Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 94

Wolfram F. Richter

Taxation According to Ability to Pay

April 1980

H. G. Bergenthal

Taxation According to Ability to Pay

Wolfram F. Richter

ABSTRACT:

The classical question how to justify progressive taxation on purely normative grounds is raised within a social choice theoretic framework. Although a fully convincing axiomatic justification is still missing this approach allows 1) to formalize *ability-to-pay* 2) to axiomatize concepts of *equal sacrifice* and 3) to derive extensive statements on progressivity. Finally, a new sacrifice concept is proposed implying "moderate" progressivity for all neo-classical utility functions.

Adress:
Institute of Math. Economics
Postfach 8640
Universität
4800 BIELEFELD 1
F.R.G.

INTRODUCTION

The normative justification of progressive tax schedules
is one of the old - though still unsettled - topics that
makes the core of the theory of public finance. The rele-
vant key-words are *taxation according to ability to pay* and
the infliction of *equal sacrifices*. The theme is so old
and widely discussed that I may safely refrain from re-
producing the vain efforts to justify progressive tax
schedules along these lines. The interested reader is
asked instead to consult Musgrave's (1959) standard work
on public finance which includes a highly recommendable
presentation of the problem.

This paper resumes the question whether progressive taxa-
tion can be justified on purely normative grounds. We
start from the presumption that progressivity should be
a derivative and intrinsic property of all distributions
of tax payments that are socially considered to be equitable
and just. This view allows to conceive the problem as one
of *social choice theory*. More specifically we shall have
to deal with the problem of aggregating   tax
payers' preferences to a socially acceptable outcome. In
the first instance we shall thus discuss axiomatic charac-
terizations of social preference relations, social welfare
functionals, choice correspondences, and the like.

Having determined such socially acceptable aggregation
procedures we shall then investigate their performance
when specifically applied to the allotment of tax shares.
The normative justification of progressive taxation will
thus present itself as a nice exemplary case for the
application of social choice theory.

Benefiting from relatively late findings we shall be able
to derive the following results:

1) The concept of *equal marginal sacrifice* can be axioma-
   tically characterized.

However, this axiomatic characterization will be rejected
for several reasons among which the argument is that this
approach fails to exlpain the role of the ability-to-pay
principle.

2) We shall propose and axiomatize a class of *generalized*
   sacrifice concepts that includes the *equal marginal* (MS)
   and the *equal absolute* (AS) ones as extreme special
   cases.

This uniform treatment is noteworthy in itself as the pre-
vailing literature is apt to view (MS) and (AS) as two in-
comparable concepts being justified on two different grounds:
welfare economics versus distributive justice.

3) Among the considered axioms is one that truely de-
serves the name of an *ability-to-pay axiom*.

This approach thus unambiguously relates the ability-to-
pay *principle* to the *axiomatic* foundation which as a
whole demands the infliction of equal sacrifices.

4) We shall derive extensive statements about progressivity
depending on the utility functions and sacrifice con-
cepts under consideration. It will be surprising to
observe that the question of progressivity is relatively
insensitive to variations of the sacrifice concept.

An important shortcoming of the presented theory should be
mentioned. Although the   class of considered sacrifice
concepts is widened we shall not be able to designate a
specific member that generically, i.e. for reasonably nice
individual utility functions implies "moderate" progressivity.
Moderate progressivity is meant to refer to progressive tax
schedules that show marginal tax rates smaller than one.
The equal marginal sacrifice (MS), e.g., implies immoderate
progressivity as it generically leads to an equalization of
incomes after taxes.

Up to now we have not been speaking about *equal proportional
sacrifice* (PS). The reason is that this concept implicitely
assumes an *informational basis* which is different from that

which is satisfied by (MS) and (AS). The term informational basis (Sen, 1974) relates to the set of assumptions that are met with respect to measureability and interpersonal comparability of individual utilities. We shall neither axiomatically characterize (PS) itself nor the class of all sacrifice concepts that share the informational basis of (PS).

On the other hand we shall propose a new sacrifice concept - different from (AS), (MS), and (PS) - that shows moderate progressivity for *all* utility functions under rather mild conditions. This non-classical sacrifice concept amounts to applying the utilitarian aggregation procedure to suitably normalized utility functions. The normalization is such that the resulting tax distribution is invariant under affine transformations of utilities. The old question is thus settled positively whether some sacrifice concept *exists* at *all* yielding moderate progressivity for any utility function. This aggregation procedure - let it be called *proportional utilitarianism* is intuitively appealing. What is lacking, however, is an axiomatic characterization.

THE MODEL

We explicitely distinguish n tax payers, $N = \{1,..,n\}$, with fixed income vector $y = (y_1,...,y_n)$. Thus individual $i \in N$ disposes of *income* $y_i \in \mathbb{R}_+$ *before taxes*. $i \in N$ derives utility from *income after taxes* $y_i - t_i$ according to $u_i(\cdot)$ which exhibits decreasing positive marginal increments: $u_i' > 0$, $u_i'' \leq 0$. The n *utility functions* are summarized by $u = (u_1,..,u_n) : \mathbb{R} \rightarrow \mathbb{R}^n$.

$t = (t_1,..,t_n)$ denotes a distribution of tax shares. The *set of all feasible tax distributions* is assumed to be

$$X := \{t \in \mathbb{R}^n \mid \Sigma t_i = g, \ t_i \geq 0\}$$

where the tax yield g is fixed by an exogenously determined expenditure side of the budget. The restriction $t_i \geq 0$ renders impossible subsidies to incomes. The scope for redistributional targets is hence rather limited. It is not so much a correction of the personal income distribution that we aim at but an equitable distribution of tax shares according to individual abilities-to-pay.

