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Directed and Weighted Majority Games

by

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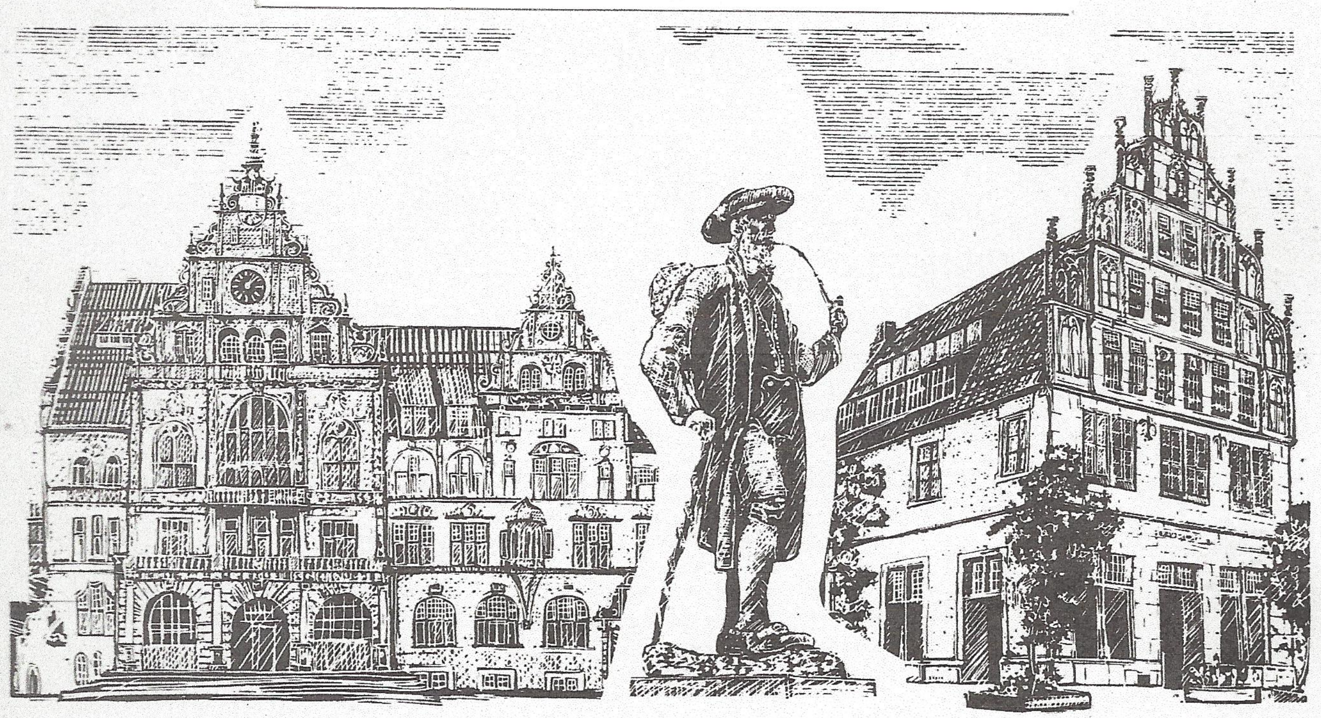

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Abstract

Algorithms are presented which generate two certain subsets of the simple games, namely the directed and the directed zero–sum games with n players.

Both classes of games are ranked partially ordered sets in a natural way, the first being additionally a rank symmetric and unitary modular lattice.

The subclass of n–person weighted majority games is characterized by the n+1–person weighted majority zero–sum games, being a subset of the directed zero–sum games.

Using methods of linear programming Algorithms, especially styled for the shift minimal coalitions of directed zero–sum games, are presented, which generate a representation of such a game, if and only if this game is a weighted majority game. This representation often is a minimal one. Additionally, some examples are offered which illustrate the theoretical results.

§1 Basic notation and preliminary results

During this paper let n be a natural number and $\Omega = \Omega_n = \{1,\ldots,n\}$. A simple n–person game is a mapping $v : \mathscr{P}(\Omega) \rightarrow \{0,1\}$. The elements of Ω are the players and those of $\mathscr{P}(\Omega)$, i.e., the subsets of Ω, are the coalitions. A coalition S is often identified with the indicator function 1_S, considered as n–vector. A coalition S is winning, if $v(S) = 1$, and losing otherwise. The set of winning coalitions is abbreviated by W_v.

In a monotone simple game all subcoalitions of the losing coalitions are losing. If each proper subcoalition of a winning coalition is losing, this coalition is a minimal winning coalition. It should be noted that a monotone simple game is completely determined by the set of its minimal winning coalitions, denoted by W^m or W^m_v, if the dependence of the game is to be stressed.

Each coalition S can be canonically considered as a number in the dual system, which can be decoded as usual. Let D(S) denote the corresponding number in the decimal system $\sum_{i \in S} 2^{n-i}$ and $\tilde{S} = (\tilde{S}_1,\ldots,\tilde{S}_n)$ be the n–vector defined by $\tilde{S}_j = | S \cap \Omega_j |$, i.e., the number of elements of S having indices less or equal to j, for all $j \in \Omega$.

From now on all considered simple games are assumed to be monotonous.

The matrix with n columns

$$I := I(v) := (S)_{S \in W^m_v}$$

and with rows ordered with respect to D, i.e., $D(I_{j.}) > D(I_{k.})$ whenever $j < k$, is called incidence matrix of v.

Two simple n–person games v and v' are equivalent, if there is a permutation π of Ω such that $v \circ \pi = v'$. As our interest is restricted to these equivalence classes of simple games only, we will choose a canonical representative of each class. The formal notation is given in

Definition 1.1: If the equivalence class of a simple game v is denoted [v], then v^c is defined to be the representative of [v], such that

$$\sum_{S \in W_{v^c}} 2^{2^n - D(S)} = \min \left\{ \sum_{S \in W_{v'}} 2^{2^n - D(S)} \mid v' \in [v] \right\}$$

is satisfied.

Therefore the canonical representative v^c of [v] is the first in some lexicographical ordering of $W_{v'}$, considered as subset of $\mathcal{P}(\Omega)$.

Let v be a simple game. The relation $\preceq \subseteq \Omega^2$, defined by $i \preceq j$, if $v(\{i\} \cup S) \leq v(\{j\} \cup S)$, for all coalition S satisfying $\{i,j\} \cap S = \emptyset$, is called desirability relation of v (see Maschler and Peleg (1966), Einy (1985)).

The simple game v is called an ordered game if its desirability relation is complete and a directed game if additionally $1 \succeq 2 \succeq \ldots \succeq n$ is valid. Concerning this notation we also refer to Ostmann (1987, 1989) and Sudhölter (1989).

Two players i and j are of the same type iff $i \succeq j$ and $j \succeq i$, which is abbreviated by $i \sim j$. Besides we recall that i is a dummy, if $v(S \cup \{i\}) = v(S)$ for all $S \in \mathcal{P}(\Omega)$.

Lemma 1.2: Let v be an ordered game. Then the following assertions are valid.

(1) The equivalence class [v] of v contains a unique directed game v'.

(2) If v is directed, then $v^c = v$.

Proof: The first assertion is a trivial consequence of the definitions of ordered and directed games. It remains to show: if v is directed, then $v^c = v$.

Consider the game v^c. We show for all $1 \leq k \leq n-1$ that $i \succeq j$ for all $i \in \Omega_k$, $j \in \Omega_n \setminus \Omega_k$. Assume on the contrary that there is a k and $i \in \Omega_k$, $j \notin \Omega_k$, such that $i \preceq j$, $j \not\preceq i$. Let π be the transposition on Ω defined by $\pi(i) = j$. Then $D(S) \geq D(\pi(S))$ $(i \in S \subseteq \Omega)$, thus

$$\sum_{S \in W_{v^c}} 2^{2^n - D(S)} > \sum_{S \in W_{v^c \circ \pi}} 2^{2^n - D(S)} ,$$

a contradiction. q.e.d.

A <u>weighted majority game</u> v is a simple game having a <u>representation</u> $(\lambda;m)$, i.e. a <u>level</u> $\lambda \in \mathbb{N}_0$ and a vector of <u>weights</u> $m \in \mathbb{N}_0^n$ such that

$$v(S) = \begin{cases} 1, \text{ if } m(S) \geq \lambda \\ 0, \text{ if } m(S) < \lambda \end{cases}, \text{ where } m(S) = \sum_{i \in S} m_i \ (S \in \mathcal{P}(\Omega))$$

is the <u>weight</u> of coalition S.

A representation is called <u>minimal</u>, if it is minimal w.r.t. the weight of the grand coalition Ω. Each weighted majority game is ordered and thus directed, iff it has a representation satisfying $m_1 \geq m_2 \geq \ldots \geq m_n$. Note that $m_i \geq m_j$ implies $i \succeq j$. For these definitions and assertions we refer to Ostmann (1987). The terms "simple" and "weighted majority" were introduced by von Neumann and Morgenstern (1944). They, of course, assumed the zero–sum property.

A simple game is a <u>zero–sum</u> game, if either S or $\Omega \backslash S$ is winning and is a <u>super-additive</u> game, if at most S or $\Omega \backslash S$ is winning for each coalition S. The <u>dual game</u> v^* is defined by $v^*(S) = 1$, iff $v(\Omega \backslash S) = 0$ (see e.g. Shapley (1962)). The game v is <u>dual superadditive</u> iff v^* is superadditive. Note that both the classes of weighted majority games and of directed games are closed under duality. Moreover, $v^* = v$, iff v is a zero–sum game, and each weighted majority game is dual– or superadditive. At last observe that * is an involution, i.e. $v^{**} = v$. Using Lemma 1.2 and some of the preceding assertions we obtain that $[v] = [v^*]$ enforces v to be a zero–sum game in the case of directed games. This is no longer true in general, if v is only monotonous (see e.g. Dubey and Shapley (1978)).

A directed game is completely determined by a subset of its minimal winning coalitions. In order to specify this subset we need some more notation.

Definition 1.3: The span of a coalition S is the set $<S> = \{T \subseteq \Omega \mid \tilde{T} \geq \tilde{S}\}$. Moreover, define the span of a subset $A \subseteq \mathcal{P}(\Omega)$ by

$$<A> = \bigcup_{S \in A} <S>.$$

It is known that v is a directed game, iff $<W_v> = W_v$. Moreover, in this case there is a unique minimal subset $W_v^s \subseteq W_v$ such that $<W_v^s> = W_v$. The elements of W_v^s are the shift minimal coalitions of v, which are automatically minimal winning coalitions. The directed game v is completely determined by W_v^s. The corresponding submatrix of the incidence matrix is the shift minimal matrix of v, abbreviated

$$I^s := I^s(v) = (S)_{S \in W_v^s}.$$

For this notation we again refer to Ostmann (1987).

Definition 1.4: Two coalitions S,T are defined to satisfy $S \preceq T$, if $\tilde{S} \leq \tilde{T}$; and $S \prec\!\cdot\; T$ if $S \neq T$, $S \preceq T$ and additionally $S \preceq R \preceq T$ implies $R \in \{S,T\}$.
The relations $\preceq$ and $\prec\!\cdot$ are called order relation and cover relation respectively.

With this notation $(\mathcal{P}(\Omega), \preceq)$ is a partially ordered set and the order relation is the reflexive and transitive closure of the cover relation. This partially ordered set can be illustrated by its Hasse diagram, i.e. by the directed graph, whose vertex set is $\mathcal{P}(\Omega)$ and whose edge set consists of all pairs (S,T) with $S \prec\!\cdot\; T$. In Fig.1 S and T are joined by an edge and T lies above S, iff $S \prec\!\cdot\; T$ $(n = 4)$.

