Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 143

Homogeneous Games
with
Countably many Players

Joachim Rosenmüller

December 1985

H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

Abstract

A finite homogeneous n-person cooperative game allows for classifying the players (as well as fellowships and types) according to characters, called "dummy", "sum", and "step". Homogeneous representations of the game are (uniquely) defined by assigning arbitrary rates to the dummies and arbitrary surplus rates to the steps by which to exceed the total weight of their satellites. This way in particular the unique (minimal) representation of a finite homogeneous game can be defined (see [5], [9]).

This paper shows: for games with countably many players, there are five characters, as sums and steps split into improper and proper representatives respectively. However, games with dummies or improper steps are essentially finite. It is then verified that homogeneous representations are obtained as in the finite case, essentially by specifying the surplus weights of the (proper) steps.

Thus, the finite theory of homogeneous games has a countable counterpart.

0 Introduction, Notations

Let $k = (k_1,\dots,k_{r+1}) \in \mathbb{N}_0^{r+1}$ be such that

(1) $$k_1,\dots,k_r \geq 1,\ k_{r+1} \geq 0\ ;$$

a vector $s \in \mathbb{N}_0^{r+1}$ is a profile feasible for k if $s \leq k$.
A characteristic function (cf.) (for k) is a mapping on the feasible profiles of k:

$$v : \{s \mid s \in \mathbb{N}_0^{r+1},\ s \leq k\} \to \{0,1\}$$

The extension of k is the vector

$$\tilde{k} = (1,\dots,1)$$

of length $\sum_{i=1}^{r+1} k_i$. To any profile $\tilde{s}$ feasible for $\tilde{k}$ there corresponds a profile s feasible for k which is defined by

(2) $$s_i := \sum_{k_{i-1} < \rho \leq k_i} \tilde{s}_\rho \qquad (i=1,\dots,r+1)$$

(where $k_0 := 0$) and the extension $\tilde{v}$ of v is the cf. (for $\tilde{k}$) defined by

(3) $$\tilde{v}(\tilde{s}) = v(s) \qquad (\tilde{s} \leq \tilde{k}).$$

Whenever $\hat{v}$ is a cf. for some vector $e = (1,\dots,1) \in \mathbb{N}^n$, then $(\hat{v},e)$ is called a game and any (v,k) such that $(\tilde{v},\tilde{k}) = (\hat{v},e)$ extends to $(\hat{v},e)$ or has the extension $(\hat{v},e)$.

Next, suppose that $g = (g_1,\dots,g_{r+1}) \in \mathbb{N}_0^{r+1}$ satisfies

(4) $$g_1 \neq 0;\ g_{r+1} = 0;\ \ g_1 \geq g_2 \geq \dots \geq g_{r+1}\ .$$

Then, if k satisfies (1), we consider a function also denoted by g, defined on the profiles feasible for k by

(5) $$g(s) = \sum_{i=1}^{r+1} s_i g_i \qquad (s \leq k)$$

g is "additive", i.e. $g(s) + g(t) = g(s+t)$ whenever s,t and $s+t$ are feasible, and thus sometimes called a <u>measure</u>. However, the term measure will also be used (somewhat sloppily) for the pair $M = (g,k)$.

Clearly, the <u>extension</u> of g is given by

(6) $$\tilde{g} = (\underbrace{g_1,\ldots,g_1}_{k_1}, \underbrace{g_2,\ldots,g_2}_{k_2},\ldots,\underbrace{g_r,\ldots,g_r}_{k_r},\underbrace{0,\ldots,0}_{k_{r+1}})$$

A function corresponds to $\tilde{g}$ via a procedure indicated by (5); it is defined on the feasible profiles of $\tilde{k}$ and clearly can as well be obtained by

(7) $$\tilde{g}(\tilde{s}) = g(s) \qquad (\tilde{s} \leq \tilde{k}) ,$$

cf. (2) and (3). Accordingly, we call $\tilde{M} = (\tilde{g},\tilde{k})$ the extension of $M = (g,k)$.

Let k satisfy (1) and let g satisfy (4) and suppose that $\lambda \in \mathbb{N}$ is such that $\lambda \leq g(k)$.

Then $(g;k;\lambda) = (M;\lambda)$ <u>generates</u> a cf. $v = v_\lambda^M$ for k by

$$v(s) = \begin{cases} 1 & g(s) \geq \lambda \\ 0 & g(s) < \lambda \end{cases} \qquad (s \leq k)$$

and a cf. generated in this way by some (M,λ) is a "weighted majority" cf. Various pairs (M,λ) may generate the same cf., but it is easy to see that for the extensions we have

(8) $$\widetilde{v_\lambda^M} = v_\lambda^{\tilde{M}} .$$

Thus, there is the equivalence class of all (M,λ) such that (v_λ^M,k) extends to the same game, say $(\hat{v},e)$; any element of this equivalence class is said to be a <u>representation</u> of the game $(\hat{v},e)$.

The term "weighted majority" may be attached to games having a representation (M,λ), however, we shall drop this term alltogether as we are only concerned with weighted majority games.

Thus, any two representations of a game have the properties that the extensions of the generated cf.'s equal the game and that their extensions generate the game (cf. (9)).

There is a natural partial order defined on all representations of a game as follows:
Write

(9) $$(g,k,\lambda) \precsim (g',k',\lambda')$$

if $r \leq r'$ and $\tilde{g} \leq \tilde{g}'$ (coordinate-wise).

A <u>minimal representation</u> is then a representation such that no smaller one (in the sense of $\precsim$) exists.

The familiar framework of n-person cooperative game theory is obtained by identifying the feasible profiles for some $e = (1,\dots,1)$ with subsets of $\{1,\dots,n\}$. Then $\{1,\dots,n\}$ represents the "players" and $S \subseteq \{1,\dots,n\}$ denotes a coalition while $\hat{v}$ yields the usual characteristic function $\hat{w}$ defined on the coalitions by

$$\hat{w}(S) = \hat{v}(1_S)$$

where 1_S is the indicator profile of S; $1_S(i) = 1$ $(i \in S)$, $1_S(i) = 0$ $(i \notin S)$.

If k is arbitrary but satisfies (1), then we may take $n = \sum_{i=1}^{r+1} k_i$ and put

$$\{1,\dots,n\} = \bigcup_{i=1}^{r+1} K_i$$

with $K_j := \{\rho \mid \sum_{i=0}^{j-1} k_i < \rho \leq \sum_{i=0}^{j} k_i\} \subseteq \{1,\dots,n\}$ $(k_0 := 0)$. Then

$$w(S) = v(|S \cap K_1|,\dots, |S \cap K_{r+1}|)$$

provides a cf. in the usual sense and this function depends essentially on the game only.

Refering to this framework, given $k \in \mathbb{N}_0^{r+1}$ such that (1) is satisfied, we call the indices $i = 1,\dots,r+1$ "fellowships" (and k is interpreted as a distribution of players over the fellowships : there are k_i players of fellowship i). k has length $n = \sum_{i=1}^{r+1} k_i$ and $\omega = 1,\dots,n$ are the <u>players</u>; thus, for k players and fellowships coincide. If $g \in \mathbb{N}_0^{r+1}$ satisfies (4), then g_i is the <u>weight</u> of fellowship i. We also say that players ω, $\sum_{i=1}^{j-1} k_i < \omega \leq \sum_{i=1}^{j} k_i$ "belong" to fellowship i. Of course, given various representations of a game a player may belong to different fellowships.

A <u>type</u> is a subset of fellowships, consistently defined for all (v,k) extending to the same game. More precisely, i and j belong to the same <u>type</u> <u>w.r.t.</u> v if, for any profile s such that $s_i > 0$, $s_j < k_j$ (or $s_i < k_i$, $s_j > 0$) we have

$$v(s) = v(s - e^i + e^j)$$

$$(\text{and } v(s) = v(s - e^j + e^i) \text{ respectively}).$$

It is not hard to see that this is an equivalence releation (we consider only weighted majority games!). In particular, players with the same weight belong to the same type w.r.t. $\tilde{v}$ and moreover, two players belong to the same type (w.r.t. $\tilde{v}$) if and only if their fellowships belong to the same type (w.r.t. v); i.e. the decomposition into types depends only on the game.

It is well known that the representations induce an ordering of the types, that is, if two players belong to a different type and, for some representation (M,λ) one player has a smaller weight than the other one, then this will be the case for <u>all</u> representations. This is the reason for restricting our attention to representations (M,λ) satisfying (1) and (3). Let us, therefore, introduce the notation

$$\mathfrak{M}^r := \{(g,k) \in \mathbb{N}_0^{2(r+1)} \mid g \text{ satisfies (3)}, \ k \text{ satisfies (1)}\} \quad (r \in \mathbb{N})$$

(10) $$\mathfrak{M}^o := \{(0,k_0) \in \mathbb{N}_0^2\}$$

$$\mathfrak{M} = \bigcup_{r=0}^{\infty} \mathfrak{M}^r$$

(the measure corresponding to $(0,0) \in \mathfrak{M}^o$ is interpreted as the trivial measure on the empty set).

Certain projections will be denoted as follows.
If $M = (g,k) \in \mathfrak{M}^r$, then for $1 \leq i_0 \leq r$ and $1 < c < k_{i_0}$ let

(11) $$M_{i_0}^c = (g,k)_{i_0}^c = (g_{i_0},\dots,g_{r+1};\ k_{i_0}-c,\dots,k_{r+1}) \in \mathfrak{M}^{r-i_0+1}$$

while for $c = k_{i_0}$

(12) $$M_{i_0+1} = M_{i_0}^{k_{i_0}} = (g_{i_0+1},\dots,g_{r+1};\ k_{i_0+1},\dots,k_{r+1}) \in \mathfrak{M}^{r-i_0} \quad .$$

The corresponding "additive functions" will be denoted accordingly, thus e.g. $g^c_{i_o}$ corresponds to $(g_{i_o},\dots,g_{r+1})$ and is always understood to live on $(k_{i_o}-c,\dots,k_{r+1})$, i.e., to be defined for the feasible profiles of this vector.

