Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 138

Homogeneous games: Recursive structure and computation

Joachim Rosenmüller

October 1984

H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

Abstract

The structure of a homogeneous weighted majority game (in general not constant-sum) is analyzed via the concepts of characters of types and of satellite games being played by smaller players in order to replace larger ones. Two proofs for the existence of the minimal representation (see OSTMANN [5]) are given. An algorithm to construct the satellite games, the characters and the minimal representation directly from any homogeneous representation is described.

1. Introduction, Notations, and the Basic Lemma

Let $\Omega = \{1,\ldots,n\}$ denote the "set of players". A pair of vectors

$$(g\,;k) = (g_0,\ldots,g_r\,;\,k_0,\ldots,k_r) \in \mathbb{N}_0^{2(r+1)}$$

induces an additive set function M (a measure) on the subsets of Ω (the "coalitions") in a natural way provided

$$\sum_{i=0}^{r} k_i = n\ .$$

Indeed, put $K_\rho := \{\omega \in \Omega \mid k_{\rho-1} < \omega \leq k_\rho\}$ $(k_{-1} := -1)$ such that

$$\Omega = K_0 + K_1 + \ldots + K_r$$

is a decomposition of Ω ("+" is used instead of "$\cup$" iff the union is disjoint) and define, for $S \subseteq \Omega$,

$$\text{(1)} \qquad M(S) = \sum_{i=0}^{r} |\, S \cap K_i \,|\, g_i \quad .$$

Thus $M : \mathfrak{P}(\Omega) \to \mathbb{N}_0$ is a mapping defined on the power set of Ω ("the coalitions"). Clearly, any $\mathbb{N}_0$-valued measure M may be represented by a suitable $(g\,;\,k) \in \mathbb{N}_0^{2\,(r+1)}$, possibly after reordering Ω.

A <u>(simple)</u> <u>game</u> is a mapping

$$v : \mathfrak{P}(\Omega) \to \{0,1\}$$

such that $v(\emptyset) = 0$ (and, in general, $v(\Omega) = 1$) The term game throughout this paper refers to simple games. A game is a <u>weighted majority</u> if there exists a measure M and a number $\lambda \in \mathbb{N}$ such that for $S \subseteq \Omega$

$$\text{(2)} \qquad v(S) = \begin{cases} 1 & M(S) \geq \lambda \\ 0 & M(S) < \lambda \end{cases}$$

In this case, (M,λ) is called a <u>representation</u> of v and the relation (2) is indicated by writing $v = v_\lambda^M$. If $\omega \in K_i$, then g_i is the <u>weight</u> of ω.

Of course, a game may have various representations. Let us discuss the symmetry-properties of a game and its representation: if $\omega, \eta \in \Omega$ are players with equal weight, then (M,λ) (and v) is not affected by exchanging ω and η.

In particular, exchanging players inside the same K_i does not affect v; we shall call the elements of K_i <u>fellows</u> (w.r.t. (M,λ)) and i a <u>fellowship</u>.

For any game v, the symmetry group

$$\Pi^v = \{\pi : \Omega \to \Omega \mid \pi \text{ a permutation which does not affect } v\}$$

describes the symmetry properties of v. Π^v decomposes Ω into transivity domains, the "<u>types</u>".

If v is a weighted majority then ω and η belong to the same type if and only if exchanging ω and η leaves v unaffected.

It is also easy to verify (see OSTMANN [5]) that the <u>representations induce an ordering of the types</u>, i.e., if ω and η belong to different types and, for some representation (M,λ) of v, we have $M(\omega) < M(\eta)$, then, for any other representation (M', λ'), it follows that $M'(\omega) < M'(\eta)$.

It is, therefore, no loss of generality to assume that players are ordered in advance and that this ordering is provided by any representation to start out with.

We shall assume that "smaller players are recognized by smaller numbers" and that players with weights zero (if any) are first in our ordering. Thus we restrict the term "representation" as follows.

Let $\mathfrak{M}^r$ denote the set of all vectors

$$(g,k) = (g_0, g_1, \ldots, g_r \,;\, k_0, k_1, \ldots, k_r) \in \mathbb{N}_0^{2(r+1)}$$

such that

(3) $$0 = g_0 \,;\, 1 \leq g_1 \leq \ldots \leq g_r$$

(4) $$0 \leq k_0 \,;\, 1 \leq k_1, \ldots, k_r \,;\, \sum_{i=0}^{r} k_i = n$$

is satisfied. Let $\mathfrak{M} = \bigcup_{r=1}^{\infty} \mathfrak{M}^r$. Any $M = (g,k) \in \mathfrak{M}^r$ is always interpreted as an measure on Ω. Therefore, given $M \in \mathfrak{M}^r$, the projections

(5) $$M_{i_0-1} = (g_0, \ldots, g_{i_0-1} \,;\, k_0, \ldots, k_{i_0-1}) \in {}^{i_0-1}$$

(6) $$M_{i_0}^{c} = (g_0, \ldots, g_{i_0-1}, g_{i_0} \,;\, k_0, \ldots, k_{i_0-1}, k_{i_0}-c) \in {}^{i_0}$$

(for $1 \leq i_0 \leq r$ and $1 \leq c \leq k_{i_0}$) may be interpreted as restrictions of M, that is, measures which are regarded to live on an appropriate subset of Ω. E.g., in case of (5), this subset is of the form

$$K_0 + \ldots + K_{i_0-1} + D$$

where $D \subseteq K_{i_0}$, $|D| = k_{i_0}-c$.

As a notational convention, the total mass of M is always denoted by m (indices are carried through appropriately) i.e.

$$m = \sum_{i=1}^{r} k_i \, g_i = M(\Omega)$$

(7) $$m_{i_0}^{c} = \sum_{i=1}^{i_0-1} k_i \, g_i + (k_{i_0} - c) g_{i_0} = M_{i_0}^{c}(\Omega) = M_{i_0}^{c}(C)$$

$$m_{i_0-1} = \sum_{i=1}^{i_0-1} k_i \, g_i = M_{i_0-1}(\Omega)$$

etc.

Next, it will be necessary to compare measures (vectors (g,k)) of different length; the prerequisits for this procedure are provided by

Definition 1.1. 1. $k' \in \mathbb{N}_0^{r'+1}$ extends $k \in \mathbb{N}_0^{r+1}$ if $r' \geq r$ and there is $l = (l_0,\ldots,l_r) \in \mathbb{N}_0^{r+1}$ satisfying $-1 =: l_{-1} < l_0 < l_1 < \ldots < l_r$ and

(8) $$\sum_{l_{\rho-1} < i \leq l} k_i' = k$$

2. Let $(g,k) \in \mathfrak{M}^r$ and $(g',k') \in \mathfrak{M}^{r'}$. We shall say that (g',k') extends (g,k) if k' ext k and

(9) $$g_i' = g_\rho \quad (l_{\rho-1} + 1 \leq i \leq l_\rho) ,$$

where l is specified by 1. If $(g,k) \in {}^r$ and, for some $k' \in \mathbb{N}_0^{r'+1}$, k' ext k, then there is a unique g' such that (g',k') ext (g,k); let us write

$$g_\rho' = \text{ext}\big|_{k'} \; g$$

(extension of g w.r.t. k').

3. A half ordering $\precsim$ on $\mathfrak{M}$ is defined by "$(g,k) \precsim (\tilde{g},\tilde{k})$ if and only if $\tilde{k}$ ext k and $\text{ext}_{\tilde{k}}\, g \leq \tilde{g}$".

Remark 1.2.

1. For any $(g',k') \in \mathfrak{M}$ there is a unique minimal (w.r.t. $\precsim$) element $(g,k) \in \mathfrak{M}$ such that (g',k') ext (g,k) ("grouping fellowships of equal weight together"); g satisfies

(10) $$0 = g_0 < g_1 < \ldots < g_r \ .$$

We call (g,k) the reduction of (g',k') and any (g,k) satisfying (10) is said to be reduced.

2. Clearly, whenever k' ext k , then

$$(g,k) \quad (\text{ext}|_{k'}\ g,\ k').$$

3. If $\mathfrak{h} \subseteq \mathfrak{M} \times \mathbb{N}$ is a family of representations of a game v, then the term minimal refers to an element $(\tilde{M},\tilde{\lambda}) \in \mathfrak{h}$ such that for $(M,\lambda) \in \mathfrak{h}$, we have $\tilde{M} \precsim M$ and $\tilde{\lambda} \leq \lambda$.

If $M \in \mathfrak{M}^r$, then a vector $s \in \mathbb{N}_0^{r+1}$ is a profile feasible for M if $s \leq k$.

Profiles correspond to coalitions $S \subseteq \Omega$ such that $s_i = | S \cap K_i |$; we have

(11) $$M(S) = \sum_{i=0}^{r} | S \cap K_i |\, g_i = \sum_{i=0}^{r} s_i\, g_i =: M(s),$$

and thus we shall frequently regard M as a (additive) function on profiles.

Similarly, $s' \in \mathbb{N}_0^{i_0+1}$ is a profile feasible for $M^c_{i_0}$ if

$$s' \leq (k_0, \ldots, k_{i_0-1}, k_{i_0} - c) \ ,$$

thus, s' corresponds to a coalition in $\Omega_0 = K_0 + \ldots + K_{i_0-1} + D$.

However, as s' may also describe a coalition in Ω we may as well regard

$$s'' = (s'_0, \ldots, s'_{i_0}, 0, \ldots, 0) \in \mathbb{N}_0^{r+1}$$

as to be feasible for $M^c_{i_0}$. Frequently it is not necessary to distinguish between s' and s'' and we will use both notations freely (and switch between them) in particular as we decompose Ω in order to construct subgames.

The "largest" profile feasible for M is

$$C(M) := (k_0, \ldots, k_r)$$

which is (somewhat sloppily) called the "carrier" of M.

Note: if (g',k') ext (g,k), then a profile s' w.r.t. (g',k') corresponds naturally to a profile s (w.r.t. (g,k)) $(s_\rho = \sum_{l_{\rho-1} < i \leq l_\rho} s'_i)$ and we have

$$M'(s') = M(s) .$$

If $(g,k) \lesssim (g',k')$ then $s' \to s$ is also well defined and we have

$$M'(s') \geq M(s)$$

in this case.