The objective of the following discussion is first to axiomatically justify specific *choice functions*

$$t^* : \{(u,y)\} \rightarrow X$$

and secondly to investigate their implications on the

marginal tax rates. What we would like to obtain but fail to reach is the following result: There exists some specific choice function t^* which can be axiomatically justified such that for all u, y and $i \in N$ we generically obtain

$$1 > \frac{\partial t_i^*}{\partial y_i} > \frac{t_i^*}{y_i} . \tag{1}$$

The latter inequality defines *progressivity* in a local sense.

As noted before the results will crucially depend on the informational basis we are willing to assume. In accordance with the classical sacrifice concepts (MS) and (AS) we shall require *cardinal unit comparability* (cf. Sen, 1977) if nothing else is stated. This means that

$$t^*(u,y) = t^*(\alpha u + v, y) \tag{2}$$

should hold for any vector $v \in \mathbb{R}^n$ and any positive scalar $\alpha > 0$. As v_i is allowed to differ from v_j the rescaling factor α yet has to be independent of $i \in N$ (2) implies that interpersonal comparisons of welfare *gains* but not of welfare *levels* are permitted.

An Axiomatization of Equal Marginal Sacrifice

There is a well-known result in social choice theory which axiomatically characterizes the utilitarian aggregation procedure when cardinal unit comparability is given. This theorem is outlined below.[1/] It is suggested to interpret X as an arbitrary set of social alternatives evaluated by individuals according to $u_i : X \to \mathbb{R}$. $R(u)$ denotes some hypothetical planner's social preference relation obtained from $u = (u_1, \ldots, u_n)$ by aggregation. Deviating from our more special assumptions on utility functions we let $\mathcal{U}$ be the set of *all* $u : X \to \mathbb{R}^n$ that are *bounded* on the fixed arbitrary set X.

Theorem 1: Let $R : \mathcal{U} \to X \times X$ denote a social welfare functional which means that for all $u \in \mathcal{U}$ $R(u)$ defines some reflexive, transitive and complete relation on X. In accordance with (2) let

$$R(\alpha u + v) = R(u) \quad \text{for all } \alpha > 0, \ v \in \mathbb{R}^n.$$

If furthermore strong Pareto efficiency, anonymity, and Arrow's independence of irrelevant alternatives are met then R must be utilitarian:

$$x^0 R(u) x^1 \quad \text{iff} \quad \sum_{i=1}^{n} u_i(x^0) \geq \sum_{i=1}^{n} u_i(x^1)$$

$(x^0, x^1 \in X)$.

[1/] For a full statement of the assumptions confer d'Aspremont and Gevers (1977).

Applying this theorem to the tax issue suggests to maximize the sum total of utilities derived from incomes after taxes:

$$\max \sum u_i(y_i - t_i) \qquad \text{for} \quad t \in X. \qquad (3)$$

Let $t^* = t^*(u,y)$ be an optimal solution. To rule out boundary problems we assume $t^* \gg 0$ (i.e. $t_i^* > 0$ for all $i \in N$). Hence no individual goes without paying. Then (3) is equivalent to

$$u_i'(y_i - t_i^*) = \text{const.} \quad \text{for} \quad i \in N. \qquad (MS)$$

(MS) states the condition which we like to invoke when speaking of *equal marginal sacrifice*. (MS) allows the utility functions to differ among individuals. The more conventional analysis would define (MS) by means of

$$U'(Y - T(Y)) = \text{const.}, \qquad (4)$$

where Y denotes income, T(Y) taxes, and U some utility function applying to all individuals alike. The version (4) might be preferred by some theorists because equality before the law *in practice* demands some unique tax schedule $T(\cdot)$ applying to all individuals and to all incomes. This objective is not compatible with the utilitarian approach to progressive taxation. If tax payers are unequal by their utility functions they should be taxed unequally. The best we can hope in this frame-work is a persoanlized and with respect to income local assertion on progressivity - such as the following:

If we keep $u = (u_1,\ldots,u_n)$ and $y_2,\ldots,y_n$ fixed and
write $U(\cdot)$ for $u_1(\cdot)$ then the tax schedule

$$T(Y) = T_U(Y) := t_1^*((U,u_2,\ldots,u_n),(Y,y_2,\ldots,y_n))$$

(which is *implicitely* defined by (MS) in a neighbourhood
$\mathcal{O}(y_1)$ of y_1) is progressive *at* $Y = y_1$ - or not.

In accordance with this remark let the *equal absolute*
and the *equal proportional sacrifice* be defined by:

$$u_i(y_i) - u_i(y_i - t_i^*) = \text{const. for } i \in N \qquad \text{(AS)}$$

$$\frac{u_i(y_i) - u_i(y_i - t_i^*)}{u_i(y_i)} = \text{const. for } i \in N. \qquad \text{(PS)}$$

It is obvious that t^* defined by (AS) and (MS) but not
by (PS) satisfies (2), cardinal unit comparability.
We postpone the discussion of (PS) to a later stage
and close this section by appraising the above axiomatic
characterization of (MS).

Several points have to be made. On one hand the axiomatic
basis seems to be rather restrictive because (AS) fails
to pass it. Yet (AS) is the classical definition of sacrifice
which probably conveys the intuitively most plausible
notion of distributive justice. On the other hand the
theorem makes no use of informations economists are
readily prepared to exploite. The condition put on $\mathcal{U}$ is
surprisingly weak. The theorem makes no use of the fact

that utility functions display positive or decreasing
marginal increments.

Another point must be raised against the above axiomatization.
It fails to explain the role of ability-to-pay. There is
no axiom which reminds of this principle classically so
important to the whole discussion.

