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Wolfram Richter
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SOCIAL CHOICE FOR BLISS-POINT PROBLEMS

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ABSTRACT:

This paper's concern is the axiomatic determination of social choice correspondences Ψ for a class of n-person problems $\mathbf{V}$ that are characterized by some - generally - non-feasible bliss-point $u(V)$ ($\notin V \in \mathbf{V}$). Meeting appropriate assumptions of "planner's rationality" it is shown that Ψ is necessarily norm-induced, i.e. one can find some norm $\|\cdot\|$ in $\mathbb{R}^n$ s.t. $\Psi(V) = \{u \in V \mid \|u-u(V)\| = \min\{\|v-u(V)\| \mid v \in V\}\}$. The mathematical problem of recovering $\|\cdot\|$ from Ψ is one of integration which has its well-known parallel in the theory of revealed preference.

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1. INTRODUCTION

Let U be the set of *social alternatives* and **V** a subset of 2^U which is the power-set of U. A *social choice correspondance* (SCC) $\psi : \mathbf{V} \to U$ is a set-valued choice function[0] that selects some non-empty *choice-set* $\psi(V) \subseteq V \subseteq U$ from any *(feasible) environment* $V \in \mathbf{V}$.

The axiomatic framework we shall adopt suggests to restrict the interpretation of $\psi : \mathbf{V} \to U$ to *social* choices. Thus ψ is meant to describe the choice behaviour of some fictitious planner. In a different context, such as consumption theory, ψ might describe the market behaviour of some private consumer.

As a matter of fact the basic question of this paper was first raised and answered by revealed preference theory: When does there exist some ordering $R = R^\psi$ on U *rationalizing* ψ in the following sense:

$$u \in \psi(V) \quad \text{iff} \quad uRv \quad \text{for all} \quad v \in V, V \in \mathbf{V}. \qquad (1)$$

An ordering is a complete, reflexive, and transitive relation.

It was K.J. ARROW who observed that the existence question

of rationalizing orderings "would be greatly simplified if the range over which the choice functions are considered to be determined is broadened to include all finite sets." ([2], p.122) This statement has found strong support by a subsequent paper of A.K. SEN [22]. All what we actually need is a straight-forward generalization of SAMUELSON's Weak Axiom of Revealed Preference [19] such that the case of (demand) correspondences Ψ is subsumed. Then a rationalizing ordering R is easily shown to exist for Ψ if we only allow $\mathbf{V}$ to comprise all subsets V of U with $|V| \leq 3$. (Cf. [2], [22] or theorem 1 below.)

This paper's object is to give more information about the structure of Ψ and its rationalizing social ordering. Surprisingly enough and in contrast with expectations that might be raised by ARROW's remark the strongest results only obtain if finite environments are infeasible.

To get a rough idea of the kind of results we are aiming at let U be $\mathbb{R}^n$, the EUCLIDean space of dimension n. n refers to the number of individuals $\Omega = \{1,..,n\}$ the welfare of whom the planner has to consider. $u \in U$ stands for a vector $(u_1,...,u_n)$ of personal welfare (= utility) levels u_i. The planner is thus conceived to make social

choices among utility payoffs rather than welfare affec-
ting physical states.

The more conventional analysis would focus on the latter.
Let accordingly X denote a set of alternative physical
states. Individual i's welfare at $x \in X$ is measured by
some utility function $d_i : X \to \mathbb{R}$. Let D be the set of all
feasible $d : X \to \mathbb{R}^n$. Physical environments that are
feasible with respect to X are denoted by $Y \in \mathbf{Y} \subseteq 2^X$.
P. FISHBURN would call $\Psi : \mathbf{V} \to U$ a "choice function" ([5],
p.190). A "social choice function" in his terminology is
a set-valued mapping

$$\Psi : \mathbf{Y} \times D \to X \ , \quad \emptyset \neq \Psi(Y,d) \subseteq Y \quad (\subseteq X). \quad ([5], \ p.4)$$

Note that the above definition of $\Psi : \mathbf{V} \to U$ is the more
general concept if $\mathbf{V}$ can be chosen sufficiently large in
2^U. This follows as any conflict of the "real world" can
naturally be embedded into the utility space:

$$U := \mathbb{R}^n, \ (Y,d) \mapsto d(Y) =: V \subseteq U \ , \ \Psi(Y,d) := d^{-1}(\Psi(d(Y))) \cap Y.$$

Hence any results which we derive for $\Psi : \mathbf{V} \to \mathbb{R}^n$ and a
sufficiently large $\mathbf{V} \subseteq 2^{\mathbb{R}^n}$ immediately carry over to a
whole class of conflicts of the "real world". On the con-
trary characterization theorems for $\Psi : \mathbf{Y} \times D \to X$ only ob-
tain for the single set X of social alternatives under
consideration.

There is one distinguished utility payoff associated with
$d : X \to \mathbb{R}^n$, namely

$$\bar{u} = \bar{u}(X,d) , \quad \bar{u}_i := \sup d_i(X) ,$$

where we tacitely - and in contrast with subsequent prac-
tice - assumed that agent $i \in \Omega$ weakly prefers x to y if
$d_i(x) \geq d_i(y)$. $\bar{u}$ is called *bliss-point (payoff)*, a term
which is borrowed from [28]. Bliss-points naturally pre-
sent themselves as refernce payoffs for interpersonal
comparisons of welfare gains. If we assume that the
function d is bounded in the direction of increasing
utility then

$$\bar{u} \in \mathbb{R}^n . \tag{2}$$

Furthermore by definition

$$\bar{u}_i \geq v_i \quad \text{for all } i \in \Omega, \ v \in d(Y), \ Y \subseteq X ; \tag{3}$$

$$\bar{u}(X,td+v) = t\bar{u}(X,d) + v \quad \text{for all } 0 < t \in \mathbb{R} \tag{4}$$

and $v \in \mathbb{R}^n$.

Once we have derived these crucial properties (2-4) of
bliss-points we can safely forget the real-world conflict
(X,Y,d) fictitiously underlying some $V \in \mathbf{V}$. We assume in-
stead that there exists some mapping $\bar{u} : \mathbf{V} \to \mathbb{R}^n$ such that

$$\bar{u}_i(V) \geq v_i \quad \text{for all } i \in \Omega, \ v \in V \in \mathbf{V} \quad \text{and} \tag{3'}$$

$$\bar{u}(tV+v) = t\bar{u}(V) + v \quad \text{for all } 0 < t \in \mathbb{R}, \ v \in \mathbb{R}^n. \tag{4'}$$

Theorem 2 below states that under appropriate assumptions
every $\Psi : \mathbf{V} \to U$ is necessarily induced by norm-minimization.
This means that we can always find some norm $\|\cdot\| = \|\cdot\|^{\Psi}$
in $\mathbb{R}^n$ such that

$$u \in \Psi(V) \quad \text{iff} \quad \|u - \bar{u}(V)\| \leq \|v - \bar{u}(V)\| \quad \text{for all} \quad (5)$$

$v \in V$ and $V \in \mathbf{V}$.

This result is noteworthy in several respects. In its
full extent ("iff") it is only valid if finite environments
are *excluded* from $\mathbf{V}$. Thus in order to prove (5) we are
really back to the classic problem of "path-independent
choices" that makes the mathematical core of revealed
preference theory. It is this integration problem that
ARROW meant to circumvent by including finite environments.
On the other hand SEN [22] convincingly argues that in the
framework of social choice we can hardly dispense with
finite environments by mere reasons of interpretation. If
the planner is conceived to make choices from infinite sets
than he should all the more be able to decide on finitely
many alternatives. However, if we accept this point of view
we have to be content with weaker results without being
able to simplify the mathematics involved. The norm $\|\cdot\|$
which as a corollary to theorem 2 we still prove to exist
for Ψ then only holds

$$\|u - \bar{u}(V)\| \leq \|v - \bar{u}(V)\| \quad \text{if} \quad u \in \Psi(V) \quad \text{and} \quad v \in V \in \mathbf{V} .$$

Theorem 2 can be looked upon as the non-trivial part of a
theorem characterizing aggregation procedures. There do
exist characterization theorems for SCCs which should
however be kept apart. MAY [12], FISHBURN [5], and YOUNG
[27] deserve to be mentioned. Their results share as common
feature that they only apply to a *single* set of social
alternatives X. X is fixed and its cardinality finite.
MAY and FISHBURN even allow for only two alternatives and
a fixed number of individuals (voters). YOUNG's results
heavily rely on the assumption that every finite number of
voters is conceivable. His powerful "consistency assumption"
makes his theory applicable to committee decisions rather
than to genuine social choice.