The term <u>homogeneous</u> for a (simple) game has been introduced by VON NEUMANN and MORGENSTERN ([4]).

Let $(M,\lambda) \in \mathfrak{M} \times \mathbb{N}$. M is said to be <u>homogeneous</u> w.r.t. λ if

1. $M(\Omega) \geq \lambda$
2. For $S \subseteq \Omega$, $M(S) > \lambda$ there is $T \subseteq S$ such that $M(T) = \lambda$.

We write <u>M hom λ</u> as an abbreviation; also, <u>M hom$_0$ λ</u> means that either M hom λ or $M(\Omega) < \lambda$.

A game v is homogeneous if there exists a homogeneous representation, i.e., if there is $(M,\lambda) \in \mathfrak{M} \times \mathbb{N}$ s.t.

$$v = v_\lambda^M \text{ , } M \text{ hom } \lambda .$$

Essentially, a game is described by its minimal winning coalitions (the min-win coalitions) and in a homogeneous game the min-win coalitions have exactly weight λ.

Homogeneous games are of special interest, because they allow for "nice" solution concepts. (see [6])

It is the aim of this paper to exactly describe the structure of all homogeneous representations of such a game. The construction of homogeneous games with arbitrarily prescribed weights is indicated in [9]. From this paper we take the following Lemma (Theorem 1.4., [9]), we shall refer to it as to the

BASIC LEMMA Let $M = (g,k)$ be reduced, $\lambda \in \mathbb{N}$, and assume

$$\lambda \leq m = M(\Omega) = \sum_{i=1}^{r} k_i \, g_i \, .$$

Then M hom λ if and only if there is $i_0 \in \{1,\ldots,r\}$ and $c \in \mathbb{N}$, $1 \leq c \leq k_{i_0}$, such that

(12) $$\lambda = c g_{i_0} + \sum_{i=i_0+1}^{r} k_i \, g_i$$

(13) $$M_{i_0}^c \text{ hom}_0 \, g_j \; (i_0 + 1 \leq j \leq r)$$

(14) $$M_{i_0-1} \text{ hom}_0 \, g_{i_0}$$

Intuitively, the Basic Lemma states that, given a homogeneous representation of a game, the measure of the largest min-win coalition (when collecting players according to rank) must exactly hit the majority level. Moreover, the remaining players - collecting their weights according to the measure $M^c_{i_0}$ - are going to play a series of homogeneous "satellite games" in order to replace the "large players" ($j \geq i_0 + 1$) and the medium players (i_0) - or rather, the members of the large and medium fellowships. (cf.Fig. 1)

The fact, that [9] deals only with reduced representations may be neglected. This is verified at once by reducing and extending representations at will.

The term "largest coalition", suggested by the Basic Lemma leads to an ordering of profiles. As the largest fellows have weight g_r, it is reasonable to introduce the lexicographic order backwards on vectors (profiles) $s \in \mathbb{N}^r$; thus the last coordinate s_r is the first to be considered for the lexicographic ordering.

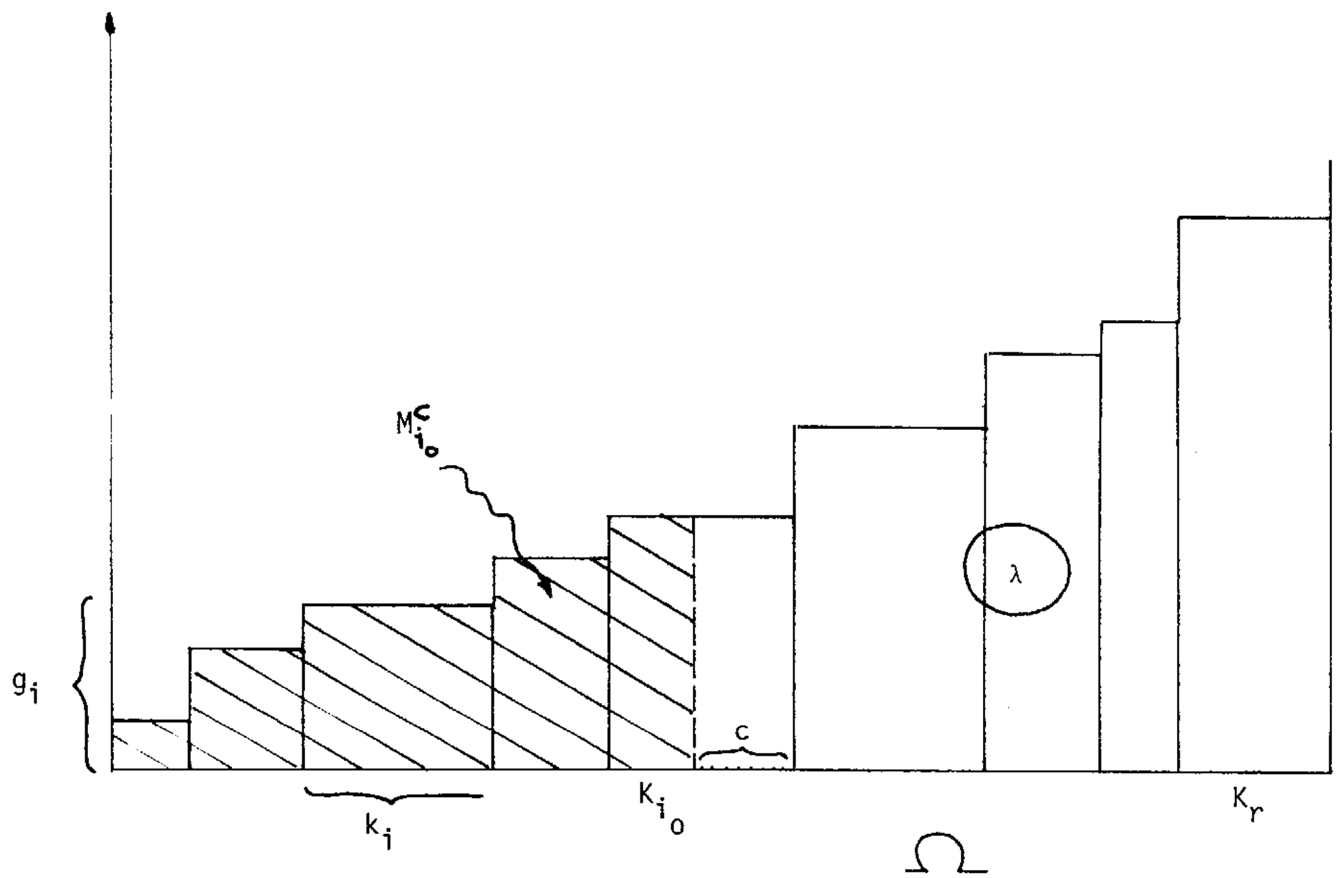

Fig. 1 : "The Basic Lemma"

According to (15) (16) (17), the profile of the "lexicographically largest coalition" (the lex-max coalition) is uniquely determined by M and λ and given by

$$s^{\lambda} = s^{\lambda}_{M} = (0,\ldots,0,\ c,\ k_{i_0+1},\ldots,k_r)$$

The Basic Lemma permits to define (recursively) a test for homogeneity of a pair (M,). In addition, it provides a method of computing "all homogeneous games" via the "matrix of homogeneity" - the details may be found in [9] .

OSTMANN [5] proves the following

Theorem ("The smallest committee") For any homogeneous game there is a unique representation $(M,\lambda) \in \mathfrak{M} \times \mathbb{N}$ such that $M(\Omega)$ is minimal.

The present paper offers the following results.

The fellowships of a homogeneous game may be classified according to their character, we know three characters called "sum", "step", and "dummy". These characters are defined inductively via the satellite games as suggested by the Basic Lemma. Whenever a fellowship is a sum, it induces in turn its satellite game to be played by smaller players in order to substitute the sum. Thus, the first aim of this paper is to clear up the relationship between the BASIC LEMMA and the theory of the minimal representation presented in [9] (where the characters apply to players).

The second aim is to provide a completely independent proof for the existence of the minimal representation which essentially runs by induction via the satellite games.

The third aim is to supply an algorithm which, given any homogeneous representation, computes the minimal one. This is done in a direct way and without referring to the game v (i.e. to the incidence matrix of the minimal winning coalitions).

2. Games with few non-dummies

In view of the Basic Lemma, the measures M_{i_0-1} and $M^c_{i_0}$ play a decisive rôle concerning the relations between smaller fellowships ($i < i_0$), the medium one ($i = i_0$) and larger fellowships ($i > i_0$).

In Section 3, dummies, sums, and steps will be defined inductively as to be one of three possible characters of any fellowship. The present section provides the induction beginning by dealing with some degenerate cases (small fellowships are dummies and large ones constitute one type).

In this context let us write $M = 0$ for a measure which is understood to live on K_0 (thus corresponding to $(0,g_0)$), even if $K_0 = \emptyset$ ($k_0 = 0$), in which case $M = 0$ may be regarded as the trivial measure on the empty set.

Remark 2.1. Let $M \in \mathfrak{M}^1$ and $\lambda \in \mathbb{N}$ be such that M hom λ and consider

$$v = v^M_\lambda .$$

Then there is $c \in \mathbb{N}$, $1 \leq c \leq k_1$ such that $\lambda = cg_1$. Thus, a typical game of this class (r=1) is specified by

(1) $(g;k;\lambda) = (0,g_1;\ k_0,k_1;\ cg_1) \quad (k_0+k_1 = n,\ g_1 \geq 1,\ 1 \leq c \leq k_1)$

Let us define the characters of the two fellowships involved as fellows.

(2) Fellowship $i = 0$ is a dummy.

(3) Fellowship $i = 1$ is a step.
Its satellite measure is $M^1_\lambda = 0$.

Note, that there is a unique minimal representation of v^M_λ given by

(4) $(\bar{g};\bar{k};\bar{\lambda}) = (0,1;\ k_0,k_1;\ c)$
$(k_0 + k_1 = n,\ 1 \leq c \leq k_1)$

Remark 2.2. Let $M \in \mathfrak{M}^r$, $r \geq 2$, and $\lambda \in \mathbb{N}$ be such that M hom λ. Let $i_0 \in \{1,\dots,r\}$ and $c \in \mathbb{N}$, $1 \leq c \leq k_{i_0}$ be specified by the Basic Lemma. Consider the case that

(5) $$m_{i_0-1} < g_{i_0} \, , \; m_{i_0}^c < g_{i_0+1}$$

(Recall $m_{i_0-1} = M_{i_0-1}(\Omega)$ etc.) and let us analyze the game $v = v_\lambda^M$.