If all utility functions are identical and if marginal
utility strictly decreases ($u_i'' > 0$) then (MS) is
equivalent to $y_i - t_i^*$ being constant for all $i \in N$.
This radical tendency to eqalitarianism will hardly meet
broad social agreement. However, it might be debatable
whether the reasons ought to be seen on normative grounds
or preferably somewhere else. In case that equal net in-
comes would never be socially acceptable on purely normative
grounds theorem 1 were falsified and we consequently had
to reject one of its underlying axioms.

As a matter of fact the next characterization of sacrifice
concepts will disregard Arrow's axiom of independence of
irrelevant alternatives. It is the only axiom among the
ones stated by theorem 1 which is not passed by (AS).
According to (AS) the utility vector of incomes before taxes defines
a natural reference point becoming *relevant* to the pair-

wise social ranking of alternative tax distributions.

An Axiomatically Determined Class of Generalized Sacrifice Concepts

The set of feasible tax distributions X will be mapped into the utility space $\mathbb{R}^n$:

$$X \rightarrow \bar{V} := V(u,y) := \{v \in \mathbb{R}^n \mid \exists t \in X \; \forall i \in N : v_i = u_i(y_i - t_i)\}.$$

Instead of determining an equitable choice t^* among the "objective" social alternatives X we shall now focus on the "subjective" valuation by the concerned individuals. The following theorem, shortly named *norm-theorem*, aims at the axiomatic characterization of just and equitable choices in situations for which $\bar{V}$ is generic. One might think of a some hypothetical ethical planner who chooses $v^* = (u_i(y_i - t_i^*))_{i \in N}$ among the feasible distributions of utility levels. The equitable distribution of tax shares t^* is then derived from v^* as inverse image.

Note that theorems 1 and 2 differ with respect to the kind of information which is assumed to be known of the planner. The difference might be better known from consumer's theory. We can explain consumer's - or as in our case: planner's - behaviour on the basis of a given utility function or -

in accordance with the revealed preference approach - on the basis of some demand function.

As the $u_i(\cdot)$ are assumed to be concave $\bar{V}$ is *convex from above* which means that

$$\forall\ v^0, v^1 \in \bar{V}\ \exists\ v \in \bar{V} :\quad v \geq \alpha v^1 + (1-\alpha)v^0 \quad (\alpha \in [0,1]).$$

As tax shares are non-negative the utility derived from income before taxes is distinguished.

$$\bar{u}(\bar{V}) \quad \text{with} \quad \bar{u}_i(\bar{V}) := \sup_{v \in \bar{V}} v_i = u_i(y_i)$$

will be called *bliss-* or *ideal-point*.

All informations which for the tax issue are *assumed* to be relevant are summarized by

$$(\bar{V},\ \bar{u}(\bar{V}),\ C(\bar{V})) \tag{5}$$

where $C(\bar{V}) := v^*$ denotes the planner's choice. The informational structure of (5) will be generalized to arbitrary feasibility sets $V \subseteq \mathbb{R}^n$.

A0) Let $(\mathbb{V}, \bar{u}, C)$ be such that

1) $V \subseteq \mathbb{R}^n$ is convex from above whenever $V \in \mathbb{V}$;

2) $\bar{u} : \mathbb{V} \to \mathbb{R}^n$ is a bliss-point mapping, i.e.

$$\bar{u}(V) \geq v \quad \text{for all} \quad v \in V \in \mathbb{V};$$

3) $C : \mathbb{V} \Rightarrow \mathbb{R}^n$ is a choice correspondence, i.e.

$$\emptyset \neq C(V) \subseteq V \quad \text{for all} \quad V \in \mathbb{V}.$$

A1) $(\mathbb{V},\bar{u},C)$ satisfies *cardinal unit comparability*, i.e.

for all $\alpha \in \mathbb{R}$, > 0 and $v \in \mathbb{R}^n$

0) $\alpha V + v \in \mathbb{V}$ whenever $V \in \mathbb{V}$;

1) $\bar{u}(\alpha V + v) = \alpha \bar{u}(V) + v$;

2) $C(\alpha V + v) = \alpha C(V) + v$.

Theorem 2: (Norm-theorem, Richter, 1979) Let $(\mathbb{V},\bar{u},C)$

satisfy A0,1) and some further axioms which will

be discussed at a later stage. Then there is

some norm $\|\cdot\|$ in $\mathbb{R}^n$ such that

$$C(V) = C^{\|\cdot\|}(V) \quad \text{for all} \quad V \in \mathbb{V}$$

where $\bar{v} \in C^{\|\cdot\|}(V)$ iff $\|\bar{u}(V) - \bar{v}\| = \min_{v \in V} \|\bar{u}(V) - v\|$.

The message of the norm-theorem is such that if the planner

does not only want to settle one special tax conflict

in a just and equitable fashion but is confronted with

similar problems again and again then he should proceed

as follows: Fix some arbitrary norm - once and for all -

and apply it to all occuring conflicts in the same way.

Minimize the distance from the bliss-point to the Pareto

surface. This procedure is the only one which guarantees

distributive justice and logic consistency.

Before analysing the implications of the above theorem

for the tax issue let us consider some prominent norms in $\mathbb{R}^n$.