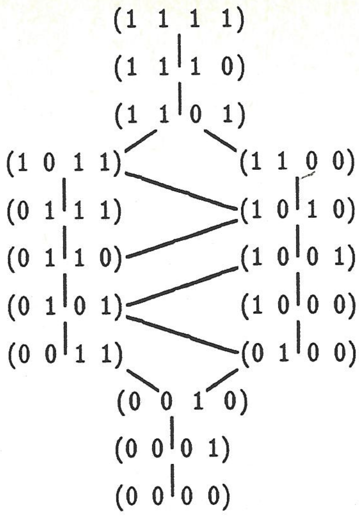

Figure 1

The partially ordered set $(\mathscr{P}(\Omega), \preceq)$ is isomorphic to the famous partially ordered set of "partitions" $(M(n), \leq)$, where

$$M(n) = \{a = (a_1,\dots,a_n) \in \mathbb{N}_0^n \mid 0 = a_1 = a_2 = \dots = a_h < a_{h+1} < \dots < a_n \leq n$$
$$\text{for some } h \in \{0\} \cup \Omega_n\}$$

The isomorphism is obviously induced by the bijective mapping on the corresponding vertex sets

$$\mathscr{P}(\Omega) \to M(n), \quad S \mapsto (0,\dots,0,n+1-i_1,\ n+1-i_2,\dots,n+1-i_{|S|}),$$

where

$$i_1 > \dots > i_{|S|} \text{ and } S = \{i_1,\dots,i_{|S|}\}.$$

This partially ordered set $(M(n), \leq)$ was introduced by Euler (1750) and it can easily be seen that it has a unique rank function (given by $a \mapsto \sum_{i=1}^{n} a_i$) with maximal rank $\binom{n+1}{2}$, that it is a lattice, i.e., to each two elements a,b there is a unique minimal element covering both and a unique maximal element covered by both a and b (which can be seen in $\mathscr{P}(\Omega)$ by observing that $R = \max(\min)\{S,T\}$, where $\tilde{R}_j = \max(\min)\{\tilde{S}_j, \tilde{T}_j\}$ componentwise), and finally that it is rank symmetric (which is seen in $(\mathscr{P}(\Omega),\preceq)$ using the map $S \mapsto \Omega\setminus S$). Proctor (1982b) proved that

$(M(n),\leq)$ is strongly Sperner and rank unimodal, which – besides – shows a famous conjecture of Erdös and Moser (1965). For these properties we refer also to Engel and Gronau (1985) and Proctor (1982a). Now it is clear that the directed n–person games are exactly the filters of $(\mathcal{P}(\Omega), \preceq)$, i.e. if v is directed, then W_v is a filter and vice versa. Moreover each filter is spanned by its minimal elements, which are exactly the shift minimal coalitions of the corresponding game.

Let α^n_k denote the number of elements in the k–th rank of $(M(n),\leq)$. Then it is known that $\alpha^{n+1}_k = \alpha^n_k + \alpha^n_{k-n-1}$ holds true. Thus especially the number $\alpha^n_{\left[\frac{\binom{n+1}{2}}{2}\right]}$ can easily be computed recursively. Using the Sperner property, the rank unimodality and symmetry of the lattice $(M(n),\leq)$, we easily obtain the following interesting result.

Proposition 1.5: $\max\{|W^s(v)| \mid v \text{ is a directed n–person game}\}$
$= \alpha^n_{\left[\frac{\binom{n+1}{2}}{2}\right]}$.

Now we come back to the directed games, considered as filters of $(\mathcal{P}(\Omega), \preceq)$ or $(M(n), \leq)$. These filters are ordered by inclusion and it easily turns out that $(\{W_v \mid v \text{ is a directed n–person game}\}, \supseteq)$ again is a ranked partially ordered set (with rank function r, defined by $r(W_v) = 2^n - |W_v|$) and total rank 2^n. The case $n = 4$ is illustrated in Fig.2, where $I^s(v)$ is written instead of W_v. In order to distinguish these partially ordered sets from the sets $(\mathcal{P}(\Omega), \preceq)$, we sketch the corresponding Hasse diagrams in such a way that the larger elements are on the right hand side of the smaller elements (not above as in the sketches of the $(\mathcal{P}(\Omega), \preceq)$).

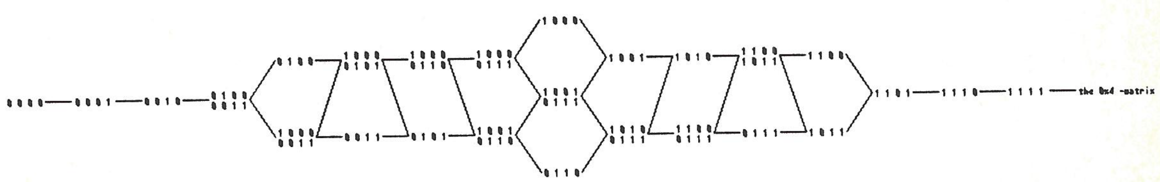

Figure 2

This partially ordered set is a lattice, since $<W_v \cup W_{v'}> = W_v \cup W_{v'}$ and $<W_v \cap W_{v'}> = W_v \cap W_{v'}$ for each pair of directed games (v,v'). Moreover the rank symmetry is easily checked by applying the mapping $v \mapsto v^*$ and observing that the restriction on the k–th rank is bijective on the 2^n – k–th rank. Moreover we conjecture that it is rank unimodal, although the linear algebra methods used by Proctor (1982a,b) cannot solve this problem. The set of filters ordered by inclusion is indeed unitary modular but there is no edge labeling for general n as it exists in the lattice $(M(n), \leq)$.

Up to the end of this chapter we show that the knowledge of the (n+1)–person zero–sum weighted majority games is, in some way, sufficient and necessary for the knowledge of all n–person weighted majority games as suggested by e.g. Wolsey (1976). At first we need some notation.

Definition 1.6: Let v be a directed superadditive n–person game and let $\hat{v}$ be the n+1–person game, defined by $\hat{v}(S) = 1$, iff $(S \in W_v)$ or $(n+1 \in S$ and $S \backslash \{n+1\} \in W_{v^*})$. Then $v^0 := \hat{v}^c$ is called the zero–sum extension of v.

For a more general definition we refer to e.g. Einy and Lehrer (1989).

Lemma 1.7: Let v be a superadditive directed n–person game. Then

(i) v^0 is a monotone simple n+1–person zero–sum game, not necessarily ordered, i.e. directed.

(ii) If v is a weighted majority game, then v^0 is. In this case both of the following assertions are valid:

(a) If $(\lambda;m)$ represents v and

$$i_0 = \max \{0\} \cup \{i \mid m_i > 2\lambda - m(\Omega) - 1\},$$

then

$$(\lambda; m_1,\dots,m_{i_0}, 2\lambda - m(\Omega) - 1, m_{i_0+1},\dots,m_n)$$

is a representation of v^0.

(b) If $(\lambda; m_1,\dots,m_{n+1})$ represents v^0, then

$$(\lambda; m_1,\dots,m_{i_0}, m_{i_0+2},\dots,m_{n+1})$$

represents v.

Proof: ad (i): Let $S,T \subseteq \Omega_{n+1}$ and $S \subseteq T$. Three cases are distinguished to show monotonicity:

(α) $n+1 \notin T$: Then $\hat{v}(S) = v(S)$ and $\hat{v}(T) = v(T)$, thus $\hat{v}(S) \leq \hat{v}(T)$, since v is monotonous.

(β) $n+1 \in S$: Then $\hat{v}(S) = v^*(S\backslash\{n+1\})$ and $\hat{v}(T) = v^*(T\backslash\{n+1\})$, thus $\hat{v}(S) \leq \hat{v}(T)$, since v^* is monotonous.

(γ) $S \not\ni n+1 \in T$: Then $\hat{v}(S) = v(S)$ and $\hat{v}(T) = v^*(T\backslash\{n+1\})$, i.e. $\hat{v}(T) = 1 - v(\Omega_{n+1}\backslash T)$. Assume $\hat{v}(S) > \hat{v}(T)$, i.e. $v(S) = 1$ and $v(\Omega_{n+1}\backslash T) = 1$. Furthermore, $\Omega_n\backslash S \supseteq \Omega_{n+1}\backslash T$, showing that S and $\Omega_n\backslash S$ are winning with respect to v. This fact contradicts the superadditivity of v.

The zero–sum property is a trivial consequence of the definition of $\hat{v}$.

The second part of this assertion is shown by an example: Let v be the 7–person game which has the shift minimal matrix

$$I^S(v) = \begin{bmatrix} 1\,1\,0\,0\,0\,1\,0 \\ 1\,0\,0\,0\,1\,1\,1 \\ 0\,1\,1\,1\,0\,0\,1 \\ 0\,1\,0\,1\,1\,1\,1 \end{bmatrix}.$$

Then it can easily be verified that v is superadditive and that

$$I^S(v^*) = \begin{bmatrix} 1\,1\,0\,0\,0\,0\,0 \\ 1\,0\,0\,1\,0\,0\,1 \\ 0\,1\,0\,0\,1\,1\,1 \\ 0\,0\,1\,1\,1\,1\,0 \end{bmatrix}.$$

From the construction of $\hat{v}$ it is clear that

$$(1\,1\,0\,0\,0\,0\,0\,1) \in W_{\hat{v}}\,,\ (1\,1\,0\,0\,0\,0\,1\,0) \notin W_{\hat{v}}\,;$$
$$(1\,0\,0\,0\,1\,1\,1\,0) \in W_{\hat{v}}\,,\ (1\,0\,0\,0\,1\,1\,0\,1) \notin W_{\hat{v}}\,.$$

Thus, for the players 7,8 it follows $7 \not\succeq 8$ and $8 \not\succeq 7$, i.e. the desirability relation is not complete.

ad(ii):

If v represented by $(\lambda;m)$, then it is well-known that v^* is represented by $(m(\Omega)+1-\lambda;m)$ (and vice versa).

In view of Definition 1.6 it is clear that $\hat{v}$ can be represented by

$$(\lambda;\tilde{m})) := (\lambda;\ m_1,\dots,m_n,\ 2\lambda-m(\Omega)-1) \text{ since}$$

then $\tilde{m}(S) = m(S)$, if $n+1 \notin S$, and

$$\tilde{m}(S) = m(S\backslash\{n+1\}) + 2\lambda-m(\Omega)-1, \text{ if } n+1 \in S,$$

which means

$$\tilde{m}(S) \geq \lambda, \text{ iff } (m(S) \geq \lambda \text{ and } n+1 \notin S(v(S) = 1)) \text{ or}$$

$$(m(S\backslash\{n+1\}) \geq m(\Omega)+1-\lambda \text{ and } n+1 \in S\ (v^*(S) = 1)).$$

Consequently the game represented by $(\lambda;\ m_1,\dots,m_{i_0},\ 2\lambda-m(\Omega)-1,\ m_{i_0+1},\dots,m_n)$ is the directed representative of $[\hat{v}]$, thus part (a) of assertion (ii) follows by Lemma 1.2.

Now take another representation of v^0, let us say $(\bar{\lambda};\ \bar{m}_1,\dots,\bar{m}_{n+1})$. Then $\bar{m}(S) \geq \bar{\lambda}$, iff $\tilde{m}(\pi(S)) \geq \lambda$; where π is the following permutation of Ω_{n+1}:

$$\pi(i) = \begin{cases} i, & \text{if } i \leq i_0 \\ i_0+1, & \text{if } i=n+1 \\ i+1, & \text{otherwise} \end{cases}$$

This assertion (ii) is shown. q.e.d.

There is a converse statement to Lemma 1.7 in the case of a weighted majority game, which is formulated in Proposition 1.10 with the help of

Definition 1.8: Let v be a directed (n+1)-person zero-sum game and $T_1,\dots,T_{t(v)} \subseteq \Omega_{n+1}$ which satisfy:

$$\text{(a)} \quad \bigcup_{k=1}^{t(v)} T_k = \Omega_{n+1}$$

(b) $i,j \in T_k$ implies i and j are of the same type for all $1 \lesssim k \lesssim t(v)$

(c) $i \in T_k$, $j \in T_{k+1}$ implies $j \preceq i$, $j \not\sim i$ for all $1 \lesssim k < t(v)$.

(The sets T_k are the types of the game.) Let $\tilde{t}(v)$ be the number of non dummy types, i.e. $\tilde{t}(v) = t(v)$, if n+1 is not a dummy, and $\tilde{t}(v) = t(v) - 1$ otherwise. For each $k \in \Omega_{t(v)}$ we define the k–th underlying game of v to be an n–person game, denoted $v^{(k)}$, defined by

$$v^{(k)}(S) = 1, \text{ iff } v(\{i \in \Omega_{n+1} \mid (i < i_0 \text{ and } i \in S) \text{ or } (i > i_0 \text{ and } i_0 - 1 \in S)\} = 1 \text{ for some } i_0 \in T_k.$$

It should be noted that the k–th underlying game of v is the game which arises from v by dropping an arbitrary player of the k–th type T_k, and considering only the winning coalitions not containing this player to be the winning coalitions of the new game.