By way of contrast this paper provides an axiomatic charac-
terization of SCCs for a setting where the number of indivi-
duals is fixed to any positive integer n but the set
of physical social alternatives X may vary. The more
restictive assumption that actually characterizes our frame-
work concerns the informational basis: In SEN's terminology
[24] we let personal welfares be *cardinal unit comparable*.

There is another strand of social choice theory which is
related to our topic. It is the work of d'ASPREMONT and
GEVERS [3], ROBERTS [18] and others who derive axiomatications
of *social welfare functionals* (SWFl). As an SWFl is a more
structured mapping by assumption this approach can be regarded
to be less basic.

ROBERT's theorem 8 comes nearest to our results. On one side his theorem is "stronger" as he is able to restrict the set of eligible social norms $||\cdot||$ to the class of p-norms $||\cdot||_p$ $(p < \infty)$[1]. As a matter of fact his results are "a bit too strong" as just the excluded social ordering induced by $||\cdot||_\infty$ plays an important rôle in application. (The notion of the *equal absolute sacrifice* known from public finance corresponds to some $||\cdot||_\infty$-minimization. See [17].) On the other hand ROBERTS' results are "weaker" than ours. The strength of his theorem 8 amounts to that of our corollary to theorem 2 and not to theorem 2 itself.

Actually, the respective sets of assumptions rule out any comparison of results. E.g. ROBERTS relies on a weakened version of ARROW's *indepencence axiom of irrelevant alternatives* [1]. There is no such axiom in our set-up.

In [17] theorem 2 is applied to the sacrifice theoretic approach to taxation. It is shown that our axiomatication allows a nice and convincing interpretation in the spirit of the *ability-to-pay* principle. On the contrary ROBERTS axiomatication is not too convincing in this important application.

This paper is organized as follows: In section 2 the most relevant axioms of rational choice are reproduced. Section 3 presents the main theorem 2 together with a discussion of the axioms that are met in its context. Section 4 collects proofs. Section 5 is devoted to illustations. The proof of theorem 2 involves a problem of integration which has its well-known counterpart in the theory of revealed preference. In section 6 we therefore relate our results to the revealed preference literature.

2. RATIONAL CHOICE BEHAVIOUR

In this section there will be no restriction put on U. Let $\Psi : \mathbf{V} \to U$ be a SCC. We shall have to consider various relations on U. These are all depending on Ψ and the class of feasible environments $\mathbf{V}$ although the employed notation will not make it explicite. v, u, u^r will always be elements of U.

Definition:

a) $v \succ^* u$ iff $v \neq u$ and $v \in \bigcup\limits_{\substack{V \in \mathbf{V}: \\ u \in \Psi(V)}} V$;

b) $v \succ\succ^* u$ iff $v \in \bigcup\limits_{u \in \Psi(V)} V \setminus \Psi(V)$;

c) Let $\succ/\succ\succ$ denote the transitive hull of $\succ^*/\succ\succ^*$

i.e. $v \succ u$ iff $v \succ^* u^1 \succ^* \ldots \succ^* u^r \succ^* u$ for some

$r \in \mathbb{N} \cup \{0\}$.

The relations $\succ/\succ^*$ coincide with $\succ\succ/\succ\succ^*$ whenever Ψ is a funcion, i.e. single-valued. Note that the standard inter-pretation of relations has to be reversed. The planner is supposed to "prefer" u to v if $u \prec v$, $u \prec\prec v$, etc.

Two relations are of further interest and justify a symbol of their own:

$v \succsim\succsim u$ if not $u \succ\succ v$ and

$v \succeq u$ if $v \succ u$ or $v = u$.

In case that Ψ defines a function SAMUELSON's Weak Axiom of Revealed Preference (WARP) might be stated as asymmetry of $\succ\succ^* = \succ^*$, i.e. $v \succ\succ^* u$ implying $u \not\succ\succ^* v$. Allowing Ψ to be set-valued ARROW [2] defined [2)]

WARP) $v \succ\succ^* u$ implies $u \not\succ^* v$.

v $\gg^*$ u   says that there is a conflict V $\in$ **V**   for which u
is considered to be a fair solution contrary to v which is
feasible though not accepted as fair payoff. If, then, for a
different conflict V' $\in$ **V**   v is chosen as fair solution u
should not have been feasible. The weak axiom thus is a re-
quirement of rationality. This notion of rationality is
stronger than the one which was defined by M.K.RICHTER [15].
According to him Ψ is a "rational choice"iff Ψ meets his

V-*axiom*: $\Psi(V)$ = $\{u \in V \mid v \in V$ implies $u = v$ or $v \gg^* u\}$
            for all $V \in$ **V** .

This axiom requires that $\Psi(V)$ always consists of the $\gg^*$-
minimal elements of V. One easily shows that for corres-
pondences the V-axiom is implied by our weak one. The
following example is to demonstrate that the reversal is
not true: **V** := $\{V,V'\}$, $V = \{u,v,w\}$, $V' = \{u,v\}$, $\Psi(V) = \{u\}$,
$\Psi(V') = \{u,v\}$.

There exist other equivalent formulations of WARP such as
SEN's "Weak Congruence Axiom" [22]. One version seems to
me of special interest since it can be stated without
resorting to relations and since it was originally defined
in a completely different context. The following axiom
generalizes NASH's (later so-called) *axiom of independence*

of irrelevant alternatives [13] such that the case of correspondences is covered.

NASH' *independence of irrelevant alternatives*:

$$V \subseteq V', \ \Psi(V') \cap V \neq \emptyset \quad \text{implies} \quad \Psi(V') \cap V = \Psi(V) . \quad (6)$$

Remark 1: Let $\Psi : \mathbf{V} \to U$ be a SCC. WARP then implies (6).
Vice versa (6) implies WARP if $V \cap V' \in \mathbf{V}$
for all $V, V' \in \mathbf{V}$.

Proof: Let $u^0 \succ\succ^* u^1 \succ^* u^0$ and $V^i \in \mathbf{V}$ $(i=0,1)$ such that $u^i \in \Psi(V^i)$ and $u^0 \in V^1 \backslash \Psi(V^1)$, $u^1 \in V^0$. Then $u^0, u^1 \in V :=$ $V^0 \cap V^1 \in \mathbf{V}$. Hence $\Psi(V^i) \cap V \neq \emptyset$, $V \subseteq V^i$. According to (6) $\Psi(V^i) \cap V = \Psi(V)$ which implies $u^0, u^1 \in \Psi(V^1)$ which contradicts $u^0 \succ\succ^* u^1$.

For the reversed statement assume

$$V^1 \subseteq V^0, \quad \Psi(V^0) \cap V^1 \ni u^0. \quad (7)$$

Suppose $u^0 \notin \Psi(V^1)$. As Ψ is SCC we can find $u^1 \in \Psi(V^1) \subseteq$ $V^1 \subseteq V^0$. Hence $u^0 \succ\succ^* u^1$ or $u^1 \not\succ^* u^0$ by the weak axiom. Yet the latter contradicts $u^1 \in V^0$. We thus obtain

$$\Psi(V^0) \cap V^1 \subseteq \Psi(V^1).$$

For the reversed inclusion let $u^1 \in \Psi(V^1) \subseteq V^1 \subseteq V^0$. Suppose $u^1 \notin \Psi(V^0)$ then $u^1 \succ\succ^* u^0$ or $u^0 \not\succ^* u^1$ by the weak axiom. Hence $u^0 \notin V^1$ which contradicts (7). q.e.d.

We can now state the result which was mentioned before.

Theorem 1: ([2], [22]) Let $\Psi : \mathbf{V} \to U$ be a SCC such that

    a) $V \in \mathbf{V}$ for all $V \subseteq U$ with $|V| \leq 3$,

    b)   WARP holds.

   Then $\prec^*$ is transitive hence $\prec^* = \prec$. $\precsim$ is an

   ordering rationalizing Ψ in the sense of (1).

In case that assumption a) is or shall not be met WARP
has to be strengthened:

A1) (*Strong Axiom of Revealed Preference*: SARP)

   $v \succ\!\!\succ u$ implies $u \not\succ v$ $(u, v \in U)$.

SARP clearly implies WARP. Hence the justification for the
latter can be extended to the former by repeating the
arguments involved. There exist non-strictly equivalent
versions of the strong axiom for the case where corres-
pondences Ψ are taken into regard. Thus A1) slightly
differs from the formulation found in [22]. Yet A1) is
strictly equivalent to M.K. RICHTER's [14]

Congruence Axiom: Whenever $u \succ v$ and $u \in \Psi(V)$, $v \in V$

      for some $V \in \mathbf{V}$ then $v \in \Psi(V)$.

The latter is again equivalent to

   $v \succ\!\!\succ^* u$ implying $u \not\succ v$ (8)

By which A1) implies the Congruence Axiom. The reversal is

true as $u \gtrsim v \gg u$ is not compatible with (8):

$u \gtrsim v \gg u$ implies $u \gtrsim v \gg^* \ldots \gg^* w \gg^* u$ thus

$u \gtrsim w \gg^* u$.