Finally, as a matter of convenience, we shall use the letter m to indicate "total mass", i.e., if $M = (g,k)$ then $m = g(k) = \sum_{i=1}^{r+1} k_i g_i$. Indices are carried accordingly, thus, e.g.,

$$m^c_{i_o} = (k_{i_o}-c)\, g_{i_o} + \sum_{i=i_o+1}^{r+1} k_i g_i$$

(13)

$$m_{i_o-1} = m^{k_{i_o}}_{i_o} = \sum_{i=i_o+1}^{r+1} k_i g_i \; .$$

1 Homogeneous Games

A pair $M = (g,k)$ is said to be homogeneous w.r.t. $\lambda \in \mathbb{N}$ if

(1) $m \geq \lambda$

(2) For any $s \leq k$, $g(s) > \lambda$ there is $t \leq s$ such that $g(t) = \lambda$

We write $M \text{ hom } \lambda$ in this case; also $M \text{ hom}_0 \lambda$ means that either $M \text{ hom } \lambda$ or $m < \lambda$.

A game is homogeneous if there exists a homogeneous representation, i.e., a representation $(M,\lambda) = (g,k,\lambda)$ s.t. $M \text{ hom } \lambda$. This term (in the framework of Game Theory) has been introducted by von NEUMANN-MORGENSTERN.

We are now going to shortly review the main results of [8] for our present purpose in order to use the structure of homogeneous games exhibited there.

The following characterization of homogeneity we shall refer to as the "BASIC LEMMA".

BASIC LEMMA 1.1. Let $M = (g,k) \in \mathfrak{M}^r$ and $\mathbb{N} \ni \lambda \leq m$. Then $M \text{ hom } \lambda$ if and only if there is $i_0 \in \{1,\dots,r\}$ and $c \in \mathbb{N}$, $1 \leq c \leq k_{i_0}$, such that the following holds true:

(3) $$\lambda = \sum_{i=1}^{i_0-1} k_i g_i + c g_{i_0}$$

(4) $$M_{i_0}^c \text{ hom}_0 g_j \quad (1 \leq j \leq i_0-1)$$

(5) $$M_{i_0+1} \text{ hom}_0 g_{i_0}$$

A characteristic function is essentially determined by the minimal-winning profiles. In particular, if $(M,\lambda) = (g,k,\lambda)$ is a homogeneous representation, then the min-win profiles have exactly weight λ and are minimal with this property. The BASIC LEMMA may be interpreted as follows: if we start collecting players according to weight (i.e., members of large fellowships first), then the weight of the resulting profile must exactly hit the majority level λ. Moreover, the remaining followships, having total mass $m^c_{i_0}$ available, are engaged in a series of homogeneous "replacement" or "satellite" games, represented by $(M^c_{i_0}, g_j)$ $(j=1,\dots,i_0-1)$ and (M_{i_0-1}, g_{i_0}). By this procedure, smaller players will "substitute" larger ones thus entering successively min-win coalitions.

In particular, the profile of the lexicographically first min win profile (the lex-max-profile) is uniquely determined by $M = (g,k) \in \mathfrak{M}^r$ and λ, this profile is

(6) $$s^\lambda = s^\lambda_M = (k_1,\dots,k_{i_0-1}, k_{i_0}-c, 0,\dots,0)$$

The BASIC LEMMA, among other properties, enables us to define the characters of fellowships and the satellite measures (and satellite games) for certain fellowships with appropriate character. This is performed as follows by an induction procedure.

We proceed by inductively defining two mappings κ and $\mathcal{M}$ on the domain

(7) $$\{(M,\lambda) \in \mathfrak{M} \times \mathbb{N} \mid M \in \mathfrak{M}^r, r \geq 1, M \text{ hom } \lambda\}$$

The range of κ is $\mathbb{N}$ while the range of $\mathcal{M}$ is $\bigcup_{j=1}^{\infty} \underbrace{\mathfrak{M} \times \dots \times \mathfrak{M}}_{j}$,

more precisely, we shall require $\kappa(M,\lambda) \in \{i_1,\dots,i_{r+1}\}$ if

$M = (g_{i_1},\dots,g_{i_{r+1}};\ k_{i_1},\dots,k_{i_{r+1}})$ and

(8) $$\mathcal{M}(M,\lambda) = (\mathcal{M}^{(i_1)}(M,\lambda),\dots,\ \mathcal{M}^{\kappa(M,\lambda)}(M,\lambda))$$
with $\mathcal{M}^{i_1}(M,\lambda) \in \mathfrak{M}$.

(Note that $M \in \mathfrak{M}^r$ may have coordinates that are not necessarily indexed by $1,\dots,r$; e.g., $M^c_{i_o} \in \mathfrak{M}^{r-i_o+1}$ carries indices $i_o,\dots,r+1$.)

<u>1st STEP:</u> For $r = 1$ put $\kappa(M,\lambda) = 1$ and
$\mathcal{M}(M,\lambda) = \mathcal{M}^{(1)}(M,\lambda) = (g_2,k_2) \in \mathfrak{M}^{(0)}$
if $M = (g_1,g_2;\ k_1,k_2)$. (Analogously, if $M = (g_{i_1},g_{i_2};\ k_{i_1},k_{i_2})$).

<u>2nd STEP:</u> Let $M = (g_1,\dots,g_{r+1};\ k_1,\dots,k_{r+1}) \in \mathfrak{M}^r$, $r \geq 2$
and $\lambda \in \mathbb{N}$; suppose M hom λ. Let $i_o \in \{1,\dots,r\}$
and $c \in \mathbb{N}$, $1 \leq c \leq k_{i_o}$ be specified by the BASIC LEMMA. Put

(9) $$\mathcal{M}^{(j)}(M,\lambda) = M^c_{i_o} \qquad 1 \leq j < i_o$$

(10) $$\mathcal{M}^{(j)}(M,\lambda) = M_{i_o+1} \qquad j = i_o$$

<u>3rd STEP:</u> Let us write $M^{(j)} := \mathcal{M}^{(j)}(M,\lambda)$ $(j \leq i_o)$, recall that $\dot{m}^{(j)}$ denotes "total mass" of $M^{(j)}$.

Now, if

$$m^{(j)} < g_j \qquad (1 \leq g_j \leq i_o)$$

then put

$$\kappa(M,\lambda) := i_o$$

$$\mathcal{M}(M,\lambda) = (\mathcal{M}^{(1)}(M,\lambda),\ldots,\mathcal{M}^{i_o}(M,\lambda))$$
$$= (M^{(1)},\ldots,M^{(i_o)})$$

and our definition is complete. (If the coordinates of M are not indexed by $1,\ldots,r+1$, the generalization is obvious)

<u>4th STEP:</u> Otherwise let

(11) $$J = J(M,\lambda) = \{j \mid 1 \leq j \leq i_o,\ m^{(j)} \geq g_j\} \neq \emptyset$$

Now, by induction hypothesis, κ and $\mathcal{M}$ are defined for $(M^{(j)}, g_j)$ $(j \in J)$ (as $M^{(j)}$ hom g_j by the BASIC LEMMA).

As (for $j < i_o$ and $c < k_{i_o}$)

$$M^{(j)} = (g_{i_o},\ldots,g_{r+1};\ k_{i_o}-c,\ldots,k_{r+1})\ ,$$

we may write

(12) $$\mathcal{M}(M^{(j)},g_j) = (\mathcal{M}^{(i_o)}(M^{(j)},g_j),\ldots,\ \mathcal{M}^{\kappa^{(j)}}(M^{(j)},g_j))$$
$$=: (M^{(i_o,j)},\ldots,\ M^{(\kappa^{(j)},j)})$$

where $\kappa^{(j)} = \kappa(M^{(j)},g_j) \in \{i_o,\ldots,r+1\}$, that is

(13) $$M^{(i,j)} := \mathcal{M}^{(i)}(M^{(j)},g_j) \qquad (1 \leq j < i_o,\ i_o \leq i \leq \kappa^{(j)})\ .$$

Similarly, for $j = i_o$, we have

(14) $$\mathcal{M}(M^{(j)}, g_j) =: (M^{(i_o+1,j)}, \ldots, M^{(\kappa^{(j)},j)})$$

with

(15) $$M^{(i,j)} := \mathcal{M}^{(i)}(M^{(j)}, g_j) \quad (j=i_o,\ i_o+1 \leq i \leq \kappa^{(j)})$$

Now, define

(16) $$\kappa(M,\lambda) := \max \{\kappa^{(j)} \mid j \in J\,(M,\lambda)\}$$

and for $i_o < i \leq \kappa\,(M,\lambda)$

(17) $$J^{(i)} = J^{(i)}(M,\lambda) = \{j \in J \mid i \leq \kappa^{(j)}\}$$

(18) $$\mathcal{M}^{(i)}(M,\lambda) := \max \{M^{(i,j)} \mid j \in J^{(i)}\}$$

where the last max is to be interpreted w.r.t. either the lexicographic or the coordinatewise partial ordering. This completes the definition of κ and $\mathcal{M}$.

5th STEP: Again write $M^{(i)} := \mathcal{M}^{(i)}(M,\lambda)$. If $i > \kappa\,(M,\lambda)$ then the character of i is dummy. If $i \leq \kappa\,(M,\lambda)$, then i is step whenever $m^{(i)} < g_i$ and a sum whenever $m^{(i)} \geq g_i$.

$M^{(i)} = \mathcal{M}^{(i)}(M,\lambda)$ is the satellite measure of i and in case that i is a sum, the game represented by $(M^{(i)}, g_i)$ is the satellite game of i. By the BASIC LEMMA (and induction) it is seen that $M^{(i)}\ \mathrm{hom}_o\ g_i$ (see [9]), thus, satellite games are homogeneous.

(For the sake of completeness we have now to add that the generalization of the procedure is obvious in case that $M = (g_{i_1},\dots,g_{i_{r+1}}; k_{i_1},\dots,k_{i_{r+1}}) \in \mathfrak{M}^r$).

If i is a sum, we denote by $s^{(i)} := s^{M^{(i)}}_{g_i}$ the lexicographically first min-win coalition in $(M^{(i)}, g_i)$ (cf.(6)), $s^{(i)}$ denotes the <u>substitutes</u> of i . If i is a step and $M^{(i)} = g^{(i)}, k^{(i)})$ then we put $s^{(i)} := k^{(i)}$. Thus, $g(s^{(i)}) = g_i$ if i is a sum and $g(s^{(i)}) + 1 \leq g_i$ if i is a step.

