpha,\hat{q},\delta) := \{f \in F \ / \ |f \cdot H^\alpha - \hat{q}| \leq \delta\} \text{ for } \hat{q} \in Q, \ y \in Y, \ \delta > 0$$

$$\xi^\alpha(\hat{q},y,\delta) := \begin{cases} \max\limits_{f \in F(\alpha,\hat{q},\delta)} f \cdot G^\alpha \cdot y & \text{if } F(\alpha,\hat{q},\delta) \neq \emptyset \\ -\infty & \text{else.} \end{cases}$$

Further let $\tilde{q}_k$ be defined on $(\Omega, \hat{R}, pr_{p,3,\sigma,\tau})$ like on page 31, then also $\xi^\alpha(\tilde{q}_k,y,\delta)$ is defined.

Before quoting two theorems proved by KOHLBERG[1976 p 19 ff.] we will still define stationary strategies of Player II here,

$$\tau_k(3,h^\alpha_{i_1,j_1},\ldots,h^\alpha_{i_k,j_{k-1}}) = \tau(3) = y \in Y \ \forall k \in \mathbb{N} \text{ whereby}$$

$$(h^\alpha_{i_1,j_1},\ldots,h^\alpha_{i_k,j_{k-1}}) \in \prod_{n=1}^{k-1} H \text{ arbitrary, that means a stationary}$$

strategy of Player II is a probability distribution for the choice of a column at each repetition, which does not depend on the information of the repetitions but only on the sequence of move "3"

__Theorem 5.2:__ If $\tau := ST(y)$ with $y^j > 0$ for all $j \in S$, then for all $\eta \geq 0$, $\delta \geq 0$ there exists $K := K(\eta, \delta)$ such that for all $\sigma \in \Sigma$, $\alpha \in M$

$$pr_{p,3,\sigma,\tau}(\widetilde{\widetilde{g}}_k > \xi^{\widetilde{\alpha}}(\widetilde{q}_k, y, \delta) + \eta, \ \xi^{\widetilde{\alpha}}(\widetilde{q}_k, y, \delta) > -\infty) < \eta \ \text{for} \ k \geq K$$

__Remarks:__ The requirement $y^j > 0$ is necessary as can be taken from the proof.

__Theorem 5.3:__ For every $\varepsilon > 0$ there exists $\delta := \delta(\varepsilon) > 0$ and a $N := N(\varepsilon)$, so that for every sequence $(\hat{q}_n)$ with $\hat{q}_n \in Q$ for all n one can construct a sequence (y_n) - $y_1 \in Y \ni y_n$ $:= y(\hat{q}_1, \ldots, \hat{q}_{n-1})$ for $n \geq 2$ - with:

$$\xi^\alpha_n := \frac{1}{n} \sum_{k=1}^{n} \xi^\alpha_k := \frac{1}{n} \sum_{k=1}^{n} \xi^\alpha(\hat{q}_k, y_k, \delta) \leq \varepsilon \quad \text{for all } \alpha \in M, \ n \geq N.$$

__Remarks:__ 1) KOHLBERG wrote "$y_n := y(\hat{q}_1, \ldots, \hat{q}_n)$" instead of "$y_n = y(\hat{q}_1, \ldots, \hat{q}_{n-1})$". But as can be seen from the proof (first you construct $p_{n+1} := p_{n+1}(\hat{q}_1, \ldots, \hat{q}_n, y_1, \ldots, y_n)$ and then y_{n+1} optimal $\Delta(p_{n+1}, (0, 0, 1)))$, y_n depends merely on $\hat{q}_1, \ldots, \hat{q}_{n-1}$.

2) In Theorem 5.3 y_n can be chosen in such a manner that $y_n^j > 0$ for all $j \in S$, $n \in \mathbb{N}$, as for the only nontrivial case $F(\alpha, q_n, \delta) \neq \emptyset$, $q_n \in Q$, the continuity of $\xi^\alpha(q_n, \delta): Y \to \mathbb{R}$, $\xi^\alpha(\hat{q}_n, \delta)(y) := \xi^\alpha(\hat{q}_n, y, \delta)$ - where $y = y(q_1, \ldots, q_{n-1})$ is constructed according to the principle given by KOHLBERG with $|y' - y_n|$

sufficiently small - implies, as $F(\alpha, \hat{q}_n, \delta)$ compact:

$$|\xi^\alpha(\hat{q}_n, y_n, \delta) - \xi^\alpha(\hat{q}_n, y_n', \delta)| < \varepsilon,$$

hence $\dfrac{1}{n} \displaystyle\sum_{k=1}^{n} \xi^\alpha(\hat{q}_k, y_k', \delta) \leq \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} \xi^\alpha(\hat{q}_k, y_k, \delta) + \varepsilon \leq 2\varepsilon.$

Applying Theorem 5.2 and 5.3 we get an ε-good strategy. The principle of construction can be described as in section 4, but only the role of Theorem 4.4 plays Theorem 5.2, those of Theorem 4.3 Theorem 5.3. Further attention must be payed that for $\varepsilon > 0$ also $\delta(\varepsilon)$ must be determined; the length of a block depends on δ.

