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Repeated Bertrand–Edgeworth–Competition with Increasing Marginal Costs

by

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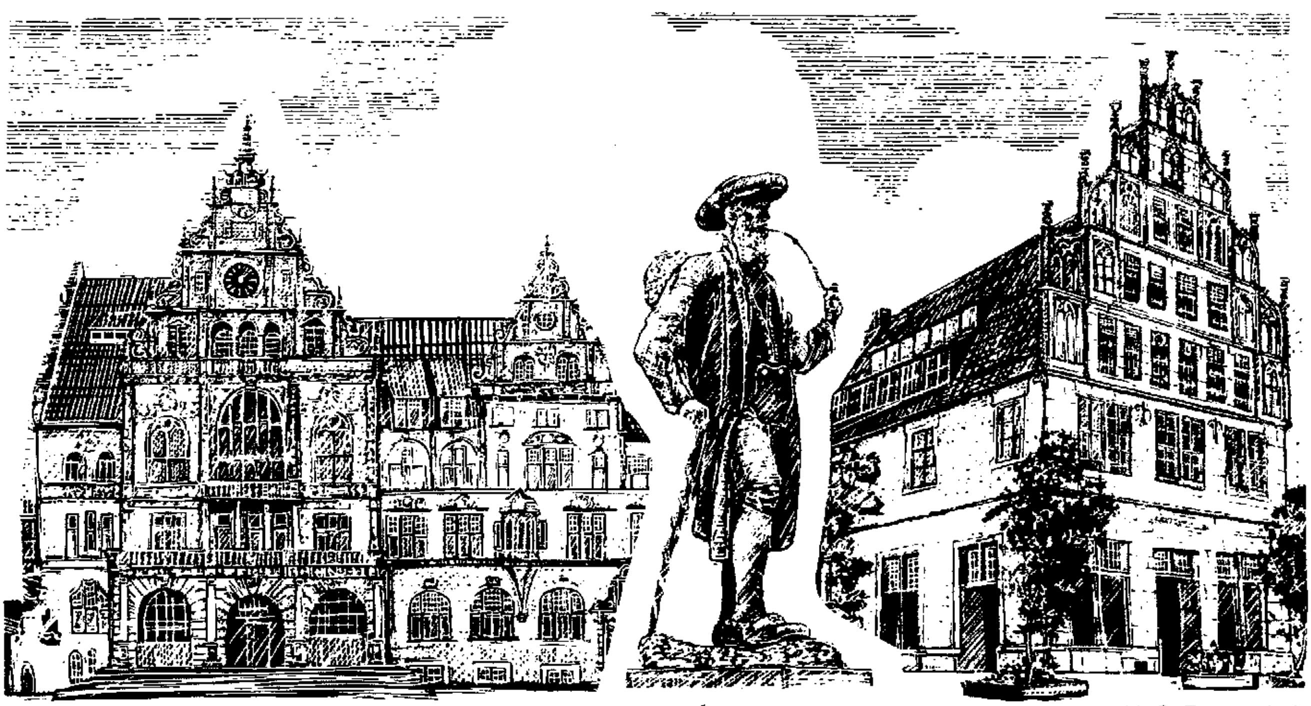

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Abstract

We consider a repeated price setting game with firms facing increasing marginal costs and positive fixed costs. Besides setting prices, firms may decide to be not active, which is different from selling nothing in this model. Since it is well known that there is no Nash–equilibrium in pure strategies in the stage game, we look for pure strategy equilibria in the repeated game and give a full characterization of all stationary symmetric equilibrium outcomes, sustained by optimal penal codes (in pure strategies).

Contents

1 Introduction

Bertrand–Edgeworth Competition has attracted much attention during the last decades, this is above all due to the nonexistence of equilibria in pure strategies, apart from special examples. Much work has been done on proving existence of and characterizing equilibria in mixed strategies. For this issue see above all the works of LEVITHAN–SHUBIK [12], BECKMAN [5], ALLEN–HELLWIG [4,3], DASGUPTA–MASKIN [7,8] and MASKIN [13] and DIXON [9].