Lemma 1.9: If v is a directed (n+1)–person zero–sum game and k, $\bar{k} \in \Omega_{t(v)}$, then

(i) $v^{(k)}$ is a superadditive directed game,

(ii) $v^{(k)} = v^{(\bar{k})}$ if and only if $k = \bar{k}$,

(iii) $(v^{(k)})^0 = v$,

(iv) if $v^{(k)}$ is a zero–sum game then $k = t(v) > \tilde{t}(v)$, i.e. player n+1 is a dummy of v.

A proof is skipped, as all necessary arguments are straightforward and almost trivial. Using the last two lemmata we get our proclaimed result.

Proposition 1.10: The set of directed superadditive n–person weighted majority games is the union of all underlying games of the directed (n+1)–person zero–sum weighted majority games.

The missing assertion concerning dual superadditive n–person weighted majority games follows especially from Lemma 1.9 (iv) by looking at dual games and is therefore not stated in detail. We only formulate the exact result concerning the cardinalities of these sets of games.

Let Z_n and Z_n^r denote the set of directed n–person zero–sum games and those having a representation respectively. Moreover let R_n be the set of directed n–person weighted majority games.

From the fact that R_n can be partitioned into its superadditive and dual superadditive, not zero–sum games, formally written

$$R_n = \{v \in R_n \mid v \text{ superadditive}\} \cup \{v \in R_n \mid v^* \in R_n, v^* \text{ superadditive}, v^* \notin Z_n^r\},$$

we obtain the following result, concerning the cardinality of R_n.

Corollary 1.11: (i) $| R_n \cap \{v \mid v \text{ is superadditive}\} | = \sum_{v \in Z_{n+1}^r} t(v)$

$$= | R_n \cap \{v \mid \text{is dual superadditive}\} |$$

(ii) $| Z_n^r | = \sum_{v \in Z_{n+1}^r} t(v) - \tilde{t}(v)$

(iii) $| R_n | = \sum_{v \in Z_{n+1}^r} t(v) + \tilde{t}(v) = 2 \cdot \sum_{v \in Z_{n+1}^r} t(v) - | Z_n^r |.$

In the next chapter we construct an algorithm which generates the directed n–person games and the directed n–person zero–sum games respectively.

§2 Generation of directed and directed zero-sum games

The procedures to generate these subclasses of the simple games presented in this chapter have been used explicitly to enumerate the games with the help of a computer. For detailed results we refer to the Appendix.

In the last chapter we showed that the directed games can be considered as a certain lattice. Moreover it turns out that the directed n-person zero-sum games form a partially ordered set in a canonical way, though no lattice. The algorithms of generating are very fast ones, but they do not reveal the structures of the corresponding ordered set. Of course we also know procedures to generate the Hasse diagrams, but these algorithms are quite slow ones, because the sets of edges (having large cardinalities compared with the sets of vertices) must be computed in addition.

Definition 2.1: Let $v_1,\dots,v_l$ be the lexicographic enumeration of the directed n-person games, i.e.

$$W_{v_1} = \emptyset \text{ and } W_{v_l} = \mathcal{P}(\Omega) ,$$

$$\sum_{S \in W_{v_j}} 2^{2^n - D(S)} < \sum_{S \in W_{v_{j+1}}} 2^{2^n - D(S)} \text{ for all } j \in \Omega_{l-1} .$$

Define the <u>successor</u> of a directed game v, let us say v_j, to be v_{j+1}, if $j \neq l$; written $\sigma(v) := v_{j+1}$.

We shall construct an algorithm which starts with v_1 and generates the chain of games $v_1,\dots,v_l$ recursively. To do this the successor of a game v is characterized in terms of v.

Lemma 2.2: Let v be a directed n-person game with $W_v \neq \mathcal{P}(\Omega)$. Take the coalition $S \in \mathcal{P}(\Omega)\backslash W_v$, which maximizes D(S). Then

$$W^s_{\sigma(v)} = \{T \in W^s_v \mid D(T) < D(S)\} \cup \{S\} .$$

Proof: We have $W_{\sigma(v)} \subsetneq W_v$ by the definition of the successor. If $S^1 \in W_{\sigma(v)} \backslash W_{v'}$, which is the coalition minimizing $D(S^1)$, then it is clear by the definition of S that $D(S^1) \leq D(S)$. Let

$$A := \{T \in W_v^S \mid D(T) < D(S)\} \cup \{S\} .$$

The $\langle A \rangle$ is the set of winning coalitions of a certain directed n–person game v'. It remains to show that $v' = \sigma(v)$ and that A contains no proper subset spanning $\langle A \rangle$. The second assertion is obvious.

For the first assertion assume $D(S^1) < D(S)$, thus

$$\sum_{T \in \langle A \rangle} 2^{2^n - D(T)} = \sum_{\substack{T \in \langle A \rangle \\ D(T) < D(S^1)}} 2^{2^n - D(T)} + \sum_{\substack{T \in \langle A \rangle \\ D(T) > D(S^1)}} 2^{2^n - D(T)}$$

$$< \sum_{\substack{T \in \langle A \rangle \cup \{S^1\} \\ D(T) \leq D(S^1)}} 2^{2^n - D(T)} \leq \sum_{T \in W_{\sigma(v)}} 2^{2^n - D(T)} ,$$

a contradiction. Thus $W_{\sigma(v)} = \langle A \rangle$, which finishes the proof. q.e.d.

From Lemma 2.2 we obtain the desired algorithm:

Start with $W_{v_1}^S = \emptyset = W_{v_1}$.

If $W_{v_j}^S$ and W_{v_j} are known and $W_{v_j}^S \neq \{\emptyset\}$ (i.e. $j \neq 1$), take the lexicographically maximal losing coalition S and observe that

$$W_{v_{j+1}}^S = \{S\} \cup \{T \in W_j^S \mid D(T) < S\}.$$

Moreover $W_{v_{j+1}} = \langle W_{v_{j+1}}^S \rangle$.

Note that $W_{v_{j+1}} = \langle S \rangle \cup W_{v''}$, where v' is already constructed, since

$$W_{v'} = \langle \{T \in W_{v_j}^S \mid D(T) < S\} \rangle.$$

If only the number of directed n–person games is to be computed, this algorithm should be simplified as follows: Before starting the proper algorithm the principal filters <T>, $T \in \mathcal{P}(\Omega)$, are computed.

If W_{v_j}, $W^S_{v_j}$ is known and if S is as above, then $W_{v_{j+1}}$, $W^S_{v_{j+1}}$ are computed as indicated before. Now all games v_i already constructed, which satisfy $W^S_{v_i} \cap \{T \mid D(T) > S\}$, are dropped. If $j+1 < l$, the successor of v_{j+1} can be computed easily only using the computed principal filters and the present games.

The rest of this chapter is used to establish an algorithm, which generates the directed n–person zero–sum games Z_n. It does not seem to be natural to generate these games recursively w.r.t. the lexicographic order in view of the fact that the cardinalities of the sets of winning coalitions are constant.

At first we define a relation on Z_n, where $n \geq 2$ for the rest of this chapter.

Definition 2.3: For games $v, v_1, v_2 \in Z_n$ define $v_1 \preceq v_2$, iff $W_{v_1}(1) \subseteq W_{v_2}(1)$.

Besides, notice that there is a canonical bijection from $W_v(1)$ to $W_v \cap \{S \subseteq \Omega \mid 1 \notin S\}$, given by $(S_1,\ldots,S_{n-1}) \mapsto (0,S_1,\ldots,S_{n-1})$. Define a partition of Z_n by $Z_{n,i} = \{v \in Z_n \mid |W_v(1)| = i\}$ for all $i \in \mathbb{N}_0$. Moreover, let $T^{max}(v)$ be the lexicographically maximal losing coalition of v, i.e., the losing coalition satisfying

$$D(T^{max}(v)) = \max \{D(T) \mid T \in \mathcal{P}(\Omega)\backslash W_v\}.$$

At last define the set of <u>large coalitions</u> of v to be

$$W^{sl}_v := \{S \in W_v \mid T \in W_v, \text{ if } T \subseteq \Omega \text{ and } D(T) \geq D(S)\}.$$

It should be remarked that $(Z_n, \preceq)$ is a ranked partially ordered set, where the rank function $Z_n \to \mathbb{N}_0$ is given by $v \mapsto |W_v(1)|$. Fig.3 sketches the corresponding

Hasse diagram in the case n=5 and shows that $(Z_n, \preceq)$ is not a lattice in general, since e.g. (0 0 1 1 1) and (0 1 1 0 0) have no supremum.

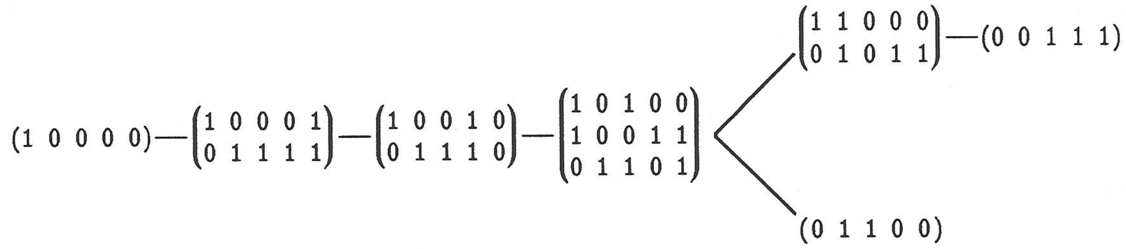

Figure 3

It is clear that $Z_{n,0}$ contains the unique game v characterized by $W_v^s = \{\{1\}\}$. Moreover $Z_{n,i}$ vanishes for $i > 2^{n-2}$, because the injective mapping $\{S \subseteq \Omega | 1 \notin S\} \to \{S \subseteq \Omega | 1 \in S\}$, $S \mapsto S \cup \{1\}$, shows that $|W_v(1)| \leq |W_v(2)|$ $(v \in Z_n)$, but $|W_v(1)| + |W_v(2)| = |W_v| = 2^{n-1}$.

For the sake of completeness we proof an exact result concerning the proper total rank of the partially ordered set $(Z_n, \preceq)$.

Lemma 2.4: $Z_{n,i} \neq \emptyset$, iff $i \in \{0\} \cup \Omega_{r_n}$, where $r_n = 2^{n-2} - \binom{n-2}{[n-2/2]}$. Moreover Z_{n,r_n} contains a unique game, characterized by the unique shift-minimal coalition

$$\begin{cases} (0,\ldots,0,\underbrace{1,\ldots,1}_{n+1/2 \text{ times}}) & \text{, if n is odd} \\ (0,\ldots,0,\underbrace{1,\ldots,1}_{n/2 \text{ times}},0) & \text{, if n is even} \end{cases}$$

Proof: Define $i_0 := \max \{i \in \mathbb{N}_0 \mid Z_{n,i} \neq \emptyset\}$.

(1) At first it will be shown by induction that $Z_{n,i} \neq \emptyset$ for all $0 \leq i \leq i_0$. Z_{n,i_0} is nonempty by definition. Let v be an element of $Z_{n,i+1}$ and S be the lexicographically minimal winning coalition of v, thus $S \not\ni 1$ and S is shiftminimal.

Moreover $W_v \backslash \{S\}$ characterizes a directed game v' with $2^{n-1} - 1$ winning coalitions. Obviously the coalition $\Omega \backslash S$ is the lexicographically maximal losing coalition of v'. Therefore $W_{v'} \cup \{\Omega \backslash S\}$ characterizes a zero–sum game $\tilde{v} \in Z_{n,i}$, thus

$$W_v \cup \{T^{max}(v)\} \backslash \{\Omega \backslash T^{max}(v)\}$$

characterizes $\tilde{v}$.

(2) An easy argument of the elementary theory of combinations concerning Pascal's triangle verifies that Z_{n,r_n} indeed contains the game, defined in the second assertion of the lemma.

Therefore it remains to show that $i_0 \leq r_n$ and $|Z_{n,r_n}| = 1$.

(3) In order to complete the proof of this lemma it suffices to show the following:
Let v be a directed (n–1)–person game (not necessarily a zero–sum game) and $U_v^i = W_v \cap \{S \mid \Omega_i \subseteq S\}$, for all $0 \leq i < n$. Let S be one of the elements

$$(0,\ldots,0,\underbrace{1,\ldots,1}_{k \text{ times}}),\ (0,\ldots,0,\underbrace{1,\ldots,1}_{k-1 \text{ times}},0)$$

for some $k \in \Omega_{n-1}$ and $\bar{v}$ be defined by the prime filter $W_{\bar{v}} = \langle S \rangle$.