3. ASSUMPTIONS AND RESULTS

In this section U will be a subset of $\mathbb{R}^n$. $\Omega = \{1, \ldots i \ldots, n\}$ denotes the set of individuals. It will be convenient to reverse the direction of increasing utility. Thus for the rest of the paper $i \in \Omega$ is assumed to prefer u_i to v_i if $u_i < v_i$. We meet the following notational conventions. $\mathbb{R}^n_+ = \{u \in \mathbb{R}^n \mid u \geq 0\}$ where $u \geq v$ iff $u_i \geq v_i$ $(i \in \Omega)$. We shall write $u > v$ iff $u \geq v$ and $u \neq v$; $u \gg v$ iff $u_i > v_i$ $(i \in \Omega)$; $u \in \mathbb{R}^n_+ \backslash 0$ iff $u > 0$.

Let $\underline{u} : \mathbf{V} \rightarrow \mathbb{R}^n$ be a bliss-point mapping, i.e. we assume (3') and (4') where the inequality in (3') is reversed to accord our sign convention. Let $\Psi : \mathbf{V} \rightarrow \mathbb{R}^n$ be a SCC. Suppose that Ψ satisfies

$$\Psi(u + V) = u + \Psi(V) \quad (u \in \mathbb{R}^n, V \subseteq \mathbb{R}^n). \tag{9}$$

Then Ψ is uniquely determined on

$$\mathbf{V}_0 := \{V \subseteq \mathbb{R}^n_+ \mid \underline{u}(V) = 0\}$$

because of

$$\Psi(V) = \Psi(V - \underline{u}(V)) + \underline{u}(V) \ ,$$

$$V' := V - \underline{u}(V) \subseteq \mathbb{R}_+^n \ , \quad \underline{u}(V') = 0 \ .$$

Hence from now on

$$\Psi : \mathbf{V} \to U := \mathbb{R}_+^n \ , \quad \mathbf{V} \subseteq \mathbf{V}_0 \ .$$

Some $\quad u \in P_R = P_R^V := \{u \in V \mid \not\exists \ v \in V : vRu\}$

is called *weak/strong PARETO payoff* if $R = \ll/<$. Both

versions of PARETO efficiency are standard in the theory of

collective choice. We actually need some intermediate case:

A2) (*PARETO efficiency*) $\Psi(V) \subseteq P_{\bar{R}}^V$ $(V \in \mathbf{V})$ where $v\bar{R}u$ is

defined by $v < u$ and $v_i < u_i$ for all $i \in \Omega$ s.t. $u_i \neq 0$.

Thus $P_< \subseteq P_{\bar{R}} \subseteq P_{\ll}$ and $(1,0) \ \bar{R} \ (1,1)$ but $(1,0) \ \bar{\bar{R}} \ (2,0)$.

According to A2) the weak PARETO principle is only applied

to those individuals that do not receive their bliss-point

payoff. A2) implies

$$\Psi(V) = \{0\} \quad \text{whenever} \quad 0 \in V \ . \tag{10}$$

Hence the bliss-point is chosen whenever it is feasible.

Note that $t V \in \mathbf{V}_0$ if $V \in \mathbf{V}_0$, $0 < t \in \mathbb{R}$.

A3) (invariance under dilatations)

$$t\Psi(V) \subseteq \Psi(tV) \quad \text{for all} \quad 0 < t \in \mathbb{R}, \text{ whenever } V, tV \in \mathbf{V}.$$

Because of $t^{-1} \Psi(tV) \subseteq \Psi(V)$ A3) clearly implies $t\Psi(V) = \Psi(tV)$. A3) in connection with (9) implies that the informational basis we are working with corresponds to *cardinal unit comparability* in SEN's terminology ([24], p.1542). There is no doubt that a less demanding informational set-up would be desirable, i.e. some invariance of Ψ with respect to a larger class of transformations. Unfortunatelly there are limitations on which we shall soon comment.

We next meet some "regularity conditions". Denote by

$$V(\lambda,a) := \{u \in \mathbb{R}_+^n \mid \lambda \cdot u \geq a\} ,$$
$$\mathbb{W} := \{V(\lambda,a) \mid \lambda \in \mathbb{R}_+^n \backslash 0, \ a \in \mathbb{R}_+\} .$$

A4) $\mathbb{W} \subseteq \mathbb{V}$.

A5) The correspondence $\Psi \cdot V(\cdot,\cdot) : (\mathbb{R}_+^n \backslash 0) \times \mathbb{R}_+ \to \mathbb{R}_+^n$ is

   a) surjective (i.e. "onto") and

   b) closed-valued.

   c) (zero-continuity) $u^i \in \Psi(V(\lambda^i,a^i))$, $u^i \to u'$, $\lambda^i \to \lambda' \neq 0$

      ($i \in \mathbb{N}$) and $\lambda' \cdot u' = 0$ imply $u' = 0$.

Note that by A2) $\lambda \cdot u = a$ whenever $u \in \Psi(V(\lambda,a))$, $(\lambda,a) \in \mathbb{R}_+^{n+1}$. This follows for $a = 0$ from (10). Suppose on the other hand $0 < a < \lambda \cdot u$. Then $tu \in V(\lambda,a)$ for some $t \in (0,1)$. As $tu \bar{R} u$ we obtain a contradiction to A2).

Condition A5c) can be given a normative justification. Consider the following axiom a more demanding version of which SEN meets for a different context. ([23], p.399).

Weighting equity: Let $u \in \Psi(V)$, $\lambda \cdot u = \min \lambda \cdot V$ for some
$\lambda = \lambda^V \in \mathbb{R}_+^n \setminus 0$. Then $\lambda_j \geq \lambda_k$ whenever
$u_j > u_k$, $j,k \in \Omega$.

$u \in \Psi(V)$ means that u is the planner's choice for the environment V. $\lambda \cdot u = \min \lambda \cdot V$ means that the planner's choice can be "rationalized" by minimizing some social wefare function $w(v) = \lambda \cdot v$ with respect to V. λ_i reveals the social weight that the planner attaches to the welfare of individual i. The latter takes a primary interest in its realized welfare level u_i and in λ_i, itself, only so far as the distribution of welfare levels is implied by the distribution of weights. In comparative terms large welfare weights lead to small payoffs thus reflecting high consideration by the planner. For assume
$\lambda_1^1 > \lambda_1^2$, $\lambda_i^1 = \lambda_i^2$ ($i = 2,..,n$) and $\lambda^j \cdot u^j = \min \lambda^j \cdot V$, $u^j \in V$ ($j = 1,2$).
Then $0 \leq \lambda^1 \cdot (u^2 - u^1) + \lambda^2 \cdot (u^1 - u^2) = (\lambda^1 - \lambda^2) \cdot (u^2 - u^1) = (\lambda_1^1 - \lambda_1^2)(u_1^2 - u_1^1)$ and hence $u_1^2 \geq u_1^1$.

Suppose now that the assumptions of the weighting equity

axiom are fulfilled. As $u_j > u_k$ implies $\lambda_j \geq \lambda_k$ individual j can raise no objections that his personal interests were not sufficienty safeguarded. $\lambda_j \geq \lambda_k$ guarantees him an at least equally high regard by the planner. Note that inter-personal comparisons of this kind are compatible with our informational setting.

Remark 2: PARETO efficiency (A2) and weighting equity imply zero-continuity (A5c).

Proof: Suppose $(i \in \mathbb{N})$ $u^i \in \Psi(V(\lambda^i, \lambda^i \cdot u^i))$, $u^i \to u'$, $\lambda^i \to \lambda'$ $\in \mathbb{R}^n_+ \backslash 0$, $\lambda' \cdot u' = 0$ and $u' \neq 0$. For some $j \in \Omega$ $u'_j > 0$ and thus $\lambda'_j = 0$. Vice versa $\lambda'_k > 0$ and $u'_k = 0$ for some $k \in \Omega$. For all $i \geq i_0$ we obtain $\lambda^i_j < \lambda^i_k$, $u^i_j > u^i_k$ which contra-dicts weighting equity.

$V \subseteq \mathbb{R}^n$ is called *convex from below* if for all $u, v \in V$ there is some $w \in V$ such that $2w \leq u + v$.

Theorem 2: Let $\Psi : \mathbf{V} \to U = \mathbb{R}^n_+$, $\mathbf{V} \subseteq \mathbf{V}_0$, hold:

    a) A1-5)

    b) $tV \in \mathbf{V}$ for all $V \in \mathbf{V}$, $0 < t \in \mathbb{R}$,

    c) V is convex from below for all $V \in \mathbf{V}$.