It is seen at once that $s^\lambda = (0,\dots,0, c, k_{i_0+1},\dots,k_r)$ is the only min-win profile. Fellowships $i_0+1,\dots,r$ are "inevitable" while exactly c players of fellowships i_0 are entering a min-win coalition with profile s^λ.

Let us define characters of the various fellowships by distinguishing two cases.

If $r = i_0$, we have $\lambda = cg_r$ and

$m_{i_0-1} = m_{r-1} = \sum_{i=1}^{r-1} k_i g_i < g_r$. Then, fellowships $0,1,\dots,r-1$ are dummies. Fellowship r is called a step. It's satellite measure is $M^{(1)} = M_{r-1}$ which has total mass $m^{(1)} = m_{r-1}$.

There is a unique minimal representation of v_λ^M given by

(6) $$(\bar{g},\bar{k},\bar{\lambda}) = (0,1; \sum_{i=0}^{r-1} k_i, k_r; c)$$

hence these games are members of the class described in Remark 2.1.. Obviously, there are two types involved in the game v.

If $i_0 < r$, then we have in general (i.e. $c < k_{i_0}$) 3 types involved in the game. Fellowships $0,\dots,i_0-1$ are **dummies**, fellowship i_0 is a <u>step</u>, fellowships $i_o+1,\dots,r$ are <u>steps</u> as well (and belong to one type!). The <u>satellite measures</u> are $M^{(i_o)} = M_{i_o-1}$ and $M^{(j)} = M^c_{i_o}$ $(j \geq i_o + 1)$. Again, v^M_λ has a minimal homogeneous representation given by

$$(7) \qquad (\bar{g},\bar{k},\bar{\lambda}) = (0,1,k_i-c+1;\ \sum_{i=o}^{i_o-1} k_i, k_{i_o},\ \sum_{i=i_o+1}^{r} k_i,\ c + \sum_{i=i_o+1}^{r} k_i(k_i-c+1))$$

provided $c < k_{i_o}$. For $c = k_{i_o}$, the game is again seen to be an element of the class discussed in Remark 2.1.

3. Characters

During this section we assume that $M \in \mathfrak{M}^r$ and $\lambda \in \mathbb{N}$ satisfy M hom λ. Hence there is $i_o \in \{1,\dots,r\}$ and c, $1 \leq c \leq k_{i_o}$ as specified by the Basic Lemma.

Using the results of section 2, we now start out to define characters and satellite measures for the general case simultaneously by induction.

Definition 3.1. Let $M \in \mathfrak{M}^r$ and $\lambda \in \mathbb{N}$

1. For $j \geq i_o + 1$ let $M^{(j)} := M^c_{i_o}$, while $M^{(i_o)} := M_{i_o - 1}$.

   $M^{(j)}$ $(j = i_o,\dots,r)$ is the satellite measure of fellowship j.

2. If $m^{(j)} \geq g_j$, then j is a sum.
   If $m^{(j)} < g_j$, then j is a step.

3. Let $J = J(M,\lambda) = \{j \geq i_o \mid m^{(j)} \geq g_j\}$

4. For $r = 1$ or $J = \emptyset$, characters and satellite measures are given by Remarks 2.1. and 2.2.. Assume $r \geq 2$ and $J \neq \emptyset$. Let

$$D^{(j)} = D(M^{(j)}, g_j) \qquad (j \in J)$$

denote the dummies of $(M^{(j)}, g_j)$ (being defined by induction hypothesis). Then

$$D = D(M,\lambda) := \bigcap_{j \in J} D^{(j)} \tag{1}$$

is the set of dummies of (M,λ).

5. For $i \notin D$, $i < i_o$ let

$$J^{(i)} := \{j \in J \mid i \notin D^{(j)}\} \neq \emptyset$$

and, for $j \in J^{(i)}$, let $M^{(i,j)}$ be the satellite measure of i w.r.t. $(M^{(j)}, g_j)$ which is defined by induction hypothesis. This

(2) $$M^{(i)} := \max_{j \in J^{(i)}} M^{(i,j)}$$

is the satellite measure of i .

6. For $i \notin D$, $i < i_0$ we specify the character as follows:
if $m^{(i)} \geq g_i$ then i is a sum;
if $m^{(i)} < g_i$ then i is a step.

Thus
$$\{0,\ldots,r\} = D + \Sigma + \mathbb{T}$$

(all quantities presently defined w.r.t. (M,λ), i.e. $\Sigma = \Sigma\ (M,\lambda)$ denoting the steps etc.)

Remark 3.2. 1. $M^{(i)}$ is a projection of M and corresponds to a certain
$$(0,g_1,\ldots,g_i;\ k_0,k_1,\ldots,k_{i_1} - d) \in \mathfrak{M}^{i_1}$$
where $i_1 < i$. In addition, we have

(3) $$M^{(i)} \ \mathrm{hom}_0\ g_i \ .$$

To see this, observe that it is true for $j \geq i_0$ by the Basic Lemma and follows for $i < i_0$ at once by induction. Let us write

(4) $$C^{(i)} := (k_0,k_1,\ldots,k_{i_1} - d)$$
(the "carrier" of $M^{(i)}$) .

2. In particular, $M^{(i,j)}$ corresponds to a certain $(g_0,\ldots,g_1;\ k_0,\ldots,k_1) \in \mathfrak{M}^1$. Hence, the maximizer in (2) can be understood either to define the largest vector (measure) coordinate-wise (coalition-wise) or to be given by the lexicographically (backwards) largest carrier.

We may, therefore, (regarding $M^{(i,j)}$ as a measure or vector) also write (with suitable j_0)

(5) $$M^{(i)} = M^{(i,j_0)} \geq M^{(i,j)} \qquad (j \in J^i)$$

or

(6) $$M^{(i)} = \text{lex max } \{M^{(i,j)} \mid j \in J^i\}$$

Definition 3.3. 1. Let $v = v_\lambda^M$ and $i \notin D = D(M,\lambda)$.

2. If i is a sum, then

$$v^{(i)} := v_{g_i}^{M^{(i)}}$$

is the satellite game of i (and $(M^{(i)}, g_i)$ is hom representation of $v^{(i)}$). The lexicographically largest profile of a min win coalition (in $v^{(i)}$)

$$s^{(i)} := s_{g_i}^{M^{(i)}}$$

is said to represent the substitutes of i , $s^{(i)}$ has the shape

(7) $$s^{(i)} = (0,\ldots,e,\ k_{i_2+1},\ldots,k_{i_1-1},\ k_{i_1} - d)$$

and satisfies, of course

(8) $$M^{(i)}\ (s^{(i)}) = g_i$$

3. If i is a step, then the carrier of $M^{(i)}$

(9) $$s^{(i)} := C^{(i)} = (k_1,\ldots,k_{i_1} - d)$$

is said to represent the pseudo substitutes of i; clearly we have in this case:

(10) $$M^{(i)}\ (s^{(i)}) < g_i\ .$$

Thus, a fellowship must have one of three characters. If it is a dummy, it plays no essential rôle in the game; it's satellite measure is not defined at all. If it is a step, then it's satellite measure is well defined; but as it is too small, there is no satellite game. If it is a sum, then it's satellite measure is sufficiently large, the smaller players can combine their weights in order to play the satellite game and the substitutes are the largest (lex) coalition to replace it in a min win coalition of v. (cf. Fig. 2)

Note that steps of $j \geq i_0$ are inevitable players: they all show up in any min win coalition. On the other hand, the only way smaller fellows ($i < i_0$) may enter a min win coalition is via replacing successively sums by playing satellite games. This will become obvious during the later development.

Remark 3.4. 1. Fellowship $i < i_0$ is a sum (w.r.t. (M,λ) if and only if it is a sum w.r.t. $(M^{(j)}, g_j)$ for at least one $j \geq i_0$.

2. Let $i < i_0$ be a sum and let $j_0 \geq i_0$ be such that

$$M(i) = M(i,j_0)$$

Then i is a sum w.r.t. $(M^{(j_0)}, g_{j_0})$

3. In this case clearly $s^{(i,j_0)} = s^i$ (with obvious notation) i.e., the substitutes of i w.r.t. (M,λ) and w.r.t. $(M^{(j_0)}, g_{j_0})$ are the same. (Remark 3.2.2.)

Our first aim is to show that the characters "dummy", "sum", and "step" are awarded by the game v and not by the representation (M,λ).

Remark 3.5. ("The canonical decomposition")

Let s be a min win coalition w.r.t. (M,λ). Given $i_0 \in \{1,\dots,r\}$ and $c \in \mathbb{N}$, $1 \leq c \leq k_{i_0}$ (by the Basic Lemma) consider the profiles

$$\overset{\circ}{\underset{\circ}{s}} := (s_0,\dots, s_{i_0-1}, (s_{i_0}-c)^+, 0,\dots,0)$$

$$\overset{\circ}{s} := (s_0,\dots, s_{i_0-1}, 0, 0,\dots,0),$$

which are regarded to be feasible for $M^c_{i_0}$ and M_{i_0-1} respectively. Note that $\overset{\circ}{\underset{\circ}{s}} = (s-s^\lambda)^+$. In view of

$$s = (s - s^\lambda)^+ + (c \wedge s_{i_0})e^{i_0} + \sum_{j=i_0+1}^{r} s_j e^j$$

we have

$$M(s) = M^c_{i_0}(\overset{\circ}{\underset{\circ}{s}}) + (c \wedge s_{i_0})g_{i_0} + \sum_{j=i_0+1}^{r} s_j g_j .$$

As s is min-win, we have also

$$M(s) = \lambda = cg_{i_0} + \sum_{j=i_0+1}^{r} k_j g_j$$

and it follows that

$$\text{(11)} \qquad M^c_{i_0}(\overset{\circ}{\underset{\circ}{s}}) = (c - s_{i_0})^+ g_{i_0} + \sum_{j=i_0+1}^{r} (k_j-s_j) g_j$$