If $v \neq \bar{v}$ and $|W_v| \geq |W_{\bar{v}}|$, then $|U_v^i| \geq |U_{\bar{v}}^i|$ for all $i \in \Omega_{n-1}$ and there is some i_1, such that $|U_v^{i_1}| > |U_{\bar{v}}^{i_1}|$.

For n=2 these assertions are trivially satisfied (there are exactly three directed 1–person games which behave in the desired way). Assume the assertions are valid for some $n \geq 2$ and define v, $\bar{v}$ to be n–person games.

The following three disjoint cases are distinguished.

(α) (*) There is a coalition $S^1 \in W_v$ such $D(S^1) < D(S)$. Then it is obvious by the definition of S that $<S^1> \ni S$ and thus $<S^1> \supseteq S^1 \cup <S>$. This completes the proof in this case. Now assume (*) is not valid.

(β)
$$|W_{v^{(1)}}| \geq |W_{\bar{v}^{(1)}}|, \text{ i.e. } |W_v \cap \{S | 1 \notin S\}| \geq |W_{\bar{v}} \cap \{S | 1 \notin S\}|.$$

$\bar{v}^{(1)}$ is again characterized by one coalition, namely $S^1 = S\backslash\{1\}$, since $W_{\bar{v}^{(1)}} = (W_{\bar{v}} \cap \{S \mid 1 \notin S\})\backslash\{1\}$. Thus we can apply the inductive hypothesis to $\bar{v}^{(1)}$, $v^{(1)}$ and obtain:

(**) $|U^i_{v^{(1)}}| \geq |U^i_{\bar{v}^{(1)}}| \quad (0 \leq i \leq n-1)$

(***) $|U^{i_1}_{v^{(1)}}| > |U^{i_1}_{\bar{v}^{(1)}}|$ for some i_1.

If $T \subseteq \Omega_n$, $1 \notin T$ and $(\Omega_t\backslash\{1\}) \subseteq T$ $(2 \leq t \leq n)$, define

$$\alpha(T) = \Omega_{t-1} \cup (T\backslash\Omega_t), \text{ thus } \alpha(T) \in <T>.$$

From the obvious fact that $U^{t-1}_{v^{(1)}}$ corresponds bijectively to

$$W_v \cap \{T \mid (\Omega_t\backslash\{1\}) \subseteq T \not\ni 1\}$$

we conclude that

$$|U^j_v| \geq \sum_{t=j+1}^{n} |U^{t-1}_{v^{(1)}}| + 1$$

and

$$|U^j_{\bar{v}}| = \sum_{t=j+1}^{n} |U^{t-1}_{\bar{v}^{(1)}}| + 1, \text{ since } D(S) < 2^{n-1}.$$

Therefore this case is finished by using (**) and (***).

(γ) If neither the prerequisites of (α) nor those of (β) are satisfied, then clearly $U^1_v > U^1_{\bar{v}}$.

Now it is obvious that $U^1_{\bar{v}} = <S^1>$, where

$$S^1 = (1,0,...,0,\underbrace{1,...,1}_{k-1 \text{ times}}) \text{ or } S^1 = (1,0,...,0,\underbrace{1,...,1}_{k-2 \text{ times}},0)$$

respectively (we do not have to consider the case k=1 since this case is trivial). If there is a coalition $\bar{S} \in U^1_v$ with $D(\bar{S}) \leq D(S^1)$, then the proof is finished by an argument completely analog to the one of case (α).

Otherwise consider $v^{(1)*}$ and $\bar{v}^{(1)*}$ respectively and observe that

$$|U^i_{v^{(1)*}}| = |U^{i+1}_v| \, , \, |U^i_{\bar{v}^{(1)*}}| = |U^{i+1}_{\bar{v}}| \, .$$

Consequently the assertions follow from the inductive hypothesis applied to the dual games $v^{(1)*}$, $\bar{v}^{(1)*}$. q.e.d.

Next a result is formulated which directly leads to the algorithm.

Lemma 2.5: Let v be a game in $Z_{n,i}$ and $n \geq 3$ for some $0 \leq i \leq r_n$.

(i) The game $\tilde{v}$, characterized by $W_{\tilde{v}} = (W_v \backslash \{\Omega \backslash T^{max}_{(v)}\}) \cup \{T^{max}_{(v)}\}$, is an element of $Z_{n,i-1}$, if $i > 0$, and of $Z_{n,1}$, if $i = 0$.

(ii) If S is a large coalition of v, then $\bar{v} \in Z_{n,i+1}$, where $\bar{v}$ is the game, characterized by $W_{\bar{v}} = (W_v \backslash \{S\}) \cup \{\Omega \backslash S\}$, and $T^{max}_{\bar{v}} = \Omega \backslash S$.

Proof: The same arguments as in the proof of Lemma 2.4, part (1), show that $\tilde{v}$ is a directed and thus zero–sum game, if $i \neq 0$. But in the case i=0 we also see that $\tilde{v}$ is directed, because $n \geq 3$. It remains to show that $\bar{v}$ is directed.

Assume on the contrary that $\bar{v}$ is not directed and put $T := \Omega \backslash S$, thus $<T> \backslash (W_v \backslash \{S\} \cup \{T\}) \neq \emptyset$. Take a coalition T^1 of this nonvoid subset of all coalitions, then $\tilde{T} \leq \tilde{T}^1$, $T^1 \notin W_v$, thus $\widetilde{\Omega \backslash T^1} \leq \widetilde{\Omega \backslash T} = \tilde{S}$ and $\Omega \backslash T^1 \in W_v$. Therefore T^1 must coincide with S, because S is shift minimal.

If $|T| = 1$, then $S =$ $(1...1\ 0\ 1...1) \in W_v^s$,

$(0\ 1...1\ 1...1) \notin W_v^s$,

$(1\ 0...0\ 0...0) \notin W_v^s$ by $n \geq 3$.

The union of these last two coalitions is Ω, a contradiction to the zero–sum property of v. Therefore define: $t_1 = \min T$, $t_2 = \min T \setminus \{t_1\}$.

If $t_2 = t_1+1$, then

$$\langle T\rangle \ni T^2 := T \cup \{1\}\setminus\{t_1\} \neq T^3 := T \cup \{1\}\setminus\{t_2\} \in \langle T\rangle$$

and clearly S covers both of these coalitions, i.e. $S \in \langle T^2\rangle \cap \langle T^3\rangle$, which contradicts the shift minimality of S. In the remaining case, i.e. $t_2 > t_1+1$, T^3 can be substituted by $T \cup \{t_2-1\}\setminus\{t_2\}$ and the same arguments lead to a contradiction. q.e.d.

Definition 2.6: Let $\varphi : Z_n \setminus Z_{n,0} \to Z_n$ be defined by $\varphi(v) = \tilde{v}$, where $\tilde{v}$ is the game given by Lemma 2.5 (i). Note that $\{\varphi^j(v)\} = Z_{n,0}$, if $v \in Z_{n,i}$. If S is a large coalition of $v \in Z_n$, i.e. $S \in W_v^{sl}$, then define v_S to be the game $\bar{v}$ of Lemma 2.5 (ii) and

$$\rho : Z_n \to \mathcal{P}(Z_n)$$
$$v \mapsto \{v_S \mid S \in W_v^{sl}\}.$$

Combining the last definitions and results we obtain

Proposition 2.7:

(i) $\varphi(\rho(v)) = \{v\}$ for all $v \in Z_n$ with $\rho(v) \neq \emptyset$

(ii) $\rho(\varphi(v)) \ni v$ for all $v \in Z_n \setminus Z_{n,0}$

(iii) $|\rho(v)| = |W_v^{sl}|$ for all $v \in Z_n$

(iv) $Z_{n,i+1} = \bigcup_{v \in Z_{n,i}} \rho(v)$, for all $0 \leq i \leq r_n$

(v) $\rho(v) = \emptyset$, if $v \in Z_{n,r_n}$.

Now the algorithm to generate the directed n–person zero–sum games proceeds as follows:
Starting with the unique game of $Z_{n,0}$ and applying ρ yields $Z_{n,1}$. If $Z_{n,i}$ is constructed and $i < r_n$, then $Z_{n,i+1}$ is obtained by applying ρ to each element of $Z_{n,i}$.

It should be remarked that this algorithm can be modified in such a way that the arising procedure computes all edges of the partially ordered set $(Z_n, \preceq)$:
If $v \in Z_n$ and $v' \in \rho(v)$, then v and v' are joined by an edge. But there is a canonical extension $\bar{\rho}$ of ρ, which considers all shiftminimal coalitions S of v with $D(S) \geq 2^{n-1}$ instead of the large coalitions only and we obtain that $v \preceq v'$ and v, v' are joined by an edge, iff $v' \in \bar{\rho}(v)$. The proof is analogous to the one of Lemma 2.5. The first algorithm is clearly faster, since it generates a subgraph of $(Z_n, \preceq)$, which contains all vertices and which is a tree.

For an example we refer to the Appendix.

The last chapter gives an answer to the question how the games of Z_n can be tested on representability.

§3 Weighted majority zero-sum games

If $(\lambda;m)$ is a representation of an n-person weighted majority zero-sum game, $m(T) < m(\Omega)/2 < m(S)$ for all coalitions $T \notin W_v$, $S \in W_v$. Therefore the game remains unchanged if λ is substituted by $[(1+m(\Omega))/2]$.

For the sake of brevity we will drop the level λ in the zero-sum case, i.e. $(\lambda;m)$ is identified with $m = (m_1,\dots,m_n)$. Moreover $\overline{m} = \left[\frac{m_1}{m(\Omega)},\dots,\frac{m_n}{m(\Omega)}\right]$ is called a <u>normed</u> representation of v.

Conversely, a payoff vector $\tilde{m} = (\tilde{m}_1,\dots,\tilde{m}_n)$ is the normed representation of a weighted majority zero-sum game, if there is no coalition S with $\tilde{m}(S) = \frac{1}{2}$.

Additionally it is known that a simple zero-sum game v is a weighted majority game, iff the nucleolus of v is a normed representation of v (see Peleg (1968) and Schmeidler (1966)).

In what follows we use an approach similar to the first step of the algorithm computing the nucleolus considered by Wolsey (1976) (we also refer to Kopelowitz (1967)), to compute a payoff vector to each directed zero-sum game, which is a representation in the case of a weighted majority game.

Definition 3.1: If $v \in Z_n$, then define $X_v :=$

$$\left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} x \geq 0,\ x(\Omega) = 1 \text{ and } x(S) \geq \\ \max\{\min\{y(T) \mid T \in W_v\} \mid y \geq 0, y(\Omega) = 1\} \text{ for all } S \in W_v \end{array} \right\}$$

$$q_v := \min\{x(S) \mid S \in W_v^m\} \text{ for each } x \in X_v$$

$$\overline{X}_v := \{x \in X_v \mid x_1 \geq \dots \geq x_n\}$$

Note that the set X_v remains unchanged, if W_v is substituted by W_v^m at all places, and that this set is the least core in the sense of Maschler, Peleg and Shapley (1978).

Observe that X_v and $\overline{X}_v$ are convex polyeders, containing the nucleolus of v and being subsets of the set of normed representations of v in the weighted majority case.

We want to compute an extreme point of X_v or $\overline{X}_v$ using the equilibrium concept of a non–cooperative matrix game which is characterized, roughly speaking, by W_v^m or W_v^s respectively.

Now we come to the detailed description of the matrix games.
Let Γ_v be the matrix game characterized by the transpose matrix of the incidence matrix of the directed n–person zero–sum game v, namely $A := I(v)^t$ (i.e. if $k = |W_v^m|$, then $Y = \{y | y$ is a payoff k–vector$\}$ and $X = \{x | x$ is a payoff n–vector$\}$ are the sets of strategies for player II and player I respectively. A tupel of strategies $(x,y) \in X \times Y$ leads to the payoff xAy for player I and to –xAy for player II). Moreover $\bar{x} \in X$ is an optimal strategy for player I, iff $\bar{x} \in X_v$. For this property we refer to e.g. Rosenmüller (1981), chapter 1. The second matrix game Γ_v^* is characterized by the matrix

$$A^* := E_{k,n} - I(v), \text{ where } E_{k,n} = \begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{bmatrix}$$

is a $k \times n$ matrix. We conclude that $\tilde{x}$ is an optimal strategy for player II w.r.t. Γ^*, iff $\tilde{x} \in X_v$. Let e_n denote the n–vector (1,...,1).
An arbitrary $k \times n$ matrix B is defined to have property (P), if each entry of B is nonnegative and B has no all–zero column.