As in the proof given by KOHLBERG there is a fundamental mistake we will execute:

The Construction of an ε-good Strategy

Let $\varepsilon > 0$, $\delta := \delta(\varepsilon)$ and $N := N(\varepsilon)$ determined according to Theorem 5.2; $K := K(\eta, \delta)$ according to Theorem 5.3 and $(\Omega, \mathcal{A}, pr_{p,3,\sigma,\tau})$, $\tilde{Q}(u,w)$ as defined on page 40 and finally τ_ε by

$$\tau_{\varepsilon_k}(3, h^\alpha_{i_1, j_1}, \ldots, h^\alpha_{i_{k-1}, j_{k-1}}) := y_{u,1} \in Y, \; y^j_{u,1} > 0$$

for all $j \in S$ and $(u-1) \, KN < k \leq (u-1)KN + K$

$$\tau_{\varepsilon_k}(3, h^\alpha_{i_1, j_1}, \ldots, h^\alpha_{i_{k-1}, j_{k-1}}) := y_{u,w} := y(\tilde{Q}(u,1), \ldots \tilde{Q}(u, w-1))$$

according to the construction of Theorem 5.3 with $y_{u,w}^j > 0$

for all $j \in S$ and $(u-1)KN+(w-1)K < k \le (u-1)KN+wK$;

$u \in \mathbb{N}$, $2 \le w \le N$, and $(h_{i_1,j_1}^\alpha, \ldots, h_{i_{k-1},j_{k-1}}^\alpha) \in \prod_{n=1}^{k-1} H$.

Then with $\widetilde{G}(u,w)$ as on page 40 whereby only "$\widetilde{g}_1'$" instead of $\widetilde{g}_1$

$$pr_{p,3,\sigma,\tau_\varepsilon} (\widetilde{G}(u,w) > \xi^{\widetilde{\alpha}}(\widetilde{Q}(u,w), y_{u,w}, \delta) + \varepsilon ,$$

$\xi^{\widetilde{\alpha}}(\widetilde{Q}(u,w), y_{u,w}, \delta) > -\infty) < \dfrac{\varepsilon}{N}$ and with $\overline{\widetilde{G}}(u) := \dfrac{1}{N} \sum\limits_{w=1}^{N} \widetilde{G}(u,w)$.

$$\overline{\xi}^{\widetilde{\alpha}}(u) := \frac{1}{N} \sum_{w=1}^{N} \xi^{\widetilde{\alpha}}(\widetilde{Q}(u,w), y_{u,w}, \delta) \quad \text{then}$$

$$(+) pr_{p,3,\sigma,\tau_\varepsilon} (\overline{\widetilde{G}}(u) > \overline{\xi}^{\widetilde{\alpha}}(u) + \varepsilon, \quad \overline{\xi}^{\widetilde{\alpha}}(u) > -\infty) < \varepsilon \text{ for all } \sigma \in \Sigma.$$

But according to the construction for $\omega \in \Omega$, $u \in \mathbb{N}$ arbitrary

(w.l.o.g. here $\omega := (3, i_1, j_1, \ldots, 3, i_n, j_n, \ldots)$)

$\overline{\xi}^{\widetilde{\alpha}}(u)(\omega) < \varepsilon \; \forall \; \widetilde{\alpha} \; (=\widetilde{\alpha}(\omega)) \in M$, therefore

$E_{p,3,\sigma,\tau_\varepsilon} \overline{\widetilde{G}}(u) \le \varepsilon \cdot 1 + (1-\varepsilon) 2\varepsilon \le 3\varepsilon \quad \forall \; \sigma \in \Sigma$.

Hence again with $k > k_0 > u_0 \cdot KN$, $u_0 > \dfrac{1}{\varepsilon}$ whereby

$k = u_k \cdot KN + w_k K + r_k$ (comp. page 41)

$E_{p,3,\sigma,\tau} \widetilde{\widetilde{g}}_k \le 4\varepsilon$ for all $\sigma \in \Sigma$, therefore τ_ε is an ε-strategy.

<u>Theorem 5.4</u>: The value $v_\infty^3(p)$ of $\Gamma_\infty(p,3)$ exists:

$$v^3(p) = \text{cav } u^3(p) \qquad \forall p \in P.$$

<u>Remarks</u>: In KOHLBERG's paper [1975, p.22] one concludes as follows (if we take into consideration what is remarked above on Theorem 5.3.): Let τ_ϵ' be defined by

$$\tau_{\epsilon_k}'(3, h_{i_1, j_1}^\alpha, \ldots, h_{i_{k-1}, j_{k-1}}^\alpha) := y_n$$

for $k = (n-1) \cdot K + r$, $n \in \mathbb{N}$, $0 \leq r < K$ and

$y_n := y(\widetilde{Q}_1, \ldots, \widetilde{Q}_{n-1})$ constructed according to Theorem 5.3

with $2 \leq n \leq (u_n-1)N + w_n$, $0 \leq w_n < N$ and $\widetilde{Q}_n := \widetilde{Q}(u_n, w_n)$

then $\text{pr}_{p,3,\sigma,\tau_\epsilon'}(\widetilde{G}_n > \xi^{\widetilde{\alpha}}(\widetilde{Q}_n, y_n, \delta) + \epsilon, \xi^{\widetilde{\alpha}}(\widetilde{Q}_n, y_n, \delta) > -\infty) < \frac{\epsilon}{N}$

for all $\alpha \in M$, whereby $\widetilde{G}_n := \frac{1}{K} \sum_{k=(n-1)K+1}^{nK} \widetilde{g}_k$ which should

imply: $\text{pr}_{p,3,\sigma,\tau_\epsilon'}(\frac{1}{n} \sum_{k=1}^{n} \widetilde{G}_n > \frac{1}{n} \sum_{k=1}^{n} \xi^{\widetilde{\alpha}}(\widetilde{Q}_k, y_k, \delta) + \epsilon) < \epsilon$

for all $\alpha \in M$ and $n > N$.