For $u \in \mathbb{R}^n_+$ define

$$\|u\|_p \; := \; \begin{cases} \max_{i \in N} u_i & \text{if} \quad p = +\infty \\[2mm] \min_{i \in N} u_i & \text{if} \quad p = -\infty \\[2mm] \prod_{i \in N} u_i & \text{if} \quad p = 0 \\[2mm] \sqrt[p]{\sum_{i \in N} u_i^p} & \text{if} \quad p \in \mathrm{I\!R}, \; \neq 0. \end{cases}$$

$\|\cdot\|_p$ defines a norm in $\mathrm{I\!R}^n$ iff $p \geq 1$. These norms are called p-norms. The reason why we defined $\|\cdot\|_p : \mathrm{I\!R}_+^n \to \mathrm{I\!R}_+$ also for $p < 1$ is that we like to point out a difference which lies between the determination of equitable tax distributions and the measurement of income inequality. According to Atkinson (1970)

$$I_p(u) \; := \; 1 - \|u\|_p / \|u\|_1 \qquad \text{(all } p \leq 1)$$

defines an important class of indeces for measuring the inequality of income distributions. Note that the domaine of parameters $\mathrm{I\!R} \cup \{\pm\infty\}$ falls into two parts. One is relevant for measuring the distribution of "goods" like income and one for measuring the distribution of "bads" like taxes. The border line, $p = 1$, is marked by utilitarianism.

It might be temptative to determine just tax shares by considering the impact on the personal distribution of incomes. It is highly suggestive to define a tax distribution as fair if the income distribution shows a certain socially desirable change. The decomposition of the domaine of

parameters obtained above is a strong indication that
such procedure might be inadequate - at least from a
normative point of view. The formal calculus draws a
sharp line between the purely fiscal and the purely
re-distributional function of taxation.

Let us now apply the norm-theorem to determine equitable
tax shares. Fix some norm $\|\cdot\|$ and solve the minimization

$$\left\{ \begin{array}{l} \|(u_i(y_i) - u_i(y_i - t_i))_{i \in N}\| \to \min , \\ \Sigma t_i = g , t_i \geq 0 . \end{array} \right\} \quad (6)$$

Let $t^* = t^{\|\cdot\|} = t^{\|\cdot\|}(u,y)$ denote an optimal solution.
For simplicity of presentation let us assume throughout
the following discussion $t^{\|\cdot\|}(u,y) \gg 0$.

Trivial calculations show that (6) is equivalent to (MO)/
(AO) in case that $\|\cdot\|$ equals $\|\cdot\|_1 / \|\cdot\|_\infty$. (MO) and (AO)
actually are the first-order conditions of the corres-
ponding minimizations. Under the stated assumptions these
first-order conditions are necessary as well as sufficient.
(See below.) The classical sacrifice concepts thus turn
up as special cases in a uniform axiomatically justified
theory. Note that the nature of the norm-theorem does not
allow to discriminate against any p-norm. There is no
indication that the classical sacrifice concepts (AO) and
(MO) define preferable standards of equity and distributive
justice. It is merely accidental that the 1-norm (as mini-

mization) and the ∞-norm (as first-order condition) allow more suggestive interpretations.

Let us now consider some arbitrary norm which is twice continuously differentiable in the interior of IR_+^n and which there has non-negative partial derivatives: $\partial_i \| \cdot \| \geq 0$. The latter assumption is easily shown to hold whenever $C(\cdot)$ selects Pareto efficient utility payoffs.

Introducing Lagrangeian multipliers we obtain
$$\partial_i \|(u_i(y_i) - u_i(y_i - t_i^{\| \cdot \|}))_{i \in N} \| \, u_i'(y_i - t_i^{\| \cdot \|}) = \text{const.} \quad (7)$$
as the first-order conditions. As the utility functions $u_i(\cdot)$ are concave and because of $\partial_i \| \cdot \| \geq 0$ $\|(u_i(y_i) - u_i(y_i - \cdot))_{i \in N}\|$ is convex in $t \gg 0$. From consumer's theory it is known that the first-order condition is sufficient if some convex (concave) function is minimized (maximized) with respect to the "budget constraint" $\Sigma t_i = g$, $t \gg 0$. The preceding discussion suggests to call (7) $\| \cdot \| - \textit{sacrifice}$. This concept generalizes the classical equal absolute and equal marginal sacrifices in a most natural way. The $\| \cdot \|$-sacrifice uniformly refers to the first-order condition of some minimization parametrized by $\| \cdot \|$.

On Progressivity

Having determined equitable tax distributions which are suggested from a social choice theoretic point of view we now turn to the question of progressivity. For this purpose the implicite function theorem is applied to condition (7). The employed notation

$$U(\cdot) := u_1(\cdot) \ , \quad T_U^{\|\cdot\|}(Y) := t_1^{\|\cdot\|}(u,(Y,y_2,..,y_n))$$

is meant to express that individuals $2,..,n$ have exogenously fixed incomes and utility functions. We focus on the amount of taxes individual 1 has to pay because of his income y_1 and his utility function $U(\cdot)$. Due to the very nature of the implicite function theorem the derived statements only obtain local validity.

Let it be recalled that the tax schedule $T(\cdot)$ is called *progressive* (respectively *regressive* or *proportional*) at y_1 if for $Y = y_1 \ \frac{d}{dY} \frac{T(Y)}{Y} > 0$ (respectively < 0 or $=0$). As Samuelson noted before (1947, p.227) the key to progressivity is the *elasticity of marginal utility*[2]

$$E_U(Y) \ := \ \frac{Y \, U''(Y)}{U'(Y)} \ .$$

Note the following elementary properties:

$$E_U(Y) \leq 0 \quad \text{whenever} \quad Y \geq 0.$$

---

[2] Atkinson (1970) interprets E_U as a measure of *proportional inequality aversion*.

$$E_U = E_{\alpha U + \beta} \quad \text{for all } \alpha, \beta \in \mathbb{R};$$

$$E_{\log} = -1, \quad E_{p\sqrt{}} = \frac{1}{p} - 1 \quad \text{for } p \in (0, \infty);$$

$$E_{U^1}(Y) < E_{U^2}(Y) \quad \text{iff some } \bar{U} \text{ exists with } \bar{U}' > 0,$$
$$\bar{U}'' < 0 \quad \text{such that } U^1(Y) = \bar{U}(U^2(Y)).$$

(Cf. Pratt, 1964, p.128.)