<u>Remark 1.2.</u> (see [9] for the details)

1. Given a homogeneous representation (M,λ) of a game, fellowships decompose into characters, say

$$\{1,\dots,r+1\} = \Sigma + T + D = \Sigma(M,\lambda) + T(M,\lambda) + D(M,\lambda),$$

where Σ denotes the set of sums, T steps, and D dummies. In particular,

$$D = D(M,\lambda) = \{i \mid i \geq \kappa(M,\lambda) + 1\}$$

Hence, if $D^{(j)}$ denotes the dummies of $(M^{(j)}, g_j)$, then e.g. (17) reads

$$J^{(i)} = \{j \in J \mid i \notin D^{(j)}\}$$

and (16) is interpreted as

$$D = \bigcap_{j \in J} D^{(j)}.$$

That is, a fellowship is a dummy if and only if it is a dummy w.r.t. every $(M^{(j)}, g_j)$ such that $m^{(j)} \geq g_j$.

Similarly we note that $i > i_0$ is a sum if and only if it is a sum w.r. to some $(M^{(j)}, g_j)$ with $j \in J$. i is a step if and only if it is a step w.r.t. any $(M^{(j)}, g_j)$ $(j \in J)$ where it is no dummy, and if there is at least some j with this property.

2. It follows that $\kappa(\widetilde{M},\lambda) = \sum_{i=1}^{\kappa(M,\lambda)} k_i$; i.e.,

   a player is a dummy if and only if his fellowship is a dummy in any (hom) representation. (Note that players are assigned to characters as in $(\widetilde{M},\lambda)$ all fellowships and players coincide.) Dummy players are exactly the dummy players in the ordinary sense. (i.e., those i for which there is no min-win profile s with $s_i > 0$).

3. A player is a step if and only if in any (hom) representation his fellowship is a step.

4. If the smallest player of a type is a sum, then so are all players of this type.

5. The dummy fellowships in any (hom) representation form a type. This type is suitably called a dummy as well.

6. A type is called a sum if all fellowships (w.r.t. any hom representation) are sums. In particular, if different fellowships have different weights ("reduced representation") then a fellowship with character sum constitutes a type and vice versa.

7. The remaining types are called steps. Thus, types can as well be classified according to the characters dummy, step, and sum.

8. Suppose, (M,λ) is a hom representation of some game. Suppose also, that the following quantities are given: for $i \in D = D(M,\lambda)$ an arbitrary

weight $g_i' \in \mathbb{N}_0$ $(i \in D)$ (decreasing in i) and for $i \in T = T(M,\lambda)$ a natural number $\Delta_i \in \mathbb{N}$ $(i \in T)$ (decreasing in i). Then, we may define $(M',\lambda') \in \mathfrak{M} \times \mathbb{N}$ recursively by assigning weights g_i' to the non-dummies via

(19) $$g_i' := g'(s^{(i)}) \qquad (i \in \Sigma = \Sigma(M,\lambda))$$

(20) $$g_i' := g'(s^{(i)}) + \Delta_i \qquad (i \in T = T(M,\lambda))$$

and by putting

(21) $$\lambda' := g'(s_\lambda^M) \ , \quad k' = k$$

In this case, (M',λ') is a homogeneous and monotone representation of the same game, i.e.

$$v_{\lambda'}^{M'} = v_\lambda^M \ .$$

(M',λ') is said to be compatible with (M,λ).

9. In particular, by putting $\bar{g}_i = 0$ $(i \in D)$, $\Delta_i = 1$ $(i \in T)$ we obtain the minimal (homogeneous) representation by grouping fellowships of equal weight together. It is uniquely defined by either to be minimal w.r.t. the partial order (cf. (9) in SEC.1) or w.r.t. total mass $m = g(k)$. Given the minimal representation, a type equals a fellowship and the characters of a type and the corresponding fellowship coincide. The unique minimal representation is also defined by the requirement that

(22) $$\kappa(\bar{M},\bar{\lambda}) = \bar{r}$$

(23) $$\bar{g}_i = \bar{g}(\bar{s}^{(i)}) \qquad (i \in \Sigma\ (\bar{M},\bar{\lambda}))$$

(24) $$\bar{g}_i = \bar{g}(\bar{s}^{(i)}) + 1$$

(26) $$\bar{\lambda} = \bar{g}(s_{\bar{\lambda}}^{\bar{M}})$$

(with $\bar{s}^{(i)} = s_{\bar{g}_i}^{\bar{M}^{(i)}}$ is referring to $(\bar{M},\bar{\lambda})$ etc.)

10. Finally, let us note that multiplication with a constant does not change characters, satellite measures, substitutes, etc....

2 The Projection Lemma

During this section we want to study the effect of adding a fellowship. To this end, a few auxiliary statements are necessary which will we treated in 2.1. - 2.4. Finally, Theorem 2.5 shows, that "cutting off the smallest fellowship" does not affect certain characters given the appropriate conditions. Of course, this means also, that adding a smallest fellowship does not affect certain characters.

We shall write $\alpha^+ = \min(\alpha, 0)$ for $\alpha \in \mathbb{R}$.

As we want to deal with measures M (or g) and their projections e.g. $M^c_{i_o}$ simultaneously a slight changement w.r.t. our conventions is necessary. E.g., if

$$s = (s_1, \ldots, s_{r+1})$$

is a min-win profile of a representation (M, λ) then

$$\text{(1)} \qquad ((s_{i_o} - c)^+, s_{i_o+1}, \ldots, s_{r+1})$$

is a feasible profile for the k-coordinates of $M^c_{i_o}$ while

$$\text{(2)} \qquad (s - s^\lambda)^+ = (0, \ldots, 0, (s_{i_o} - c)^+, s_{i_o+1}, \ldots, s_{r+1})$$

formally is not. However, as we regard the measure $g^c_{i_o}$ to be a restriction of g, we sometimes want it to be defined on the profile (2) as well. Thus, we shall also consider (2) to be "feasible for $M^c_{i_o}$" - a slight inconsistency which saves some formalities.

As a further notational convenience we shall generally write $M^{(i)} = M^{(i)}(M, \lambda)$ if the argument (M, λ) is fixed.

Having this in mind, we state

Lemma 2.1. ("The canonical decomposition")

Let $M \in \mathfrak{M}^r$ and M hom λ. Let i_0 and c be specified by the BASIC LEMMA. Also, let s be a min-win profile of (M,λ) and put $\mathring{s} := (s-s^\lambda)^+$ as in (2) such that

$$(3) \qquad s = \sum_{i=1}^{i_0-1} s_i e^i + (c \wedge s_{i_0}) e^{i_0} + \mathring{s}$$

where e^i is the i'th unit vector. If $s \neq s^\lambda$, then there is $d = (d_j)_{j \in J(M,\lambda)}$, $d_j \in \mathbb{N}_0$, $d \neq 0$, and for every $j \in J$ with $d_j \neq 0$ a set of profiles $s^{j\kappa}$, $\kappa = 1,\dots,d_j$, such that

$$(4) \qquad \mathring{s} = \sum_{j \in J(M,\lambda)} \sum_{\kappa=1}^{d_j} s^{j\kappa}$$

and

(5) $\qquad s^{j\kappa}$ is a min-win profile for $(M^{(j)}, g_j)$

i.e., in particular $g^{(j)}(s^{j\kappa}) = g_j$.

Thus, if a min-win profile is not lex-max, then some members of the larger fellowships are missing, but the mass of the smaller players must appear in suitable multiples of the weights g_j of the larger players. That is, $(s-s^\lambda)^+ = \mathring{s}$ is decomposed into min-win profiles of certain satellites.

The proof is easy (see Remark 3.5 of [9]).

Lemma 2.2.

Let $\mathfrak{M}^r \ni M$ hom $\lambda \in \mathbb{N}_0$ and let $i \notin D(M,\lambda)$ (i.e., $i \leq \kappa(M,\lambda)$).

If, for every min-win profile s with $s_i > 0$, we have necessarily $s_r > 0$, then

(6) $$M^{(i)} = (g_r, g_{r+1}; d, k_{r+1})$$

with $d < k_r$ or

(7) $$M^{(i)} = (g_{r+1}, k_{r+1}).$$

That is, as r "cannot be separated from i", not all members of this fellowship can participate in i's satellite measure.

The proof proceeds by induction; the lemma is trivial for $r = 1$ or for $r \geq 2$ and $J(M,\lambda) = \emptyset$. Assume $r \geq 2$ and $J(M,\lambda) \neq \emptyset$.

Given M and λ, let i_0 and c be as usual. First of all, consider the case that $i \leq i_0$. Then, in particular, the lex-max coalition $s = s^M$ satisfies $s_i > 0$, thus $s_r > 0$ and hence $i_0 = r$. As $M^{(i)} = M_{i_0-1}$ or $M^{(i)} = M^c_{i_0}$, (6) or (7) is obviously true.

Now, let $i > i_0$. By induction hypothesis, the statement is true for shorter measures. As i is no dummy, there is $j \in J(i)$ such that is no dummy for $(M^{(j)}, g_j)$ and (say, for $j < i_0$)

$$M^{(i,j)} = M^{(i)}\ (M^{(j)}, g_j)$$

is well defined (cf. SEC. 1). Consider an arbitrary min-win profile s'^j for $(M^{(j)}, g_j)$ s.t. $s'^j_i > 0$. As the profile

$$s'' = s^{\lambda} - e^j + (0,\dots,0, s'^j)$$

(e^j = j-th unit vector) is min-win for (M,λ) and satisfies $s''_i > 0$, we conclude that $s'^j_r > 0$. By induction hypothesis, $M^{(i,j)}$ must, therefore satisfy (6) or (7), suitably rewritten. As this is so for every $j \in J(i)$, $M^{(i)}$ as the max over all $M^{(i,j)}$ $(j \in J(i))$ (cf. (18) of SEC. 1) has to satisfy (6) or (7) as well, q.e.d.

Lemma 2.3.

Let $\mathfrak{M}^r \ni M$ hom λ and let $i \notin D(M,\lambda)$ (i.e., $i \leq \kappa(M,\lambda)$). If there is a min-win profile s with $s_i > 0$ and $s_r = 0$, then

(8) $$M^{(i)} = (g_1,\ldots,g_{r+1};\ d,\ldots,k_{r+1})$$

with $1 < r$ or $1 = r$ and $d = k_r$.

Proof: For $r = 1$ or $J = \emptyset$ our claim is trivial. For $r \geq 2$ and $J \neq \emptyset$ it is also trivial for $i \leq i_0$ and for $i > i_0$ we proceed again by an inductive argument.

