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Limited Rationality and Structural Uncertainty

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The description of human decision behavior requires theories of limited rationality (Simon 1959, Sauermann-Selten 1962, Selten 1978). An important aspect of limited rationality is the lack of complete overview over all possible states of nature. The term structural uncertainty will be used for this incompleteness of the decision maker's knowledge.

It is the aim of this paper to develop a theory of decision making under structural uncertainty which differs as little as possible from Bayesianism. The principle of expected subjective utility maximization is accepted as a satisfactory description of complete rationality. This view is based on the convincing result of simultaneous axiomization of utility and subjective probability (Savage 1954).

Structural uncertainty is the only aspect of limited rationality which will be considered. This is a rather small step towards a realistic description of practical decision situations but, nevertheless, an important one. Even under the most favorable conditions for the application of Bayesian reasoning, calculations will be based on models whose adequacy cannot be presumed with absolute certainty.

In view of this fact, it seems to be of interest that a small set of plausible axioms is sufficient to characterize the extension of the principle of expected subjective utility maximization to decision situations with structural uncertainty. Moreover, it turns out that the axioms force us to use a very simple rule of dealing with structural uncertainty: The unknown states of nature must be treated as if a constant utility level h were attached to every one of them. h may be interpreted as a parameter expressing the degree of optimism with respect to unknown states of nature.

Prospects: The theory is based on the following conceptualization of a decision situation under structural uncertainty: The decision maker assigns subjective probabilities $p_1,...,p_n$ to n mutually exclusive known states of nature $S_1,...,S_n$, respectively; the sum $p = p_1+ ... +p_n$ may be smaller than 1, since unknown states of nature are expected with probability 1-p. For any possible decision alternative the decision maker knows the utility levels $u_1,...,u_n$ attached to each of the known states of nature $S_1,...,S_n$, but nothing is known on the utilities of unknown states of nature. Thus every possible choice leads to a "prospect"

$$(1) \qquad P = (p_1,...,p_n; u_1,...,u_n)$$

with $0 \leq p_1 \leq 1$ and $p = p_1+ ...+p_n \leq 1$. The probability 1-p of the unknown states of nature may be thought of as the structural uncertainty of the prospect.

The decision maker must be able to make preference comparisons between any two prospects P and Q which may differ with respect to structural uncertainty and the number of known states of nature. The need for such comparisons can easily arise in connection with multilevel decision problems where earlier choices influence later states of the world.

The decision maker's utility function may be bounded from below or above. Therefore, it is assumed that the values of the utility levels u_i are restricted to an open, closed or half open interval I which contains more than one point; borders at $-\infty$ or $+\infty$ are not excluded.

The set of all prospects P of the form (1) where n is an arbitrary positive integer and $u_i \varepsilon I$ holds for $i = 1,...,n$ is denoted by X. The axioms to be introduced concern the decision maker's weak preference relation $\succsim$ over X. The corresponding strong preference and equivalence relations are expressed by $\succ$ and $\sim$,respectively.

Prospects of the form $(p;u)$ corresponding to situations with just one state of nature will play a special role in the theory. Such prospects are called binary. The set of all binary prospects $(p;u)$ with $0\le p\le 1$ and $u\varepsilon I$ is denoted by Y.

<u>Axiom I (reduction to binary prospects)</u>: Let $P = (p_1,\ldots,p_n;u_1,\ldots,u_n)$ be a prospect in X with $p = p_1+\ldots+p_n>0$; define

$$(2) \qquad u = \frac{1}{p}\sum_{i=1}^{n} p_i u_i$$

Then we have $P\sim(p;u)$

<u>Interpretation</u>: u is the conditional expected utility for the known states of nature in P. It is reasonable to require that the decision maker's preferences should not depend on anything else but this conditional utility u and the structural uncertainty $1-p$.

<u>Axiom II (utility monotonicity)</u>: Let $P = (p;u)$ and $Q = (p;v)$ be two prospects in Y with $p>0$ and $u>v$. Then we have $P\succ Q$.