Theorem 3: (Samuelson, 1947, p.227)

Case $\|\cdot\| = \|\cdot\|_\infty$. Put $T_U := T_U^{\|\cdot\|_\infty}$,

$\emptyset := (y_1 - T_U(y_1), y_1)$.

a) $E_U(Y) < -1$ for all $Y \in \emptyset$ implies progressivity

of T_U at y_1;

b) $E_U(Y) = -1$ for all $Y \in \emptyset$ implies proportionality;

c) $E_U(Y) > -1$ for all $Y \in \emptyset$ implies regressivity.

For a proof see appendix AI. Theorem 3 can easily be extended to an arbitrary p-norm.

Theorem 4: Case $\|\cdot\| = \|\cdot\|_p$, $p \in (1, \infty)$. The above assertion

a) holds without further restriction;

c) holds if $E_U(Y) > \frac{1}{p} - 1$ for all $Y \in \emptyset$.

For a proof see AII.

It might be surprising to note the uniformity of results we obtain for alternative p-norms. Appropriate additional assumptions allow some extensions to more general norms. However, the tendency is clear enough such

that we can safely refrain from going into more technical details.

By appropriate interpretation theorem 4 may be stated for p including 1 and ∞. As $E_U(Y) \leq 0$ generally holds the condition of c) can never be met for $p = 1$. This observation is in line with the fact that the equal marginal sacrifice always leads to (maximal) progressivity. The classical sacrifice concepts are both obtained as extreme special cases ($p = 1$ and $p = \infty$) of the class of p-norm sacrifices. However, with regard to the contents (MS) is an isolated boundary case. It is the only p-norm sacrifice which never leads to regressivity. On the other hand (AS) admits the strongest assertions which are uniformly approached by p-norm sacrifices for $p \rightarrow \infty$.

There is one severe shortcoming of the results obtained so far. Even in the class of generalized $\| \cdot \|$-sacrifices no example can be made out that definitely leads to moderate progressivity independently of the specific utility functions under consideration. This is why we have to extend the analysis below and introduce a further class of generalized sacrifice concepts.

THE ROLE OF ABILITY TO PAY

It would certainly lead to far to formulate and discuss
all assumptions in detail which allow to prove the norm-
theorem. The interested reader is asked instead to confer
the original text (Richter, 1979). Let us only point out
some structural relationship to the theory of revealed
preference. From that theory the question is well-known
which conditions allow to construct some utility function
"rationalizing" some pre-given demand function. This
question mathematically involves the problem of integrating
vector fields. This makes it necessary to introduce an
integrability condition such as the *strong axiom of revealed
preference*.

In our context we have to find some "social utility function"
$\| \cdot \|$ which allows to "rationalize" the planner's choice
behaviour revealed by $C(\cdot)$. The strong axiom of revealed
preference guarantees logic consistency of finite sequences
of social choices.

Among the assumptions underlying the norm-theorem there
is one which is highly worthy of special notice. It allows
to be intepreted in the spirit of *ability-to-pay*. This
axiom is   called *weighting equity* as it shows strong

similarity to one which has so been called by Sen (1974, p. 399).

<u>Weighting equity axiom</u> (WE)

Let $\bar{v} \in C(V)$ and $\lambda \cdot \bar{v} = \min \lambda \cdot V$ for some $\lambda = \lambda^V \in IR_+^n$, $\neq 0$. Then $\lambda_j \geq \lambda_k$ whenever $\bar{u}_j(V) - \bar{v}_j > \bar{u}_k(V) - \bar{v}_k$ $(j,k \in N)$.

$\bar{v} \in C(V)$ states that $\bar{v}$ is the planner's choice for the environment V. This choice can be justified by minimizing the social welfare function $w(v) = \lambda \cdot v$ with respect to V. λ_i measures the social weight attached to the welfare of $i \in N$. The weight vector λ is endogenously determined and thus depends on the specific environment under consideration. According to our informational assumptions the interpersonal comparison of welfare gains - or losses - are permitted. More specifically the comparison of the hypothetical welfare losses $\bar{u}_i(V) - \bar{v}_i$ is feasible and even more suggestive. Individuals make comparisons relative to their bliss-points. Suppose, now, that j is worse off in comparison to k: $\bar{u}_j(V) - \bar{v}_j > \bar{u}_k(V) - \bar{v}_k$. Then j cannot complain that his personal interests were not sufficiently safeguarded by the planner's choice. $\lambda_j \geq \lambda_k$ tells him that the social valuation of his welfare with respect to the environment V has been at least as high as the one of $k \in N$.

The weighting equity axiom is formulated for arbitrary feasibility sets $V \in \mathbb{W}$. If specialized to tax cases a further reaching interpretation is possible. The weighting equity can then be related to the ability-to-pay principle.

<u>Definition:</u>

a) The tax distribution $t^* \in X$ meets *ability-to-pay* (ATP) if $u_k(y_k) - u_k(y_k - t_k^*) \leq u_j(y_j) - u_j(y_j - t_j^*)$ whenever $y_k \leq y_j$.

b) t^* *preserves the order of incomes* (OP) if $y_k - t_k^* \leq y_j - t_j^*$ whenever $y_k < y_j$.

These definitions are straight forward and need no further comment. The ability-to-pay axiom formalizes the idea traditionally attached by public finance to the equally named principle. On the other hand the order of incomes will be preserved whenever there is a tax schedule with marginal tax rates less or equal than one uniformly applying to all individuals. (OP) is a minimal requirement of vertical distributive equity. Note that horizontal equity ($y_j = y_k$ implying $y_j - t_j^* = y_k - t_k^*$) would only follow if (OP) were enforced to $y_j \leq y_k$ implying $y_j - t_j^* \leq y_k - t_k^*$.