Let s be a min-win profile such that $s_i > 0$, $s_r = 0$. As $i > i_0$, we have $s \neq s^{\lambda}$. Decompose $\overset{\circ}{s} = (s-s^{\lambda})^+$ canonically according to Lemma 2.1., that is, find d and $(s^{j\kappa})\ldots$ such that

$$\overset{\circ}{s} = \sum_{j \in J} \sum_{\kappa=1}^{d_j} s^{j\kappa}$$

(assume $s^{j\kappa}$ to be augmented by 0's so the length is r).
As $s_r = 0$, $s_r^{j\kappa} = 0$ for all j,κ and, as $s_i > 0$, there must be some $j \in J$ and $\hat{\kappa}$, $1 \leq \hat{\kappa} \leq d_j$ such that $s_i^{j\hat{\kappa}} > 0$. As $g^{(j)}(s^{j\kappa}) = g_j$, and $s^{j\kappa}$ is min-win for $(M^{(j)}, g_j)$, i is no dummy for $(M^{(j)}, g_j)$. Therefore, we may apply the induction hypothesis for $(M^{(j)}, g_j)$ and $M^{(i,j)}$ satisfies the statement of the lemma. But then $M^{(i)} = \max_{j \in J(i)} M^{(i,j)}$ does so a fortiori, q.e.d.

Definition 2.4. The projection $P : \mathfrak{M}^r \to \mathfrak{M}^{r-1}$ is defined for $r \geq 1$ by

(9) $$P\,(g_1,\ldots,g_r,0;\ k_1,\ldots,k_r,\ k_{r+1}) = (g_1,\ldots,g_{r-1},0;\ k_1,\ldots,k_r)$$

Theorem 2.5. (The projection lemma)

For $r \geq 2$ let $M \in \mathfrak{M}^r$ and $\lambda \in \mathbb{N}$ be such that M hom λ and PM hom λ. Asumme $i \notin D(PM,\lambda)$. Then $i \notin D(M,\lambda)$ and

$$(10) \qquad M^{(i)}(PM,\lambda) = P(M^{(i)}(M,\lambda))$$

Proof:

1st STEP: For $r = 2$ the result is obvious. We may therefore assume $r > 0$ and use an inductive argument at some stage of our proof. Note also that $i \notin D(M,\lambda)$ is trivial in any case.

2nd STEP: Let i_0 and c be determined by the BASIC LEMMA w.r.t. (M,λ). We must necessarily have $i_0 \leq r-1$. For, if $i_0 = r$ the total mass of PM would be

$$\sum_{i=1}^{r-1} k_i g_i < \sum_{i=1}^{r-1} k_i g_i + cg_{i_0} = \lambda$$

as $c \geq 1$, which contradicts PM hom λ. It is then seen at once that i_0 and c, when determined by means of the BASIC LEMMA w.r.t. (PM,λ) are the same quantities.

3rd STEP: Now, pick j such that $1 \leq j \leq i_0$. Then we have by definition

$$M^{(j)}(M,\lambda) = \begin{cases} M^c_{i_0} \\ M_{i_0-1} \end{cases}$$

But as i_0 and c are the same w.r.t. (M,λ) and (PM,λ) it is seen at once that omitting the smallest fellowship $r + 1$ and replacing the weight of fellowship r by 0 commutes with the formation of $M^c_{i_0}$ and M_{i_0-1}, i.e., we have (for $j < i_0$)

(11) $$\mathcal{PM}^{(j)}(M,\lambda) = \mathcal{P}M^{c}_{i_o} = \mathcal{M}^{(j)}(PM,\lambda)$$

and similarly for $j = i_o$.

Now, in case that $J(M,\lambda) = \emptyset$, we are done with the proof. Assume $J \neq \emptyset$ for the remaining part.

4th STEP: Assume now $i > i_o$. As $i \notin D(PM,\lambda)$ there is j with $i \notin D(\mathcal{M}^{(j)}(PM,\lambda), g_j)$ (i.e., $j \in J(i)$)

Now we have the following line of equations:

$$\mathcal{M}^{(i)}(PM,\lambda) = \max_{\substack{j \leq i_o \\ i \notin D(\mathcal{M}^{(j)}(PM,\lambda),\, g_j)}} \mathcal{M}^{(i)}(\mathcal{M}^{(j)}(PM,\lambda), g_j)$$

$$= \max_{\substack{j \leq i_o \\ i \notin D(\mathcal{PM}^{(j)}(M,\lambda),\, g_j)}} \mathcal{M}^{(i)}(\mathcal{PM}^{(j)}(M,\lambda), g_j)$$

in view of the 3rd STEP, see (11). Simplifying the notation yields

$$\ldots = \max_{\substack{j \leq i_o \\ i \notin D(\mathcal{P}M^{(j)},\, g_j)}} \mathcal{M}^{(i)}(\mathcal{P}M^{(j)}, g_j)$$

$$= \ldots$$

Next, as each $M^{(j)}$ is shorter then M, we may use induction hypothesis, which yields

$$\ldots = \max_{\substack{j \leq i_o \\ i \notin D(PM^{(j)}, g_j)}} PM^{(i)}(M^{(j)}, g_j)$$

(12) $$= \max_{\substack{j \leq i_o \\ i \notin D(PM^{(j)}, g_j)}} PM^{(i,j)} = P\max_{\substack{j \leq i_o \\ i \notin D(PM^{(j)}, g_j)}} M^{(i,j)}$$

$$\leq P\max_{\substack{j \leq i_o \\ i \notin D(M^{(j)}, g_j)}} M^{(i,j)} ,$$

here we have written $M^{(i,j)}$ for $M^{(i)}(M^{(j)}, g_j)$ and for the last inequality used the fact that a nondummy in $(PM^{(j)}, g_j)$ is certainly a nondummy in $(M^{(j)}, g_j)$.

For every j such that $i \notin D(PM^{(j)}, g_j)$ there is a min-win profile w.r.t. $(PM^{(j)}, g_j)$ such that the i-coordinate is positive. Augmenting this by a 0 we obtain a min-win profile w.r.t. $(M^{(j)}, g_j)$, say s'^{j} such that $s'^{j}_i > 0$ and $s'^{j}_r = 0$ (coordinates being indexted in agreement with the coordinates of $M^{(j)}$). Applying Lemma 2.3. (to $(M^{(j)}, g_j)$ we observe that

(13) $$M^{(i,j)} = (g_1, \ldots, g_{r+1}; d, \ldots, k_{r+1})$$

with $1 < r$ or $1 = r$ and $d = k_r$.

On the other hand, let $j_o \in J(i)$ be such that

(14) $$M^{(i)} = M^{(i,j_o)} = \max_{\substack{j \leq i_o \\ i \notin D(M^{(j)}, g_j)}} M^{(i,j)}$$

The assumption $i \in D\,(PM^{(j_o)}, g_{i_o})$ leads to a contradiction. For, in this case every min-win profile s' w.r.t. $(M^{(j_o)}, g_{j_o})$ with $s'_i > 0$ has to satisfy $s'_r > 0$, and by Lemma 2.2. this means

(15) $$M^{(i,j_o)} = (g_r, g_{r+1}; d, k_{r+1})$$

with $d < k_r$ or

(16) $$M^{(i,j_o)} = (g_{r+1}, k_{r+1}).$$

But neither (15) nor (16) is compatible with (13), as $M^{(i,j_o)}$ is the max over $M^{(i,j)}$, $j \in J(i)$. We conclude that $i \notin D\,(PM^{(j_o)}, g_{j_o})$.

Hence, the last inequality in (12) is in fact an equation, and we may continue in (12) by

$$\mathcal{M}^{(i)}\,(PM,\lambda) = P \max_{\substack{j \le i_o \\ i \notin D\,(M^{(j)},\, g_j)}} M^{(i,j)}$$

$$= P\,\mathcal{M}^{(i)}\,(M,\lambda)$$

q.e.d.

3 Countably many fellowships: the definition of characters

For games with countably many players and fellowships most of the basic definitions as presented in SEC. 0 and SEC. 1 may be generalized in a straight forward manner.

Thus, we consider profiles $s = (s_1, s_2, \ldots) \in \mathbb{N}_0^{\mathbb{N}}$ to be feasible for $k = (k_1, k_2, ..) \in \mathbb{N}^{\mathbb{N}}$ if $s \leq k$. A cf. v for k is defined on the feasible profiles of k taking values 0 and 1. The sequence $\tilde{k} = (1,1,\ldots)$ =: e extends k and formula (2) of SEC. 0 (for $i = 1,2,\ldots$) serves in order to define s if $\tilde{s}$ is specified; this also explains the extension $\tilde{v}$ of v by means of formula (3) of SEC. 0. Accordingly, a pair $(\tilde{v}, e)$ is called a game.

In order to define (homogeneous) representations of (weighted majority) games, we shall restrict ourselves to <u>rational</u> sequences of weights.

Let $\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathbb{Q}^{\mathbb{N}}$ satisfy

(1) $$0 \neq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0,$$

and

(2) $$\sum_{i=1}^{\infty} k_i \gamma_i = \gamma(k) < \infty ,$$

Then the corresponding set function γ on the feasible profiles of k $(\gamma(s) = \sum_{i=1}^{\infty} s_i \gamma_i)$ is called a measure (and so is the pair $\mu = (\gamma, k)$).

Introduce

(3) $$\mathfrak{M}^{\infty} := \{\mu = (\gamma, k) \in \mathbb{Q}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \mid \gamma \text{ satisfies (1) and (2)}\}$$

If $\alpha \in \mathbb{Q}$, $\alpha > 0$ then "μ hom α" is defined as in the integer territory.

A pair $(\mu,\alpha) \in \mathfrak{M}^{\infty} \times \mathbb{Q}$ generates a cf. $v = v^{\mu}_{\alpha}$ thus representing a game $(\widetilde{v^{\mu}_{\alpha}}, \tilde{k})$ as previously.

The BASIC LEMMA also holds true mutatis mutandis for (μ,α) if μ hom α; in particular the lexicographically first min-win profile

$s^{\mu}_{\alpha} = (k_1,k_2,\ldots, k_{i_0-1}, c, 0,0,\ldots)$ is well defined (and equals k if and only if $\gamma(k) = \alpha$).

Examples 3.1. The following are straight forward examples for homogeneous representations of a game.

1. $\mu = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8},\ldots; 1, 1, 1,\ldots); \quad \alpha = 1$

2. $\mu = (\frac{1}{4}, \frac{1}{8}, \frac{1}{16},\ldots; 3,1,1,\ldots) ; \quad \alpha = 1/2$

3. $\mu = (\frac{1}{4}, \frac{1}{8}, \frac{1}{16},\ldots; 2,2,2,\ldots) ; \quad \alpha = 1/2$

4. $\mu = (3,1, \frac{3}{4}, \frac{1}{4}, \frac{3}{16}, \frac{1}{16},\ldots; 1,1,1,1,1,1,\ldots) ; \quad \alpha = 4$

5. An obvious method to provide examples is described as follows: replace some small fellowships in a finite representation by a finite measure having the same total weight and being hom w.r.t. every replaced fellowship. Repeat this procedure ad infinitum.