<u>Notation</u>: Let $P = (p;u)$ and $Q = (q;v)$ be two binary prospects. Imagine a lottery which yields P with probability r and Q with probability $1-r$. The probability that a known state of nature is reached by this lottery is

$$(3) \qquad \bar{p} = rp + (1-r)q$$

assume $\bar{p} > 0$. Then

$$(4) \qquad \bar{u} = \frac{rpu + (1-r)qv}{p}$$

is the conditional expected utility of the lottery for the case that a known state of nature is reached. With this interpretation in mind we use the notational convention

$$(5) \qquad rP + (1-r)Q = (\bar{p},\bar{u})$$

The expression on the left side of (5) will be referred to as well defined for $\bar{p}>0$. For $\bar{p} = 0$ a conditional expected utility cannot be formed.

<u>Axiom III (lottery neutrality)</u>: Let P,Q,T,R be four binary lotteries in Y with

(6) $P \sim T$

and

(7) $Q \sim R$

Then for $0 \leq r \leq 1$

(8) $rP + (1-r)Q \sim rT + (1-r)R$

holds, wherever both sides of (8) are well defined.

<u>Interpretation</u>: Axiom III combines two requirements which are reminiscent of similar postulates in axiomizations of the von Neumann-Morgenstern utility (Luce-Raiffa 1957). One requirement concerns the reduction of a lottery between two binary prospects P and Q to a binary prospect $rP + (1-r)Q$. The second requirement says that in any such lottery between P and Q both P and Q can be substituted by equivalent prospects $T \sim P$ and $R \sim Q$, without changing the preference evaluation. Since for the sake of simplicity we avoid the introduction of lotteries between prospects as formal objects of the axiomatic theory, the two requirements are not formalized directly. Instead of this, an immediate consequence of both of them together is expressed by axiom III. Since both requirements are very natural and near to the Bayesian spirit, an extension of Bayesianism to decision situations under structural uncertainty should have the property of axiom III.

<u>Axiom IV (continuity)</u>: The restriction of the preference relation $\succsim$ to Y can be represented numerically by a continuous function f(p,u) in the sense that $(p;u) \succsim (q;v)$ holds if and only if we have $f(p,u) \geq f(q,v)$.

Interpretation: The conditions for continuous numerical representability are well known (Debreu 1954). The continuity requirement excludes the possibility that $\succsim$ is a lexicographic ordering over the (p,u)-space, a case of only marginal theoretical interest.

The class H: In order to prepare the statement of a theorem we introduce a one parameter family H of preference relations $\succsim_h$ on X which will turn out to be exactly those satisfying axioms I to IV. For every real number h a preference relation $\succsim_h$ is defined as follows: For any two prospects $P = (p_1,\ldots,p_n;\ u_1,\ldots,u_n)$ and $Q = (q_1,\ldots,q_m;\ v_1,\ldots,v_m)$ in X define

$$(9)\qquad u = \begin{cases} \frac{1}{p}\sum_{i=1}^{n} p_i u_i & \text{for } p = p_1 + \ldots + p_n > 0 \\ 0 & \text{for } p = p_1 + \ldots + p_n = 0 \end{cases}$$

and analogously

$$(10)\qquad v = \begin{cases} \frac{1}{q}\sum_{i=1}^{m} q_i v_i & \text{for } q = q_1 + \ldots + q_m > 0 \\ 0 & \text{for } q = q_1 + \ldots + q_m = 0 \end{cases}$$

Then

$$(11)\qquad P \succsim_h Q$$

holds if and only if we have

$$(12)\qquad p\mu + (1-p)h \geq qv + (1-q)h$$

The parameter h can be interpreted as the utility evaluation of the unknown states of nature. All preference relations in H extend the principle of subjective expected utility maximization to situations under structural uncertainty in essentially the same way. The unknown states of nature are treated as if they were known states of nature with utility h. With this interpretation in mind we shall call

$$(13)\qquad U_h(P) = pu + (1-p)h$$

the expected utility of P with respect to $\succsim_h$. Obviously $P \succsim_h Q$ holds if and only if we have $U_h(P) \geq U_h(Q)$.

Theorem: A preference relation "$\succsim$" on X satisfies axioms I to IV, if and only if it is in class H.

In order to prove the theorem we shall first prove lemma 1 which covers the if-part of the theorem.