Then Ψ can be rationalized by some norm $\| \cdot \| = \| \cdot \|^{\Psi}$, restricted to U:

$$\Psi(V) = \Psi^{\| \cdot \|}(V) := \{u \in V \mid \|u\| = \min_{v \in V} \|v\|\}$$

for all $V \in \mathbf{V}$. $\| \cdot \|$ is uniquely determined up to a positvie scalar multiple and satisfies

$$u \precsim v \quad iff \quad \|u\| \leq \|v\| . \tag{11}$$

Furthermore $u \leq v$ implies $\|u\| \leq \|v\|$.

The core of theorem 2 is a problem of integration and thus closely related to the work of SAMUELSON [20], HOUTHAKKER [7], UZAWA [26], HURWICZ and/or M.K. RICHTER ([8], [9], [14], [15]), MAS-COLELL [11], to name just a few. A1) provides us with a "global integrability condition". We shall come back to a thorough discussion of the revealed preference literature in section 6, below. The analogy is obvious if we interpret $V \in \mathbf{V}$ as generalized "budget", $V(\lambda,a)$ as budget defined by "income" a and "price vector" λ. Ψ is the "demand correspondence", $\succsim*$ the "revealed" and $\succsim$ the "indirectly revealed preference".

Let us briefly comment on the assumptions of theorem 2 which are relevant from a social-choice-theoretic point of view. It should first be clear that the invariance axiom A3) cannot essentially be strengthened.

Remark 3: $\psi^{\|\cdot\|} : \mathbf{V} \to \mathbb{R}_+^2$ is not invariant with respect to linear transformations of utilities if $\mathbf{W} \subseteq \mathbf{V}$.

Proof: Put
$$T := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad TV := \{T(v) \mid v \in V\}.$$
We shall lead
$$T\psi^{\|\cdot\|}(V) \subseteq \psi^{\|\cdot\|}(TV) \quad (V \in \mathbf{W}) \tag{12}$$
to a contradiction. Fix some $u = (0, u_2) > 0$ and choose $v \in \mathbb{R}_+^2$ such that $v_1 > 0$ and $\|v\| < \|u\|$. Such a v exists since otherwise we could construct a sequence $v^i \to 0$, $v_1^i > 0$ for which $0 < \|u\| \leq \|v^i\| \to 0$. W.l.o.g. $u_2 > v_2$. For $\lambda := (u_2 - v_2, v_1)$ we obtain $\lambda \cdot v = \lambda \cdot u > 0$. Let $V' := V(\lambda, \lambda \cdot u)$ and $v' \in \psi^{\|\cdot\|}(V')$. Denote $v^i := T^i v' = (2^i v_1', v_2')$ $(i \in \mathbb{N})$. Because of (12) $v^i \in \psi^{\|\cdot\|}(T^i V')$. As $Tu = u$, $u \in T^i V'$. Hence $\|v^i\| \leq \|u\|$ $(i \in \mathbb{N})$ and a contradiction follows by considering
$$0 < \|(v_1^i, 0)\| = 2^{1-i} \|((2^i - 2^{i-1})v_1', 0)\| = 2^{1-i} \|v^i - v^{i-1}\| \leq 2^{-i} \|u\|$$
which tends to zero for $i \to \infty$. q.e.d.

In some vague sense there is a "dual" relationship between NASH's bargaining problem [13] and the $V \in \mathbf{V}$ that satisfy convexity from below. Cf. figures 1 and 2 below. The direction of convexity is reversed and the role of the threat-point is taken over by the bliss-point. Convexity from
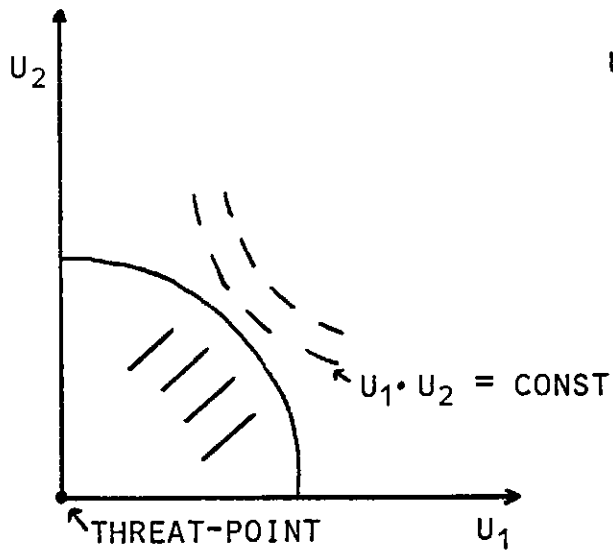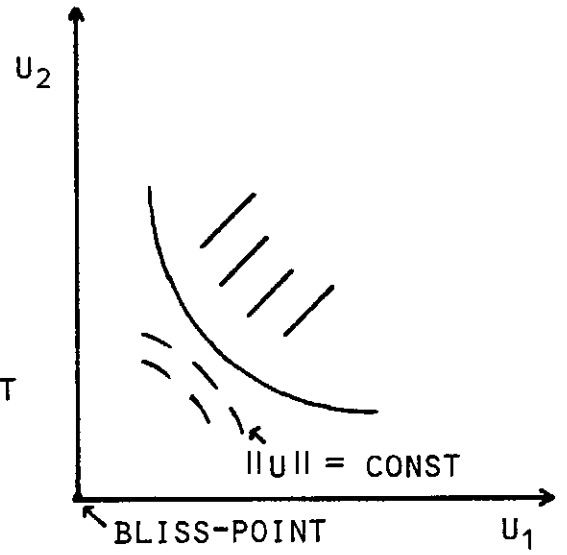
FIGURE 1

FIGURE 2

below clearly excludes all non-trivial finite environments from $\mathbf{V}$. As SEN [22] argues the social-choice-theoretic interpretation hardly justifies to rule out finite environments.

<u>Corollary</u>: Let all assumptions of theorem 2 be met except for c). Then some norm $\|\cdot\|$ exists such that

$$\Psi(V) \subseteq \Psi^{\|\cdot\|}(V) \quad \text{for all } V \in \mathbf{V}. \tag{13}$$

$\|\cdot\|$ is unique up to a positve scalar multiple.

<u>Proof</u>: By A4) $\mathbf{W}$ is a subset of $\mathbf{V}$. According to theorem 2 some norm $\|\cdot\|$ exists such that $\|u\| < \|v\|$ iff $u \lessless v$ where $\lessless = \lessless(\mathbf{W})$ is defined with respect to $\mathbf{W}$. Suppose, now, that in contradiction to (13) some $v \in V \in \mathbf{V}$ and $u \in \Psi(V)$ exist such that $\|u\| > \|v\|$. Then $u \gtrgtr(\mathbf{W}) v$ and hence $u \gtrgtr(\mathbf{V}) v$. SARP requires that $u \lessless(\mathbf{V}) v$ is not true which clearly contradicts $v \in V$, $u \in \Psi(V)$.

We can easily construct SCCs for which the inclusion (13) is strict though all assumptions are fulfilled. Put

$$\mathbf{V}^{fin} := \{V \in \mathbf{V}_0 \mid |V| < \infty\}$$

$$\mathbf{V}' := \mathbf{W} \cup \mathbf{V}^{fin}, \qquad \psi^{|\cdot|,n} : \mathbf{V}' \to \mathbb{R}_+^n,$$

$$\psi^{|\cdot|,n}(V) := \begin{cases} \{u\} & \text{if } u \in \mathbb{R}_+(0,..,0,1) \text{ and} \\ \psi^{|\cdot|}(V) & \text{else,} \end{cases} \quad u \in \psi^{|\cdot|}(V),$$

whereby $|\cdot|$ denotes the EUCLIDean norm of $\mathbb{R}^n$.

The norm which according to the above corollary is associated to $\psi^{|\cdot|,n}$ must be a scalar multiple of $|\cdot|$. Strict inclusion in (13) is obtained for, say, $V' := \{(1,0,..,0), (0,..,0,1)\}$. The difference between theorem 2 and its corollary comes from the fact that SCCs defined on $\mathbf{W} \cup \mathbf{V}^{fin}$ are not uniquely determined on the restricted domain W.

When proving the corollary we cannot dispense with A5a) (surjectivity). This is best seen by the following example: Let $n = 2$ and R^{min} denote the ordering which is induced by the "dictatorship of the most favoured non- indifferent rank" ([3], p.204). In the special case of $n = 2$ we obtain:

$$u \, R^{min} \, v \quad \text{iff} \quad \{\min(u_1,u_2) \leq \min(v_1,v_2) \quad \text{and in case of}$$
$$\text{equality} \quad \max(u_1,u_2) \leq \max(u_1,u_2)\}.$$

Let $\psi^{min} := \psi^{R^{min}}$ be the SCC engendered by R^{min} via (1) and $\|\cdot\|$ any norm in $\mathbb{R}^2$. Choose $0 < t \in \mathbb{R}$ such that $t \, \|(1,0)\| > \|(1,1)\|$. Put $V' := \{(t,0),(1,1)\}$. Then $\{(t,0)\} = \psi^{min}(V')$ whereas $\{(1,1)\} = \psi^{\|\cdot\|}(V')$.