E.g. $M = (50,21,8,5,2,1,0; 1,2,2,1,2,3,0)$ hom $\lambda = 71$ holds true (cf. [9], Example 3.17.). Add players of the smallest non-dummy fellowship ($g_6 = 1$) such that the total weight is $2 \cdot 71$, i.e., consider

$$M' = (50,21,8,5,2,1,0; 1,2,2,1,2,25,0) ; \quad \lambda = 71$$

where $m' = 142 = 2\lambda$.

Now, replace two players of fellowship 6 (with total weight $2 \cdot g_6 = 2$) by $\frac{1}{71} M'$ which has total weight $\frac{m'}{71} = 2$ and satisfies $\frac{1}{71} M'$ hom 1. Then

$$\mu' = (50,21,\ldots,1, \frac{50}{71}, \frac{21}{71},\ldots, \frac{1}{71}, 0; 1,2,\ldots,23,1,2,\ldots,25,0)$$

satisfies μ' hom 71. Proceeding this way we obtain the example

$$\mu = (50,21,\ldots,1, \frac{50}{71}, \frac{21}{71},\ldots, \frac{1}{71}, \frac{50}{(71)^2}, \frac{21}{(71)^2},\ldots, \frac{1}{(71)^2}, \frac{50}{(71)^3},\ldots;$$

$$1,2,\ldots,23,1,2,\ldots,23,1\ldots) \; ; \; \alpha = 71 \; .$$

Remark 3.2.

1. Finite games may of course be treated within the countable framework, e.g., but putting weights $\gamma_i = 0$ for i exceeding some large number N (or admitting $k_i = 0$ for $i \geq N$ or both). Profiles will be called "finite" if $s_i = 0$ for $i \geq N$ for some N holds true.

2. Let μ hom α. Whenever $\gamma(s) > \alpha$, then there is a finite profile $\overset{\circ}{s} \leq s$ s.t. $\gamma(\overset{\circ}{s}) = \alpha$; this follows from the BASIC LEMMA.

3. If s is a maximal losing profile, then there is N such that

$$s = (s_1,\ldots,s_N, k_{N+1}, k_{N+2},\ldots).$$

The proof is obvious.

By these remarks it is suggested that the structure of the finite subgames obtained by cutting off tails of a measure γ plays an important role in the infinite game.

We shall therefore attempt to define characters by way of a limiting procedure and it is not surprising that the projection lemma of SEC. 2 provides the clue for the success of this approach.

For $\mu \in \mathfrak{M}^\infty$ the notation

(4) $\quad {}^r\mu := {}^rp_\mu = (\gamma_1,\dots,\gamma_r,0;\ k_1,\dots,k_{r+1})$

defines a projection ${}^rp$ and for large suitable integers t clearly $t\,{}^r\mu \in \mathfrak{M}^r$.

In order to define characters for $(\mu,\alpha) \in \mathfrak{M}^\infty \times \mathbb{Q}$, fix an integer ${}^rt$ for $r = 1,2,\dots$ such that

(5) $\quad {}^rM := {}^rt\,{}^r\mu \in \mathfrak{M}^r \qquad$ (i.e., ${}^rt\,{}^r\gamma \in \mathbb{N}_0^{r+1}$),

(6) $\quad {}^r\lambda := {}^rt\,\alpha \in \mathbb{N}$

(7) $\quad {}^rt \mid {}^{r+1}t$

Assume $\gamma(k) > \alpha$. For sufficiently large r the total mass ${}^rm$ satisfies ${}^rm \geq {}^r\lambda$ and ${}^rM$ hom ${}^r\lambda$; thus characters w.r.t. $({}^rM, {}^r\lambda)$ may be defined according to the finite theory. This means that $\{1,\dots,r+1\}$ is decomposed, say

$$\{1,\dots,r+1\} = {}^r\Sigma + {}^rT + {}^rD$$

This decomposition is independent of the choice of the ${}^rt$, as multiplication with a constant does not affect the characters (see Remark 1.2.9.).

Also, whenever $i \notin {}^rD$, then we may define a satellite measure for i, this is

$$M^{(i)}\,({}^rM,\ {}^r\lambda) = M^{(i)}\,({}^rt\,{}^r\mu,\ {}^r\lambda) =: {}^rM^{(i)}$$

Again, ${}^rm^{(i)}$ is the total mass of ${}^rM^{(i)}$ and ${}^rs^{(i)}$ denotes the substitutes of i (cf. the 5th STEP in SEC. 1).

<u>Lemma 3.3.</u> Let $(\mu,\alpha) \in \mathfrak{M}^\infty \times \mathbb{Q}$ and let (4), (5), (6), (7) describe the situation as explained above.

Let ${}^r q := {}^{r+1}t \,/\, {}^r t$ (r=1,2,...). Now, if $i \notin {}^r D$, then $i \notin {}^s D$ for all $s \geq r$ and

(8) $$ {}^r q \; M^{(i)} ({}^r M, {}^r \lambda) = M^{(i)} ({}^r q \, {}^r M, {}^r q \, {}^r \lambda) = P(M^{(i)} ({}^{r+1}M, {}^{r+1}\lambda)) $$

or, for short

(9) $$ {}^r q \; {}^r M^{(i)} = P \; {}^{r+1}M^{(i)} $$

(where P is given by Definition 2.4.).

In particular, if $i \in {}^r\Sigma$, then $i \in {}^s\Sigma$ for $s \geq r$ and ${}^s s^{(i)} = ({}^r s^{(i)},0,\ldots,0)$.

<u>Proof:</u> Consider the case that, for some fixed r, we have ${}^r t = {}^{r+1}t$, thus

(10) $$ {}^r M = P \; {}^{r+1}M, \quad {}^{r+1}\lambda = {}^r\lambda $$

where P is defined in 2.4.

Now, i is no dummy w.r.t. $({}^r M, {}^r \lambda)$ and by the Projection Lemma (Theorem 2.5.), i is no dummy w.r.t. $({}^{r+1}M, {}^{r+1}\lambda)$ such that

(11) $$ M^{(i)} (P \; {}^{r+1}M, {}^{r+1}\lambda) = P \; (M^{(i)} ({}^{r+1}M, {}^{r+1}\lambda)) $$

is true. Obviously, (8) follows from (10) and (11).

If $i \in {}^r\Sigma$ then $i \in {}^{r+1}\Sigma$ in view of

$$ {}^r g_i = {}^{r+1}g_i \leq {}^r m^{(i)} \leq {}^{r+1}m^{(i)} $$

and the statement concerning the substitutes is obvious. Finally, if

${}^{r}t \neq {}^{r+1}t$, note that operations like ${}^{r}M^{(i)}$, P, ${}^{r}\Sigma$,... behave nicely under multiplication with constants, thus (8) is obtained by a suitable multiplication of both sides with ${}^{r+1}t$ and ${}^{r}t$ respectively. q.e.d.

Remark 3.4. Suppose, for some $i, r \in \mathbb{N}$, $i \leq r$, we have $i \notin {}^{r}D$.

Consider the finite vector or measure ${}^{r}\mu^{(i)} := \frac{{}^{r}M^{(i)}}{{}^{r}t}$, a quantity that does not depend on the choice of ${}^{r}t$ at all. Because of (9) we have

$$(12) \qquad {}^{r}\mu^{(i)} = P\,{}^{r+1}\mu^{(i)}$$

Therefore, the sequence ${}^{r}\mu^{(i)}, {}^{r+1}\mu^{(i)},\ldots$ defines a measure $\mu^{(i)} \in \mathfrak{M}^{\infty}$, which is a certain "tail of μ" (one might also think of convergence of ${}^{r}\gamma^{(i)}$ towards $\gamma^{(i)}$ taking place in the sense of 1^{1}). We write

$$(13) \qquad \mu^{(i)} := \lim_{r\in\mathbb{N}} {}^{r}\mu^{(i)}$$

Definition 3.5. If $i \notin {}^{r}D$ for some r, then $\mu^{(i)}$ is the satellite measure of i (w.r.t. (μ,α)). If the total mass of $\mu^{(i)}$, say $m^{(i)}$, is at least γ_i, then $(\mu^{(i)}, \gamma_i)$ represents the satellite game of i (Note that in this case $\mu^{(i)}$ hom γ_i, this follows from Lemma 3.3., Remark 3.4., Remark 3.2.2. and the BASIC LEMMA). In particular, the substitutes of i, $s^{(i)}$, are defined to be either the lex-max min-win profile $s^{\mu^{(i)}}_{\gamma_i}$ if $m^{(i)} \geq \gamma_i$ or to be equal to $k^{(i)}$, i.e., the projection $\mu^{(i)} = (\gamma^{(i)}; k^{(i)})$.

Definition 3.6. The characters w.r.t. (μ,α) are defined as follows:

1. i is a dummy if there is no min-win profile s such that $s_i > 0$.

2. i is an <u>improper step</u>, if i is a nondummy but a dummy for every $({}^rM, {}^r\lambda)$ $(r = i, i+1, \ldots)$.

   For the remaining cases we may assume that i is a nondummy for some $({}^rM, {}^r\lambda)$ (and hence for all $({}^lM, {}^l\lambda)$, $l \geq r$) and for (μ, α); thus $\mu^{(i)}$ is well defined. Again, $m^{(i)}$ denotes the total mass of $\mu^{(i)}$.

3. i is a <u>proper step</u>, if $\mu^{(i)} < \gamma_i$.

4. i is an <u>improper sum</u>, if $m^{(i)} = \gamma_i$ and ${}^rm^{(i)} < \gamma_i$ for all r such that ${}^r\mu^{(i)}$ is defined. (Thus, i is a step for all $({}^rM, {}^r\lambda)$ such that ${}^rM^{(i)}$ is defined.)

5. i is a <u>proper sum</u>, if $m^{(i)} \geq \gamma_i$ and ${}^rm^{(i)} \geq \gamma_i$ for some r (and thus ${}^lm^{(i)} \geq \gamma_i$ for all $l \geq r$).