Assume now that $s \neq s^\lambda$. Then at least one of the terms $d_j := s_j$ $(j > i_0)$, $d_{i_0} := (c - s_{i_0})^+$ on the right hand side of (11) is positive. For each $j \geq i_0$ with $d_j > 0$ we have $M^{(j)}$ hom g_j by the Basic Lemma. Therefore we can take successively d_j subcoalitions represented by profiles $s^{j\kappa}$ $(\kappa = 1,\dots,d_j)$ out of the coalition represented by $\overset{\circ}{\underset{\circ}{s}}$ such that

$$M^j (s^{j\kappa}) = g_j \qquad (\kappa = 1,\dots,d_j) .$$

In doing so we may start with an arbitrary j s.t. $d_j > 0$ but continue by collecting the largest coordinates of $\overset{o}{s}$ first (i.e., collecting from right to left) (Basic Lemma). Thus $\overset{o}{s}$ is decomposed

$$\overset{o}{s} = (\overbrace{s_0, s_1, s_2, \ldots}^{s^{j'\kappa'}}\overbrace{\ldots}^{s^{j\kappa}} \ldots \overbrace{\ldots, s_{i_0} - c}^{s^{j'\kappa'}})$$

or

(12) $$\overset{o}{s} = \sum_{j=i_0}^{r} \sum_{\kappa=1}^{d_j} s^{j\kappa}$$

where e.g.

$$s^{j\kappa} = (0\ldots.0,\ e,\ s_{p+1},\ldots,s_{q-1},\ f,\ 0,\ldots,0)$$

$(1 \leq e \leq s_p,\ 1 \leq f \leq s_q)$. Of course, for $d_j > 0$, j is a sum and $s^{j\kappa}$ is min win "in $v^{(j)}$" or w.r.t. $(M^{(j)}, g_j)$ and feasible for $M^{(j)}$.

<u>Lemma 3.6.</u> Let $v = v_\lambda^M$. Then D is independent of the representation. More precisely:

1. $j \in D = D(M,\lambda)$ iff each $\omega \in K_j$ is a v-Dummy in the ordinary sense (see e.g. [8], Ch.III, SEC.2).

2. If (M',λ') is a further homogenous representation of v we may (after suitable extension (cf. section 1) assume that $M, M' \in \mathfrak{M}^r$. Then

$$D(M,\lambda) = D(M',\lambda').$$

<u>Proof:</u> It suffices to check that the dummies given by Definition 3.1. are exactly the ones of v in the ordinary sense. This is performed by induction.

1. For $r = 1$ or $I = \emptyset$ our statement is obvious.

2. For $r \geq 0$ and $I \neq \emptyset$ our statement is obvious for $j \geq i_0$. Let $\bar{i} \leq i_0-1$. Suppose $\bar{i} \notin \bigcup_{j \in I} D^{(j)}$, say $\bar{i} \notin D^{(\bar{j})}$;

   Clearly, $g_{\bar{i}} > 0$, i.e., $\bar{i} > 0$. By induction $\omega \in K_{\bar{i}}$ is not a dummy in $v^{(\bar{j})}$ (in the ordinary sense). Assume $\bar{j} \geq i_0 + 1$ ($\bar{j} = i_0$ runs analogously). There is a profile

$$s^0 = (s_1^0, \ldots, s_{i_0}^0)$$

   of a min-win coalition in $v^{(\bar{j})}$ ("feasible for $M^{(j)}$") such that $s_{\bar{i}}^0 > 0$ and

$$M^{(\bar{j})}(s^0) = M_{i_0}^c(s^0) = g_{\bar{j}} .$$

Note that $s_{i_0}^0 \leq k_{i_0} - c$.

The profile

$$\bar{s} = (s_1^0, \ldots, s_{i_0-1}^0, s_{i_0}^0 + c, k_{i_0+1}, \ldots, k_{\bar{j}} - 1, \ldots, k_r)$$

reflects a min-win coalition in Ω since

$$M(\bar{s}) = \lambda$$

and as $\bar{s}_{\bar{i}} = s_{\bar{i}}^0 > 0$, $\omega \in K_{\bar{i}}$ is not a v-Dummy, which completes the first part of the proof.

On the other hand, suppose that $\bar{i} \leq i_0-1$ and $\omega \in K_{\bar{i}}$ is not a v-dummy; again $g_{\bar{i}} > 0$ is necessarily true. Let $\tilde{s}$ be the profile of a min-win coalition (in Ω) s.t. $\tilde{s}_{\bar{i}} > 0$.

Since $s \neq s^\lambda$, we may consider a canonical decomposition of $\overset{\circ}{s} = (s - s^\lambda)^+$ (cf. Remark 3.5.), say

$$s = \sum_{j=i_0}^{r} \sum_{\kappa=1}^{d_j} s^{j\kappa}$$

such that $s^{j\kappa}$ is feasible for $M^{(j)}$. Now, as $\overset{\circ}{s}_{\bar{i}} > 0$, at least one of the terms $s^{j\kappa}$ is positive and hence $\omega \in K_{\bar{i}}$ is a member of a min-win coalition in some $v^{(j)}$.

By induction, $i \notin D^{(j)}$ and by Definition 3.1.4., $i \notin D = D(M,\lambda)$, q.e.d.

Theorem 3.7. Assume M to be reduced. Let $v = v_\lambda^M$ and let i be a sum (w.r.t. (M,λ)). Suppose i' is of the same type as i (w.r.t. v). The $i = i'$.

Proof: For $r = 1$ and for $r \geq 2$, $J = \emptyset$, there is nothing to be proved as there are no sums (Remarks 2.1., 2.2.). Assume therefore, $r \geq 2$ and $J \neq \emptyset$.

Next observe that $i \geq i_0$ implies $i' \geq i_0$ and vice versa for otherwise it is easily verified that i and i' cannot be of the same type.

1st CASE: $i, i' \geq i_0$.

Suppose $i < i'$. As i is a sum, one player of fellowship i may be replaced by his substitutes in order to change the profile of the lex max coalition; thus there is a min win coalition with profile

$$\begin{aligned}\tilde{s} &= (s_1,\ldots,s_{i_0},\ldots,k_i - 1,\ldots,k_r)\\ &= s^\lambda - e^i g_i + s^{(i)}\end{aligned}$$

(say, the tacit assumption $i > i_0$ made by writing down (11) is unimportant). Here, e^i is the i'th coordinate vector and $s^{(i)}$ the lex max profile in $(M^{(i)}, g_i)$, i.e.

$$s^{(i)} := (s_1,\ldots, s_{i_0} - c).$$

However, the profile

$$\hat{s} := \tilde{s} + e^{i} g_i - e^{i'} g_{i'}$$

has to total weight $M(\hat{s}) < \lambda$, thus i and i' are different types if $g_{i'} < g_i$, a contradiction.

On the other hand, if $i > i'$, then

$$m^{(i')} = m^c_{i_0} = m^{(i)} \geq g_i > g_{i'} ,$$

thus i' is a sum. We may then repeat the above argument, exchanging i and i'. This settles the first case.

<u>2^{nd} CASE:</u> $i, i' < i_0$. As i is a sum (w.r.t. (M,λ)) he is a sum w.r.t. <u>some</u> $(M^{(j)}, g_j)$ $(j \geq i_0)$. By induction, we have $i = i'$, q.e.d.

<u>Remark 3.8.</u> 1. In any reduced representation of a homogeneous game v, sums of the same type have the same weight.

2. Dummies, steps and sums are defined w.r.t. v. For a precise version of this statement observe that with respect to any reduced representation, a fellowship which is a sum equals a type; this type is suitably called a "sum" as well. The dummy type may be decomposed into several fellowships but it is well defined by Lemma 3.6. Any of the remaining types may be decomposed into several fellowships - however, by Theorem 3.7., they are all steps and thus this kind of type is called a step. Thus, types are classified to belong to one of the three <u>characters</u>.

3. Consider two reduced homogeneous representations (M,λ) and (M',λ') of v. If they are reduced and of the same length, then $\Sigma\ (M,\lambda) = \Sigma(M',\lambda')$ etc.

4. For a non-reduced representation, a type which is a step might be decomposed into fellowships, some of which could be sums. Consider $M = (0,1,\ldots,1; 1,\ldots,1) \in \mathfrak{M}^r$, $\lambda \in \mathbb{N}$, $\lambda < r$ for an example. This representation has sums while the reduced version $(0,1; 1,r)$ has none, λ being unchanged.

<u>Lemma 3.9.</u> Let $v = v_\lambda^M$ and $i, i' < i_0$. Then i and i' belong to the same type (w.r.t. v) if and only if they belong to the same type w.r.t. each $v^{(j)}$ $(j \in J)$.

<u>Proof:</u> For $r = 1$ or $J = \emptyset$, nothing has to be proved: both, i and i' are dummies (Remarks 2.1., 2.2.). We may therefore, assume that $r \geq 2$ and $J \neq \emptyset$ for the remaining part of the proof.

<u>1^{st} STEP:</u> Assume that i and i' belong to the same type w.r. to each $v^{(j)}$ $(j \in J)$.

Let $s \in \mathbb{N}_0^{r+1}$ be the profile of a min win coalition such that $s_i > 0$ and $s_{i'} < k_{i'}$; we have to show that

$$s' = s - e^i + e^{i'}$$

is also a min win coalition (e^i denoting the i'th basis vector).

Now, as $s_i > 0$ clearly $s \neq s^\lambda$, therefore we may by Remark 3.5. decompose $s^0 = (s - s^\lambda)^+$ canonically, say

$$(13) \qquad s^0 = \sum_j \sum_\kappa s^{j\kappa}$$

Profiles $s^{j\kappa}$ (if $\neq 0$) are min-win w.r.t. v^j and, for at least on $\bar{j}$, there is $\bar{\kappa}$ such that

$$s_i^{\bar{j}\bar{\kappa}} > 0 \ , \ s_{i'}^{\bar{j}\bar{\kappa}} < k_{i'} \quad .$$

Replace one player of fellowship i by one player of fellowship i' in $s^{\hat{J}\hat{K}}$, i.e., consider

$$\tilde{s}^{\hat{J}} := s^{\hat{J}\hat{K}} + e^{i'} - e^{i}.$$

As i and i' are of the same type w.r.t. $v^{(\hat{J})}$, $\tilde{s}^{\hat{J}}$ has the same weight as $s^{\hat{J}\hat{K}}$. Replacing $s^{\hat{J}\hat{K}}$ within the sum (13) amounts to forming $s^{0} + e^{i'} - e^{i} = \mathring{s} + \tilde{s}^{\hat{J}} - s^{\hat{J}\hat{K}}$, which obviously has the same weight as $\mathring{s}$. From this it is easily inferred that $s + e^{i'} - e^{i}$ has the same weight (i.e. λ) as s, which completes the first step.