This conclusion i.g. is not allowed, as following simple example demonstrates

<u>Example 5.1:</u>   If $\Omega := (0,1)$, $\mathcal{R} := \mathcal{B}_{(0,1)}$ and p the rectang.ular

distribution $\xi_n : \Omega \to \mathbb{R}$, $\xi_n(\omega) = 0, \forall \omega \in \Omega$ and $\epsilon < \frac{1}{6}$ fixed,

$$G_n(\omega) := \begin{cases} 1 \text{ for } \omega \in (0,\frac{\epsilon}{2}) \\ 0 \text{ else} \end{cases} \text{ for } n=3k-3 \quad k \in \mathbb{N}$$

$$G_n(\omega) := \begin{cases} 1 \text{ for } \omega \in (\frac{1}{2},\frac{1}{2}+\epsilon) \\ 0 \text{ else} \end{cases} \text{ for } n=3k-2 \quad k \in \mathbb{N}$$

$$G_n(\omega) := \begin{cases} 1 \text{ for } \omega \in (1-\frac{\epsilon}{2},1) \\ 0 \text{ else} \end{cases} \text{ for } n = 3k-1 \quad k \in \mathbb{N}$$

then

(i) $\forall \epsilon > 0 \; \exists N(\epsilon) \forall \; n \geq N \;\; \bar{\xi}_n \leq \epsilon$, choose $N(\epsilon) = 1$ and

(ii) $p(G_n > \xi_n + \epsilon) < \epsilon$

then also $p(G_n > \xi_n + \epsilon) = \frac{\epsilon}{2}$

but not $\forall n > N(\epsilon) \;\;\; p(\bar{G}_n > \bar{\xi}_n + \epsilon) < \epsilon$ as for $n > 3$ follows

$p(\bar{G}_n > \bar{\xi}_n + \epsilon) = \frac{3}{2} \frac{\epsilon}{} \neq \epsilon.$

To avoid the conclusion above which i.g. is wrong, as the Example
5.1 demonstrates and because the loocking for specifications for
which the conclusions made by KOHLBERG remains valid is too com-
plicated caused by Theorem 5.3 and 5.4 (and probably hardly fea-
sable) is given here a modificated proof.

Finally one can get optimal strategies now completely analogous
to section 4.

6. Value and Optimal Strategies of Games with Incidental Sequence of Moves

First again is given a strategy of Player I which guarantees him however under certain conditions, $\sum_{l=1}^{3} q^l$ cav $u^l(p)$.

To do this we define again stationary strategies $\sigma =: ST(x^{(1)}, x^{(2)}, x^{(3)})$ where $x^{(1)} := (x^{(1)}(1), \ldots, x^{(1)}(m))$ for $l = 1,3$; $x^{(2)} := (x^{(2)}(1,1), \ldots, x^{(2)}(m,s))$, $x^2(\alpha, j)$, $x^1(\alpha) \in F$ for all $\alpha \in M$, $j \in S$ by: $\sigma_k(\alpha, a_1, b_1, c_1, \ldots, c_{k-1}, l) := \sigma_1(\alpha, l) := x^{(1)}(\alpha)$ for $l=1,3$, $k \geq 2$

$$\sigma_k(\alpha, a_1, b_1, c_1, \ldots, 2, j) = x^{(2)}(\alpha, j) \quad \text{for all } j \in S, k \in \mathbb{N},$$

for $\alpha; a_1, b_1, c_1, \ldots, c_{k-1}$ arbitrary.

Let further h_k denote both:

$$h_k := h^\alpha_{i_{k_0}, j_{k_0}} \quad \text{and} \quad h_k := h^{\alpha_0}(i_{k_0}); j_{k_0} \quad \text{if } a_k=1 \text{ for } k \in \mathbb{N},$$

$i_{k_0} \in R$, $j_{k_0} \in S$, $\alpha_k \in M$, similar to $\tilde{h}_k$, then we get the

Theorem 6.1: Let $\sigma := ST(x^{(1)}, x^{(2)}, x^{(3)})$ with $x^l \in Ns^l(p)$ $\forall l \in \{1,2,3\}$ for $q^l > 0$ and $a_1, h_1, \ldots, a_k, h_k$ arbitrary with

$$pr_{p,q,\sigma,\tau}(\tilde{a}_1=a_1, \tilde{h}_1=h_1, \ldots, \tilde{a}_k=a_k, \tilde{h}_k=h_k) > 0 \quad \text{for } \tau \in T \text{ then:}$$

$$pr_{p,q,\sigma,\tau}(\tilde{\alpha}=\alpha/\tilde{a}_1=a_1, \tilde{h}_1=h_1, \ldots, \tilde{a}_k=a_k, \tilde{h}_k=h_k) = p^\alpha \; \forall \alpha \in M, k \in \mathbb{N}.$$

Proof: The proof may be again conducted with induction on k essentially analogous to the proof of Theorem 3.2 and will be submitted. Here it will be only noted, that treating step k to k+1 it is most practicable to seperate the cases, then to factorize out corresponding variables--especially q^{a_k}--and to divide in order to reduce the term for k, so that it is possible to apply the induction.