If Γ is the matrix game characterized by B, then the following lemma is well known (see Brickmann (1989) and again Rosenmüller (1981)).

Lemma 3.2: $\max \{y(\Omega) \mid y \geq 0 \text{ and } By \leq e_k\}$ and
$\min \{x(\Omega) \mid x \geq 0 \text{ and } B \geq e_n\}$ exist.
If $\bar{y}$ and $\bar{x}$ is a maximizer and minimizer respectively, then $(\bar{x},\bar{y})/\bar{y}(\Omega)$ is an optimal pair of strategies for Γ. Conversely, if $\tilde{x}$ or $\tilde{y}$ is an optimal pair of strategy for player I or II, then there is a vector $\bar{x}$ or $\bar{y}$ such that $\tilde{x} = \bar{x}/\bar{x}(\Omega_k)$ or $\tilde{y} = \bar{y}/\bar{y}(\Omega)$ respectively.

The maximization problem of Lemma 3.2 is the dual of the minimization problem, thus $\bar{x}(\Omega_k) = \bar{y}(\Omega)$.

The matrix A (transpose of I(v)) trivially has property (P), since no minimal winning coalition is empty. Moreover A^* (consisting of all maximal losing coalitions) has property (P), as long as $\{v\} \neq Z_{n,0}$ is valid (see the last chapter). It is sufficient to test the elements of $Z_n \backslash Z_{n,0}$ on representability, because the only remaining game is trivially a weighted majority game ((1,0,...,0) is a (normed) representation). Therefore v is assumed to be an element of $Z_n \backslash Z_{n,0}$ from now on.

Corollary 3.3: Let v be a game of $Z_n \backslash Z_{n,0}$ and $k = |W_v^m|$. The following assertions are equivalent:

(i) v is a weighted majority game.

(ii) $\max \{y(\Omega_k) \mid 0 \leq y \text{ is a k-vector and } y \cdot I(v) \leq e_n\} < 2.$

(iii) $\max \{x(\Omega_n) \mid 0 \leq x \text{ is an n-vector, } (E_{k,n} - I(v)) \cdot x \leq e_k\} > 2$

The maximization problems (ii), (iii) of Corollary 3.3 can be solved by the Simplex Method.

Algorithm 1: Let $v \in Z_n$, $A = I(v)^t$, $k = |W_v^m|$.

First step: Start with the initial tableau (see Brickmann (1989))

0	n+1 · · · n+k	0
1		1
⋮	A	⋮
n		1
0	–1 · · · –1	0

Second step: Apply the Simplex Method by choosing the pivot element according to e.g. Bland's Rule. If the entry p in the last row and column is not smaller than 2, then continue with the fourth step. If no optimum is reached, take this new tableau and continue with the second step.

Third step: Define for each $i \in \Omega_n$

$$m_i = \begin{cases} 0, & \text{if } i \text{ is not contained in the first row of the tableau} \\ \text{the last entry of the column with first entry } i, & \text{otherwise} \end{cases}$$

and observe that $(m_1,\ldots,m_n)/p$ is a normed representation of v. Now stop the algorithm.

Fourth step: Conclude that v is no weighted majority game (by (ii) of Corollary 3.3).

Algorithm 2: Let $v \in Z_n \setminus Z_{n,0}$ and $A^* = E_{k,n} - I(v)$, where $k = |W_v^m|$.

(1) Start with the initial tableau

0	1 ⋯ n	0
n+1 ⋮ n+k	A^*	1 ⋮ 1
0	−1 ⋯ −1	0

(2) Apply the Simplex Method by choosing the pivot element according to Bland's Rule. If the entry p in the last row and column exceeds 2, continue with (4). If no optimum is reached, take this new tableau and continue with (2).

(3) Conclude that v is no weighted majority game (by (iii) of Corollary 3.3) and stop this algorithm.

(4) By (iii) of Corollary 3.3 v is a weighted majority game.

This Algorithm 2 can be modified to

Algorithm 2a: Let the steps (1a) and (3a) be exactly the steps (1) and (3) from Algorithm 2 and introduce two further steps:

(2a) Apply the Simplex Method and compute p as in (2). If no optimum is reached, take the new tableau and continue with (2a). If $p > 2$, continue with (4a).

(4a) Define

$$m_i = \begin{cases} 0, & \text{if } i \text{ is not contained in the first column} \\ \text{the last entry in the row with first entry } i, & \text{otherwise} \end{cases}$$

and conclude that $(m_1,\dots,m_n)/p$ is a normed representation of v.

It should be remarked that both algorithms, slightly modified, can be used to compute an extreme point of X_v even in the case v being no weighted majority game: Apply the Simplex method until an optimum is reached. Now define the vector m/p according to the third step or (4a) respectively and observe that this vector is an extreme point of X_v in any case.

The algorithms, just presented, work even in the case that the game started with is not directed (but still monotonous). We proceed by constructing generically faster and quite similar algorithms, which can only be applied to directed zero–sum games. It is our aim to substitute the incidence matrix by the shiftminimal matrix and a few additional rows.

Definition 3.4: For each $v \in Z_n$ define the $(n+k-1) \times n$ matrices

$$\tilde{I}(v) = \begin{bmatrix} & & & I^s(v) & & & \\ 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}, \qquad \bar{I}(v) = \begin{bmatrix} & & & E_{k,n} - I^s(v) & & & \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

where $k = |W_v^s|$.

Now the direct analogon of Corollary 3.3 is the following

Lemma 3.5: Let $v \in Z_n \backslash Z_{n,0}$ and $k = |W_v^{sm}|$. Then the following assertions are equivalent.

(i) v is a weighted majority game.

(ii) $\max \{y(\Omega_k) \mid y \in \mathbb{R}^{k+n-1}, y \geq 0 \text{ and } y \cdot \tilde{I}(v) \leq e_n\} < 2.$

(iii) $\max \{x(\Omega_n) \mid x \in \mathbb{R}^n, x \geq 0 \text{ and } \bar{I}(v) \cdot x \leq (e_k, \underbrace{0,\ldots,0}_{n-1 \text{ times}})\} > 2.$

Proof: Put $\bar{k} = |W_v^m|$. It suffices to show that

(1) $$\delta_0 := \max \{y(\Omega_{\bar{k}}) \mid 0 \leq y \in \mathbb{R}^{\bar{k}} \text{ and } y \cdot I(v) \leq e_n\} = \max \{y(\Omega_k) \mid 0 \leq y \in \mathbb{R}^k \text{ and } y \cdot \tilde{I}(v) \leq e_n\} =: \delta_1$$

and

(2) $$\gamma_0 := \max \{x(\Omega) \mid 0 \leq x \in \mathbb{R}^n \text{ and } (E_{\bar{k},n} - I(v)) \cdot x \leq e_{\bar{k}}\} = \max \{x(\Omega) \mid 0 \leq x \in \mathbb{R}^n \text{ and } \bar{I}(v) \cdot x \leq (e_k, 0, \ldots, 0)\} =: \gamma_1.$$

ad(2): For each $x \in \mathbb{R}^n$ define

$i(x) := \max \{i \in \{0\} \cup \Omega_n \mid x_1 \geq \ldots \geq x_i \geq \max \{x_j \mid i < j \leq n\}\}.$

Take $\bar{x} \in \mathbb{R}^n$ such that

(α) $\bar{x} \geq 0$, $(E_{\bar{k},n} - I(v)) \cdot x \leq e_{\bar{k}}$ and $x(\Omega) = \gamma_0$

is valid and $i(\bar{x})$ is maximal. Now $\bar{x}_1 \geq \ldots \geq \bar{x}_n$ is to be verified. Assume, on the contrary, $i(\bar{x}) < n$, let us say $\bar{x}_{i_0} = \max \{\bar{x}_i \mid i > i_0\}$, thus $i_0 > i(x)+1$. Therefore $i(x) > i(\bar{x})$, where

$x = (\bar{x}_1, \ldots, \bar{x}_{i(\bar{x})}, \bar{x}_{i_0}, \bar{x}_{i(\bar{x})+2}, \ldots, \bar{x}_{i_0-1}, \bar{x}_{i(\bar{x})+1}, \bar{x}_{i_0+1}, \ldots, \bar{x}_n).$

Moreover there is a maximal losing coalition T, i.e. $\Omega \backslash T \in W_v^m$, with $x(T) > 1$ (because of the maximality of $i(\bar{x})$). Thus $i_0 \notin T$, $i(\bar{x})+1 \in T$. Therefore $T' = T \cup \{i_0\} \backslash \{i(\bar{x})+1\}$ is a losing coalition, which satisfies $\bar{x}(T') = x(T) > 1$, a contradiction.

These arguments directly imply $\gamma_1 \geq \gamma_0$.

Conversely take $x \in \{x \in \mathbb{R}^n \mid x \geq 0 \text{ and } \bar{I}(v) \cdot x \leq (e_k, 0, \dots, 0)\}$, thus ($\beta$) $x_1 \geq x_2 \geq \dots \geq x_n$ by the definition of $\bar{I}(v)$.

If S is a minimal winning coalition of v, then there is a shiftminimal coalition S' such that $\tilde{S}' \leq \tilde{S}$. Let T be a row of $E_{\bar{k},n} - I(v)$. Then $S = \Omega \backslash T$ is a minimal winning coalition, showing that $T' = \Omega \backslash S'$ is a row in $E_{k,n} - I^s(v)$ and $\tilde{T}' \geq \tilde{T}$. Thus $x(T) \leq x(T') \leq 1$ (by (β)), implying $\gamma_0 \geq \gamma_1$.

ad (1): Look at the dual problems:
Let $x \in \mathbb{R}^n$, $x \geq 0$, $I(v)\, x \leq e_{\bar{k}}$ and $x(\Omega) = \delta_0$. Then analog arguments as in ad (2) show that w.l.o.g. $x_1 \geq \dots \geq x_n$, meaning $\tilde{I}(v) \cdot x \leq (e_k, 0, \dots, 0)$, thus $\delta_1 \geq \delta_0$ by looking at the dual problems.
Conversely take $x \in \mathbb{R}^n$, $x \geq 0$ and $\tilde{I}(v) \cdot x \leq e_k$. Then $I(v) \cdot x \leq e_{\bar{k}}$, because of the fact $x_1, \geq \dots \geq x_n$, thus $\delta_0 \geq \delta_1$. q.e.d.

Clearly the extreme points of X_v and $\bar{X}_v$ are the normed extreme points of the sets of maximizers of the problems (ii) and (iii) of Corollary 3.3 and Lemma 3.5 respectively. In view of the proof of the last lemma we obtain the following

Corollary 3.6:

$$\bar{X}_v = \{x \in X_v \mid x_1 \geq \dots \geq x_n\}$$
$$= \{x \in \mathbb{R}^n \mid x_1 \geq \dots \geq x_n \text{ and there is an } \bar{x} \in X_v \text{ and a permutation } \pi \text{ of } \Omega \text{ such that } x = \bar{x} \circ \pi\}$$
$$\neq \emptyset$$

Now the analogon of Algorithm 1 is **Algorithm I:**

Let $v \in Z_n$, $A = \tilde{I}(v)^t$, $k = |W_v^s|$. Here is the initial tableau:

0	n+1 ... n+k	n+k+1 ... 2n+k–1	0
1			1
⋮	A		⋮
n			1
0	–1 ... –1	0 ... 0	0

The following steps are exactly those of Algorithm 1.

Algorithm II and IIa respectively: Let $v \in \mathbb{Z}_n \backslash Z_{n,0}$, $A^* = \bar{I}(v)$, $k = |W_v^s|$. Here is the initial tableau:

0	1 ... n	0
n+1		1
⋮		⋮
n+k	A^*	1
n+k+1		0
⋮		⋮
2n+k–1		0
0	–1 ... –1	0

The other steps are exactly those of Algorithm 2 and 2a respectively.

An example is given in the Appendix.

Concluding Remarks:

(1) Let v be an element of Z_n or $Z_n \backslash Z_{n,0}$ respectively, which is a weighted majority game.

Then each of the algorithm 1 and I or 2a and IIa generates a normed representation $(\frac{m_1}{p},\ldots,\frac{m_n}{p})$ respectively. A representation $(\bar{m}_1,\ldots,\bar{m}_n) \in \mathbb{N}_0^n$ is obtained by the following procedure:

$\tilde{m}_i = m_i \cdot q$ ($i \in \Omega$), where q is the product of the pivot elements. Indeed the fact that $\tilde{m}_i$ is a nonnegative integer can easily be verified by an inductive argument.