Lemma 1: The preference relations in H satisfy axioms I to IV.

Proof: Obviously axiom I holds by definition. II is an immediate consequence of (12). The continuous numerical representation required by IV is given by (13). It remains to show III. We observe that (3) and (4) have the following consequence

$$
\begin{aligned}
(14) \qquad U_h(rP+(1-r)Q) &= u\bar{p} + (1-\bar{p})H \\
&= rpu + (1-r)qv + (1-\bar{p})h \\
&= r(pu+(1-p)h) + (1-r)(qv+(1-q)h)
\end{aligned}
$$

This yields

$$
(15) \qquad U_h(rP+(1-r)Q) = rU_h(P)+(1-r)U_h(Q)
$$

in view of (6) and (7) we have $U_h(P) = U_h(T)$ and $U_h(Q) = U_h(R)$. This together with (15) and (16) yields (8).

Regular binary prospects: Let I_o be the interor of interval I to which the utility levels are restricted. According to our assumptions I_o is non-empty. A binary prospect $P = (p;u)$ is called regular, if we have $p>0$ and $u \varepsilon I_o$. Let Y_o be the set of all regular binary prospects in Y. In order to prove the only-if part of the theorem it will be useful to investigate preference relations on Y_o. If in axioms II, III, and IV the symbol Y is replaced by Y_o we receive axioms II_o, III_o, IV_o which can be applied to such preference relations. Once it will have been established that a preference relation on Y_o satisfying II_o, III_o, and IV_o coincides with the restriction of a preference relation in H to Y_o it will be possible to extend the result first to Y and then to X in order to prove the only-if part of the theorem.

The (p,pu)-representation: For the investigation of preference relations over Y_o it will be convenient to represent binary prospects $P = (p;u)$ as points (p,pu) in a rectangular coordinate system with p on the horizontal and pu on the vertical axis. Let Z be the set of all points (p,pu) representing prospects $P = (p;u)$ in Y_o and let φ be the mapping from Y_o onto Z defined by

$$(16) \qquad \varphi((p;u)) = (p,pu)$$

Obviously φ is a one-to-one mapping from Y_o onto Z. Moreover, both φ and its inverse are continuous.

In the following we shall first look at some properties of φ and Z. One of the advantages of working with Y_o rather than Y is the fact that for $P,Q\varepsilon Y_o$ the expression $rP+(1-r)Q$ is always well defined for $0\leq r\leq 1$.

Lemma 2: The mapping φ has the following property

$$(17) \qquad \varphi(rP+(1-r)Q) = r\varphi(P)+(1-r)\varphi(Q)$$

for $0\leq r\leq 1$ and $P,Q\varepsilon Y_o$

Proof: Equation (4) yields

$$(18) \qquad \overline{pu} = rpu+(1-r)qv$$

This together with (3) leads to (18).

Lemma 3: The set Z is convex.

Proof: Obviously for $P,Q\varepsilon Y_o$ also $rP+(1-r)Q$ is in Y_o. Therefore lemma 3 is an immediate consequence of lemma 2.

Indifference curves: Consider the equivalence classes generated by a preference relation over Y_o. The image of an equivalence class under the mapping φ will be called an indifference curve. It will be important to look at the shape of indifference curves.

Lemma 4: Let C be an indifference curve generated by a preference relation on Y_o which satisfies II_o, III_o and IV_o. Then C is convex subset of a straight line L in the (p,pu)-plane. Moreover C is relatively closed in Z.

Proof: Let P and Q be two prospects in the equivalence class corresponding to C. The application of axiom III_o to special case P = T = R together with lemma 2 and lemma 3 shows that the straight line segment connecting $\varphi(P)$ and $\varphi(Q)$ belongs to C. Therefore C is convex.

It follows by axiom II_o that for every p there can be at most one point (p,pu) in C, since one of two different regular binary prospects with the same p must be preferred to the other. Therefore C must be a convex subset of a straight line L.

In view of axiom IV_o the equivalence class corresponding to C can be described as the set of all prospects $(p;u)\varepsilon Y_o$ for which a continuous function f(p,u) is equal to a certain constant. Therefore this equivalence class is relatively closed in Y_o. Since φ is continuous C is relatively closed in Z.