Theorem 6.2: If $\sigma := ST(x^{(1)}, x^{(2)}, x^{(3)})$, $x^{(1)} \in Ns^1(p) \neq \emptyset$, x^1 optimal in $\Delta(p,l)$ for all $l \in \{1,2,3\}$ with $q^1 > 0$, then for the game $\Gamma_\infty(p,q)$:

$$E_{p,q,\sigma,\tau} \, \widetilde{\overline{g}}_n \geq \sum_{l=1}^{3} q^1 \cdot u^1(p) \quad \forall \tau \in T, \ n \in \mathbb{N}.$$

Proof: Theorem 6.1 implies for $a_k \in Z$, $i_{k_0} \in R$, $j_{k_0} \in S$, $k \in \mathbb{N}$ with $pr_{p,q,\sigma,\tau} \, (\widetilde{a}_1 = a_1, \widetilde{h}_1 = h_1, \ldots, \widetilde{a}_{n-1} = a_{n-1}, \widetilde{h}_{n-1} = h_{n-1}) > 0$ and $\tau \in T$ arbitrary:

$$E_{p,q,\sigma,\tau} \, (\widetilde{g}_n / \widetilde{a}_1 = a_1, \widetilde{h}_1 = h_1, \ldots, \widetilde{h}_{n-1} = h_{n-1})$$

$$= q^1 \sum_{\alpha=1}^{m} \sum_{j=1}^{s} \sum_{i=1}^{r} p^\alpha x^{i,(1)}(\alpha) \ g^\alpha_{i,j} \tau(a_1, h_1, \ldots, 1, h^\alpha(i))^j$$

$$+ \, q^2 \sum_{\alpha=1}^{m} \sum_{j=1}^{s} \sum_{i=1}^{r} p^\alpha x^{i,(2)}(\alpha,j) \ g^\alpha_{i,j} \tau(a_1, h_1, \ldots, h_{n-1}, 2)^j$$

$$+ \, q^3 \sum_{\alpha=1}^{m} \sum_{j=1}^{s} \sum_{i=1}^{r} p^\alpha x^{i,(3)}(\alpha) \ g^\alpha_{i,j} \tau(a_1, h_1, \ldots, a_{n-1}, h_{n-1}, 3)^j$$

$$\geq \sum_{l=1}^{3} q^1 u^1(p) \qquad \text{by Theorem 3.3, 4.1, 5.1.}$$

__Theorem 6.3__: If there exist for the game $\Gamma_\infty(p,q)$ $p_\beta \in P$, $\lambda_\beta > 0$,

$\beta=1,\ldots,m+1$ with $\displaystyle\sum_{\beta=1}^{m+1} \lambda_\beta = 1$ $\displaystyle\sum_{\beta=1}^{m+1} \lambda_\beta u^l(p_\beta) := \operatorname{cav} u^l(p_\beta)$ and

$Ns^l(p_\beta) \neq \emptyset$ for all l with $q^l > 0$ then:

if Player I uses a random mechanism which chooses, if $\alpha \in M$ is

selected, the strategy $\sigma_\beta := ST(x_\beta^{(1)}, x_\beta^{(2)}, x_\beta^{(3)})$ with

$x_\beta^l \in Ns^l(p_\beta)$ optimal in $\Delta(p_\beta,l)$ for l with $q^l > 0$ with

probability $\dfrac{\lambda_\beta \cdot p_\beta^\alpha}{p^\alpha}$, that means $pr(\sigma_0 = \sigma_\beta \,/\, \tilde\alpha = \alpha) = \dfrac{\lambda_\beta \cdot p_\beta^\alpha}{p^\alpha}, \beta=1,\ldots,m+1$

then for a strategy defined on this way:

$$E_{p,q,\sigma_0,\tau}\, \tilde g_k \geq \sum_{l=1}^{3} q^l \cdot \operatorname{cav} u^l(p)$$