2nd STEP: Assume now, that i and i' belong to the same type as v is concerned. Consider a min win profile for some $v^{(j)}$ ($j \in J$), say $\mathring{s}$, such that

$$\mathring{s}_{i} > 0 \, , \, \mathring{s}_{i'} < k_{i'} \, .$$

Then

$$s = (\mathring{s} + ce^{i_0}, k_{i_0+1}, \ldots, k_j - 1, \ldots, k_r)$$

is a min win profile for v (treating the case $j \geq i_0+1$, which is analogous to $j = i_0$).

Therefore

$$s' = s - e^{i} + e^{i'}$$

has the same weight $M(s') = \lambda$ which implies that

$$\mathring{s}' = \mathring{s} - e^{i} + e^{i'}$$

has the same weight

$$M^{c}_{i_0}(\mathring{s}') = g_j \, , \qquad \text{q.e.d.}$$

<u>Corollary 3.11.</u> Steps i and i' belong to the same type if and only if they have the same satellite measure (i.e., the same pseudo substitutes).

<u>Proof:</u> For $r = 1$ or $J = \emptyset$ the statement is obviously true. Let $r \geq 2$ and $J \neq \emptyset$. Again, for $i, i' \geq i_0$ or $i \geq i_0 > i'$, the corollary is verified at once. It remains to consider the case that i and i' both satisfy $< i_0$; this is treated by an inductive argument.

As i and i' are steps, they are steps or dummies in any $v^{(j)} (j \in J)$ and this character they share <u>simultaneously</u> by Lemma 3.9.

For any $j \in J^{(i)}, i$ and i' are steps in $v^{(j)}$ and by induction they have the same satellite measure, i.e.

$$M^{(i,j)} = M^{(i',j)} \qquad (j \in J^{(i)} = J^{(i')})$$

By Definition 3.1. it follows immediately that $M^{(i)} = M^{(i')}$, q.e.d.

<u>Lemma 3.11.</u> Let $v = v_\lambda^M$ and let $i \notin D$ be a nondummy fellowship. If $i < j$ then ($j \notin D$ and) $M^{(i)} \leq M^{(j)}$.

<u>Proof:</u> For $j \geq i_0$ the statement is obvious. For $j < i_0$ we have also $i < i_0$ and we may proceed by induction.

Indeed, whenever $j^* \geq i_0$ and $M^{(i,j^*)}$ denotes the satellite measure of i w.r.t. $(M^{(j^*)}, g_{j*})$ (i.e. $M^{(j^*)} = M^c_{i_0}$ or $M^{(j^*)} \doteq M_{i_0-1}$ respectively), then we may by induction assume that

$$M^{(i,j^*)} \leq M^{(j,j^*)}$$

holds true whenever both terms are defined, i.e. whenever i (and, consequently j) is not a dummy "in v^{j^*}". Clearly (see Definition 3.1.)

$$J^i \subseteq J^j$$

and hence

$$M^{(i)} = \max_{j^* \in I^i} M^{(i,j^*)} \leq \max_{j^* \in I^j} M^{(i,j^*)} = M^{(j)}, \qquad \text{q.e.d.}$$

<u>Lemma 3.12.</u> Let $v = v_\lambda^M$ and let $i < j$ be fellowships such that i is not a dummy in $v^{(j)}$. Also, let $M^{(i,j)}$ denote the satellite measure of i w.r.t. $(M^{(j)}, g_j)$. Then

$$M^{(i,j)} \leq M^{(i)} \quad .$$

<u>Proof:</u> For $j \geq i_o$ we have necessarily $i < i_o$ and, by Definition 3.4.:

$$M^{(i,j)} \leq \operatorname{lex\,max}_{j^* \in J^i} M^{(i,j^*)} = M^{(i)}$$

Now let $j < i_o$. Observing that $M^{(i)}$ is defined w.r.t. (M,λ) write $M^{(i)} = M^{(i)}(M,\lambda)$ for the moment, such that

$$M^{(i,j)} = M^{(i)}(M^{(j)}, g_j)$$

and

$$M^{(i,j,1)} := M^{(i)}(M^{(j)}(M^{(1)}, g_1), g_j)$$

whenever all measures are defined. Choose $j_o \geq i_o$ such that

$$M^{(j)} = M^{(j)}(M^{(j_o)}, g_{j_o}) = M^{(j,j_o)}$$

Then

$$M^{(i,j)} = M^{(i)}(M^{(j)}, g_j) = M^{(i,j,j_o)}$$

Consider $(M^{(j_o)}, g_{j_o})$; by induction we have

$$M^{(i,j,j_o)} \leq M^{(i,j_o)}$$

and hence

$$M^{(i,j)} = M^{(i,j,j_o)} \leq M^{(i,j_o)}$$

$$\leq \max_{j^* \in I^i} M^{(i,j\ )} = M^{(i)} , \qquad \text{q.e.d.}$$

<u>Theorem 3.13.</u> Let $v = v_\lambda^M$, $i < j$ and assume that i is a sum in $v^{(j)}$. Then i is a sum in v.

<u>Proof:</u> By applying (8) in Definition 3.3. to $(M^{(j)}, g_j)$ we have

(14) $$M^{(i,j)}\ (s^{(i,j)}) = M^{(j)}\ (s^{i,j}) = g_i$$

and hence

$$m^{(i)} \geq M^{(i)}\ (s^{(i,j)} = M^{(i,j)}\ (s^{(i,j)}) = g_i\ ,$$

this shows $i \in \Sigma$.

<u>Corollary 3.14.</u> Let $v = v_\lambda^M$. For $J \neq 0$

(15) $$D = \bigcap_{j \in J} D^{(j)} = \bigcap_{j \in \Sigma} D^{(j)}$$

(16) $$\Sigma = J \cup \bigcup_{j \in J} \Sigma^{(j)} = J \cup \bigcup_{j \in \Sigma} \Sigma^{(j)}$$

(17) $$T = (\{i_0, \ldots, r\} - J) \cup (\bigcup_{j \in J} T^{(j)} \cap (\bigcup_{j \in J} \Sigma^{(j)})^c)$$

$$= (\{i_0, \ldots, r\} - J) \cup (\bigcup_{j \in \Sigma} T^{(j)} \cap (\bigcup_{j \in \Sigma} \Sigma^{(j)})^c)$$

<u>Corollary 3.15.</u> For $i < i_0$, $i \notin D$:

$$M^{(i)} = \max \{M^{(i,j)} \mid i < j,\ j \in \Sigma,\ i \notin D^{(j)}\}$$

<u>Corollary 3.16.</u> Assume M to be reduced. Let $v = v_\lambda^M$. For any $i \notin D$, $v^{(i)}$ is independent of the representation (M,λ) but depends solely on v.

The details of these statements are left to the reader.

4. The minimal representation

Within this section we characterize all homogeneous representations at a homogeneous game $v = v_\lambda^M$. In particular, we show the existence of a unique minimal hom representation; the proof rests on an inductive argument based on the satellite games.

Definition 4.1. Let $(M,\lambda), (M',\lambda') \in \mathfrak{M}^r \times \mathbb{N}$ such that $k = k'$. Assume that M hom λ such that $v = v_\lambda^M$ is a homogeneous game. For $i \notin D = D(M,\lambda)$, let $C^{(i)} = (k_1,\ldots,k_{i_1}-d)$ be the carrier of $M^{(i)}$ (cf. 3.2.) and define a family

$$M'^{(i)} \qquad (i \notin D)$$

by

$$M'^{(i)} = M'|_{C^{(i)}} ,$$

the restriction on $C^{(i)}$, or, equivalently, the projection on the first coordinates).

(M',λ') is said to be compatible with (M,λ) if the following conditions are satisfied:

(1) $$g'_i \geq m'^{(i)} + 1 \qquad i \in T = T(M,\lambda)$$

(2) $$g'_i = M'^{(i)}(s^{(i)}) \qquad i \in \Sigma = \Sigma(M,\lambda)$$

(3) $$\lambda' = M'(s^\lambda)$$

Theorem 4.8.

1. If (M',λ') is compatible with (M,λ), then (M',λ') is a homogeneous representation of v, i.e.

$$v = v_\lambda^M = v_{\lambda'}^{M'}$$

2. Any two homogeneous representations are compatible with each other.

3. There is a unique minimal (homogeneous) representation of any homogeneous v; this is obtained by requiring an equation in any inequality (1). (and awarding dummies the weight 0).

4. The minimal representation may be computed by starting with the smallest nondummy and proceeding according to (1), (2), and (3).

5. For the minimal representation types and fellowships coincide.

Proof: Put $M^{r+1} := M$, $g_{r+1} :=$ and $v^{(r+1)} := v = v_{\lambda}^{M}$. Similarly for the quantities M', λ'. $r+1$ is formally called a "sum". Suppose $\underline{i}$ is the first nondummy fellowship; we are going to show by induction from $\underline{i}$ to $r+1$:

(4) "If $i \in \Sigma$, then $v^{i} = v_{g'_i}^{M'(i)}$".

Now, for $i = \underline{i}$ there is nothing to show because $\underline{i}$ is a step.

Therefore, fix some j, $\underline{i} < j \leq r+1$ and assume that (4) is true for all $i < j$. We shall show that (4) holds true for j. We proceed by two steps assuming that j is a sum.

1st Step: Let us check for $i < j$:

a) If $i \in \Sigma^{j}$ then

$$v_{g_i}^{M^{(i,j)}} = v_{g'_i}^{M'^{(i,j)}} .$$

b) If $i \in T^{j}$ then $m'^{(i,j)} + 1 \leq g'_i$.

Here, $M'^{(i,j)}$ denotes the projection of M' on $C^{(i,j)}$ etc.