The next remark directly relates the ability-to-pay principle to the axiomatic foundation of the generalized sacrifice concepts.

Remark: Let u_j'' , $u_k'' < 0$.

a) If t^* preserves the order of incomes (OP) and satisfies the ability-to-pay axiom (ATP) then weighting equity is met.

b) The reversal - (WE) implying (ATP) and (OP) - holds for identical utility functions u_j and u_k.

For a proof see AIII.

THE PROPORTIONAL UTILITARIAN SACRIFICE

As has been announced in the introduction a sacrifice concept will be proposed that implies *moderate progressivity for all utility functions under consideration*. The defect is that I do not know of any truely convincing normative justification. Actually, to be precise, this new sacrifice concept admits an axiomatization on some *meta-level* as a "compromise" on Nash's and Kalai-Smorodinsky's solutions for n-person bargaining problems (Richter, 1980). This axiomatization will convince the game theorist though probably not the theorist in public finance. The latter might be expecting axioms that have normative strength in themselves and do not recur to different solution concepts of general bargaining theory.

The further discussion will therefore be confined to some heuristic motivation. We start from the equal proportional sacrifice (PS) and note that the implicitely defined $t^*(u,y)$ is invariant under linear transformations of utilities:

$$t^*(u,y) = t^*((\alpha_i u_i)_{i \in N}, y) \quad \text{for all} \quad \alpha_i \neq 0.$$

One might wonder that t^* is not invariant under affine transformations. The more transformation invariance is given the less informational requirements have to be met. Hence there is good reason to alter (PS) slightly:

$$\frac{u_i(y_i) - u_i(y_i - t_i^*)}{u_i(y_i) - u_i(0)} = \text{const.} =: k \qquad (8)$$

This is no real alteration if the assumption is correct that the classics tacitely used to visualize utility functions as passing through the origin $(u(0) = 0)$. The advantage of (8) is to imply invariance under affine transformations:

$$t^*(u,y) = t^*((\alpha_i u_i + \beta_i)_{i \in N}, y) \quad \text{for all} \quad \alpha_i, \beta_i \in \mathbb{R}, \ \alpha_i \neq 0.$$

Furthermore (8) sheds more light on the very difference between (AS) and (MS) on one hand and the equal proportional sacrifice on the other. (PS) - and more explicitely (8) - takes into regard the existence of an "*anti-bliss-point*".

To make this more precise consider the altered set of feasible tax distributions

$$X^0 := \{t \in \mathbb{R}^n \mid \Sigma t_i = g, \ 0 \leq t \leq y\} \ .$$

The additional restriction $t \leq y$ introduces zero income as "subsistence level". A zero income after taxes is the worst which can happen to individuals. Hence in analogy to the bliss-point call $\underline{u} := u(0)$ *anti-bliss-point*. I hesitate to call $\underline{u}$ "threat-point" - which would seem to be suggested by game theory - as our institutional assumptions specify no threats.

Writing v_i^* for $u_i(y_i - t_i^*)$ and assuming $v^* \gg \underline{u}$ (8) can be transformed into

$$\bar{u} - v^* = k(\bar{u} - \underline{u}) \iff \bar{u} - v^* = \frac{k}{1-k}(v^* - \underline{u}) \ .$$

The latter formula is taken to define

$$\frac{u_i(y_i) - u_i(y_i - t_i^*)}{u_i(y_i - t_i^*) - u_i(0)} \ = \ \text{const.} \quad (i \in N) \qquad (PS^0)$$

as *adapted equal proportional sacrifice* (PS^0). Consider now the following minimizations:

$$\left\| \left(\frac{u_i(y_i) - u_i(y_i - t_i)}{u_i(y_i - t_i) - u_i(0)} \right)_{i \in N} \right\| \to \text{min} \quad \text{where} \quad t \in X^0 .$$

Let $t^{\|/\|} = t^{\|/\|}(u,y)$ be optimal. $t^{\|/\|}_\infty$ satisfies (PS^0) which is the first-order condition of the corresponding minimization if $0 \ll t^{\|/\|}_\infty \ll y$. The similarity to $\{t^{\|\cdot\|} \mid \|\cdot\|\}$ suggests to analyse $\{t^{\|/\|}_p \mid p \in [1,\infty]\}$. Unfortunatelly there is little evidence that similar results to theorems 3 and 4 can be obtained.

However, $t^{\|/\|_1}$ deserves closer inspection. Thus consider

$$\sum_{i \in N} \frac{u_i(y_i) - u_i(y_i - t_i)}{u_i(y_i - t_i)} \rightarrow min, \quad t \in X^0 \tag{9}$$

where $u(0) = 0$ by appropriate transformation. As before we make use of the shortened notation: $U(\cdot) = u_1(\cdot)$, $T(Y) = T_U(Y) := t_1^{\|/\|_1}(u,(Y,y_2,..,y_n))$. The first-order condition of (9) turns out as

$$\frac{U(Y)\ U'(Y - T(Y))}{[U(Y - T(Y))]^2} = const. \quad (Y \in \mathcal{O}(y_1)) \tag{PU}$$

Let (PU) be called *proportional utilitarian sacrifice*. It does not allow any straight forward interpretation which may explain why the classics never considered it. However, (PU) enjoys the properties the classics were looking for in vain:

Theorem 5: For all $Y \in \mathcal{O}(y_1)$ and all $U(\cdot)$ such that $U' > 0$, $U'' \leq 0$:

$$1 > \frac{d}{dY} T_U(Y) > \frac{T_U(Y)}{Y}$$

where the right-hand inequality is subject to the condition $0 < {}^{T_U(Y)}/_Y \leq {}^1/_2$.