__Proof__: Analogously to the proof of Theorem 3.5 one can show

for $\beta=1,\ldots,m+1$, $\alpha \in M$

1.) $pr(\sigma_0 = \sigma_\beta) = \lambda_\beta$

2.) $pr(\tilde\alpha = \alpha \,/\, \sigma_0 = \sigma_\beta) = p_\beta^\alpha$ hence again

$$E_{p,q,\sigma_0,\tau}\, \tilde g_k = \sum_{\beta=1}^{m+1} pr(\sigma_0 = \sigma_\beta) \cdot E_{p,q,\sigma_0,\tau}\,(\tilde g_k / \sigma_0 = \sigma_\beta)$$

$$= \sum_{\beta=1}^{m+1} \lambda_\beta \cdot E_{p_\beta,q,\sigma_\beta,\tau}\, \tilde g_k \geq \sum_{\beta=1}^{m+1} \lambda_\beta \sum_{l=1}^{3} q^l \cdot u^l(p_\beta)$$

$$= \sum_{l=1}^{3} q^l \cdot \sum_{\beta=1}^{m+1} \lambda_\beta \cdot u^l(p_\beta) = \sum_{l=1}^{3} q^l \operatorname{cav} u^l(p)$$

For the above given construction of a strategy, which guar-

antees Player I $\sum_{l=1}^{3} q^l$ cav $u^l(p)$ in $\Gamma_\infty(p,q)$ is decisive

that:

(·) the λ_β^l's and p_β^l's needed to determine cav $u^l(p)$ with

$u^l(p_\beta^l)$ for l with $q^l > 0$ are equal, that means that they

can be chosen independent of l.

This is not always possible.

Example 6.1: If $\{R, S, \{G^1, G^2\}, \{H^1, H^2\}, p, q\}$ is given by

$$R := S := \{1,2\}, \quad G^1 = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}, \quad G^2 = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix},$$

$$H^1 = H^2 = \begin{pmatrix} \mu_1 & \mu_3 \\ \mu_2 & \mu_4 \end{pmatrix}, \quad \mu_k \text{ unit vectors in } \mathbb{R}^4, \ 1 \le k \le 4.$$

then as here $Ns^1(p) = \{(x(1), x(2)) \mid x(1) = x(2)\}$ for $0 < p^1 < 1$

$Ns^1(p) = F \times F$ for $p^1 = 0$ or $p^1 = 1$

$$u^1(p) = \max_{\sigma_1 \in Ns^1(p)} \min_{\tau \in T_1} E_p^1 (\sigma_1, \tau_1)$$

$$= \max_{x \in F} \min_{\tau_1 \in T_1} \sum_{i=1}^{2} \sum_{j=1}^{2} x^i y^j(h(i)) \sum_{\alpha=1}^{2} p^\alpha g_{i,j}^\alpha$$

$$= \max_{i \in R} \min_{j \in S} \left(\sum_{\alpha=1}^{2} p^\alpha g_{i,j}^\alpha \right).$$
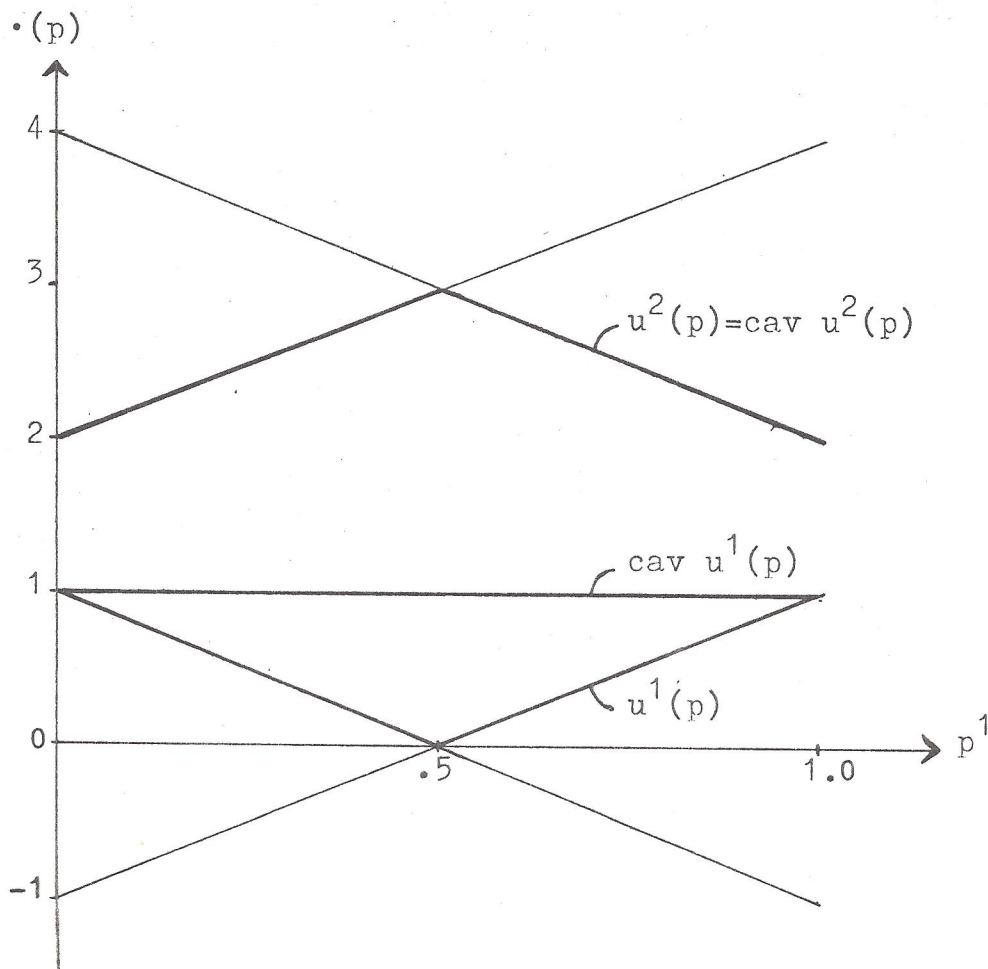
And further as $p^2 = 1 - p^1$