(2) In each case the vector $m = (m_1,\ldots,m_n)$ together with p has the interesting property

$$\min\{m(S) \mid S \in W_v\} - \max\{m(T) \mid T \notin W_v\} = \begin{cases} 2-p, & \text{if Algorithm 1 or I is used} \\ p-2, & \text{if Algorithm 2a or IIa is used} \end{cases}$$

This fact is shown for Algorithm 1, I by observing that $\min\{m(S) \mid S \in W_v\} = 1$ and $m(\Omega) = p$, thus $\max\{(T) \mid T \notin W_v\} = p-1$ (v is a zero–sum game), and for Algorithm 2a, IIa analogously by interchanging the roles of S and T.

Therefore $m/|2-p|$ is a minimal representation in the weighted majority case, if $m_i/|2-p| \in \mathbb{N}_0$ is satisfied. Surprisingly it turns out that this vector is indeed an integer vector in many cases. To be more precise each weighted majority zero–sum game with less than 9 persons has an extreme point of X_v and $\overline{X}_v$ which is a normed minimal representation of v (due to e.g. Algorithm II, I). In the 9–person case the algorithms of the last chapter generates 319,124 directed zero–sum games, from which exactly 175,428 are weighted majority games and exactly two of which are "counter examples".

Here is the first game v_1:

This game is represented by $\bar{m} = (15\ 13\ 10\ 8\ 6\ 4\ 4\ 2\ 1)$, but the normed representation $\bar{m}/63$ cannot be an element of X_{v_1} or $\overline{X}_{v_1}$, since each of the preceding algorithms yields $\tilde{m} = (14.5\ 12.5\ 9.5\ 7.5\ 6\ 4\ 4\ 1.5\ 1.5)/61$ thus $\tilde{m}(S) \geq 31/61 > 32/63 = \bar{m}(S_0)/63$ for all $S \in W_{v_1}$ and $S_0 = \{1,2,6\}$.

It remains to show that $\bar{m}$ is a minimal representation of v. Let m be a minimal representation. Then $m_9 \geq 1$, since this game has no dummies. If $m_8 \geq 2$ is presumed, then we can prove 7 lemmata which successively show that $m_7 \geq 4$, $m_6 \geq 4$, $m_5 \geq 6$, $m_4 \geq 8$, $m_3 \geq 10$, $m_2 \geq 13$, $m_1 \geq 15$. We only have to exclude w.l.o.g. that $m_8 = m_9 = 1$. In this case each coalition $S \in W^m_{v_1}$ with $\{8,9\} \cap S \neq \emptyset$ would satisfy $m(S) = \lambda := \min \{m(S) | S \in W_{v_1}\}$. Using these coalitions we successively obtain $m_7 = m_6$, $m_5 = 2m_7 - 1 = m_7 + 1$, thus $m_7 = 2$, $m_4 = 2m_7$, $m_3 = 3m_7 - 1$, $m_1 = m_2 + 1$; therefore $(m_3,\ldots,m_9) = (5\ 4\ 3\ 2\ 2\ 1\ 1)$. Since $(1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0)$ is a minimal winning coalition, we additionally obtain $9 \leq 2m_2+1 \leq 10$, thus $m_1 = 5$, $m_2 = 4$, $\lambda = 14$, but then $m(\{1,2,6\}) = 11 < 14$, a contradiction in view of the fact that this coalition is winning.

The second game v_2 is the one represented by (17 15 11 9 7 5 4 2 1), this representation being minimal (this can be verified analogously to the first game), and $(16.5\ 14.5\ 10.5\ 8.5\ 7\ 5\ 4\ 1.5\ 1.5)/69 \in X_{v_2}$. We conclude again that no normed minimal representation of v_2 is in X_{v_2}.

Conversely using Algorithm I we obtain additionally 12 weighted majority zero–sum games v with an extreme point in $\bar{X}_v$, which is not a normed minimal representation (we conjecture that there is no further 9–person game with this property).

This fact can be motivated heuristically as follows. All these games have two normed minimal representations which are extreme points of X_v and which are different only on one type of players (we refer to table 2 of the Appendix). One representation is in $\bar{X}_v$ but not the other and a certain pure convex combination of the representations is an extreme point of $\bar{X}_v$. The zero–sum extension of the game considered by Dubey and Shapley (1978) is an example: (13 7 6 6 4 4 4 3 2)/49 is a normed minimal representation of this game but the last two weights can be exchanged. Both normed representations are extreme points of X_v and the first is in $\bar{X}_v$ but $\bar{X}_v$ contains the midpoint of these representations as extreme point, too.

(3) Applying each algorithm to the famous 12–person weighted majority zero—sum game introduced by Isbell (1959), which has two minimal representations such that the affected players 1 and 9 are of different type, we obtain one of the normed minimal representation, i.e. both are extreme points of X_v and $\overline{X}_v$.

(4) Both Algorithms I and II(a) can be modified in such a way that the shift minimal and shift maximal coalitions (i.e. the complements of the shift minimal coalitions) are identified with the types of these coalitions or profiles:

$$S \mapsto a(S) := (a_1(S),\dots,a_{t(v)}(S)),$$

where

$$a_j(S) = |S \cap T_j| \quad (1 \le j \le t(v)),$$

T_j is defined according to Definition 1.8. Using the notation of Definition 3.4 $\tilde{I}(v)$ and $\bar{I}(v)$ must be substituted by the $(t(v)+k-1) \times t(v)$ matrices

$$\begin{bmatrix} & a(S) & & & \\ 1 & -1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix}_{S \in I^S(v)}, \quad \text{and} \quad \begin{bmatrix} & a(T) & & & \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}_{T \in E_{k,n} - I^S(v)} \quad \text{respectively}$$

Note that it is very easy to compute the partition sets T_j (see Sudhölter (1989), chapter 4) and therefore this procedure will generically diminish the initial tableau and the simplex steps. The disadvantage of the necessary computation of the T_j will thus be compensated especially if the number of players is large. These new algorithms yield an extreme point of the nonvoid convex subset

$$\{x \in X_v \mid x_i = x_j \text{ if } i \sim j\} \text{ of } \overline{X}_v .$$

Appendix

Some figures and tables are sketched as illustrations of the presented algorithms. Fig. 4 shows the lattice $(\mathscr{P}(\Omega_n),\precsim)$ (or $M(n),\precsim)$ for n = 3, 4, 5, 6, 7, 8. Fig. 5 sketches the lattice of directed n–person games (n = 4, 5, 6), considered as filters of $(\mathscr{P}(\Omega_n),\precsim)$ which are ordered by inclusion (see Chapter 1). The resuls of Table 1 have been developed with the help of a computer as follows:

The numbers of directed games (n = 1,...,8) are obtained using the corresponding generating algorithm of Chapter 2. The number of edges in the corresponding lattice are the numbers of occuring shift minimal coalitions, since two directed games are joined by an edge, iff the larger one arises from the smaller one by dropping one shift minimal coalition in the corresponding filter. Analogously, the numbers of directed n–person zero–sum games are computed using the second algorithm of Chapter 2 for n = 1,...,9. Testing these games on representability (see e.g. Algorithm II of Chapter 3) yields the numbers of directed n–person weighted majority zero–sum games (see the sixth row). The numbers of directed n–person weighted majority games (see the fourth row) are obtained by considering the types of the corresponding zero–sum extensions due to Corollary 1.11. In order to illustrate the extraordinary growth of the numbers of the games of the just mentioned classes we additionally show the numbers of homogeneous games, wich are easily computed using the recursive formulae of Sudhölter (1989).