For a proof see AIV.

The proportional utilitarian sacrifice thus leads to moderate progressivity whenever the average tax rate is not greater than one half. This assertion holds for all utility

functions under consideration. Evaluating (PU) for $U(Y) = \sqrt[p]{Y}$ yields $T_U(Y) = Y - \text{const.} \sqrt[p+1]{Y}$. In particular linear utility ($U(Y) = Y$) leads to an income after taxes which is proportional to $\sqrt{Y}$.

The question which remains open after all is the axiomatic characterization of (PU) or $t^{\|/\|_1}(u,y)$, respectively. There is a game-theoretic axiomatization of (8) going back to Kalai and Smorodinsky (1975; see also Huttel and Richter 1980) However, this axiomatization resorts to some monotony axiom which is difficult to justify when restricted to problems of taxation.

Appendix

A1.

$$E_U(Y) \lesseqgtr -1 \iff \frac{Y U''(Y)}{U'(Y)} \lesseqgtr -1$$

$$\iff 0 \gtreqless U'(Y) + Y U''(Y) = \frac{d}{dY} Y U'(Y) \quad (Y \in \emptyset).$$

Hence $E_U(Y) \lesseqgtr -1$, $t_1 = T_U(y_1)$ imply

$$(y_1 - t_1) U'(y_1 - t_1) \gtreqless y_1 U'(y_1) \quad \text{or}$$

$$\frac{U'(y_1 - t_1) - U'(y_1)}{U'(y_1 - t_1)} \gtreqless \frac{t_1}{y_1} \quad \text{, respectively.}$$

$$(10)$$

The left hand side equals $\frac{d}{dY} T_U$ at $Y = y_1$ if T_U is implicitly defined by

$$U(Y) - U(Y - T_U(Y)) = \text{const.}$$

Hence the assertion follows considering the equivalence of

$$\frac{d}{dY} T_U \gtrless \frac{T_U}{Y} \qquad \text{and} \qquad \frac{d}{dY} \frac{T_U(Y)}{Y} \gtrless 0$$

A II.

Computing (7) for p-norms ($p \in [1, \infty)$) yields

$$(u_i(y_i) - u_i(y_i - t_i))^{p-1} u_i'(y_i - t_i) = \text{const.} \qquad (11)$$

Writing $T_U := T_U^{\|\cdot\|_p}$, $t_1 = T_U(y_1)$, $U(\cdot) := u_1(\cdot)$ and implicitly differentiating (11) yields

$$\frac{d}{dY} T_U(y_1) =$$

$$\frac{(p-1)\, U'(y_1 - t_1)[U'(y_1 - t_1) - U'(y_1)] - [U(y_1) - U(y_1 - t_1)]\, U''(y_1 - t_1)}{(p-1)(U'(y_1 - t_1))^2 - [U(y_1) - U(y_1 - t_1)]\, U''(y_1 - t_1)}$$

Hence $\frac{d}{dY} T_U \gtrless \frac{T_U}{Y} \iff$

$$(p-1)\, U'(Y - T_U)[(Y - T_U)U'(Y - T_U) - YU'(Y)]$$

$$\gtrless (Y - T_U)\,[U(Y) - U(Y - T_U)]\, U''(Y - T_U) \qquad \Longleftrightarrow$$

$$(p-1)[YU'(Y) - (Y - T_U)U'(Y - T_U)]$$

$$\lessgtr - E_U(Y - T_U)[U(Y) - U(Y - T_U)]$$

The latter relation has to be evalueted at $Y = y_1$.

Under the assumptions of a) $E_U(Y) < -1$ for all $Y \in \mathcal{O}$. (10) implies that the left hand side is negative whereas right hand one is non-negative. Hence progressivity is obtained.

Under the assumptions of c) $E_U(Y) > \frac{1-p}{p}$ for all $Y \in \mathcal{O}$. To prove regressivity it is enough to show

$$y_1 U'(y_1) - (y_1 - t_1) U'(y_1 - t_1) > \frac{1}{p}[U(y_1) - U(y_1 - t_1)]$$

or

$$\int_{y_1 - t_1}^{t_1} \frac{d}{dY} Y U'(Y) \, dY > \frac{1}{p} \int_{y_1 - t_1}^{t_1} U'(Y) \, dY$$

The latter inequality holds if the same inequality is true for the integrands:

$$U'(Y) + Y U''(Y) > \frac{1}{p} U'(Y) \qquad \text{for all} \quad Y \in \mathcal{O}$$

$$\Longleftrightarrow \qquad 1 + E_U(Y) > \frac{1}{p}$$

which is true by assumption.

A$_{III}$.

Let t^* satisfy weighting equity $\Longleftrightarrow \exists \lambda \in \mathbb{R}^n, \neq 0$:

$$\sum_i \lambda_i u_i(y_i - t_i^*) = \max_{\sum t_i = g} \sum_i \lambda_i u_i(y_i - t_i) \tag{12}$$

and with the shortened notation $d_i := u_i(y_i) - u_i(y_i - t_i^*)$:

$d_j > d_k$ implying $\lambda_j \geq \lambda_k$.

(12) is equivalent to $\lambda_i u_i'(y_i - t_i^*) = $ const. $(i \in N)$.

Hence if t^* meets weighting equity we obtain

$$d_j > d_k \quad \text{implying} \quad u_j'(y_j - t_j^*) \leq u_k'(y_k - t_k^*) \ . \tag{13}$$

The reversed assertion equally holds true. To see this

set $\qquad \lambda_i := (u_i'(y_i - t_i^*))^{-1}$.