$$\min_{j \in S} \left(\sum_{\alpha=1}^{2} p^{\alpha} g_{1,j}^{\alpha} \right) = \min\ (2p^1+2;\ 2p-1) = 2p^1-1 \quad \forall\ p \in P$$

$$\min_{j \in S} \left(\sum_{\alpha=1}^{2} p^{\alpha} g_{2,j}^{\alpha} \right) = \min\ (-2p^1+1;\ -2p^1+4) = -2p^1+1 \quad \forall\ p \in P$$

Therefore (compare also Figure 1)

$$u^1(p) = \max\ (2p^1-2;\ -2p^1+1) = \begin{cases} -2p^1-2 & \text{for}\ \ 0 \le p^1 \le \tfrac{1}{2} \\[2ex] 2p^1-2 & \text{for}\ \ \tfrac{1}{2} \le p^1 \le 1. \end{cases}$$

Figure 1

The special information matrices here imply

$$u^2(p) = \min_{j \in S} \max_{i \in R} \left(\sum_{\alpha=1}^{2} p^\alpha g^\alpha_{i,j} \right) =$$

$$\max_{i \in R} \left(\sum_{\alpha=1}^{2} p^\alpha g^\alpha_{i,1} \right) = \max (2p^1+2; \; -2p^1+1) = 2p^1+2 \quad \forall \, p \in P$$

$$\max_{i \in R} \left(\sum_{\alpha=1}^{2} p^\alpha g^\alpha_{i,2} \right) = \max (2p^1-1; \; -2p^1+4) = -2p^1+4 \quad \forall \, p \in P$$

Thus

$$u^2(p) = \min (2p^1+2, \; 4-2p_1) = \begin{cases} 2p^1+2 & \text{for} \; 0 \le p^1 \le \frac{1}{2} \\[2em] 4-2p^1 & \text{for} \; \frac{1}{2} \le p^1 \le 1. \end{cases}$$

If $p_0 = (\frac{1}{2}, \frac{1}{2})$ take $p_\beta \in P$, $\lambda_\beta > 0$, $\sum_{\beta=1}^{3} \lambda_\beta = 1$ with

$$\sum_{\beta=1}^{3} \lambda_\beta \, p_\beta = p_0.$$

How one can compute (or can take from the figure):

i) $\sum_{\beta=1}^{3} \lambda_\beta \, u^1(p_\beta) < \mathrm{cav} \, u^1(p_0)$ if not $\lambda_\beta = 1$, $p_\beta = p_0$

for one $\beta \in \{1,2,3\}$

ii) $\sum_{\beta=1}^{3} \lambda_\beta \, u^2(p_\beta) < \mathrm{cav} \, u^2(p_0)$, $p_1=(0,1)$, $p_2=(1,0)$, $\lambda_1=\lambda_2 = \frac{1}{2}$

if numbered suitably.

i) and ii) imply here, that for the game $\Gamma_\infty(p_0,q)$ with

$q = (q^1,q^2,q^3)$ with $q^1 > 0$ and $q^2 > 0$ $(\cdot)$ can not be fulfilled.

The constructions of section 3,4,5 for strategies of Player II which guarantee $\text{cav } u^1(p)+\varepsilon$ in the games $\Gamma_\infty(p_0,1)$ brings up immediately a strategy which guarantees $\sum_{l=1}^{3} q^l \text{ cav } u^1(p) + \varepsilon$ in $\Gamma_\infty(p,q)$.

Let τ_ε be defined in such a manner that for $k \in \mathbb{N}, \varepsilon > 0$

if $a_k = 1$: $\tau_{\varepsilon_k}(a_1,b_1,\ldots,c_{k-1},a_k,h^{\alpha_0}(i_0)) := \hat{y}(h^{\alpha_0}(i_0))$

where $\hat{y} := (\hat{y}(\mu_1),\ldots,\hat{y}(\mu_k))$ optimal in $\Gamma_1(p,1)$, if $a_k = 2$

resp. $a_k = 3$

$\tau_{\varepsilon_k}(a_1,b_1,\ldots,c_{k-1},2) := y_{u,w} := y(\tilde{Q}(u,1),\ldots,\tilde{Q}(u,w-1))$ resp

$\tau_{\varepsilon_k}(a_1,b_1,\ldots,c_{k-1},3) := y'_{u,w} := y(\tilde{Q}'(u',1),\ldots,\tilde{Q}'(u',w'-1))$

for $a_1,b_1,c_1,\ldots,c_{k-1}$ arbitrary, whereby $0 \le w \le N$ $[0 \le w' \le N']$

N [resp. N'] is noting the number of blocks in the sections of the construction of an ε-good strategy in the games $\Gamma_\infty(p,2)$

$[\Gamma_\infty(p,3)]$.