Thus, in any countable representation (γ, α), we find 5 characters; hence $\mathbb{N}$ is decomposed

$$\mathbb{N} = \Sigma^p + \Sigma^i + T^p + T^i + D$$

(p is used for "proper" etc.). The first three characters, i.e. sums and proper steps, have satellite measures defined.

Having thus defined characters it is our aim to within the last section establish the analogue to the finite representation theorem.

4 Representations of games with countably many fellowships

As a prelude let us show that games with dummies or improper sums may be neglected: as they are "essentially finite", they are delt with by the methods of the finite theory.

To this end let us first of all rule out the case that $\gamma(k) = \alpha$ (the unanimous game of the grand coalition). For, in this case, all fellowships with $\gamma_i > 0$ are steps; improper, if $\gamma > 0$ and proper if $\gamma_l = 0$ for all l exceeding some $L \in \mathbb{N}$. In the first case, every (μ,α) with $\gamma > 0$ and $\gamma(k) = \alpha$ is a representation and in the second the game is also represented by

$$(\gamma; k;\alpha) = (1,0,0,\ldots; \sum_{l=1}^{L} k_l, 1,1,\ldots; \sum_{l=1}^{L} k_l)$$

if L is choosen smallest within the above property.

Thus, let us from now on always assume that $\gamma(k) < \alpha$.

Definition 4.1. (μ,α) is called essentially finite if there is $l \in \mathbb{N}$ such that every min-win profile s satisfying $s_l > 0$ has the shape

(1) $$s = (s_1, s_2, \ldots, s_{l-1}, k_l, k_{l+1}, \ldots)$$

Note: if a game has an essentially finite representation, then all representations are essentially finite and the term might also be applied to the game as well. Note also that the existence of dummies renders a game to be essentially finite: in this case the first dummy (and all following fellowships) satisfy the definition given above as there is no min-win profile s with positive coordinate for dummy fellowships.

More precisely, the following theorem describes the situation.

Theorem 4.2.

1. If i is a dummy for (μ,α) then so is $j \geq i$.

2. If i is an improper step for (μ,α), then so is $j \geq i$.

3. (μ,α) is essentially finite if and only if there are either dummies or improper steps.

4. If (μ,α) is essentially finite and l is the first fellowship without satellite measure (i.e., either dummy or improper step), then

(2) $$\gamma_{l-1} \geq \sum_{i=1}^{\infty} k_i \gamma_i .$$

Proof:

1. is trivial.

2. If i is an improper step, then it is a dummy in any $({}^rM, {}^r\lambda)$ with $r \geq i$. Thus, $j \geq i$ is a dummy in any $({}^rM, {}^r\lambda)$ with $r \geq j$. On the other hand, as i is not a dummy, there is a (necessarily not finite) min-win profile s such that $s_i > 0$. s has countably many nondummies following i. In view of 1., any $j \geq i$ has to be a nondummy - and hence is an improper step.

3. Let (μ,α) be essentially finite and pick l according to 4.1. Suppose l is no dummy. Then any min-win profile s for some $({}^rM, {}^r\lambda)$ with $s_l > 0$ would also constitute a min-win profile for (μ,α) with only finitely many positive coordinates - contradicting (1). Thus l is an improper step.

On the other hand the existence of dummies implies trivially that (μ,α) is essentially finite. Consider the case that l is an improper step. Let s be a min-win profile such that $s_l > 0$. Suppose, $s_i < k_i$ for some $i \geq l$ and consider

$$s' = s + e^i$$

which is winning (e^i is the "i'th unit vector")

In view of 2., i cannot be a dummy, thus $\gamma_i > 0$ and

$$\gamma(s') = \gamma(s) + \gamma_i > \alpha$$

For sufficiently large p we have

$$\gamma(s'_1,\ldots,s'_p, 0,\ldots,0) \geq \alpha$$

and by homogeneity of γ w.r.t. α, there is $s'' \leq (s'_1,\ldots,s'_p, 0,\ldots,0)$ such that $\gamma(s'') = \alpha$. Clearly, $p \geq i \geq l$ for otherwise $s'' \leq s$ (s was minimal winning). But then, s'' is a min-win profile "for l", i.e., $s''_l > 0$, with at most finetely many positive coordinates. It follows then that l is a nondummy in some $({}^rM, {}^r\lambda)$, a contradiction to l being improper step. Thus, $s_i = k_i$ $(i \geq l)$ and (μ,α) is essentially finite.

4. If l is the first fellowship without satellite measure, then $l \geq 2$, for otherwise there are no finite min-win profiles and $\gamma(k) = \alpha$, which we have excluded.

Now, $l - 1$ must provide a min-win profile s' of some $({}^rM, {}^r\lambda)$ s.t. $s'_{l-1} > 0$.

Thus, there is a min-win profile s, with $s_{l-1} > 0$ and only finitely many positive coordinates. In view of 1., 2., and 3. as well as (1), s has the shape

$$s = (s_1,\ldots,s_{l-1}, 0,0,\ldots)$$

If

$$g_{l-1} < \sum_{i=e}^{\infty} k_i \, g_i \quad \text{then}$$

$$\bar{s} = s + (0,\ldots,0,\, k_l,\, k_{l+1},\ldots) - e^{l-1}$$

is winning, in fact satisfies $\gamma(\bar{s}) > \alpha$. Proceeding as in the proof of 3., we cut off a sufficiently ar out "tail" of $\bar{s}$ thus finding (by homogeneity) a min-win profile

$$\bar{\bar{s}} = (s_1,\ldots,s_{l-2},\, s_{l-1} - 1,\, k_l,\ldots,k_p,\, 0,0,\ldots)$$

This shows that l is a nondummy in some $({}^{r}M,\ {}^{r}\lambda)$, a contradiction which proves (2).

Remark 4.3. The structure of an essentially finite representation is satisfyingly described by the finite theory.

For, let l be the first fellowship without a satellite measure. We may assume $l \geq 2$, for otherwise $\gamma(k) = \alpha$. If l is a dummy, then so is $j \geq l$ and the finite representation

(3) $$({}^{l-1}M,\ {}^{l-1}\lambda)$$

(cf. SEC. 3) serves to completely describe the game.

On the other hand, if l is an improper step, then so is $j \geq l$ and, in any min-win profile, fellowships $j \geq l$ appear either ên bloc or not at all.

Define $\hat{\mu} \in \mathfrak{M}^{\infty}$ by $\hat{k}_i = k_i$, $\hat{\gamma}_i = \gamma_i$ $(i=1,\ldots,l-1)$, $\hat{k}_i = 1$ $(i \geq l)$,

$\hat{\gamma}_l = \sum_{i=l}^{\infty} \gamma_i$, and $\hat{\gamma}_i = 0$ $(i \geq l+1)$. Then the min-win profiles of (μ,α) and $(\hat{\mu},\alpha)$ correspond in an obvious way and, although $\hat{\gamma}_l$ is not necessarily rational, the weights $\hat{\gamma}_i$ are decreasing by (2). The finite homogeneous game represented by $({}^{l}P\ \hat{\mu},\alpha)$ (homogeneously!) completely describes the situation.

Thus, the structure of essentially finite games is revealed. For the remaining part let us assume that (μ,α) is not essentially finite and, thus, for every $i \in \mathbb{N}$ the satellite measure $\mu^{(i)}$ is well defined.

Remark 4.4. 1. By 3.2.1. we recall that, whenever $\gamma(s) > \alpha$, then there is a finite profile $\overset{o}{s} \leq s$ such that $\gamma(\overset{o}{s}) = \alpha$. That is, γ hom α if and only if ${}^{r}\gamma$ hom α (or ${}^{r}M$ hom ${}^{r}\lambda$) for all r exceeding some sufficiently large r_o.

2. Recall that every maximal loosing profile has the shape $s = (s_1,\ldots,s_N, k_{N+1}, k_{N+2},\ldots)$. In addition, we have $\alpha - \gamma(s) \leq \gamma_N$.

Now, if $\alpha - \gamma(s) = \gamma_N$, then $s + e^N$ is minimal winning; thus s is dominated by a min-win profile.

If $\alpha - \gamma(s) < \gamma_N$, then there exists $N' \geq N$ such that $s' := (s_1,\ldots,s_N, k_{N+1},\ldots,k_{N'}, 0)$ is maximal lossing in $({}^{N'}M, {}^{N'}\lambda)$.

In order to verify this, choose N' such that $\gamma(0,\ldots,0,k_{N'+1},k_{N'+2},\ldots) < \gamma(s+e^N) - \alpha$. Then for $i \leq N$ we have

$\gamma(s', 0,0,\ldots) + \gamma_i = (s+e^i) - \gamma(0,\ldots,0, k_{N+2},k_{N+3},\ldots)$

$> \gamma(s+e^N) - \gamma(s+e^N) + \alpha = \alpha.$

Lemma 4.5. Suppose, a min-win profile has the shape

$$s = (s_1,\ldots,s_i, k_{i+1}, k_{i+2},\ldots)$$

such that $s_i < k_i$. Then i is a sum.

Proof: By the BASIC LEMMA find $l \geq i+1$ such that

$$s' = (s_1,\ldots,s_i+1, k_{i+1},\ldots,k_{l-1},c,0,0,\ldots)$$

is min-win. Then

$$\gamma_i = \gamma(0,\ldots,0, k_l-c, k_{l+1}) =: \gamma(\tilde{s}) \tag{4}$$

Assume that i is a step. For every large L consider

$$(s_1,\dots,s_{i+1},\ k_{i+1},k_{l-1},c,0,\dots,\underset{L+1}{0})$$

which is min-win in $({}^LM,\ {}^L\lambda)$. By Lemma 5.4. of [9], applied to this profile, we find that the substitutes profile ${}^Ls^{(i)}$ of i w.r.t. $({}^LM,\ {}^L\lambda)$ satisfies

$$^Ls^{(i)} \geq (0,\dots,0,\ k_l-c,\ k_{l+1},\dots,k_{L+1})\ ,$$

once L is large enough s.t. i is a step w.r.t. $({}^LM,\ {}^L\lambda)$.

It follows that $s^{(i)} \geq \tilde{s}$. As a consequence we find

(5) $$\gamma_i > \gamma(s^{(i)}) \geq \gamma(\tilde{s})\ ,$$

contradicting (4). Thus, i is a sum.