Now, as for statement a), we know that $i \in \Sigma^{j}$ and thus $i \in \Sigma$ and

(5) $M^{(i)} \geq M^{(i,j)}$

Moreover, using our induction hypothesis

(6) $$M^{(i)}(s) = g_i \quad \text{iff} \quad M'^{(i)}(s) = g_i'$$

We want to show

(7) $$M^{(i,j)}(s) = g_i \quad \text{iff} \quad M'^{(i,j)}(s) = g_i' \quad ,$$

which is equivalent to

(8) $$M^{(i)}(s \wedge C^{(i,j)}) = g_i \quad \text{iff} \quad M'^{(i)}(s \wedge C^{(i,j)}) = g_i'$$

in view of (5) and the projection properties of M and M'. Clearly, (8) follows from (6) and a) is checked.

As for statement b), let $i \in T^j$.

If $i \in T$, then

$$m'^{(i,j)} \leq m'^{(i)} \leq g_i' - 1$$

in view of (1) of Definition 4.1.

If $i \in \Sigma$, then $v^i = v_{g_i'}^{M'^{(i)}}$ by induction hypothesis, thus

$$g_i > m^{ij} = M^{(i,j)}\,(C^{(i,j)}) = M^{(i)}(C^{(i,j)})$$

implies

$$g_i' > M'^{(i)}(C^{(i,j)}) = m'^{(i,j)} \quad ,$$

hence we are through with b).

<u>2nd Step:</u> In view of the first step we may not only assume that $(M'^{(i)}, g_i')$ represents $v^{(i)}$ for $i < j$, but also that $(M'^{(i,j)}, g_i')$ represents $v^{(i,j)} = v_{g_i}^{M^{(i,j)}}$ for $i < j$ whenever $i \in \Sigma^j$.

Therefore we may, as a technicallity, omit the index j (thus arguing so to speak, our case for $r \to r+1$) and instead of (4) show that

(9) If $v^{(i)} = v_{g_i'}^{M'^{(i)}}$ holds true for $i \in \Sigma$ then $v = v_{\lambda'}^{M'}$.

Now, two statements have to be checked, namely

c) If $M(s) = \lambda$ then $M'(s) = \lambda'$

and

d) If $M(s) < \lambda$ then $M'(s) < \lambda'$.

Let us start out with c):

If s is a min win coalition and $s = s^\lambda$ then, nothing has to be proved as $M'(s) = \lambda'$ follows from 4.1.

Assume $s \neq s^\lambda$; let $\overset{o}{s} = (s - s^\lambda)^+$ and recall Remark 3.5. An inspection of $\overset{o}{s}$ teaches: if $s_j < k_j$ for some $j \geq i_0$, then $M^j(s) \geq (k_j - s_j)\, g_j$, i.e. $\overset{o}{s}$ contains (in view of the Basic Lemma) $k_j - s_j$ profils of $M^{(j)}$-measure g_j. These are min-win coalitions of $v^{(j)}$ (j must be a sum!) and by (9), $v^{(j)}$ is represented by $(M'^{(j)}, g_j')$. Thus any of the min-win coalitions of $v^{(j)}$ mentioned above has M'-measure g_j'! This way it is seen that $\overset{o}{s}$ decomposes such that its total M'-measure is

$$M'(\overset{o}{s}) = (c - s_{i_0})^+ g_{i_0}' + \sum_{j > i_0} (k_j - s_j)\, g_j' \quad .$$

Consequently

$$M'(s) = M'(\overset{o}{s}) + \min(s_{i_0}, c)\, g_{i_0}' + \sum_{j > i_0} s_j\, g_j' = M'(s^\lambda) = \lambda'$$

which finishes statement c).

Finally, statement d) is verified by the same methods as c) - just that the "decomposition is not a canonical one":

Let $M(s) < \lambda$ and put $\overset{o}{s} := (s-s^{\lambda})^{+}$. Similarly to the procedure in 3.5. we obtain

$$(10) \qquad M^{c}_{i_o}(s^{o}) < (c-s_{i_o})^{+}g_{i_o} + \sum_{j=i_o+1}^{r} (k_j-s_j)\, g_j = \sum_{j=i_o}^{r} d_j\, g_j$$

(compare 3.5. for the definition of d_j $(j=i_o,\ldots,r)$).

Now, for $j = i_o,\ldots,r$, if $M^{(j)}(\overset{o}{s}) \geq g_j$, then take profiles $s^{j\kappa}$, $\kappa=1,\ldots,\alpha_j$ such that $M^{(j)}(s^{j\kappa}) = g_j$ and α_j is as large as possible but not exceeding d_j.

Thus we obtain a decomposition

$$\overset{o}{s} = \sum_{j}\sum_{\kappa} s^{j\kappa} + s^{\varepsilon}$$

where $s^{j\kappa}$ is min-win for $(M^{(j)}, g_j)$ and s^{ε} is loosing for any $v^{(j)}$ $(j \in J)$ (possibly $s^{\varepsilon} = 0$) and $\alpha_j \leq d_j$ $(j \in J)$.

Because of (10), at least one α_j is strictly smaller than d_j.

Now, by (9) we conclude that

$$M'^{(j)}(s^{\kappa j}) = g'_j \;,\; M'^{(j)}(s^{\varepsilon}) < g'_j$$

and hence, it is seen that for M' an inequality corresponding to (10) holds true as well, i.e.

$$(11) \qquad M'^{c}_{i_o}(s^{o}) < (c-s_{i_o})\, g'_{i_o} + \sum_{j=i_o+1}^{r} (k_j-s_j)\, g_j \,.$$

Inequality (11), however, is at once converted to

$$M'(s) < \lambda'$$

This completes the first statement of Theorem 4.8.

The second statement follows from Theorem 3.7, Remark 3,8, Lemma 3.9., and Corollary 3.10..

Finally, statements 3, 4, and 5 are now immediate consequences.

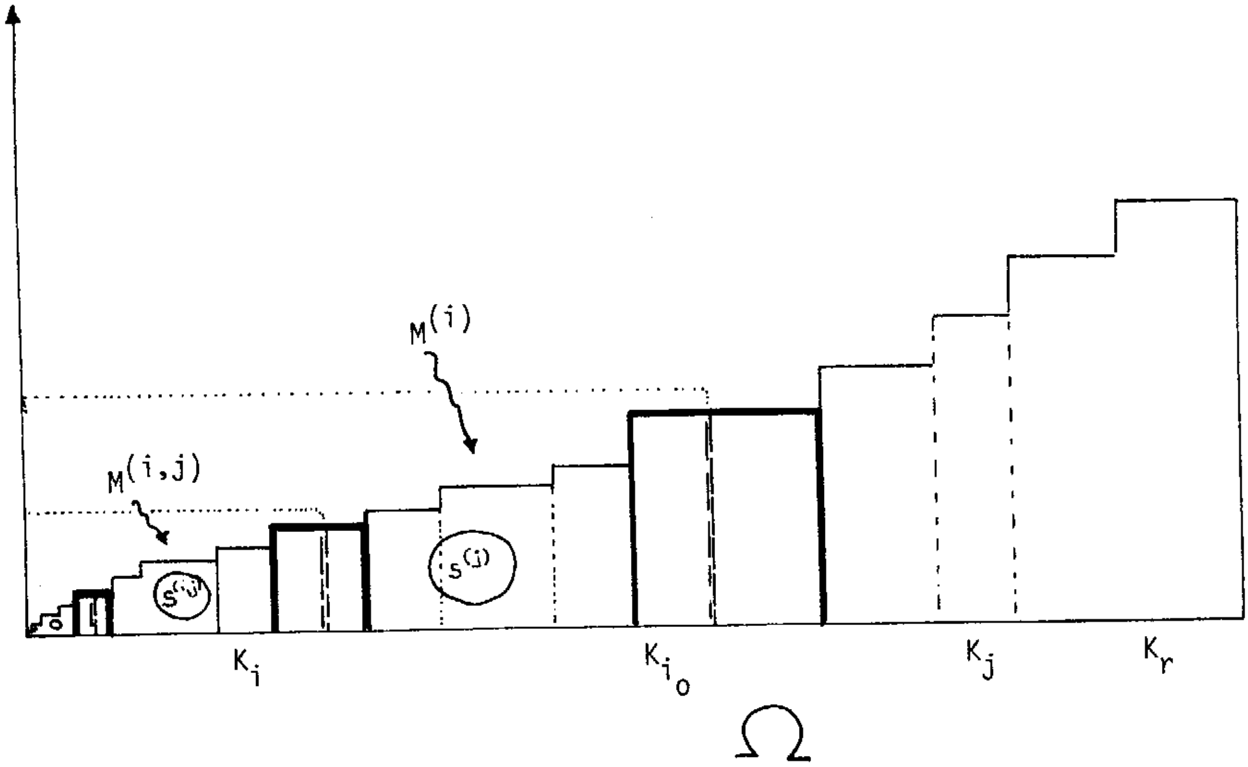

Fig. 2: Satellite measures and substitutes

5. Substitutions and the nature of min-win coalitions

During this section we study the way smaller players may enter min-win coalitions. This essentially is done by successively entering the lex-max coalition of a satellite game via the replacement of a sum. This procedure offers the access to a second existence proof for the minimal homogeneous representation. Although this proof also rests on the concept of the satellite games, it is essentially a method in the spirit of OSTMANN's [5] paper, thus, the present section also clears the connection between the setup used in [5] and our present one.

Remark 5.1. Let $s \neq s^{\lambda}$ be a min win profile and let l be the first index such that $s_l \neq 0$. Also let i be the first index larger than l s.t. $s_i < k_i$. Generically, s has the shape

(1) $$s = (0,\dots,0,c,k_{l+1},\dots,k_{i-1}, s_i, \circledast \dots \circledast)$$

where $c > 0$ and $s_i < k_i$ (of course the 0's and k_q's could not appear). i is called the smallest dropout in s.

Lemma 5.2. ("The substitution lemma") Let i be the smallest dropout of a min win profile s. Then $M^{(i)}(s) \geq g_i$. In particular, i is a sum and $s^{(i)} \leq s$.

Proof: Let $\mathring{s} = (s-s^{\lambda})^{+}$ be defined as in 3.5.. Consider the case that $i > i_0$. As $s \neq s^{\lambda}$ and $s_i < k_i$, it is seen at once that $M^i(\mathring{s}) \geq g_i$. The shape of s (consider the coordinates $i' \leq i$) implies the assertion of the Lemma. The case $i = i_0$ is handled analogously.