Lemma 5: Under the assumptions of lemma 4, the indifference curve C contains at least two points.

Proof: Let (p,pu) be a point in C. Since u is in the interior I^o of I we can find to numbers $v,w\varepsilon I_o$ with $v<u<w$. Obviously the set Z is large enough to permit the construction of a continuous curve D in Z which connects (p,pv) with (p,pw) without touching (p,pu). Let f be a continuous numerical representation as described in IV_o and consider the value which this function assigns to the inverse φ-image of points in the (p,pu)-plane. In view of II_o,IV_o this value is lower for (p,pv) than for (p,pu) and lower for (p,pu) than for (p,pw). It follows by the continuity of f and the inverse of φ that somewhere on D

there must be a point (q,qt) for which this value is the same as for (p,pu). With (q,qt) we have found a second point in C.

Lemma 6: Under the assumptions of lemma 4 the indifference curve C is the intersection of Z with a straight line in the (p,pu)-plane.

Proof: In view of lemma 5 the straight line L in lemma 4 is uniquely determined. Suppose that L contains a point b which does not belong to C. Since Z is convex and C is relative closed in Z, there must be a point in C, say ε, which is nearer to b than all other points in C. Let d be a point in C which is different from c. In view of lemma 5 such a point can be found. Obviously b,c and d are arranged on L in this order in the sense that c is between b and d. Therefore c is a convex linear combination of b and d. For some r with $0<r<1$ we have

(19) $c = rb + (1-r)d$

Let P,R,Q be the inverse φ-images of b,c,d,respectively. In view of lemma 2 we have

(20) $R = rP + (1-r)Q$

Since c and d are in C we have $R \sim Q$. It follows by axiom III_o that $T \sim R$ holds for

(21) $T = rP + (1-r)R$

Define $e = \varphi(T)$. In view of lemma 2 equation (21) yields

(22) $e = rb + (1-r)c$

consequently e is a point on L which lies between b and c. On the other hand, it follows by $T \sim R$ that e belongs to C. Therefore C contains a point which is nearer to b than c. This is a contradiction. Consequently we have $C = Z \cap L$.

Lemma 7: Under the assumptions of lemma 4 let C and D be two different indifference curves. Then C and D are parallel.

Proof: Let c be a point in C and let d be a point in D. Define

(23) $$b = \frac{1}{2} c + \frac{1}{2} d$$

Let B_1 be the set of all points $e \varepsilon Z$ of the form

(24) $$e = \frac{1}{2} c + \frac{1}{2} g \text{ with } g \varepsilon D$$

and let B_2 be the set of all points $e \varepsilon Z$ of the form

(25) $$e = \frac{1}{2} g + \frac{1}{2} d \text{ with } g \varepsilon C$$

Obviously B_1 and B_2 are intersections of Z with straight lines L_1 and L_2 intersecting in b, such that B_1 is parallel to D and B_2 is parallel to C.

Since in view of lemma 5 both C and D contain at least two points, we can conclude that both B_1 and B_2 contain at least two points. Therefore one of both sets B_1 and B_2 cannot be contained in the other unless C and D are parallel. Let B be the indifference curve with $b \varepsilon B$. We shall argue that both B_1 and B_2 are subsets of B. In view of lemma 6 this cannot be true unless C and D are parallel.

Let P be the inverse φ-image of c. The inverse φ-images of points in B_1 have the form

(26) $$T = \frac{1}{2} P + \frac{1}{2} Q$$

where Q is an inverse φ-image of a point in D. It follows by axiom III_o that all prospects of the form (26) are equivalent. Therefore B_1 is a subset of B. The proof for $B_2 \subset B$ is analogous.

Lemma 8: A preference relation on Y_o satisfies axioms II_o, III_o, and IV_o if and only if it is the restriction of a preference relation in H to Y_o.