For $k \in \mathbb{N}$ u and w [resp. u' and w'] are determined for $\omega \in \Omega$ according to the number of indices with $a_1 = 2$, $1 \le k$ [resp. $a_1 = 3$, $1 \le k$]; more exact: $\widetilde{Q}_j(u,w)(\omega)$ notes the relative frequency of the u-th block of the w-th section of the first k-1 repetitions in which sequence of move "2" happens and Player II has chosen column j (analogously $\widetilde{Q}'(u',1),\ldots,\widetilde{Q}(u',w'-1)))$.

Let $y(\widetilde{Q}(u,1),\ldots,\widetilde{Q}(u,w-1))$ be defined as in the construction of an ε-good strategy in $\Gamma_\infty(p,2)$. Accordingly take $y'(\widetilde{Q}'(w,1),\ldots,\widetilde{Q}'(u,w-1))$ for $\widetilde{Q}'(u,w)(\omega)$ in the game $\Gamma_\infty(p,3)$.

The weak law of great numbers implies:

(+) $\forall \varepsilon > 0 \ \exists k(\varepsilon) \ \forall 1 \in Z$

$$pr_{p,q,\sigma,\tau_\varepsilon} \left(\left| \frac{|\{n \le k / \widetilde{a}_n = 1\}|}{k} - q_1 \right| > \varepsilon \right) < \varepsilon \ \forall k > k(\varepsilon), \ \sigma \in \Sigma$$

As now it is easy to show, that

$$E_{p,q,\sigma,\tau_\varepsilon} (\widetilde{g}_n / \widetilde{a}_n = 1) \le \operatorname{cav} u^1(p) \quad \forall \sigma \in \Sigma$$

follows

i) $E_{p,q,\sigma,\tau_\varepsilon} \left(\dfrac{1}{k} \displaystyle\sum_{n \le k : \ \widetilde{a}_n = 1} \widetilde{g}_n \right) \le E_{p,q,\sigma,\tau_\varepsilon} \left(\dfrac{1}{k} \displaystyle\sum_{n \le k : \ \widetilde{a}_n = 1} \operatorname{cav} u^1(p) \right)$

$$= E_{p,q,\sigma,\tau_\varepsilon} \left(\frac{|\{n \le k / \widetilde{a}_n = 1\}|}{k} \cdot \operatorname{cav} u^1(p) \right) \overset{(\cdot)}{\le} \operatorname{cav} u^1(p) + \varepsilon$$

$\forall \sigma \in \Sigma$, $k > k(\varepsilon)$ as $|\operatorname{cav} u^1(p)| \le 1$

Further for k_0, according to section 4, k_0' to section 5

$\forall \varepsilon > 0 \ \exists \ k^*(\varepsilon) := k(\max\{k_0, k_0'\}, \varepsilon) \ \forall l$ with $q^l > 0 \ \forall k \geq k^*(\varepsilon)$

$pr_{p,q,\sigma,\tau_\varepsilon} \ (|\{n \leq k / \tilde{a}_n = l\}| < \max\{k_0, k_0'\}) < \varepsilon$ for $\sigma \in \Sigma$,

hence of course for $k > \hat{k}(\varepsilon) := \max (k(\varepsilon), k^*(\varepsilon))$

$$(++) \quad pr_{p,q,\sigma,\tau_\varepsilon} \left(\left| \frac{|\{n \leq k / \tilde{a}_n = l\}|}{k} - q^l \right| > \varepsilon, |\{n \leq k / \tilde{a}_n = l\}| < \max\{k_0 \cdot k_0'\} \right) < 2\varepsilon$$

for all $\sigma \in \Sigma$, $l \in \{2,3\}$ with $q^l > 0$.

Hence it follows from the definitions of τ_ε (above) and the property of τ_ε that if $|\{n \leq k / \tilde{a}_n = l\}| > \max\{k_0, k_0'\}$ then

$$(+++) \quad E_{p,q,\sigma,\tau_\varepsilon} \left(\frac{1}{|\{n \leq k / \tilde{a}_n = l\}|} \sum_{n \leq k : \tilde{a}_n = l} \tilde{g}_n \right) \leq \text{cav } u^l(p) + \varepsilon$$

for $l = 2,3$ with $q^l > 0$ even for $q^l > \varepsilon > 0$, $k > k'(\varepsilon)$:

ii) $E_{p,q,\sigma,\tau_\varepsilon} \left(\dfrac{1}{k} \displaystyle\sum_{n \leq k : \tilde{a}_n = l} \tilde{g}_n \right)$

$$= E_{p,q,\sigma,\tau_\varepsilon} \left(\underbrace{\frac{|\{n \leq k / \tilde{a}_n = l\}|}{k}}_{0 \leq \ \leq 1} \cdot \frac{1}{|\{n \leq k : \tilde{a}_n = l\}|} \cdot \sum_{n \leq k : \tilde{a}_n = l} \tilde{g}_n \right)$$

$\underset{(+++)}{\overset{(+) \ (++)}{\leq}} \quad 2\varepsilon \cdot 1 + (1 - 2\varepsilon) \underbrace{(q^l + \varepsilon \cdot \text{sgn}(\text{cav } u^l(p) + \varepsilon))}_{> 0} (\text{cav } u^l(p) + \varepsilon)$

$\leq q^l \cdot \text{cav } u^l(p) + \hat{M} \cdot \varepsilon \qquad \forall \sigma \in \Sigma$ and $1 \leq \hat{M} < \infty$ suitable.