4. PROOF OF THEOREM 2

The proof of theorem 2 is built up by a series of remarks and a central lemma. *Throughout this section the assumptions of theorem 2 are assumed to hold* without further mentioning.

Remark 4: $\gg$ holds:

      a) homogeneity: $v \gg u$ implies $tv \gg tu$ $(t > 0)$;

      b) continuity from below: $\{v \in \mathbb{R}^n_+ \mid v \gg u\}$ is open in $\mathbb{R}^n_+$ for all u.

a) follows from invariance under dilatations (A2). For b) UZAWA's coorresponding "proof of P.V" ([26], p.17) can be adapted. There is only one major change due to the fact that our $V \in \mathbf{V}$ are convex from below and not necessarily convex as UZAWA may assume.

Remember the notation $\ll$ for $\not\gg$.

Transitivity of $\gg$ yields

Remark 5: $u \gtrsim v \gg w$ or $u \gg v \gtrsim w$ imply $u \gtrsim w$.

Remark 6: $u^0 \lesssim u^1$, $t \in [0,1]$ implies $u^t := tu^1 + (1-t)u^0 \lesssim u^1$.

Proof: Let $u^t \in \Psi(V^t)$, $V^t = V(\lambda^t, \lambda^t \cdot u^t)$. If $\lambda^t \cdot u^1 \geq \lambda^t \cdot u^t$ then $u^t = u^1$ or $u^1 \succ^* u^t$. In both cases $u^1 \succsim u^t$ follows by A1). If, on the other hand, $\lambda^t \cdot u^1 < \lambda^t \cdot u^t$ then $\lambda^t \cdot u^0 > \lambda^t \cdot u^t$ and $u^0 \succ\succ^* u^t$. By assumption $u^0 \precsim u^1$. Hence $u^t \precsim u^1$ by the above remark 5.

Suppose, now, $u^0, u^1 \precsim v$. Then either $u^0 \precsim u^1$ or $u^0 \succsim u^1$. In both cases we obtain $u^t \precsim v$ by the foregoing two remarks. Continuity from below then implies

Remark 7: Let $u^0, u^1 \in cl\{u \mid v \succ\succ u\}$ where cl denotes the closure. Then $u^t := t u^1 + (1-t) u^0 \precsim v$.

Consider the set-valued mapping

$$u \mapsto \Lambda(u) := \{\lambda \in \mathbb{R}^n \mid |\lambda| = 1, u \in \Psi(V(\lambda, \lambda \cdot u))\} \ .$$

Because of A5a) $\Lambda(\cdot)$ has non-empty images and thus defines a "vector-field"-correspondence. The core of theorem 2 is that $\Lambda : \mathbb{R}^n_+ \to S^n := \{\lambda \in \mathbb{R}^n \mid |\lambda| = 1\}$ can be "integrated".

Remark 8: $\Lambda(\cdot)$ is a closed correspondence.

Proof: Let $u^i \in \Psi(V(\lambda^i, \lambda^i \cdot u^i))$, $u^i \to u'$, $\lambda^i \to \lambda'$ $(i \in \mathbb{N})$. If $u'=0$ then $u' \in \Psi(V(\lambda', \lambda' \cdot u'))$ according to (10). Hence we assume $u' \neq 0$. By A5c) $\lambda' \cdot u' \neq 0$. Clearly, $\lambda' \cdot u' > 0$. Suppose $u' \notin \Psi(V(\lambda', \lambda' \cdot u'))$ and $v \in \Psi(V(\lambda', \lambda' \cdot u'))$. Then $u' \succ\succ^* v$ and $u' \succ\succ tv$ for some $t > 1$ (remark 4). We show $tv \succ\succ u'$ which is contradictory. As $t\lambda' \cdot v > \lambda' \cdot v = \lambda' \cdot u'$

$$\lambda^i \cdot (tv) > \lambda^i \cdot u^i \quad \text{for large} \quad i \in \mathbb{N}.$$

Hence $tv \succ\succ^* u^i$ or $tv \prec\prec u^i$ by SARP. Continuity

from below then yields tv $\gtrsim\gtrsim$ u'. q.e.d.

Denote (u,v $\neq$ 0) $\underline{b}$(u,v) := sup {b $\in \mathbb{R}_+$ | bu $\lessdot\lessdot$ v}

$\qquad\qquad\qquad\qquad$ $\bar{b}$(u,v) := inf {b $\in \mathbb{R}_+$ | bu $\gtrsim\gtrsim$ v}.

As 0 $\lessdot\lessdot$ v the supremum is taken over a non-void set. Put

inf $\emptyset$:= +∞. Our aim is to prove first

$$\underline{b}(u,v) = \bar{b}(u,v) =: b(u,v) \in \mathbb{R}$$

and then that for fixed u$\neq$0 b(u,$\cdot$) - extended to zero by

b(u,0) := 0 - satisfies all norm properties on $\mathbb{R}_+^n$.

Let bu $\lessdot\lessdot$ v and suppose b > $\bar{b}$(u,v) (b $\in \mathbb{R}$). Then

bu $\gtrsim\gtrsim$ v $\gtrsim\gtrsim$ bu which contradicts A1). Hence

Remark 9: $\underline{b}$(u,v) $\leq$ $\bar{b}$(u,v) .

Lemma: Let u,v $\in \mathbb{R}_+^n\setminus 0$. Then

$$0 < \underline{b}(u,v) = \bar{b}(u,v) =: b(u,v) \in \mathbb{R}.$$

Proof: Let v^t := tv + (1-t)u for t $\in$ [0,1].

$\qquad \Lambda^v$: [0,1] $\to$ S^n , t $\mapsto \Lambda(v^t)$

defines a closed correspondence. Hence $\Lambda^v(\cdot)$ is upper-hemi-

continuous which means that for all closed sets A $\subseteq S^n$

{t $\in$ [0,1] | Λ^v(t) $\cap$ A $\neq$ 0} is closed in [0,1]. (Cf. [6], p.

22) Thus $\Lambda^v(\cdot)$ admits a BOREL-measurable everywhere selection.

(Cf. [6], Lemma 1, p. 55.) This means that we can fix a

measurable mapping

$\qquad \mu^{\bullet}$: [0,1] $\to$ S^n , t $\mapsto \mu^t \in \Lambda(v^t)$.

Fix $\delta > 0$. According to LUSIN's Theorem we find $C^\delta \subseteq [0,1]$ such that

$$\left.\begin{array}{l} [0,1] \setminus C^\delta \text{ is of LEBESQUE measure} < \delta, \\[2mm] C^\delta \text{ compact and } \mu^\bullet \text{ restricted to } C^\delta \text{ continuous.} \end{array}\right\} (14)$$

As $0 \neq v^t$, $\mu^t \cdot v^t > 0$ for all $t \in [0,1]$. Suppose for all $i \in \mathbb{N}$ we could find t^i, τ^i s.t.

$$\mu^{t^i} \cdot v^{\tau^i} \leq 0 \quad \text{and} \quad |t^i - \tau^i| < \frac{1}{i} .$$

Then for appropriate $\mathbb{N}_0 \subseteq \mathbb{N}$

$$t^i \to t' \in [0,1] , \quad \tau^i \to t' , \quad \mu^{t^i} \to \mu' \in S^n \quad (i \in \mathbb{N}_0) .$$

Thus $\mu^{t^i} \to \mu'$, $v^{t^i} \to v^{t'}$, $\mu' \cdot v^{t'} = 0$ and $v^{t'} \in \Psi(V(\mu', \mu' \cdot v^{t'}))$ By A5c) $v^{t'} = 0$ which is contradictory.

Choose $\varepsilon \leq \delta$ sufficiently small, such that

$$|t - \tau| < \varepsilon \quad \text{implies} \quad \mu^t \cdot v^\tau > 0. \tag{15}$$

Fix t^i ($0 \leq i \leq N$) $\in [0,1]$ such that

$$\left.\begin{array}{l} 0 = t^0 < t^1 < \ldots < t^N = 1 \\[2mm] t^i - t^{i-1} < \varepsilon \qquad (i = 1,\ldots,N) \end{array}\right\} (16)$$

and put $u^i := v^{t^i}$, $\lambda^i := \mu^{t^i}$ $(i = 0,\ldots,N)$

$$\left.\begin{array}{l} a^i := q_N \dfrac{\lambda^i \cdot u^{i-1}}{\lambda^i \cdot u^i} a^{i-1} \qquad (i = 1,\ldots,N) \end{array}\right\} (17)$$

where $a^0 := 1$ and $q_N := \sqrt[N]{\dfrac{N-1}{N}}$.