Proof: The if-part is an immediate consequence of lemma 1. Consider a preference relation $\succsim$ on Y_o which satisfies II_o, III_o, and IV_o. In view of lemmata 4,5,6 and 7 it is clear that the indifference curves are straight line segments with a common slope. It follows by II_o that indifference curves cannot be vertical. Let h be the common slope. Then an indifference curve can be described the set of all points $(p,pu) \epsilon Z$ with

$$(27) \qquad pu = hp + \beta$$

where β is a parameter characterizing the indifference curve. Obviously the indifference curves (27) are exactly those generated by the restriction of $\succsim_h$ to Y_o.

It remains to be shown that the order of preference between indifference curves is the same as that generated by $\succsim_h$. Let f be a continuous numerical representation as described by IV_o. Let $g(\beta)$ be the value of f attached to the inverse φ-images of the points on the indifference curves characterized by β. Let J be the set of all β which belong to indifference curves. It follows by the definition of Y_o that J is an open intervall (this does not exclude $-\infty$ or $+\infty$ as borders). We have to show that $g(\beta)$ is strictly increasing in J.

Let $P = (p;u)$ be a prospect in Y_o with $f(p,u) = g(\beta)$. For sufficiently small δ the prospect $Q = (p,u+\delta)$ will be in Y_o, too. $\varphi(Q)$ is in the indifference curve characterized by $\beta' = \beta+p\delta$. Moreover we have $P \succ Q$ and therfore $g(\beta') > g(\beta)$. Therefore for sufficiently small ε we always have $g(\beta+\varepsilon)>g(\beta)$. This shows that g is strictly increasing everywhere in J.

The (p,u)-representation: It is natural to think of binary prospects $(p;u)$ as points in a rectangular coordiate system with p on the horizontal and u on the vertical axis. This way of representing binary prospects or sets of such prospects is called (p,u)-

representation. It is convenient to identify the prospects (p;u) with the corresponding points in the (p,u)-plane. Thus an equivalence set generated by a preference relation may be thought of as a point set which is also called a (p,u)-indifference curve.

It is our intention to extend the result of lemma 8 to preference relations over Y. For this purpose we look at the (p,u)-indifference curves of the restriction of $\succsim_h$ to Y. For the sake of notational simplicity we drop the index h of U_h in (13). The (p,u)-indifference curves have the following shape

(28) $$u = h + \frac{U-h}{p} \qquad \text{for } U \neq h$$

(29) $$u = h \text{ or } p = 0 \qquad \text{for } U = h$$

The curves (28) are rectangular hyperbolic with the lines $p = 0$ and $u = h$ as asymptotes which are convex for $U>h$ and concave for $U<h$. The (p,u)-indifference curves are the non-empty intersections of Y with curves of the form (28) and (29).

<u>Lemma 9</u>: A preference relation on Y satisfies axioms II, III, and IV if and only if it is the restriction of a preference relation in H to Y.

<u>Proof</u>: As in lemma 8 it is sufficient to prove the only-if part. Let $\succsim$ be a preference relation on Y satisfying II, III, and IV and let $\succsim_h$ be the preference relation whose restriction to Y_o agrees with the restriction of $\succsim$ to Y_o according to lemma 8. Let f be a continuous numerical representation of $\succsim$ according to IV.

Consider the (p;u)-indifference curves for the restriction of $\succsim$ to Y_o. Each of these curves is characterized by a value of U and a value of f. Obviously the value of f is a strictly increasing function $\Phi(U)$ of the value U. Moreover Φ must be continuous since otherwise f would have to be discontinuous, too. Within Y_o the utility index U varies over an open intervall K which is the region for which Φ has been defined. Within Y the utility index U varies over an intervall $\bar{K}$ which is contained in the closure of K since Y is contained in the closure of Y_o. We shall show that we can continuously extend the definition of Φ to $\bar{K}$.