Hence finally i) and ii) imply for $\varepsilon > 0$ fixed, τ_ε for ε defined as above:

$$E_{p,q,\sigma,\tau_\varepsilon} \; \widetilde{\widetilde{g}}_k \;=\; \sum_{l=1}^{3} E_{p,q,\sigma \; \tau_\varepsilon} \; \left(\frac{1}{k} \sum_{n \le k:\, \widetilde{a}_n = l} \widetilde{\widetilde{g}}_k \right)$$

$$\le \sum_{l=1}^{3} q^l \; \text{cav } u^l(p) + \hat{M} \, \varepsilon \qquad \forall \, k > \hat{k}(\varepsilon)$$

So we get the summarizing

<u>Theorem 6.4</u>: If there exist $p_\beta \in P$, $\lambda_\beta \in \mathbb{R}^+$, $\beta = 1,\ldots,m+1$ with

$$\sum_{\beta=1}^{m+1} \lambda_\beta \, p_\beta = 1 \quad \text{and} \quad \sum_{\beta=1}^{m+1} \lambda_\beta \, u^l(p_\beta) = \text{cav } u^l(p) \quad \text{for all } l$$

with $q^l > 0$ then the value of $\Gamma_\infty(p,q)$ exists and

$$v = \sum_{l=1}^{3} q^l \; \text{cav } u^l(p).$$

<u>Remarks</u>: 1) The presuppositions of Theorem 6.4 especially imply that $\text{Ns}^l(p_\beta) \neq \emptyset$ for all l with $q^l > 0$ and $\beta = 1,\ldots,m+1$.

2) We get an optimal strategy of Player II if we define τ_ε for $a_k = 2, 3$ according to the optimal strategies of the games $\Gamma_\infty(p,2)$ and $\Gamma_\infty(p,3)$.

7.) Concluding Comments and Remarks for Interpretation

It could be verified in this paper that the principle of
concavication is important for the mathematical analysis of
all possible sequences of moves for zero-sum-games with
incomplete information. For games with sequences of moves
varying incidentally, we presented a sufficient mathematical
condition under which concealments of advantage in information
do not handicap each other; in these cases the value is
the sum of the values of the games in which sequences
of moves do not vary, weighted by the probabilities of
modality of sequences of moves.

Now there will be made some remarks which illustrate the
difference between these games as shown in Figure 7.1.:

	1		n		∞
1	$v_1^1(p)$	$=$	$v_n^1(p)$	$=$	$v_\infty^1(p)$
	$\text{I}\wedge$		$\text{I}\wedge$		$\text{I}\wedge$
3	$v_1^3(p)$	$\geq$	$v_n^3(p)$	$\geq$	$v_\infty^3(p)$
	$\text{I}\wedge$		$\text{I}\wedge$		$\text{I}\wedge$
2	$v_1^2(p)$	$\geq$	$v_n^2(p)$	$\geq$	$v_\infty^2(p)$

Figure 7.1.: Relation of the values of the games $\Gamma(p,l)$,
$l = 1,2,3$ where the columns $1,n,\infty$ denote the number of
r u n s   and the rows $1,2,3$ the kind of sequence
of moves.

At first the comparison between the players: while the complete informed player at least in the different games with no change in the sequence of moves and infinite repetitions can use essentially equivalent strategies, the only partially informed player - when moving first or simultaneously - has to use essentially different strategies as in games where he is the second to move. When Player II is not the second to move, then he has to try to keep track on Player I's behavior. Then there exists a strategy which is not vulnerable by any bluffs, - Player II only has to deduce his behavior from the observed frequencies of information in a certain manner. This study and observation of the opponents information, the derivation of his own behavior by what happened in advance can be interpreted as a model of a process of optimal learning. If this process took place in this way, a strategy of the informed player leads to success in which he has to pretend with certain (mathematically determined) probabilities to play other games by trying to conceal his advantage of information completely and to do best under these conditions.

Only for games in which the informed player does not have to move first there is an opportunity for him to show up better in games with finite numbers of repetitions than in games with infinite numbers. This demonstrates the relative disadvantage of a more informed player in case that he has to move first; - for, if the informed one has to move first he also has to reveal an optimal rate of information with his first move, there is no positive amount, which can perhaps be received additionally in the beginning.

Finally we come to the comparison between the different sequences of moves: what can be stated intuitively for finite games, is this: drawing first is worse than drawing simultaneously, and this again is worse than being the second to move by knowing what the other has done or at least having got some information from the other. This could be proved in this paper also for infinite games with incomplete information and incidental sequence of moves (c.f. Example 6.1). There are situations resp. games in which real differences occur, which are being expressed by the height of the value.

It is the author's hope that what could be demonstrated partially in this paper - to counterplay the process of optimal concealing and revealing of information mathematically by a smoothing resp. concavication of functions - also can be transfered even to more general situations and games, in particular non zero-sum-games.

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