As $\lambda^i \cdot u^i > 0$ and $\lambda^i \cdot u^{i-1} > 0$ by (15) we obtain recursively $a^i > 0$. The definition of a^i implies

$$\lambda^i \cdot (a^{i-1} u^{i-1}) = q_N^{-1} a^i \lambda^i \cdot u^i > \lambda^i \cdot (a^i u^i)$$

hence $\quad a^{i-1} u^{i-1} \geqslant \gtrless * a^i u^i \quad (i=1,..,N) \quad$ or

$$u = u^0 \gtrless \geqslant a^N u^N = a^N v \quad \text{with} \quad a^N > 0 . \tag{18}$$

Up to now the choice of $\{t^i \mid i=0,..,N\}$ has been free under the restriction (16). Now we choose t^i "preferably out of C^δ". We proceed inductively: Let

$$0 = t^0 < t^1 < \ldots < t^i < 1$$

be already fixed.

$$I^i := [t^i + \frac{\varepsilon}{2}, t^i + \varepsilon) , \quad |I^i| := \frac{\varepsilon}{2} .$$

Choose

$$t^{i+1} := \begin{cases} 1 & \text{if } t^i + \varepsilon > 1 \\ \in C^\delta \cap I^i & \text{if } C^\delta \cap I^i \neq \emptyset \\ \in I^i & \text{if } C^\delta \cap I^i = \emptyset . \end{cases} \tag{19}$$

Clearly, after finitely many steps $t^N = 1 \quad (N \in \mathbb{N})$.

From now on let $\{t^i \mid i=0,..,N\}$ hold (19) and thus (15).
Denote by analogy to (17):

$$\bar{b}^N := 1 ,$$
$$\bar{b}^i := q_N \frac{\lambda^i \cdot u^{i+1}}{\lambda^i \cdot u^i} \bar{b}^{i+1} \quad (i = N-1,..,0)$$
$$b^i := \bar{b}^i / \bar{b}^0 \quad (i = 0,..,N) .$$

Like above we obtain

$$u \lessgtr\lessgtr b^N v \quad \text{with} \quad b^N > 0$$

from which we conclude $0 < \underline{b}(u,v)$. $\tag{20}$

We shall prove the assertion $\underline{b}(u,v) = \bar{b}(u,v)$ by showing

$$a^N - b^N \to 0 \quad \text{for} \quad \varepsilon \to 0, \ \delta \to 0 . \tag{21}$$

$$a^N = q_N \frac{u^{N-1} \cdot \lambda^N}{\lambda^N \cdot u^N} \quad a^{N-1} = \dots = \frac{N-1}{N} \prod_{i=1}^{N} \frac{\lambda^i \cdot u^{i-1}}{\lambda^i \cdot u^i}$$

$$b^N = \frac{1}{b^0} = q_N^{-1} \frac{\lambda^0 \cdot u^0}{\lambda^0 \cdot u^1} \frac{1}{b^1} = \frac{N}{N-1} \prod_{i=0}^{N-1} \frac{\lambda^i \cdot u^i}{\lambda^i \cdot u^{i+1}} \quad .$$

$$\log \left(\frac{N}{N-1}\right)^2 \frac{a^N}{b^N} =$$

$$= \sum_{i=0}^{N-1} ([\log \lambda^i \cdot u^{i+1} - \log \lambda^i \cdot u^i] - [\log \lambda^{i+1} \cdot u^{i+1} - \log \lambda^{i+1} \cdot u^i])$$

$$= \sum_{i=0}^{N-1} \left[\frac{\lambda^i}{\lambda^i \cdot v^{\tau^i}} - \frac{\lambda^{i+1}}{\lambda^{i+1} \cdot v^{\sigma^i}} \right] \cdot (t^{i+1} - t^i)(v - u)$$

where $\tau^i, \sigma^i \in [t^i, t^{i+1}]$ are determined according to TAYLOR. we split the summation to estimate the above term.

$$N_0 := \{ i \mid 0 \le i < N, \ t^i \text{ or } t^{i+1} \notin C^\delta \}$$

$$N_1 := \{0, \dots, N-1\} \setminus N_0 \ .$$

We first note considering (19):

$$\sum_{i \in N_0} (t^{i+1} - t^i) < \sum_{N_0} \epsilon$$

$$\le \sum_{t^{i+1} \notin C^\delta} \epsilon + \sum_{t^i \notin C^\delta} \epsilon$$

$$\le 2 \sum_{t^{i+1} \notin C^\delta} \epsilon + \epsilon = 4 \sum_{t^{i+1} \notin C^\delta} |I^i| + \epsilon$$

$$= 4 \sum_{I^i \cap C^\delta = \emptyset} |I^i| + \varepsilon$$

$$= 4 \; \Big| \bigcup_{I^i \cap C^\delta = \emptyset} I^i \; \Big| + \varepsilon \; \le 5\delta \quad.$$

Hence

$$\sum_{i \in N_0} \Big| \Big[\frac{\lambda^i}{\lambda^i \cdot v^{\tau^i}} - \frac{\lambda^{i+1}}{\lambda^{i+1} \cdot v^{\sigma^i}} \Big] \cdot (t^{i+1} - t^i)(v-u) \Big|$$

$$< 5\delta \; |v-u| \; \max \Big| \frac{\lambda^i}{\lambda^i \cdot v^{\tau^i}} - \frac{\lambda^{i+1}}{\lambda^{i+1} \cdot v^{\sigma^i}} \Big|$$

Let $i = i(\varepsilon)$ be the maximizing index. If $\varepsilon \to 0$, $N = N(\varepsilon) \to \infty$. As $|\lambda^{i(\varepsilon)}| = 1$, $\lambda^{i(\varepsilon)} \cdot v^{\tau^{i(\varepsilon)}} \neq 0$ by (15) we find a constant K, independent of δ, such that for all $\varepsilon > 0$

$$\sum_{i \in N_0} |\ldots| < \delta K \quad. \tag{22}$$

On the other hand

$$\sum_{i \in N_1} \Big| \Big[\frac{\lambda^i}{\lambda^i \cdot v^{\tau^i}} - \frac{\lambda^{i+1}}{\lambda^{i+1} \cdot v^{\sigma^i}} \Big] \cdot (t^{i+1} - t^i)(v - u) \Big|$$

$$\le \max_{i \in N_1} \Big| \frac{\lambda^i}{\lambda^i \cdot v^{\tau^i}} - \frac{\lambda^{i+1}}{\lambda^{i+1} \cdot v^{\sigma^i}} \Big| \; |v-u| \sum_{j \in N_1} (t^{j+1} - t^j)$$

$$\le |v-u| \max_{i \in N_1} \Big| \frac{\lambda^i}{\lambda^i \cdot v^{\tau^i}} - \frac{\lambda^{i+1}}{\lambda^{i+1} \cdot v^{\sigma^i}} \Big| \quad.$$

Let $i = i(\varepsilon) \in N_1 = N_1(\varepsilon)$ be the maximizing index. As
$i \in N_1$ $t^i(\varepsilon) := t^{i(\varepsilon)}$, $t^{i+1}(\varepsilon) \in C^\delta$ which is compact. Let
$t^i(\varepsilon) \to t' \in C^\delta$ for $\varepsilon \to 0$.

As $\mu^{\cdot}$ is continuous on C^δ and $\lambda^i = \mu^{t^i}$

$\lambda^i(\varepsilon) - \lambda^{i(\varepsilon)+1} \to 0$. Equally

$\lambda^{i(\varepsilon)} \cdot v^{\tau^{i(\varepsilon)}} \to \mu^{t'} \cdot v^{t'} > 0$.

Considering (22) we finally obtain

$$\left| \log\left(\frac{N}{N-1}\right)^2 \frac{a^N}{b^N} \right| \to 0 \quad \text{for} \quad \varepsilon \to 0, \ \delta \to 0.$$

Hence $a^N/b^N \to 1$ for $\varepsilon, \delta \to 0$. By (18) and (20)

$$0 < \underline{b}(u,v) \leq \bar{b}(u,v) \leq (a^N)^{-1} .$$

Hence $a^N \not\to \infty$ and the assertion (21) follows. q.e.d.

On one side $b(\cdot,\cdot)$ defines a relation on $\mathbb{R}^n_+ \backslash 0$ via

$b(u,v)$ iff $\underline{b}(u,v) = \bar{b}(u,v)$.