<u>Lemma 4.6.</u> Suppose, a min-win profile s has the shape

$$s = (s_1,\dots,s_l,\ 0,0,\dots)$$

Then, for the satellite measure $\mu^{(l)} = (\gamma^{(l)},\ k^{(l)})$

$$k^{(l)} \geq (0,\dots,0,\ k_{l+1},\ k_{l+2},\dots)$$

and for $i < l$

$$k^{(i)} \geq (0,\dots,0,\ k_l-s_l,\ k_{l+1},\ k_{l+2},\dots)$$

<u>Proof:</u> s is min-win for $({}^rM,\ {}^r\lambda)$ with sufficiently large r, therefore the statement is a consequence of the corresponding one in the finite case, i.e., of Lemma 5.4. in [9].

The following theorem is the analogue of the representation theorem which is known in the finite case (cf. Remarks 1.2.7 and 1.2.8). Of course, if there are countably many steps, we cannot expect anything like a minimal or unique representation as (given rational weights) steps may surpass the total mass of their substitutes by an arbitrarily small amount. Thus, the appropriate analogue to look for is a generalization of 1.2.7. and not of 1.2.8.

It shall be useful to employ the following notation within the proof of the theorem.

For any profile $s = (s_1, s_2, s_3, \ldots)$ and $r \in \mathbb{N}$ let us write

$$^{r}s = (s_1, s_2, \ldots, s_r, 0)$$

and

$$^{\infty r}s = (0, 0, \ldots, 0, s_{r+1}, s_{r+2}, \ldots)$$

Also, if i is step (w.r.t. some (μ,α)) and $\mu^{(i)} = (\gamma^{(i)}, k^{(i)})$ his satellite measure, $s^{(i)}$ his substitutes, then put

(6) $$\varepsilon_i := \gamma_i - m^{(i)} = \gamma_i - \gamma(s^{(i)}) \qquad (i \in T)$$

where $m^{(i)}$ as usually denotes the total mass of $\mu^{(i)}$.

<u>Theorem 4.7.</u> Let $(\mu,\alpha) \in \mathfrak{M}^{\infty} \times \mathbb{Q}$ (with $\gamma(k) < \alpha$ and not essentially finite). Also, let $(\delta_i)_{i \in T}$ be a decreasing sequence of rationals such that

(7) $$0 < \delta_i \leq \varepsilon_i \qquad (i \in T)$$

holds true. Then there is $(\bar{\mu},\bar{\alpha}) \in \mathfrak{M}^{\infty} \times Q$ satisfying

(8) $$\bar{\gamma}_i = \bar{\gamma}(s^{(i)}) \qquad (i \in \Sigma\ )$$

and

(9) $$\bar{\gamma}_i = \bar{\gamma}(s^{(i)}) + \gamma_i = \bar{m}^{(i)} + \gamma_i \qquad (i \in T)$$

such that

(10) $$v^{\mu}_{\alpha} = v^{\bar{\mu}}_{\bar{\alpha}}$$

that is in particular, $(\bar{\mu},\bar{\alpha})$ represents the same game as (μ,α) (and steps and sums cooincide w.r.t. both representations).

<u>Proof:</u>

1st STEP: For $r = 1,2,\dots$ consider the finite game represented by $({}^{r}\mu,\alpha)$. W.r.t. this representation we have a decomposition of fellowships into characters, say

(11) $$\begin{aligned}\{1,\dots r+1\} &= {}^{r}\Sigma + {}^{r}T + {}^{r}D \\ &= (\Sigma \cap \{1,\dots,r+1\}) + (T \cap \{1,\dots,r+1\})\end{aligned}$$

where Σ and T refer to (μ,α). We are going to define a vector ${}^{(r)}\bar{\gamma} \in \mathbb{Q}^{r+1}$ and ${}^{(r)}\bar{\alpha} \in \mathbb{Q}$ as follows, beginning with the coordinate $r+1$ and proceeding inductively:

0. Put ${}^{(r)}\bar{\gamma}_{r+1} = 0$

1. For every dummy $i \in {}^{r}D$, $i \leq r$, put

(12) $${}^{(r)}\bar{\gamma}_i := \gamma_i .$$

2. For every sum $i \in \Sigma$, $i \notin {}^{r}D$, of (γ,α) put

(13) $${}^{(r)}\bar{\gamma}_i := {}^{(r)}\bar{\gamma}({}^{r}s^{(i)}) + \gamma({}^{\infty r}s^{(i)})$$

(note that $\bar{\gamma}_{i+1},\dots,\bar{\gamma}_{r+1}$ is defined by induction hypothesis; also $\bar{\gamma}_i$ is rational, since ${}^{(r)}\bar{\gamma}({}^{r}s^{(i)})$ is rational by induction and

$$\gamma({}^{\infty r}s^{(i)}) = \gamma(s^{(i)}) - \gamma({}^{r}s^{(i)})$$

$$= \gamma_i - \gamma({}^{r}s^{(i)})$$

is rational)

3. For every step $i \in T$, $i \notin {}^{r}D$ of (γ,α) put

(14) $${}^{(r)}\bar{\gamma}_i := {}^{(r)}\bar{\gamma}\,({}^{r}s^{(i)}) + \gamma({}^{\infty r}s^{(i)}) + \delta_i$$

4. Finally, put

(15) $${}^{(r)}\bar{\alpha} := {}^{(r)}\bar{\gamma}\,(s^{{}^{r}M}_{{}^{r}\lambda}) = {}^{(r)}\bar{\gamma}(s^{{}^{r}\mu}_{\alpha})$$

2nd STEP: Let ${}^{(r)}\bar{\mu} := ({}^{(r)}\bar{\gamma}, {}^{r}k)$.
We claim that $({}^{(r)}\bar{\mu}, {}^{(r)}\bar{\alpha})$ represents the same game as $({}^{r}\mu,\alpha)$ or $({}^{r}M, {}^{r}\lambda)$ respectively.

To this end it suffices to show that $({}^{(r)}\bar{\mu}, {}^{(r)}\bar{\alpha})$ is "compatible" with $({}^{r}M, {}^{r}\lambda)$ in the sense of Remark 1.2.7.

Now, for dummies $i \in {}^{r}D$ there is nothing to show.

Next, consider $i \in {}^{r}\Sigma$. Then, a fortiori, $i \in \Sigma$ and, in view of the projection lemma (Theorem 2.5.) we have

(16) $${}^{r}\gamma({}^{r}s^{(i)}) = \gamma(s^{(i)}) = \gamma_i$$

(and ${}^{\infty r}s^{(i)} = (0,0,\ldots)$). Thus, (13) reads

(17) $${}^{(r)}\bar{\gamma}_i = {}^{(r)}\bar{\gamma}\,({}^{r}s^{(i)}) \qquad\qquad i \in {}^{r}\Sigma$$

Finally, consider $i \in {}^{r}T$. No matter whether $i \in T$ or $i \in \Sigma$, we have $\gamma^{i}\,({}^{\infty r}s^{(i)}) > 0$ and therefore by either (13) or (14).

$$(18) \qquad {}^{(r)}\bar{\gamma}_{i} > {}^{(r)}\bar{\gamma}\,({}^{r}s^{(i)}) \qquad i \in {}^{r}T \quad .$$

Therefore, it is seen by comparing (12), (17), (18), and (15) with the conditions of 1.1.7., that $({}^{(r)}\bar{\mu}, {}^{(r)}\bar{\alpha})$ is indeed compatible with $({}^{r}\mu,\alpha)$.

3rd STEP: Let us show that for $r = 1,2,\ldots.$

$$(19) \qquad {}^{(r)}\bar{\gamma} \leq {}^{r}\gamma$$

This is certainly true for the dummies of $({}^{r}\gamma,\alpha)$ in view of (12). Hence, concerning the other characters, we proceed by induction. For, if $i \in {}^{r}\Sigma$ and ${}^{(r)}\bar{\gamma}_{j} \leq \bar{\gamma}_{j}$ $(i < j \leq r+1)$ then, by (13)

$$(20) \qquad {}^{(r)}\bar{\gamma}_{i} \leq {}^{r}\gamma({}^{r}s^{(i)}) + \gamma({}^{\infty r}s^{(i)}) = \gamma(s^{(i)}) = \gamma_{i}$$

Furthermore, if $i \in {}^{r}T$ and $i \in \Sigma$, then we may just copy (20). Finally, if $i \in T$ then

$$(21) \qquad \begin{aligned} {}^{(r)}\bar{\gamma}_{i} &\leq {}^{r}\gamma({}^{r}s^{(i)}) + \gamma({}^{\infty r}s^{(i)}) + \delta_{i} \\ &= \gamma(s^{(i)}) + \delta_{i} \leq \gamma(s^{(i)}) + \varepsilon_{i} \\ &= m^{(i)} + \varepsilon_{i} = \gamma_{i} , \end{aligned}$$

this settles (19).

4th STEP: In view of the 3rd step the limit

(22) $$\bar{\gamma}_i := \lim_{r\to\infty} {}^{(r)}\bar{\gamma}_i \leq \gamma_i$$

exists at least along a subsequence of $\mathbb{N}$ for every $i \in \mathbb{N}$ and we have

(23) $$\sum_{i=1}^{\infty} \bar{\gamma}_i \leq \sum_{i=1}^{\infty} \gamma_i < \infty$$

Analogously, $\bar{\alpha} := \lim_{r\to\infty} {}^{(r)}\alpha$, is well defined. However, it is not hard to check that ${}^{(r)}\bar{\gamma}_i$ is in fact a <u>decreasing</u> sequence in r (use the inductive method of the 3rd step).

In any case, the Lebesgue dominated convergence theorem ensures that for any profile $s \leq k$

(24) $$\bar{\gamma}(s) = \lim_{r\to\infty} {}^{(r)}\bar{\gamma}(s)$$

In particular, if $i \in \Sigma$, then (13) implies

(25) $$\bar{\gamma}_i = \bar{\gamma}(s^{(i)}),$$

and if $i \in T$, then we have analogously by (14)

(26) $$\bar{\gamma}_i = \bar{\gamma}(s^{(i)}) + \delta_i$$

That is, (8) and (9) are satisfied.

It is worthwile to note that $\bar{\gamma} > 0$ and $\bar{\gamma}_i \geq \bar{\gamma}_{i+1}$. For, the monotonicity follows from the one of the ${}^{(r)}\bar{\gamma}$ (Remark 1.2.7) and positivity is a consequence of $\bar{\gamma}_i \geq \delta_i > 0$ $(i \in \Sigma)$, if there are countably many steps, and of $\bar{\gamma}_i = \gamma_i$ (eventually) if there are finitely many steps.