Therefore it remains to treat the case $i < i_0$ which is done by induction. If $M^{(j)}(0,\dots, c, k_{l+1},\dots,k_{i-1}, s_i,0,\dots,0) \geq g_j$ then apply the Basic Lemma in order to construct

$$s^j = (0,\dots,0,\dots, c', k_{q+1},\dots,k_{i-1}, s_i,0\dots0)$$

which is a min win coalition for $v^{(j)}$, i.e.

$$M^{(j)}(s^j) = g_j , \; q \geq 1 ,$$

satisfying $s^j \leq s$. If, for all s^j, we have $M^{(j)}(s^j) < g_j$, then a canonical decomposition of $\overset{0}{s}$ (cf. Remark 3.5)

$$\overset{0}{s} = (0,\dots,0,\overbrace{c,\ k_{l+1},\dots,}^{s^{j'}}\overbrace{k_{i-1},s_i,}^{s^j}\dots)$$

serves to the same purpose.

Thus it is possible to construct a min win profile s^j s.t. i is the smallest dropout in s^j (and hence $s^j \neq s^{(j)}$). By induction, i is a sum "in $v^{(j)}$" and $M^{(i,j)}(s^j) \geq g_i$. Hence

$$M^{(i)}(s) = \max_{j^* \in J} M^{(i,j^*)}(s) \geq M^{(i,j)}(s) \geq g_i$$

q.e.d.

<u>Remark 5.3.</u> If i is the smallest dropout of a min win profile s, then

(2) $$s^+ = s - s^i + e^i$$

is a min-win profile as well. (2) shows that, on the other hand, s is obtained from s^+ by inserting i's substitutes for one fellow of fellowship i; let us call this procedure a <u>substitution</u>. (cf. Fig. 3)

Since s^λ is the only profile that has no smallest dropout, we infer that <u>any min win profile is obtained from s^λ by finitely many substitutions</u>.

<u>Lemma 5.4.</u> ("The pseudo-substitution lemma")
Let s be a min win coalition and let l be the first coordinate such that $s_l > 0$, i.e.

$$s = (0,\dots,0,\ s_l,\ \blacksquare \dots s_i \dots \blacksquare\ )$$

Then

$$C^{(1)} \geq (k_0,\dots,k_{1-1}, 0,\dots,0)$$

and for $i > 1$

$$C^{(i)} \geq (k_0,\dots,k_{1-1},k_1-s_1, 0,\dots,0).$$

Proof: Observe that we have necessarily $1 \leq i_0$; thus, for $i \geq i_0$ and $1 = i_0$ the statement of the lemma is obvious. Assume, therefore, $1 < i_0$, and proceed by induction. Again, $i \geq i_0$ is trivial, thus let $i < i_0$.

Consider a canonical decomposition of $\overset{o}{s}$ (cf. 3.5.), say

$$\overset{o}{s} = (\underbrace{0,\dots,0, s_1,\dots}_{s^{j'}},\underbrace{s_i,\dots}_{s^j} \dots)$$

There is some s^j, $M^{(j)}(s^j) = g_j$ such $s_i^j > 0$ ($i_0 \leq j \leq r$). The first nonvanishing coordinate of s^j, say $1'$ satisfies $1 \leq 1' \leq i$ and $s_{1'}^j > 0$.

Now i is no dummy in $v^{(j)}$ and by induction hypothesis we conclude that

$$C^{(i,j)} \geq (k_0,\dots,k_{1'}-s_{1'}^j, 0,\dots,0)$$
$$\geq (k_0,\dots,k_1-s_1, 0,\dots,0)$$

(in case $1' < i$, say; the other cases are treated analogously). Hence

$$C^{(i)} = \max_{j\in J} C^{(i,j)} \geq (k_0,\dots,k_1-s_1, 0,\dots,0)$$

This settles the proof of the lemma. Note that $C^{(i)} = s^{(i)}$ if i is a step. Thus, if i is a step, it follows that

$$s^{(i)} \geq (k_0,\dots,k_1-s_1) .$$

<u>Remark 5.5.</u> The term "maximal loosing profile" is supposed to be self explaining. Let s be maximal loosing and let i be the first coordinate such that $s_i < k_i$. Then s has generically the shape

$$s = (k_0, \dots, k_{i-1}, s_i, \emptyset \dots \emptyset)$$

Clearly

$$(k_0, \dots, k_{i-1}, s_i+1, \emptyset \dots \emptyset)$$

is winning, so by the Basic Lemma, we find

$$t = (0, \dots, 0, c, k_{l+1}, \dots, k_{i-1}, s_i+1, \emptyset \dots \emptyset)$$

(typically), a min win coalition. Let

$$\tilde{s} := (k_0, \dots, k_{l-1}, k_l - c, 0 \dots 0) ,$$

by the pseudo substitution lemma 5.4., applied to t we have

$$s^{(i)} \geq \tilde{s}$$

We have thus

(3) $$s = t + \tilde{s} - e^i , \quad \tilde{s} \leq s^{(i)}$$

That is, the maximal loosing coalition s is obtained from a min-win coalition t by replacing a step by a subcoalition which is at least as weak as the coalition of pseudo substitutes.

Now, alternatively let i be a sum (it cannot be a dummy!). Then obviously $s \geq s^{(i)}$. Throw out the substitutes and put in player i. The profile

$$s^+ = s - s^{(i)} + e^i$$

has the same weight as s, thus it is loosing and its first coordinate l such that $s^+_l < k_l$ is smaller then i.

We may repeat this procedure until we find the first coordinate which is smaller than k_* to be a step - and then repeat the procedure indicated above.

It follows that there is a subset I of sums such that

(4) $$s = \sum_{i \in I} (s^{(i)} - e^{i}) + t + \tilde{s} - e^{k}$$

where t is minimal winning, k is a step and $\tilde{s} \leq s^{(k)}$.

Thus, any maximal loosing coalition s is obtained from a minimal winning coalition t in the following way: replace a finite number of sums by their substitutes and replace a step by a coalition which is at least as weak as its pseudo substitutes.

Remark 5.6. (2nd proof of Theorem 4.8.)
Minimal winning and maximal loosing coalitions in a homogeneous game are obtained by a very similar procedure. This opens the path to an alternative proof for Theorem 4.8.

Proof:

1st Step: If $M(s) = \lambda$ then $M'(s) = \lambda$.
Clearly, $M'(s^{\lambda}) = \lambda'$ by 3 of section 4.

If s is min-win then, by Lemma 5.2. and Remark 5.3.

$$s = \sum_{i \in I} s^{(i)} - e^{i} + s^{\lambda}$$

whre $I \subseteq \Sigma$. Using (2) of section 4 we find

$$M'(s) = \sum_{i \in I} (M'(s^{(i)}) - g_i') + M'(s^{\lambda}) = M'(s^{\lambda}) = \lambda' .$$

2nd Step: If $M(s) < \lambda$ then $M'(s) < \lambda'$.

As s is maximal loosing. Pick I,k,t, and $\tilde{s}$ as in Remark 5 .5, formula (4), $i \in I$ is sum, k is step, t is min win and $\tilde{s} \leq s^{(k)}$. By the 2st Step $M'(t) = \lambda'$. Using (1) and (2) of section 4 we find:

$$\begin{aligned} M'(s) &= \sum_{i \in I} (M'(s^{(i)}) - M'(e^{i}) + M'(t) + M'(\tilde{s}) - g'_k \\ &= \lambda' + M'(\tilde{s}) - g'_k \\ &\leq \lambda' + M'(s^{(k)}) - g'_k < \lambda' \end{aligned}$$

This proves that $v = v_{\lambda'}^{M'}$.

The second, third, and fourth part of our Theorem follow as previously. q.e.d.

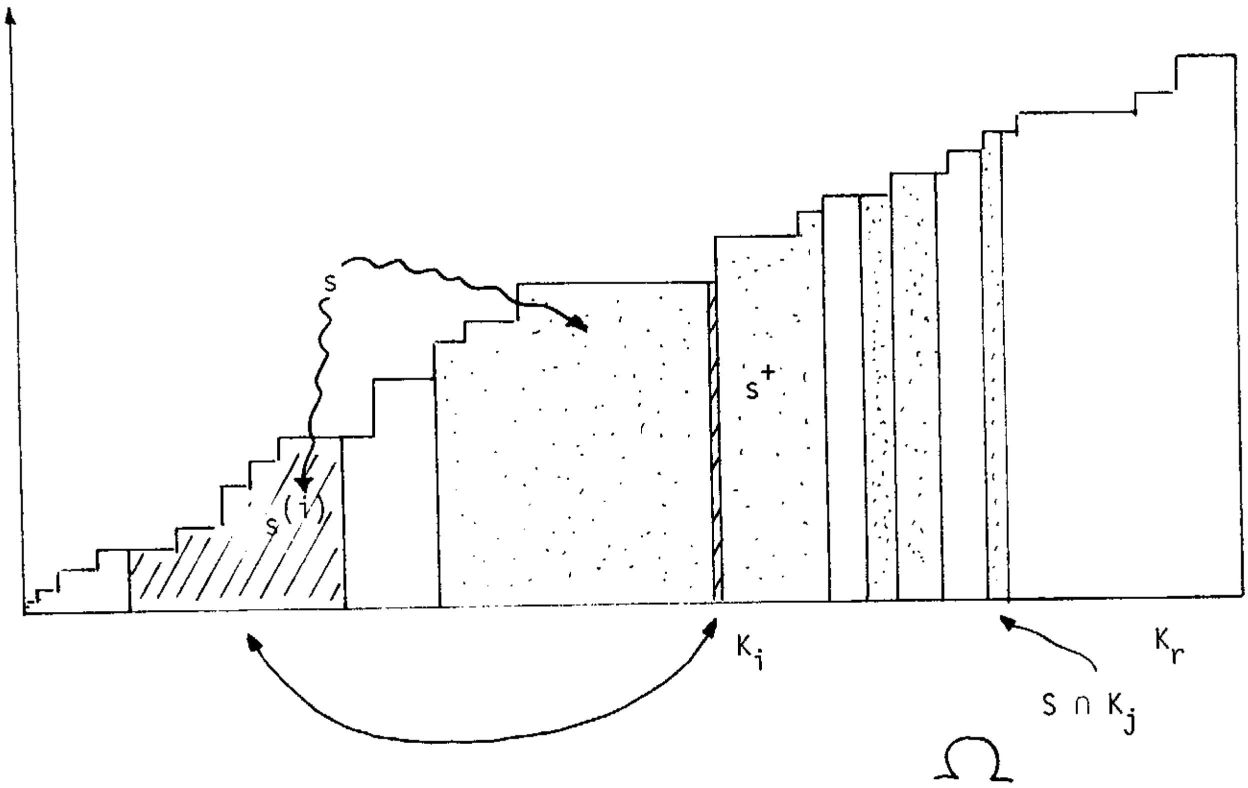

Fig. 3 : A substitution

6. Computing the minimal representation

Given any homogeneous representation (M,λ) of v, we want to compute the minimal homogeneous representation $(\bar{M},\bar{\lambda})$. We intend to construct an algorithm which does not refer to the game v (i.e., the incidence matrix of the minimal winning coalitions), thus proceeding directly from (M,λ) to $(\bar{M},\bar{\lambda})$.