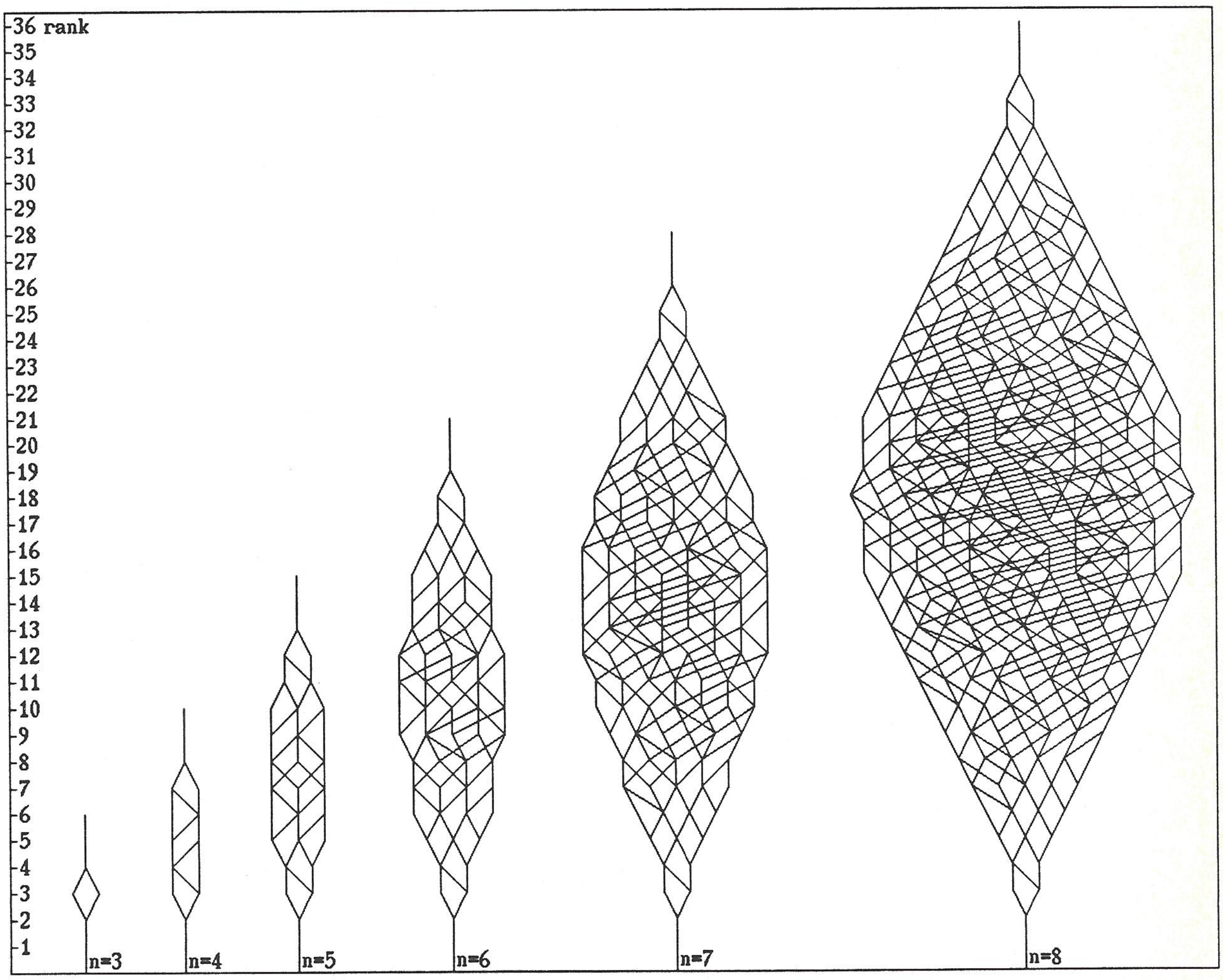

Figure 4

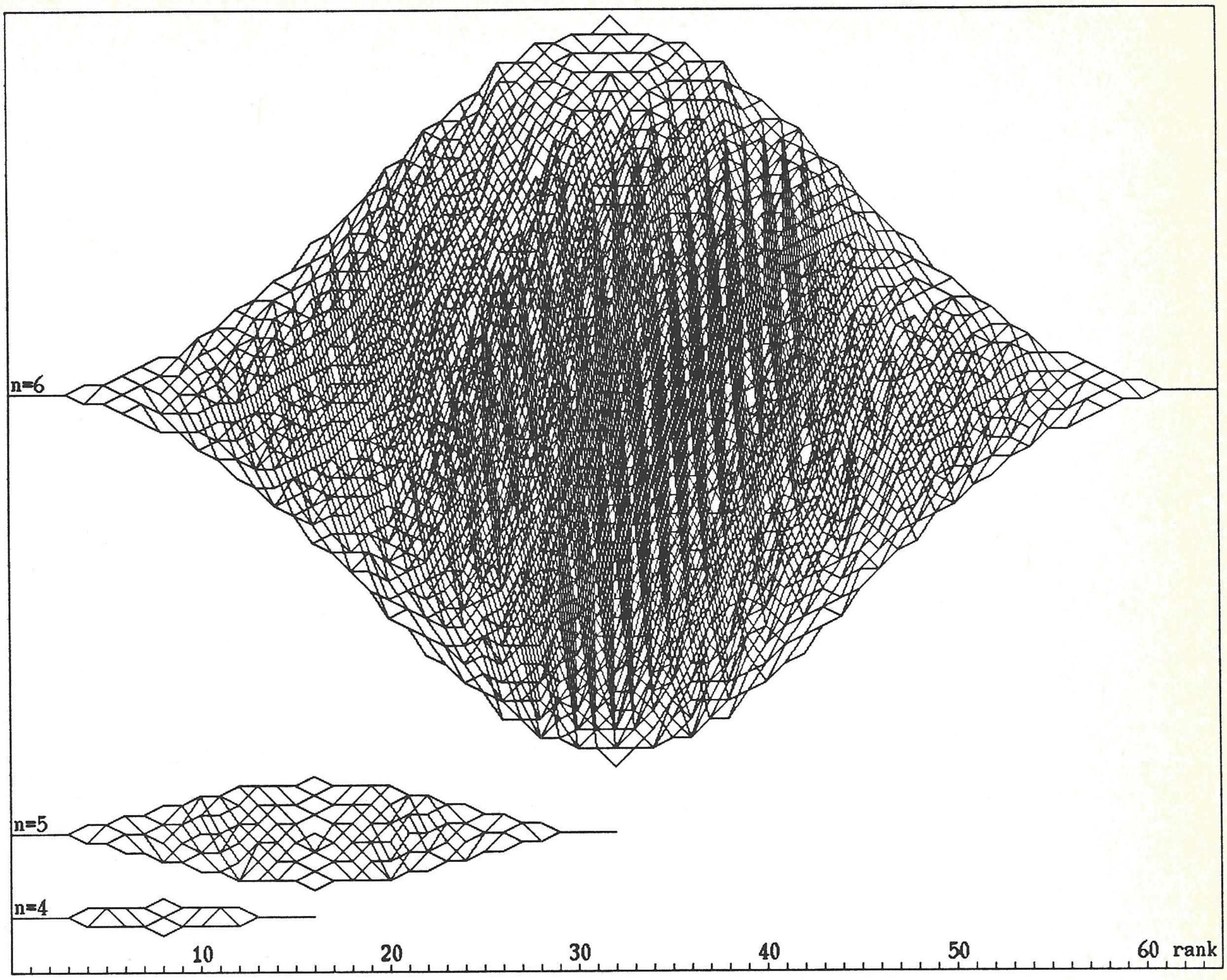

Figure 5

n	1	2	3	4	5	6	7	8	9
number of directed games	3	5	10	27	119	1173	44315	16175190	?
number of edges in directed lattice	2	4	10	36	224	3264	190162	110433364	?
number of weighted majority games	3	5	10	27	119	1113	29375	2730166	?
number of directed zero-sum games	1	1	2	3	7	21	135	2470	319124
number of games in Z_n^r	1	1	2	3	7	21	135	2470	175428
number of homogeneous games	1	3	8	23	76	293	1307	6642	37882

Table 1

Fig. 6 illustrates the lattice of directed 7–person games in the same way as Fig. 5. Both axis, the vertical and horizontal, have been deminished proportionally in order to get a one–page picture.

Fig. 7 sketches the Hasse diagram of the directed 6–person zero–sum games. The tree, consisting of all vertices, i.e. the corresponding shift minimal matrices, and the "straight line" edges, is generated by the original algorithm presented after Proposition 2.7. The additional edges result from the corresponding modified algorithm.

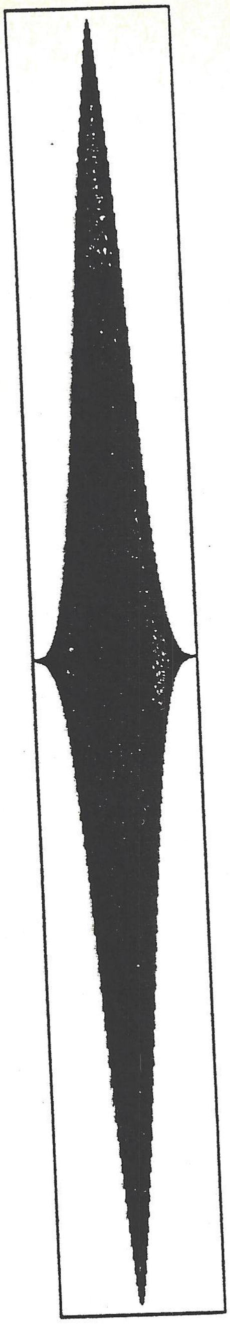

Figure 6

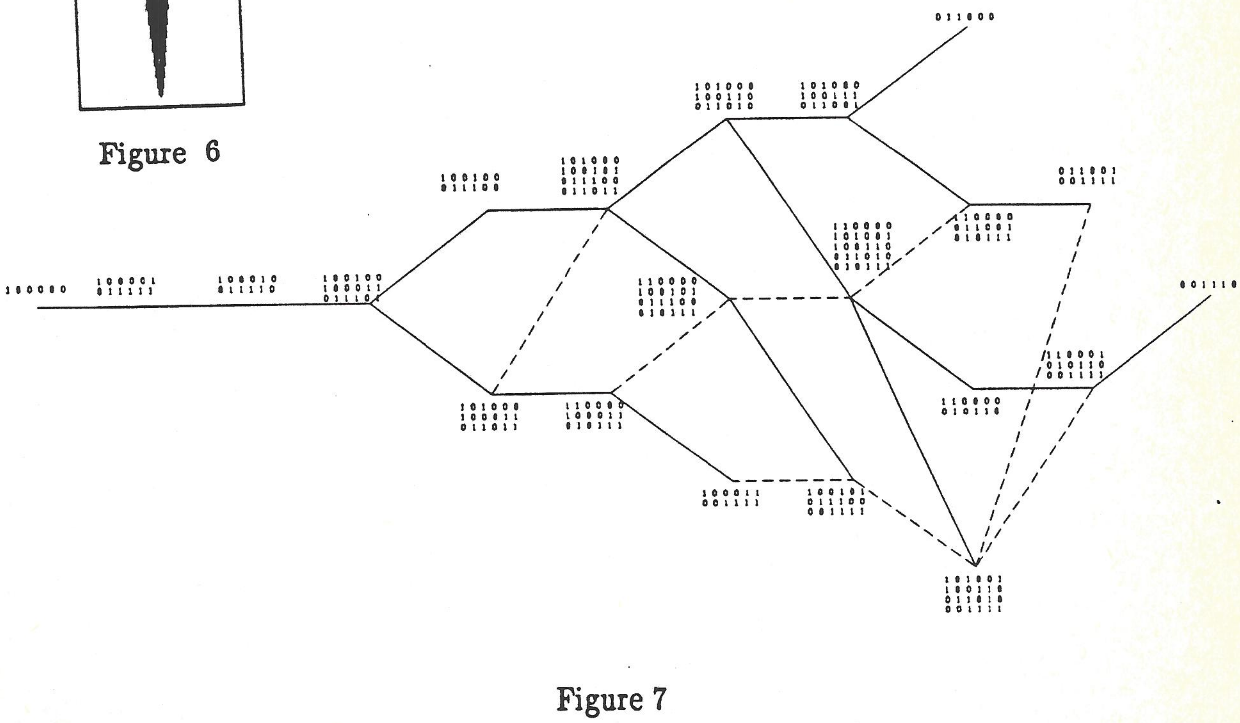

Figure 7

Two examples for the working method of Algorithm II are presented as follows:

Let v_0 be the directed 9–person zero–sum game characerized by

$$I^s(v_0)=\begin{pmatrix}1&0&0&0&1&0&0&1&1\\0&1&1&1&1&0&0&0&0\\0&0&0&1&1&1&1&1&1\end{pmatrix}$$

T1 is the initial tableau of Algorithm II and T2, T11 and T12 is the 2nd, 11th and 12th tableau respectively. Since T12 contains no negative numbers in its last row, the Simplex Method stops here.

0	1	2	3	4	5	6	7	8	9	0
10	0	1	1	1	0	1	1	0	0	1
11	1	0	0	0	0	1	1	1	1	1
12	1	1	1	0	0	0	0	0	0	1
13	¯1	1	0	0	0	0	0	0	0	0
14	0	¯1	1	0	0	0	0	0	0	0
15	0	0	¯1	1	0	0	0	0	0	0
16	0	0	0	¯1	1	0	0	0	0	0
17	0	0	0	0	¯1	1	0	0	0	0
18	0	0	0	0	0	¯1	1	0	0	0
19	0	0	0	0	0	0	¯1	1	0	0
20	0	0	0	0	0	0	0	¯1	1	0
0	¯1	¯1	¯1	¯1	¯1	¯1	¯1	¯1	¯1	0

T1

0	11	2	3	4	5	6	7	8	9	0
10	0	1	1	1	0	1	1	0	0	1
1	1	0	0	0	0	1	1	1	1	1
12	¯1	1	1	0	0	¯1	¯1	¯1	¯1	0
13	1	1	0	0	0	1	1	1	1	1
14	0	¯1	1	0	0	0	0	0	0	0
15	0	0	¯1	1	0	0	0	0	0	0
16	0	0	0	¯1	1	0	0	0	0	0
17	0	0	0	0	¯1	1	0	0	0	0
18	0	0	0	0	0	¯1	1	0	0	0
19	0	0	0	0	0	0	¯1	1	0	0
20	0	0	0	0	0	0	0	¯1	1	0
0	1	¯1	¯1	¯1	¯1	0	0	0	0	1

T2

0	11	12	17	15	16	14	10	19	20	0
7	0.308	¯0.308	¯0.385	¯0.462	¯0.385	¯0.231	0.0769	¯0.615	¯0.308	0.0769
1	0.154	0.846	0.308	0.769	0.308	0.385	¯0.462	¯0.308	¯0.154	0.538
3	¯0.0769	0.0769	¯0.154	¯0.385	¯0.154	0.308	0.231	0.154	0.0769	0.231
13	0.231	0.769	0.462	1.15	0.462	1.08	¯0.692	¯0.462	¯0.231	0.308
6	¯0.0769	0.0769	0.846	0.615	0.846	0.308	0.231	0.154	0.0769	0.231
4	¯0.0769	0.0769	¯0.154	0.615	¯0.154	0.308	0.231	0.154	0.0769	0.231
5	¯0.0769	0.0769	¯0.154	0.615	0.846	0.308	0.231	0.154	0.0769	0.231
2	¯0.0769	0.0769	¯0.154	¯0.385	¯0.154	¯0.692	0.231	0.154	0.0769	0.231
18	¯0.385	0.385	1.23	1.08	1.23	0.538	0.154	0.769	0.385	0.154
8	0.308	¯0.308	¯0.385	¯0.462	¯0.385	¯0.231	0.0769	0.385	¯0.308	0.0769
9	0.308	¯0.308	¯0.385	¯0.462	¯0.385	¯0.231	0.0769	0.385	0.692	0.0769
0	0.692	0.308	¯0.615	0.462	0.385	0.231	0.923	0.615	0.308	1.92

T11

0	11	12	18	15	16	14	10	19	20	0
7	0.187	¯0.187	0.313	¯0.125	0	¯0.0625	0.125	¯0.375	¯0.187	0.125
1	0.25	0.75	¯0.25	0.5	0	0.25	¯0.5	¯0.5	¯0.25	0.5
3	¯0.125	0.125	0.125	¯0.25	0	0.375	0.25	0.25	0.125	0.25
13	0.375	0.625	¯0.375	0.75	0	0.875	¯0.75	¯0.75	¯0.375	0.25
6	0.188	¯0.188	¯0.688	¯0.125	0	¯0.0625	0.125	¯0.375	¯0.188	0.125
4	¯0.125	0.125	0.125	0.75	0	0.375	0.25	0.25	0.125	0.25
5	¯0.125	0.125	0.125	0.75	1	0.375	0.25	0.25	0.125	0.25
2	¯0.125	0.125	0.125	¯0.25	0	¯0.625	0.25	0.25	0.125	0.25
17	¯0.312	0.312	0.813	0.875	1	0.438	0.125	0.625	0.313	0.125
8	0.188	¯0.188	0.312	¯0.125	0	¯0.0625	0.125	0.625	¯0.187	0.125
9	0.187	¯0.187	0.313	¯0.125	0	¯0.0625	0.125	0.625	0.813	0.125
0	0.5	0.5	0.5	1	1	0.5	1	1	0.5	2

T12

The entry in the last line and column does not exceed 2, thus this game is no weighted majority game.