On the other side $b(\cdot,\cdot)$ can be seen as a real-valued
mapping. Keeping both aspects in mind we note

Remark 10: $b(\cdot,\cdot)$ is

      a) transitive with $b(u,v) \, b(v,w) = b(u,w)$ and

      b) symmetric with $b(u,v) = (b(v,u))^{-1}$.

Proof:

a) Let $b^1 < \underline{b}(u,v)$, $b^2 < \underline{b}(v,w)$. Then $b^2 b^1 u \lessgtr b^2 v \lessgtr w$,
$b^1 b^2 \leq \underline{b}(u,w)$ and thus $\underline{b}(u,v) \, \underline{b}(v,w) \leq \underline{b}(u,w)$. We equally
obtain $\bar{b}(u,v) \, \bar{b}(v,w) \geq \bar{b}(u,w)$ and thus the assertion by

referring to remark 9.

b) $\underline{b}(u,v) = \sup \{ b \geq 0 \mid bu \lessdot\lessdot v \} = \sup \{ a^{-1} \geq 0 \mid av \gtrdot\gtrdot u \}$

$\qquad = (\inf \{ a \geq 0 \mid av \gtrdot\gtrdot u \})^{-1} = (\bar{b}(v,u))^{-1}.$

Remark 11: $\underline{b}(u,v^0+v^1) \leq b(u,v^0) + b(u,v^1)$.

Proof: Denote $b^i := b(u,v^i)$ (i=0,1) and $t' := b^0/(b^0+b^1)$.
As $v^i/b^i \in cl \{ v \mid v \lessdot\lessdot u \}$ we may apply remark 7 according
to which

$$u \gtrdot\gtrdot t' \frac{v^0}{b^0} + (1-t') \frac{v^1}{b^1} = \frac{v^0 + v^1}{b^0 + b^1}$$

or $(b^0+b^1)u \gtrdot\gtrdot v^0+v^1$. The assumption

$$b^0+b^1 < \sup \{ b \mid bu \lessdot\lessdot v^0+v^1 \}$$

leads to a contradiction as $\gtrdot\gtrdot$ is asymmetric. Hence
$\underline{b}(u,v^0+v^1) \leq b^0+b^1$.

As mentioned before we now fix $u \neq 0$, say $u = e^1 = (1,0,..,0)$,
and define

$$\|v\| := \begin{cases} 0 & \text{for } v = 0 \\ b(e^1,v) & \text{else.} \end{cases}$$

The foregoing remarks imply that $\| \cdot \|$ satisfies all norm
properties on the cone $\mathbb{R}^n_+$.

Proof of theorem 2:

We first show $\|v\| > \|u\|$ iff $v \gtrdot\gtrdot u$.

$\qquad \|v\| > \|u\|$ iff $b(e^1,v) > b(e^1,u)$ iff

$\qquad b(u,v) = b(u,e^1) \, b(e^1,v) > b(u,e^1) \, b(e^1,u)$

$$= b(u,u) = 1.$$

Yet $\underline{b}(u,v) > 1$ means $u \lessdot\lessdot v$. On the other hand $u \gtrdot\gtrdot v$ implies $qu \lessdot\lessdot v$ for some $q > 1$ (remark 4). Hence

$$b(u,v) = \underline{b}(u,v) > 1.$$

Next we fix $V' \in \mathbf{V}$, $u' \in V'$ and show

$$u' \in \Psi(V') \quad \text{iff} \quad u' = \min_{u \in V'} \|u\|.$$

Let $u' \in \Psi(V')$. W.l.o.g. $u' \neq 0$. Any $u \in V' \backslash \{u'\}$ implies $u \gtrdot^* u'$. Suppose $\|u\| < \|u'\|$ then $u' \gtrdot\gtrdot u$ and by the strong axiom A1) $u \gtrdot u'$. Hence $\|u'\| \leq \|u\|$.

For the reversed statement suppose $u' \notin \Psi(V')$, $u \in \Psi(V')$. Hence $u' \gtrdot\gtrdot^* u$ or $\|u'\| > \|u\|$ which contradicts $\|u'\| = \min\{\|v\| \mid v \in V'\}$.

The norm $\|\cdot\| = \|\cdot\|^{\Psi}$ given by theorem 2 is uniquely determined up to a positive scalar multiple. For assume that $\|\cdot\|'$ also holds (11). Then

$$\|v\|' > \|u\|' \quad \text{iff} \quad \|v\| > \|u\| \quad \text{or}$$

$$\|v\|' = \|u\|' \quad \text{iff} \quad \|v\| = \|u\|.$$

Fix e s.t. $\|e\| = 1$. Then $\|v/\|v\|\| = \|e\| = 1$ implies

$$\left\|\frac{v}{\|v\|}\right\|' = \|e\|' \quad \text{which yields} \quad \|\cdot\|' = \|e\|' \|\cdot\|.$$

Finally assume $u \leq v$. A5a) implies $u \in \Psi(V(\lambda,a))$. Hence $v \gtrdot^* u$. $\|u\| > \|v\|$, i.e. $u \gtrdot\gtrdot v$, contradicts A1).

$$\text{q.e.d.}$$

Let us briefly illustrate how far the consideration of finite environments would have taken us in the proofs of this section. For $\mathbf{V}^{fin} \subseteq \mathbf{V}$ we could easily enforce re-

mark 5:

Remark 5': Let $V \in \mathbf{V}$ whenever $V \subseteq U$, $|V| = 3$. Then SARP

implies transitivity of $\lesssim$.

Proof: Let $u \lesssim v \lesssim w$, $V' := \{u,v,w\} \in \mathbf{V}$. Suppose $w \in \Psi(V')$
thus $w <^* v$. As $w \gtrsim v$ v must be in $\Psi(V')$ if $w \in \Psi(V')$.
By repeating the arguement for the alternative assumption
that $v \in \Psi(V')$ holds we obtain $u \in \Psi(V')$ for all cases. Note
that $\Psi(V')$ must be non-empty. We end up with $u <^* w$ or
$u < w$ which amounts to $u \lesssim w$ by SARP.

This proof has been adapted from [22], (T.1). The importance
of remark 5' comes from the fact that transitivity of $\lesssim$
would let the central lemma melt down to the following
lines:

If $\underline{b}(u,v) < \bar{b}(u,v)$ for some $u,v \in U$ then b^1, b^2 could
be chosen holding $\underline{b}(u,v) < b^1 < b^2 < \bar{b}(u,v)$. By definition
of $\underline{b}$, $\bar{b}$ we obtain $b^1 u \gtrsim v$, $b^2 u \lesssim v$ and hence by the
above remark 5' $b^1 u \gtrsim b^2 u$. On the other hand PARETO
optimality (A2) requires $\{b^1 u\} = \Psi(\{b^1 u, b^2 u\})$ and thus
$b^1 u \lesssim^* b^2 u$ which is a contradiction.

Assuming $\mathbf{V}^{fin} \subseteq \mathbf{V}$ in this section we could still prove
equality of $\underline{b}(\cdot,\cdot)$ and $\bar{b}(\cdot,\cdot)$ on $(U\backslash 0) \times (U\backslash 0)$ and thus define
$b(\cdot,\cdot)$. But we could not derive the continuity (from below
and above) of $\lesssim$. A counter example is given by $\Psi^{|\cdot|,n}$, last
section.

5. ILLUSTRATIONS

According to theorem 2 the rational planner will make social choices as if he is minimizing the vector of individual payoffs with respect to a pre-given norm. Special interest naturally deserve those SCCs that are induced by p-norm minimization:[1]

$$\psi^p := \psi^{\|\cdot\|_p} : \mathbf{V} \to \mathbb{R}^n_+ \qquad (1 \le p \le \infty).$$

Let us take a brief look at the cases $p = 1, 2, \infty$.

$$u \in \psi^1(V) \quad \text{iff} \quad \sum_{i \in \Omega} u_i = \min_{v \in V} \sum_{i \in \Omega} v_i .$$

A planner applying ψ^1 thus reveals the classical utilitarian position, attaching equal social weights to all individuals concerned.

$$u \in \psi^\infty(V) \quad \text{iff} \quad \max_{i \in \Omega} u_i = \min_{v \in V} \max_{i \in \Omega} v_i .$$

A planner deciding according to ψ^∞ would seem to follow the RAWLSian maximin principle of distributive justice: make the worst-off best-off. Finally

$$u \in \psi^2(V) \quad \text{iff} \quad |u| = \min_{v \in V} |v|$$

the latter being equivalent to

$u \cdot u = \min u \cdot V$ whenever V is a subset of $\mathbb{R}^n_+$

that is convex from below. Applying Ψ^2 amounts to

minimizing the social welfare function $\lambda \cdot v$ $(v \in V)$ where

the weight vector $\lambda = \lambda^V$ is *endogenously* determined to be

proportionate to the solution u:

$\lambda \cdot u = \min \lambda \cdot V$, $\lambda = \text{const } u$.