Let B be a border point of K which belongs to $\bar{K}$. Obviously b must be the utility level U of at least one point $P \epsilon Y$. Let M be the set of all points P of this kind. Since Y is in the closure of Y_o, for every $P \epsilon M$ a sequence $P_1, P_2, \ldots$ with $P_i \epsilon Y_o$ can be found which converges to P. Let $U_1, U_2, \ldots$ be the sequence of utility levels corresponding to $P_1, P_2, \ldots$ according to (13). Obviously $U_1, U_2, \ldots$ converges to B. In view of the continuity of f the sequence $f(P_1)$, $f(P_2), \ldots$ converges to f(P). Moreover we have $f(P_i) = \Phi(U_i)$. Since Φ is continuous and strictly monotonic in K any sequence $\Phi(U_1), \Phi(U_2), \ldots$ with $U_i \to B$ for $i \to \infty$ behaves in the same way with respect ot its limit , i.e. the limit, if it is finite, is the same for all these sequences. Therefore f(P) assumes the same value for every $P \epsilon M$. The definition of Φ is continuously extended by attaching this value to B. Moreover, it is clear that the extended definition correctly describes the relationship of U-values and f-values for points in Y whose U-levels are not in K. It is also clear that Φ is strictly increasing in $\bar{K}$.

In view of the strict monotonicity of Φ it is sufficient to show that Φ correctly describes the relationship of U-values and f-values for all points of Y. For any point $P \epsilon Y$ a sequence $P_1, P_2, \ldots$ of points in Y_o converging to P can be found. The limit of the corresponding utility levels $U_1, U_2, \ldots$ is the utility level U of P and $f(P_i) = \Phi(U_i)$ converges to $f(P) = \Phi(U)$ for $i \to \infty$. This completes the proof.

<u>Proof of the theorem</u>: It remains to show the only - if part. Let $\succsim$ be a preference relation on Y satisfying axioms I, II, III, and IV and let $\succsim_h$ be the preference relation in H whose restriction to Y agrees with the restriction of $\succsim$ to Y according to lemma 9. Every prospect $P \epsilon X$ is equivalent to the corresponding binary prospect $(p;u) \epsilon Y$ defined in axiom 1. This is true for $\succsim$ as well as for $\succsim_h$. Therefore $\succsim$ and $\succsim_h$ agree everywhere in X.

<u>Remark on a further axiom</u>: Axiom II asserts a monotonicity property with respect to u in binary prospects (p;u). At first glance it may seem to be plausible to require a similar monotonicity property with respect to p in order to express the intuitive idea that it is desirable to reduce structural uncertainty:

<u>Axiom V (probability monotonicity)</u>: Let $P = (p;u)$ and $Q = (q;u)$ be two prospects in Y with $p>q$. Then we have $P \succsim Q$.

Interestingly this axiom is not compatible with the other axioms unless I is bounded from below. It can be seen easily with the help of (13) that we have

(30) $(0;u) \succ_h (1;u)$ for $u < h$

Only if I is bounded from below h can be chosen in such a way that $u \geq h$ holds for every $u \in I$ and (30) cannot occur.

Axiom V involves an extreme pessimism with regard to unknown states of natures. If u stands for the utility of being hanged today it is by no means unreasonable to prefer the complete structural uncertainty of (0;u) to the prospect (1;u) which corresponds to a situation where the dreadful event will arrive with certainty. Therefore, contrary to axiom V, it seems to be more plausible to assume that h is in the interior of I.

<u>Concluding remarks:</u> The result that the decision maker must treat the unknown states of nature as if they were known states of nature with a constant utility level h, is by no means obvious. There are many other rules for the evaluation of prospects under structural uncertainty which may seem to be plausible at first glance. Some possibilities which involve the minimum of the u_i in $P = (p_1,\dots,p_n;\ u_1,\dots,u_n)$ are already excluded by axiom I. Other rules could involve different combinations of u and p in the numerical representation $f(p,u)$ of the preference relation. Interestingly the simple possibility $f(p,u) = u$ of ignoring structural uncertainty is not compatible with the axioms even if it does not involve any obvious violation of the Bayesian spirit.

Generally, one thinks of utilities as the result of a mental process where consequences of imagined events are carefully compared with each other. The utility h which must be attached to unknown states of nature cannot be obtained in this way because these states are not within the reach of our imagination. Nevertheless, a decision maker who wants to obey axioms I to IV must develop a definite attitude towards structural uncertainty. This attitude finds its expression in the constant h.

Finally, it should be emphasized that it seems to be significant that Bayesianism is capable of dealing with at least one aspect of limited rationality. One may try to approach other aspects of limited rationality in a similar fashion even if it is doubtful whether this is the right approach to the problem.

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