Because of $u_i'' < 0$ (13) is equivalent to

$$d_j > d_k \quad \text{implying} \quad y_j - t_j^* \geq y_k - t_k^*. \tag{14}$$

In order to prove a) assume that $d_j > d_k$ implied $y_j - t_j^* < y_k - t_k^*$. (OP) demands $y_j < y_k$ which leads to a contradiction to (ATP).

For b) let $d_j > d_k$. For (ATP) we have to verify $y_j > y_k$. (14) implies $u(y_j - t_j) \geq u(y_k - t_k)$ where $u := u_j = u_k$. Hence $u(y_j) > u(y_k)$ or $y_j > y_k$. Finally, for (OP) assume $y_k < y_j$. If we had $y_j - t_j^* < y_k - t_k^*$ then by definition $d_k < d_j$ would follow which would imply a contradiction to (14).

Aɪᴠ.

Application of the implicite function theorem to (PU) yields

$$\frac{dT_U}{dY} = \frac{U'(Y)U'(Y-T_U)[U(Y-T_U)]^2}{U(Y)U''(Y-T_U)[U(Y-T_U)]^2 - 2U(Y)[U'(Y-T_U)]^2U(Y-T_U)} + 1 .$$

The denominator is negative which implies a marginal tax rate of less than one. For progressivity we have to show

$$\frac{dT_U}{dY} > \frac{T_U}{Y} \qquad \Longleftrightarrow$$

$$YU'(Y)U'(Y-T_U)[U(Y-T_U)]^2$$

$$< (Y-T_U)[-U(Y)U''(Y-T_U)[U(Y-T_U)]^2 + 2U(Y)[U'(Y-T_U)]^2U(Y-T_U)]$$

$$\Longleftrightarrow \quad \frac{YU'(Y)}{U(Y)} + \frac{(Y-T_U)U''(Y-T_U)}{U'(Y-T_U)} < \frac{2(Y-T_U)U'(Y-T_U)}{U(Y-T_U)}$$

As $E_U \leq 0$, $U' > 0$, $U'' < 0$ it suffices to verify

$$Y \leq 2(Y-T_U) .$$

This condition is to hold by assumption.

References

d'Aspremont, Cl. and L. Gevers, 1977, Equity and the
     informational basis of collective choice, Review
     of Economic Studies 44, 199-209.

Atkinson, A.B., 1970, On the measurement of inequality,
     Journal of Economic Theory 2, 244-263.

Huttel, G. and W.F. Richter, 1980, A note on "An impossibility
     result concerning n-person bargaining games", Disc.
     Paper, IMW, University of Bielefeld.

Kalai, E. and M. Smorodinsky, 1975, Other solutions to
     Nash's bargaining problem, Econometrica 43, 513-518.

Musgrave, R.A., 1959, The theory of public finance
     (McGraw-Hill, Tokyo) chap. 5.

Pratt, J.W., 1964, Risk aversion in the small and in the
     large, Econometrica 32, 122-136.

Richter, W.F., 1979, Social choice for bliss-point problems,
     Disc. Paper, IMW, University of Bielefeld.

Richter, W.F., 1980, A normative justification of pro-
     gressive taxation: How to compromise on Nash and
     Kalai-Smorodinsky, Disc. Paper, IMW, University of
     Bielefeld.

Samuelson, P.A., 1947, Foundations of economic analysis,
     (Harvard University Press, Cambridge).

Sen, A.K., 1974, Informational bases of alternative wel-
     fare approaches, Journal of Public Economics 3, 374-
     403.

Sen, A.K., 1977, On weights and measures: informational
     constraints in social welfare analysis, Econometrica
     45, 1539-1573.

" WIRTSCHAFTSTHEORETISCHE ENTSCHEIDUNGSFORSCHUNG"

A series of books published by the Institute of Mathematical
Economics, University of Bielefeld.

Wolfgang Rohde
Ein spieltheoretisches Modell eines Terminmarktes (A Game
Theoretical Model of a Futures Market)
The model takes the form of a multistage game with imperfect
information and strategic price formation by a specialist.
The analysis throws light on theoretically difficult empiri-
cal phenomena.

<div align="center">Vol. 1   176 pages   price: DM 24,80</div>

Klaus Binder
Oligopolistische Preisbildung und Markteintritte (Oligopolistic
Pricing and Market Entry)
The book investigates special subgame perfect equilibrium
points of a three-stage game model of oligopoly with decisions
on entry, on expenditures for market potential and on prices.

<div align="center">Vol. 2   132 pages   price: DM 22,80</div>

Karin Wagner
Ein Modell der Preisbildung in der Zementindustrie (A Model of
Pricing in the Cement Industry)
A location theory model is applied in order to explain observed
prices and quantities in the cement industry of the Federal Re-
public of Germany.

<div align="center">Vol. 3   170 pages   price: DM 24,80</div>

Rolf Stoecker
Experimentelle Untersuchung des Entscheidungsverhaltens im
Bertrand-Oligopol (Experimental Investigation of Decision-Be-
havior in Bertrand-Oligopoly Games)
The book contains laboratory experiments on repeated super-
games with two, three and five bargainers. Special emphasis is
put on the end-effect behavior of experimental subjects and
the influence of altruism on cooperation.

<div align="center">Vol. 4   197 pages   price: DM 28,80</div>

Angela Klopstech
Eingeschränkt rationale Marktprozesse (Market processes with
Bounded Rationality)
The book investigates two stochastic market models with bound-
ed rationality, one model describes an evolutionary competitive
market and the other an adaptive oligopoly market with Markovian
interaction.

<div align="center">Vol. 5   price: DM 23,-- appr.</div>

Orders should be sent to:

Pfeffersche Buchhandlung, Alter Markt 7, 4800 Bielefeld 1,
West Germany.