The second example concerns part (2) of the Concluding Remarks of Chapter 3: v_1 is the game already defined. Then $|W^s_{v_1}| = 16$. Applying Algorithm II it is seen that the 10th and 11th tableau look as follows:

0	19	25	23	20	22	14	21	12	10	0
9	0.0741	0.407	¯0.444	0.148	¯0.37	¯0.148	0.185	¯0.259	0.444	0.037
11	¯0.926	¯0.593	0.556	0.148	0.63	¯0.148	0.185	¯0.259	¯0.556	0.037
8	0.148	¯0.185	0.111	0.296	0.259	¯0.296	¯0.63	0.481	¯0.111	0.0741
13	0.0741	¯0.593	0.556	0.148	¯0.37	¯0.148	0.185	¯0.259	¯0.556	0.037
6	0.296	¯0.37	0.222	0.593	¯0.481	0.407	¯0.259	¯0.037	¯0.222	0.148
15	0.0741	¯0.593	¯0.444	0.148	0.63	¯0.148	0.185	¯0.259	¯0.556	0.037
16	0.0741	¯0.593	¯0.444	1.15	0.63	¯0.148	¯0.815	¯0.259	¯0.556	0.037
17	0.0741	¯0.593	¯0.444	1.15	¯0.37	¯0.148	0.185	¯0.259	¯0.556	0.037
18	1.07	¯0.593	¯0.444	1.15	¯0.37	¯0.148	¯0.815	¯0.259	¯0.556	0.037
1	¯0.037	0.296	0.222	¯0.0741	0.185	0.0741	0.407	¯0.37	¯0.222	0.481
4	¯0.481	¯0.148	¯0.111	0.037	0.407	¯0.037	0.296	0.185	0.111	0.259
7	0.222	0.222	¯0.333	¯0.556	¯0.111	¯0.444	0.556	0.222	0.333	0.111
5	0.37	0.037	¯0.222	¯0.259	0.148	0.259	¯0.0741	¯0.296	0.222	0.185
3	¯0.407	0.259	0.444	¯0.815	0.037	¯0.185	0.481	¯0.0741	0.556	0.296
24	0.0741	¯0.593	¯0.444	1.15	¯0.37	¯0.148	¯0.815	0.741	¯0.556	0.037
2	¯0.185	0.481	0.111	¯0.37	¯0.0741	0.37	0.037	0.148	¯0.111	0.407
26	0.148	¯0.185	0.111	0.296	0.259	¯0.296	0.37	¯0.519	¯0.111	0.0741
27	0.222	0.222	¯0.333	0.444	¯0.111	0.556	¯0.444	0.222	¯0.667	0.111
28	0.0741	0.407	0.556	¯0.852	¯0.37	¯0.148	0.185	¯0.259	0.444	0.037
29	¯0.852	¯0.185	0.111	0.296	0.259	¯0.296	0.37	0.481	¯0.111	0.0741
30	0.0741	0.407	¯0.444	¯0.852	0.63	¯0.148	0.185	¯0.259	0.444	0.037
31	0.0741	¯0.593	0.556	1.15	¯0.37	0.852	¯0.815	¯0.259	¯0.556	0.037
32	0.0741	0.407	¯0.444	¯0.852	¯0.37	¯0.148	1.19	¯0.259	0.444	0.037
33	0.0741	¯0.593	0.556	0.148	0.63	¯0.148	¯0.815	0.741	¯0.556	0.037
0	0	1	0	¯1	0	0	1	0	1	2

TT10

0	19	25	23	16	22	14	21	12	10	0
9	0.0645	0.484	¯0.387	¯0.129	¯0.452	¯0.129	0.29	¯0.226	0.516	0.0323
11	¯0.935	¯0.516	0.613	¯0.129	0.548	¯0.129	0.29	¯0.226	¯0.484	0.0323
8	0.129	¯0.0323	0.226	¯0.258	0.0968	¯0.258	¯0.419	0.548	0.0323	0.0645
13	0.0645	¯0.516	0.613	¯0.129	¯0.452	¯0.129	0.29	¯0.226	¯0.484	0.0323
6	0.258	¯0.0645	0.452	¯0.516	¯0.806	0.484	0.161	0.0968	0.0645	0.129
15	0.0645	¯0.516	¯0.387	¯0.129	0.548	¯0.129	0.29	¯0.226	¯0.484	0.0323
20	0.0645	¯0.516	¯0.387	0.871	0.548	¯0.129	¯0.71	¯0.226	¯0.484	0.0323
17	0	0	0	¯1	¯1	0	1	0	0	0
18	1	0	0	¯1	¯1	0	0	0	0	0
1	¯0.0323	0.258	0.194	0.0645	0.226	0.0645	0.355	¯0.387	¯0.258	0.484
4	¯0.484	¯0.129	¯0.0968	¯0.0323	0.387	¯0.0323	0.323	0.194	0.129	0.258
7	0.258	¯0.0645	¯0.548	0.484	0.194	¯0.516	0.161	0.0968	0.0645	0.129
5	0.387	¯0.0968	¯0.323	0.226	0.29	0.226	¯0.258	¯0.355	0.0968	0.194
3	¯0.355	¯0.161	0.129	0.71	0.484	¯0.29	¯0.0968	¯0.258	0.161	0.323
24	0	0	0	¯1	¯1	0	0	1	0	0
2	¯0.161	0.29	¯0.0323	0.323	0.129	0.323	¯0.226	0.0645	¯0.29	0.419
26	0.129	¯0.0323	0.226	¯0.258	0.0968	¯0.258	0.581	¯0.452	0.0323	0.0645
27	0.194	0.452	¯0.161	¯0.387	¯0.355	0.613	¯0.129	0.323	¯0.452	0.0968
28	0.129	¯0.0323	0.226	0.742	0.0968	¯0.258	¯0.419	¯0.452	0.0323	0.0645
29	¯0.871	¯0.0323	0.226	¯0.258	0.0968	¯0.258	0.581	0.548	0.0323	0.0645
30	0.129	¯0.0323	¯0.774	0.742	1.1	¯0.258	¯0.419	¯0.452	0.0323	0.0645
31	0	0	1	¯1	¯1	1	0	0	0	0
32	0.129	¯0.0323	¯0.774	0.742	0.0968	¯0.258	0.581	¯0.452	0.0323	0.0645
33	0.0645	¯0.516	0.613	¯0.129	0.548	¯0.129	¯0.71	0.774	¯0.484	0.0323
0	0.0645	0.484	¯0.387	0.871	0.548	¯0.129	0.29	¯0.226	0.516	2.03

TT11

The original Algorithm II stops at this stage, because the weighted majority property is shown in view of the fact that the last element in the last row exceeds 2. Algorithm II a computes four further tableaux, the last one is shown here:

0	19	25	14	16	33	31	21	24	10	0
9	0.1	0.2	¯0.15	¯0.05	0.55	0.05	¯0.1	¯0.2	0.25	0.05
11	¯0.967	¯0.267	¯0.383	¯0.35	¯0.483	¯0.317	0.633	0.6	¯0.25	0.0167
8	0.1	0.2	¯0.15	¯0.05	¯0.45	0.05	¯0.1	¯0.2	0.25	0.05
13	0.0667	¯0.533	¯0.767	0.3	0.0333	¯0.633	0.267	0.2	¯0.5	0.0333
6	0.267	¯0.133	¯0.0667	0.2	0.133	¯0.533	0.0667	¯0.2	0	0.133
15	0.0667	¯0.533	0.233	¯0.7	0.0333	0.367	0.267	0.2	¯0.5	0.0333
20	0.0667	¯0.533	0.233	0.3	0.0333	0.367	¯0.733	0.2	¯0.5	0.0333
17	0.0333	¯0.267	¯0.383	¯0.35	0.517	¯0.317	0.633	¯0.4	¯0.25	0.0167
18	1.03	¯0.267	¯0.383	¯0.35	0.517	¯0.317	¯0.367	¯0.4	¯0.25	0.0167
1	¯0.0333	0.267	¯0.117	¯0.15	¯0.0167	¯0.183	0.367	0.4	¯0.25	0.483
4	¯0.5	0	0.25	¯0.25	¯0.25	0.25	0.5	0	0.25	0.25
7	0.267	¯0.133	¯0.0667	0.2	0.133	0.467	0.0667	¯0.2	0	0.133
5	0.4	¯0.2	0.4	¯0.2	0.2	0.2	¯0.4	0.2	0	0.2
3	¯0.367	¯0.0667	¯0.283	0.35	¯0.183	¯0.0167	0.0333	0.4	0.25	0.317
12	0.0333	¯0.267	¯0.383	¯0.35	0.517	¯0.317	¯0.367	0.6	¯0.25	0.0167
2	¯0.167	0.333	0.417	0.25	¯0.0833	0.0833	¯0.167	0	¯0.25	0.417
26	0.133	¯0.0667	¯0.533	¯0.4	0.0667	¯0.267	0.533	0.4	0	0.0667
27	0.2	0.4	0.7	¯0.1	0.1	0.1	¯0.2	¯0.4	¯0.5	0.1
28	0.133	¯0.0667	¯0.533	0.6	0.0667	¯0.267	¯0.467	0.4	0	0.0667
29	¯0.9	0.2	¯0.15	¯0.05	¯0.45	0.05	0.9	¯0.2	0.25	0.05
30	0.133	¯0.0667	0.467	¯0.4	0.0667	0.733	¯0.467	0.4	0	0.0667
23	0.0333	¯0.267	0.617	¯0.35	0.517	0.683	¯0.367	¯0.4	¯0.25	0.0167
32	0.167	¯0.333	0.0833	0.25	0.583	0.417	0.167	0	¯0.25	0.0833
22	0.0333	¯0.267	¯0.383	0.65	0.517	¯0.317	¯0.367	¯0.4	¯0.25	0.0167
0	0.0667	0.467	0.233	0.3	0.0333	0.367	0.267	0.2	0.5	2.03

TT15

The Algorithm I additionally yields 13 more 9–person cases in which $\frac{m}{2-p}$ is no minimal representation (for this notation we refer to Algorithm 1, (Third step)). These 14 games are summerized in Table 2.

$\frac{1}{2-p}$; $\frac{m}{2-p}$										a minimal representation									
30 ;	17	9	8	6.5	6.5	5	3	2	2	30 ;	17	9	8	7	6	5	3	2	2
25 ;	13	7	6	6	4	4	4	2.5	2.5	25 ;	13	7	6	6	4	4	4	3	2
27 ;	14	9	6.5	6.5	5	5	3	2	2	27 ;	14	9	7	6	5	5	3	2	2
33 ;	17	12	8	8	6.5	6.5	3	2	2	33 ;	17	12	8	8	7	6	3	2	2
28 ;	13	9	7	7	6	4	4	2.5	2.5	28 ;	13	9	7	7	6	4	4	3	2
24 ;	11	9	6	6	4	4	4	1.5	1.5	24 ;	11	9	6	6	4	4	4	2	1
28 ;	13	11	8	6	6	4	4	1.5	1.5	28 ;	13	11	8	6	6	4	4	2	1
28 ;	13	11	7	7	5	5	4	1.5	1.5	28 ;	13	11	7	7	5	5	4	2	1
32 ;	15	13	9	7	7	5	4	1.5	1.5	32 ;	15	13	9	7	7	5	4	2	1
31 ;	14.5	12.5	9.5	7.5	6	4	4	1.5	1.5	32 ;	15	13	10	8	6	4	4	2	1
35 ;	16.5	14.5	10.5	8.5	7	5	4	1.5	1.5	36 ;	17	15	11	9	7	5	4	2	1
34 ;	16	14	11	9	6	4	4	1.5	1.5	34 ;	16	14	11	9	6	4	4	2	1
38 ;	18	16	12	10	7	5	4	1.5	1.5	38 ;	18	16	12	10	7	5	4	2	1
33 ;	13	11	10	8	6	6	4.5	4.5	2	33 ;	13	11	10	8	6	6	5	4	2

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