The endogenous determination of social weights is typically

non-utilitarian. The last formula reminds of NASH' bargaining

solution for which weights are inversly related to payoffs:

$\lambda_i u_i = \text{const}$ $(i \in \Omega)$.

NASH's bargaining solution derives much appeal from the

fact that it uniquely satisfies some reasonable axioms.

Compared to that result our theorem 2 is rather poor. We

only know that - under the stated assumptions - a SCC Ψ

should be induced by norm-minimization. The theorem does not

say much about specific properties of such $\| \cdot \| = \| \cdot \|^\Psi$.

One could certainly add an anonymity (= symmetry) axiom.

However, we shall remain far from uniqueness results.

The SCCs ψ^p (p = 1,2,∞) allow nice geometric interpretations if applied to EUCLIDean location conflicts. A *EUCLIDean location conflict* ([16]) in $X = \mathbb{R}^m$ is given by (Y,d) where Y is a non-empty and closed subset of $\mathbb{R}^m$ and

$$d_i(x) = |x - z^i| \quad \text{for all} \quad x \in \mathbb{R}^m, \quad i \in \Omega \quad \text{and}$$
$$\text{appropriate} \quad z^i \in \mathbb{R}^m.$$

One might think of a planner whose task it is to determine some fair location for a public project like a park or a swimming-pool. The set of all social physical alternatives is the *planning space* $\mathbb{R}^m$. $Y \in \mathbf{Y}$ stands for some specific *planning area* taking feasibility constraints into regard. The public project has to be of the kind that everybody values positively. Note that by our sign convention $i \in \Omega$ is supposed to prefer the location x to y if $d_i(x) < d_i(y)$. Furthermore $\underline{u}_i := \inf d_i(X) = 0$, i.e. $0 \in \mathbb{R}^n$ is bliss-point. Hence $d(Y) \in \mathbf{V}_0$. For convex $Y \subseteq \mathbb{R}^m$ d(Y) is convex from below. As Y is closed by assumption d(Y) is also closed and $\psi^p(d(Y)) \neq \emptyset$ for $p \in [1,\infty]$.

Let us have a closer look at

$$x \in \Psi^p(Y,d) := d^{-1}(\psi^p(d(Y))) \cap Y \quad \text{for} \quad p = 1,2,\infty.$$

Such $x \in \Psi^p(Y,d)$ shall be called ψ^p-*locations*.
ψ^∞ - locations obviously generalize the geometric concept of the circumcentre. ψ^1 - locations are better known as FERMAT or WEBERian points. (Cf., say, [10].)

Remark 12: Let (Y,d), $Y \subseteq \mathbb{R}^m$ be EUCLIDean and $g = \frac{1}{n} \sum_{i \in \Omega} z^i$ the centre of gravity. Then

$$d(g) \in \Psi^2(d(Y)) \text{ whenever } g \in Y.$$

Proof:

Let $A \subseteq \Omega$ be maximal such that $g \notin \{z^i \mid i \in A\}$. W.l.o.g. $A \neq \emptyset$.

$$0 = \sum_{i \in A} (g-z^i) = \sum_{i \in A} |g-z^i| \frac{g-z^i}{|g-z^i|}$$

$$= \sum_{i \in A} |g-z^i| \, \text{grad}(|x-z^i|)|_{x=g}$$

$$= \text{grad}(d(g) \cdot d(x))|_{x=g} \; .$$

$d(g) \cdot d(\cdot)$ being a convex function we obtain

$$d(g) \cdot d(g) = \min_{x \in \mathbb{R}^m} d(g) \cdot d(x) \leq \min_{x \in Y} |d(g)| |d(x)|$$

from which the assertion follows. q.e.d.

In case that Y is convex $d(Y)$ is convex from below and $\Psi^2(d(Y))$ single-valued. A planner adopting Ψ^2 would thus select the centre of gravity g for EUCLIDean location conflicts whenever g is feasible $(g \in Y)$ and the planning area Y convex.

6. RELATED WORK IN THE THEORY OF REVEALED PREFERENCE

As noted before theorem 2 has its parallel in the theory of revealed preference. There are two major points of contact:

a) The technique of proof which follows the work of SAMUELSON [20], HOUTHAKKER [7] and UZAWA [26];

b) The kind of assumptions, particularly the allowance for social choice *correspondences* which relates theorem 2 to the joint work of L. HURWICZ and M.K. RICHTER [9].

The latter show a slightly stronger version of the following

Theorem 3 ([9]): Let Ψ be social choice correspondence for

$\mathbf{W}' := \{V(\lambda,a) \mid \lambda \gg 0,\ a \geq 0\}$. Assume A5b), A1) and convexity of $U := \underset{V \in \mathbf{W}'}{\cup} \Psi(V)$. Then there exist a real-valued function f on U such that

1) f is upper semicontinuous;

2) $f(u^1) \leq f(u^0)$ and $u^t := tu^1 + (1-t)u^0,\ t \in (0,1)$ imply $f(u^t) < f(u^0)$ or $u^i, u^t \in \Psi(V)$ for some $V \in \mathbf{W}'$ and some $i = 0,1$;

Moreover for all $V \in \mathbf{W}'$

3) there exists $u \in V \cap U$ s.t. $f(u) = \min f(V \cap U)$;

4) $f(u) = \min f(V \cap U)$ implies $u \in \Psi(V)$.

Thus under assumptions that are significantly weaker then
ours HURWICZ and RICHTER derive weaker results. Note that
they fail to prove full continuity of f and equivalence in
statement 4). Full continuity of the rationalizing function
f is a critical property even in the theory which excusively
deals with demand functions Ψ. It makes it necessary to
introduce some axiom (LIPSCHITZ continuity with respect
to income or non-inferiority of Ψ. Cf.[11].) which goes far
beyond those of theorem 3. Such an axiom enters theorem 2
by means of dilatation invariance (A3). The latter con-
siderably stregthens the mentioned non-inferiority property
which again implies income-LIPSCHITZ continuity. HURWICZ
and RICHTER demonstrate by means of a counter-example
that full continuity of f or equivalence in 4) cannot be
expected to hold under their assumptions.

Theorem 3 suggests that theorem 2 might be valid under still
weaker assumptions. It is an open question to me whether
theorem 2 could be proved more directly by making
use of theorem 3.

The technique of proof employed for theorem 2 and 3 com-
pletely differs from one another. The idea of our proof
is the one sketched by SAMUELSON in [20]. The technique
is adapted from the HOUTHAKKER-UZAWA approach ([7] and [26]).
A few additional comments are in order.

UZAWA deals with demand functions that induce monotone re-
vealed preferences. Thus his demand pattern allows no
"point of satiation" or "bliss point" like zero in our
context. The introduction of a bliss point is in turn
responsible for A5c) which has no parallel in [26]. UZAWA
shows that continuity of $\Psi(V(\lambda,a))$ with respect to λ and
a essenttially is a consequence of SARP (cf.
his theorem 2). This is also true in our case (remark 8)
except for situations where the bliss point comes in. For
continuity reasons we thus assume A5c).

A5b) is an immediate consequence of admitting correspon-
dences. It is trivially fulfilled for functions.

There is one point where UZAWA admits greater generality and
that is A3). We assume dilatation invariance for reasons
of interpretation whereas UZAWA requires that his demand
function meets a LIPSCHITZ coninuity condition with respect
to positive income.

UZAWA is not able to prove full continuity of the
indirectly revealed preference $\gg$ except for cases where
the vector-field correspondence $\Lambda(\cdot)$ reduces to a mapping
(cf. his theorem 3). Our results obviously imply such
continuity. It is an open question whether this discrepance
is primarily due to the strengthening of the LIPSCHITZ
condition to A3) or to the altered way of proving the central
lemma.

UZAWA "integrates" the vector-field induced by the "direct" or "demand-quantity" function $\Psi(\lambda,a)$ to obtain some income compensation function from which all desired properties of the revealed preference follow.[4] Our approach, on the other hand, follows SAMUELSON's idea [20] to "integrate" the vector- field correspondence $\Lambda(\cdot)$ which corresponds to an "indirect" or "demand-price" function. Hence our integration directly ends up with the wanted rationalizing norm. The existence of these two alternative approaches is better known from the "local" theory of recovering utility functions from demand functions. (Cf. L. HURWICZ, [8].)

NOTES

0. The term *correspondence* is used for set-valued mappings the images of which are non-empty.

1. $\| u \|_p = (\Sigma |u_i|^p)^{1/P}$ $(1 \leq p < \infty)$, $\| u \|_\infty = \max |u_i|$

2. Strictly speaking an equivalent version of WARP.

4. This statement is valid up to some minor logic gaps. Cf. [25], p.411.

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