Of course, $\bar{\mu} := (\bar{\gamma},k)$ is our candidate for the proof of the theorem and we have to verify that $(\bar{\mu},\bar{\alpha})$ is indeed a representation of the game.

5th STEP: If $\gamma(s) = \alpha$, then $\bar{\gamma}(s) = \bar{\alpha}$.

a) If s_i equals zero eventually, then for large r, ${}^{(r)}\bar{\gamma}(s) = {}^{(r)}\bar{\alpha}$ (as $({}^{(r)}\bar{\mu},{}^{(r)}\bar{\alpha})$ is a representation) and our claim follows by a passage to the limit.

b) Next, if $s_i < k_i$ for infinitely many $i \in \mathbb{N}$, then, for any i with this property use homogeneity (the BASIC LEMMA) to find

$$s' = (s_1,\ldots,s_i+1,\ldots, s_{L-1}, s_L-c, 0,0,\ldots)$$

with $\gamma(s') = \alpha$. By a), $\bar{\gamma}(s') = \bar{\alpha}$, and hence

$$(26) \qquad \bar{\gamma}(s) = \bar{\gamma}(s') - \bar{\gamma}_i + \bar{\gamma}\ (0,\ldots,0,c,s_{L+1},s_{L+2},\ldots) =: \bar{\alpha} - \eta_i \ .$$

As i can be chosen arbitrarily large, the term η_i is arbitrarily small, thus $\bar{\gamma}(s) = \bar{\alpha}$.

c) It remains to study the case that $s_i < k_i$ and $s_j = k_j$ for $j \geq i+1$, say. By Lemma 4.5., i is a sum and by (25) $\bar{\gamma}_i = \bar{\gamma}(s^{(i)})$ (and, of course $\gamma_i = \gamma_i(s^{(i)})$). Clearly, $s' := s + e^i - s^{(i)}$ is a feasible profile for k and

$$(27) \qquad \begin{aligned} \bar{\gamma}(s') &= \bar{\gamma}(s) + \bar{\gamma}_i - \bar{\gamma}(s^{(i)}) = \bar{\gamma}(s) \\ \gamma(s') &= \gamma(s) \end{aligned}$$

is satisfied. Now, if i is an improper sum (and $s^{(i)}$ has coordinates k_N eventually) then we are done as s' is treated in a). Otherwise, we may repeat the procedure with some $i' > i$ and $s_{i'} < k_{i'}$, $s_j = k_j$ $(j \geq i')$. If the procedure does not terminate, then we consider the profiles

$$s^\infty := s + e^i - s^{(i)} + e^{i'} - s^{(i')} + e^{i''} - s^{(i'')} \pm \ldots$$

and

$$s^n := s + e^i - s^{(i)} \pm \ldots + e^{i^{(n)}} - s^{(i^{(n)})}$$

We have $\bar{\gamma}(s^n) = \bar{\gamma}(s)$ and $\gamma(s^n) = \gamma(s) = \alpha$. Moreover $\bar{\gamma}(s^n) \to \bar{\gamma}(s^\infty)$ and $\gamma(s^n) \to \gamma(s^\infty)$ $(n\to\infty)$ as $i^{(n)} \to \infty$ $(n\to\infty)$. Hence

$$(29) \qquad \bar{\gamma}(s) = \bar{\gamma}(s^\infty) \text{ , } \gamma(s) = \gamma(s^\infty) = \alpha$$

But s^∞ is of the type treated in b). This completes the 5th step.

6th STEP: If s is a maximal losing profile, then $\bar{\gamma}(s) < \bar{\alpha}$.

Suppose $i \in \mathbb{N}$ is such that $s_i < k_i$ and

$$s = (s_1, s_2, \ldots, s_i, k_{i+1}, k_{i+2}, \ldots)$$

(cf. Remarks 3.2.1. and 4.4.2.)

a) Let i be a step. Now, $s + e^i$ is winning, if it is min-win, then we have (by the 5th step) $\bar{\gamma}(s+e^i) = \bar{\alpha}$ and $\bar{\gamma}(s) < \bar{\gamma}(s+e^i)$, thus we are already done. Assume $\gamma(s+e^i) > \alpha$. By homogeneity, find

$$s^+ = (s_1, \ldots, s_i+1, k_{i+1}, \ldots, k_{l-1}, c, 0, 0, \ldots)$$

which is min-win, thus $\bar{\gamma}(s^+) = \bar{\alpha}$ (5th step). By Lemma 4.6., as i is a step and $k^{(i)} = s^{(i)}$,

$$s^{(i)} \geq (0, \ldots, 0, k_l - c, k_{l+1}, \ldots) =: s^0 .$$

That is

$$s = s^{+} - e^{i} + s^{o} \ , \quad s^{o} \leq s^{(i)}$$

and

$$\begin{aligned} \bar{\gamma}(s) &= \bar{\gamma}(s^{+}) - \bar{\gamma}_{i} + \bar{\gamma}(\overset{o}{s}) \\ &\leq \bar{\alpha} - \bar{\gamma}_{i} + \bar{\gamma}(s^{(i)}) \qquad (30) \\ &= \bar{\alpha} - \delta_{i} < \bar{\alpha} \ . \end{aligned}$$

b) Now let i be a sum (proper or not). The profile

$$s^{+} = s + e^{i} - s^{(i)}$$

has the same measure as s (w.r.t. γ <u>and</u> $\bar{\gamma}$) thus, it is losing, but not necessarily maximal. s^{+} has coordinates "smaller than k_{*}" to the right of i, say

$$s^{+} = (s_{1},\ldots,s_{i}+1,\ k_{i+1},\ldots,k_{l},c,0,\ldots,0,d,k_{r},k_{r+1},\ldots)$$

Filling these up from the right we obtain a maximal losing profile s^{2} with $s^{+} \leq s^{2}$, i.e.,

$$s = s^{+} - e^{i} + s^{(i)} \qquad s^{+} \leq s^{2} \qquad (31)$$

The first j with $s^{2}_{j} < k_{j}$ satisfies $j > i$. Thus, we apply the same procedure to the maximal losing coalition s^{2}.

$$\begin{aligned} s^{(2)} &= s^{++} - e^{j} + s^{(j)} \qquad & s^{++} &\leq s^{(3)} \\ &\vdots & & \qquad (32) \\ s^{(n)} &= s^{(n+)} - e^{r} + s^{(r)} \qquad & s^{(n+)} &\leq s^{(n+1)} \end{aligned}$$

where $s, s^{(2)}, s^{(3)},\ldots,s^{(n)}$ is maximal losing.

Suppose, the procedure terminates. That is, in $s^{(n+1)}$ the last coordinate, say L, such that $s_L^{(n+1)} < k_L$ is a <u>step</u>. Then, by part a) of the present 6th step

(33) $$\bar{\gamma}(s^{(n+1)}) < \bar{\alpha}$$

Hence

$$\bar{\gamma}(s) = \bar{\gamma}(s^+) \underbrace{- \bar{\gamma}_i + \bar{\gamma}(s^{(i)})}_{0} \leq \bar{\gamma}(s^2)$$

by (31) and, consequently,

$$\bar{\gamma}(s^2) \leq \ldots \leq \bar{\gamma}(s^{(n+1)}) < \bar{\alpha}$$

by (32) and (33).

Suppose, on the other hand, the procedure does not terminate, i.e., (32) may be continued for $n = 2,3,\ldots$.

Now, changing from s to $s^{(2)}$ does not affect coordinates $< i$, changing from $s^{(2)}$ to $s^{(3)}$ does not affect coordinates $< j$ etc. Thus, there is an admissible profile s^∞ such that $s_N^{(n)} = s_N^\infty$ for $N \geq N(n)$.

Obviously $\gamma(s^{(n)}) \to \gamma(s^\infty)$ and $\gamma(s) \leq \gamma(s^{(2)}) \leq \ldots \leq \gamma(s^{(n)})$; thus,

$$\gamma(s) \leq \gamma(s^2) \leq \ldots \quad \leq (s^\infty) \leq \alpha$$

On the other hand

$$\begin{aligned} \alpha &- \gamma(s) < \gamma_i \\ \alpha &- \gamma(s^{(2)} < \gamma_j \\ &\vdots \\ \alpha &- \gamma(s^{(n)}) < \gamma_r \end{aligned}$$

follows from (32) (see also Remark 4.4.2.). Hence

$$0 = \alpha - \lim_{n\to\infty} \gamma(s^{(n)}) = \alpha - \gamma(s^{\infty}),$$

i.e., s^{∞} is min-win. Therefore $\bar{\gamma}(s^{\infty}) = \bar{\alpha}$ (5th step). In view of $\gamma(s) < \gamma(s^{\infty})$ we must have

$$s^{(n+)} \lneq s^{(n+1)}$$

for some n. Hence

$$\bar{\gamma}(s) \leq \bar{\gamma}(s^{(n+1)}) < \bar{\gamma}\, s^{(n+1)} \leq \bar{\gamma}(s^{\infty}) = \bar{\alpha}$$

q.e.d.

<u>Remark 4.8.</u> 1. It is sufficient to require that $0 < \delta_i < C\,\varepsilon_i$ $(i \in T)$ for some positive constant C holds true instead of (7).

2. The reader may want to classify the fellowships of the examples provided in 3.1.

REFERENCES

[1] ISBELL, J.R.: A class of majority games. Quarterly Journal of Math. Ser. 2, 7 (1956), 183-7

[2] ISBELL, J.R.: A class of simple games. Duke Math. Journal 25 (1958), 423-39

[3] ISBELL, J.R.: On the Enumeration of Majority Games. Math. Tables Aids Comput. 13 (1959), 21-28

[4] von NEUMANN, J. and MORGENSTERN, O.: Theory of Games and Economic Behavior. Princeton Univ. Press, NJ 1944

[5] OSTMANN, A.: On the minimal representation of homogeneous games. Working Paper 124, Inst. of Math. Ec., University of Bielefeld (1983), to appear in IJGT

[6] PELEG, B.: On the kernel of constant-sum simple games with homogeneous weights. Ill. J. Math. 10 (1966), 39-48

[7] PELEG, B.: On weights of constant sum majority games. SIAM J. of Appl. Math. 16 (1968), 527 ff.

[8] ROSENMÜLLER, J.: Weighted majority games and the matrix of homogeneity. Zeitschrift für Operations Research (ZOR), 28, (1984), 123-141

[9] ROSENMÜLLER, J.: Homogeneous games: Recursive structure and computation. Working Paper 138, Inst. of Math. Ec., University of Bielefeld (1984). Submitted with MOR.