Lemma 5.1. Let $i \notin D$. Then either $s_i^\lambda > 0$ or there is $\bar{j} \in \Sigma$ such that

1. $i \notin D^{(\bar{j})}$
2. $M^{(i)} = M^{(i,\bar{j})}$
3. $s_i^{(\bar{j})} > 0$

Proof: Trivial for $i \geq i_0$; assume $i < i_0$. By 3.15. we know that

$$\bar{j} := \min \{j \mid i < j,\ j \notin \Sigma,\ i \notin D^{(j)},\ M^{(i,j)} = M^{(i)}\}$$

is well defined; we have

(1) $\quad M^{(i)} = M^{(i,\bar{j})},\ \bar{j} \in \Sigma,\ i \notin D^{(\bar{j})}.$

We are going to show that $s_i^{(\bar{j})} > 0$. Assume that, on the contrary, $s_i^{(\bar{j})} = 0$. Now, $s^{(\bar{j})}$ has generically the shape

$$s^{(\bar{j})} = (0,\ldots,0,\ c',\ k_{p+1},\ldots,k_{q-1},\ d',\ 0,\ldots,0)$$

and if $s_i^{(\bar{j})} = 0$, then $i < p$ and there is j_0, $p \leq j_0 \leq q < \bar{j}$ such that

(2) $\quad j_0 \in \Sigma^{(\bar{j})},\ i \in D^{(j_0,\bar{j})}$

and

(3) $\quad M^{(i,\bar{j})} = M^{(i,j_0,\bar{j})}$

As $j_o \in \Sigma^{(\hat{j})}$, we have $j_o \in \Sigma$. Similarly, it follows from $i \notin D^{(j_o,\hat{j})}$ that $i \notin D^{(j_o)}$ (compare 3.14.).

Moreover, we have (compare 3.12.)

$$(4) \qquad M^{(i,j_o,\hat{j})} \leq M^{(i,j_o)} \leq M^{(i)}$$

which together with (1) and (3) implies $M^{(i)} = M^{(i,j_o)}$. Thus the existence of j_o contradicts the minimality of $\hat{j}$, q.e.d.

<u>Corollary 6.3.</u> $M^{(i)} = \max \{M^{(i,j)} \mid i < j,\ j \in \Sigma,\ i \notin D^{(j)},\ s_i^{(j)} > 0\}$

$$= \max \{M^{(i,j)} \mid i < j,\ s^{(j)} \text{ is defined and } s_i^{(j)} > 0\}$$

Clearly, if $i < j$, $j \in \Sigma$, $i \in D^{(j)}$ and $s^{(j)}$ has the shape

$$s^{(j)} = (0,\dots,0,\ c',\ k_{p+1},\dots,k_{q-1},\ d',\ 0,\dots 0)$$

then

$$M^{(i)} = M^{(i,j)} = \begin{cases} (k_o,\dots,k_{p-1};\ g_o,\dots,g_{p-1}) \\ (k_o,\dots,k_{p-1},\ k_p - c';\ g_o,\dots,g_p) \end{cases}$$

according to whether $i = p$ or $i > p$.

This suggests the following algorithm. (Assume (M,λ) to be reduced, so that characters are marked correctly.)

0. Define functions I_o, C_o, MI, $S\Lambda$ on $\mathfrak{M} \times \mathbb{N}$ by

$$I_o(M,\lambda) = i_o, \quad C_o(M,\lambda) = c$$

$$S\Lambda(M,\lambda) = s^{\lambda}$$

$$MI(M,\lambda) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ M_{i_o-1} \\ M_{i_o}^c \\ \vdots \\ M_{i_o}^c \end{pmatrix}$$

1. Given a fixed (M,λ), compute i_0, c, s^λ, $M^{(i)}$ $(i \geq i_0)$ by using the functions of the first step. Also, compute $m^{(i)}$ $(i \geq i_0)$.

2. For $i \geq i_0$, let

$$CH(i) = \begin{cases} 1 & m^{(i)} \geq g_i \\ 2 & m^{(i)} < g_i \end{cases}$$

and

$$s^{(i)} = \begin{cases} c^{(i)} & CH(i) = 2 \\ S\Lambda(M^{(i)},g_i) & CH(i) = 1 \end{cases}$$

3. Let $i = i_0 - 1$ and define

$$INTV(i) = \{j > i \mid s_i^{(j)} > 0\} ;$$

If $INTV(i) = \emptyset$, then $CH(i') = 0$ for $0 \leq i' \leq i$. Proceed with STEP 6.

If $INTV(i) \neq \emptyset$ proceed with STEP 4.

4. For $j \in INTV(i)$, put $M^{(i,j)} := MI(M^{(j)},g_j)\ [i]$ (i'th row of the matrix defined via the function of STEP 1). Let

$$M^{(i)} = \max_{j \in INTV(i)} M^{(i,j)}$$

5. Repeat steps 3 and 4 for the present i, thus defining $CH(i)$ and $s^{(i)}$.

Replace i by $i - 1$ and return to 3.

6. Characters, satellite measures, substitutes and pseudosubstitutes are now defined for all i. It remains to compute the minimal representation.

To this end, put $\bar{g}_0 = 0$ and $\bar{g}_i = 0$ as long as CH(i) = 0. If

$$\bar{\imath} = \min \{i \mid CH(i) > 0\}$$

then let $\bar{g}_{\bar{\imath}} = 1$.

7. For any $i > \bar{\imath}$ assume that the vector $\bar{g}$ is defined for all coordinates $i' < i$. Put

$$\bar{g}_i = \begin{cases} \langle s^{(i)}, \bar{g} \rangle & CH(i) = 1 \\ \langle s^{(i)}, \bar{g} \rangle + 1 & CH(i) = 2 \end{cases}$$

and proceed until $i = r$.

8. Reduce $\bar{g}$, if necessary, by grouping fellowships of equal weight together.

Remark 6.4. A language like APL which handles vectors and matrices easily is capable of dealing with the function MI directly. However, is should be noted that $M^{(i)}$ is essentially described by the four quantities p, q, c, d if e.g.

$$c^{(i)} = (0,\dots,d,\ k_{p+1},\dots,k_{q-1},\ d,\ 0,\dots.0)$$

and thus, $M\Lambda$ can be viewed as a function which labels every (M,λ) accordingly.

Example:

Let $M = (2,5,9,28,56,252;\ 2,3,2,2,4,2)$.
The "matrix of homogeneity" (see [9]) is

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & \infty & 0 & 0 & 0 \\ 1 & 2 & \infty & 1 & 0 & 0 \\ 1 & 2 & \infty & 1 & 1 & 0 \\ 1 & 2 & \infty & 1 & 1 & 1 \end{pmatrix}$$

Therefore $\lambda = 672 = 3 \quad 56 + 2 \quad 252$ is a homogeneity level for M (as $c_5^6 \leq 3$) and (M,λ) represents a homogeneous $v = v_\lambda^M$. There are no dummies in this game, the character vector is

$$(2,2,1,1,1,2) \; .$$

The satellite measures $M^{(i)}$ are given by the rows of the matrix

$$MI = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 5 & 9 & 0 & 0 & 0 & 2 & 3 & 2 & 0 & 0 & 0 \\ 2 & 5 & 9 & 28 & 0 & 0 & 2 & 3 & 2 & 2 & 0 & 0 \\ 2 & 5 & 9 & 28 & 56 & 0 & 2 & 3 & 2 & 2 & 1 & 0 \end{pmatrix}$$

and the substitutes and pseudo-substitutes are the rows of the matrix

$$SI = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 2 & 3 & 2 & 2 & 1 & 0 \end{pmatrix}$$

Thus, the minimal representation is

$$\bar{M} = (1,3,5,16,32,86;\ 2,3,2,2,4,2)$$

$$\bar{\lambda} = 268$$

The game v is described by the min-win profiles which are listed as rows of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 1 & 2 & 2 \\ 2 & 3 & 1 & 1 & 2 & 2 \end{pmatrix} .$$

REFERENCES

[1] ISBELL, J.R.: A class of majority games. Quarterly Journal of Math.Ser. 2, 7 (1956), 183-7

[2] ISBELL, J.R.: A class of simple games. Duke Math. Journal 25 (1958), 423-39

[3] ISBELL, J.R.: On the Enumeration of Majority Games. Math. Tables Aids Comput. 13 (1959), 21-28

[4] von NEUMANN, J. and MORGENSTERN, O.: Theory of Games and Economic Behavior, Princeton Univ.Press, NJ 1944

[5] OSTMANN, A.: On the minimal representation of homogeneous games. Working Paper 124, Inst. of Math.Ec., University of Bielefeld (1983)

[6] PELEG, B.: On the kernel of constant-sum simple games with homogeneous weights, Ill.J.Math. 10 (1966), 39-48

[7] PELEG, B.: On Weights of Constant Sum Majority Games, SIAM J. of Appl.Math. 16 (1968), 527 ff.

[8] ROSENMOLLER, J.: The Theory of Games and Markets. North Holland Publishing Company, Amsterdam. (1981)

[9] ROSENMOLLER, J.: Weighted majority games and the matrix of homogeneity. Working Paper 128, Inst. of Math.Ec., University of Bielefeld (1984). To appear in ZOR