In this paper, we investigate the repeated price game with firms facing a very commonly used cost function. We assume the firms to have positive fixed costs and increasing marginal costs. We apply ABREU's pathbreaking developments on repeated games with discounting, that is, we look for equilibria in pure strategies, which exhibit optimal penal codes in pure strategies, also, if a firm happened to deviate from the original path. LAMBSON [11] developed optimal penal codes for price setting supergames with firms facing constant marginal costs and capacity constraints, but no fixed costs, for a broad class of rationing rules. *We* simplify on the demand side in this paper, assuming identical consumers with unit demand. By having fixed costs a firm may earn negative profits if it faces no demand. Hence we let the firms not only to choose among prices but also allow them to be not active, which always guarantees them a profit of zero. For each oligopoly size and for an arbitrary discount factor we give a complete characterization of all stationary symmertic equilibrium outcomes in pure strategies, that is, also during the punishment phase firms choose prices rather than distributions on prices. Since we assume that firms produce to order, the value a firm can be held down to depends heavily on the oligopoly size. Let m be the efficient number of firms, that is, that number a social planner would choose in order to minimize average costs. We will see that for $n > m$ a firm can always be held down to zero, whereas for $n \leq m$ it can guarantee itself a positive profit which is the higher, the smaller is the number of firms in the oligopoly. We will see further that the equilibrium conditions imply upper and lower bounds for stationary equilibrium prices. The lower bound is strictly greater than the avarage cost, that is, if an equilibrium exists, firms earn positive profits. The upper bound, on the other hand, may be strictly smaller than the monopoly price if the number of firms is high. This stands in contrast to repeated Bertrand games with constant marginal costs with and without capacity constraints.

The paper is organized as follows. In section 2 we set up the model with a brief consideration of the one shot game. Section 3 deals with the repeated game. In section 4 we apply our results to quadratic cost functions. In this case we can calculate the upper and lower bounds for stationary equilibrium prices. In section 5 we briefly discuss asymmetric stationary and nonstationary equilibrium outcomes. Most of the proofs are given in the appendix. We close with some concluding remarks in section 6.

2 The Static Game

2.1 The Model

We consider a market for a homogeneous commodity supplied by $n \geq 1$ identical firms and demanded by a continuum of identical consumers represented by the closed interval $[0,1]$. The *technology* of a typical firm is given by the (total) cost function

$$c(q) = F + v(q) \tag{2.1}$$

with $v(0) = 0, v' > 0, v'' > 0$, where $q \geq 0$ denotes the quantity produced, and $F > 0$ is the fixed cost, $v > 0$ is the variable cost function, which exhibits increasing marginal cost.

The *preferences* of a typical consumer are given by her demand function

$$d(p) = \begin{cases} 1 & \text{for } p \leq L \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

where $p \geq 0$ denotes the price to be paid for the commodity, and $L > 0$ the consumer's *reservation price.*

The *market game* is played as follows: each firm i announces a price p_i at which it is willing to sell a certain quantity. In this case its profit is given by

$$p_i q_i - c(q_i) = p_i q_i - v(q_i) - F \ , \tag{2.3}$$

where q_i is the quantity sold by firm i (which will be determined precisely below). Clearly, it will not be profitable to charge a price above the reservation price L, so that we need only consider prices in the closed interval $[0, L] \subset \mathbb{R}$. If a firm's price is too high it may happen that its demand is zero and it will produce nothing. In this case its profit equals $-F$. Hence, each firm has the option to be not active (n.a.), which yields it a profit of zero. A typical firm's strategy space can thus be written as

$$S_i := [\,0\,, L] \cup \{\text{n.a.}\} \ ,$$

where $s_i = p_i \in [\,0\,, L]$ means that firm i is active and charges price p_i, and $s_i = \text{n.a.}$ means that firm i is not active. The joint strategy space is written $S := \prod_{i=1}^{n} S_i$.

Notice that if there are no positive fixed costs, producing nothing would yield zero profits. So the difference between activity and non–activity vanishes in that case, and the extension of the strategy space by the element "n.a." would not be necessary.

The quantity sold by firm i is determined as follows. The firms announce a price and produce to order. That is, they produce as much as they can sell. At price p they are not willing to produce more than $v'^{-1}(p)$, since otherwise marginal cost exceeds the price. Thus $v'^{-1}(p)$ can be considered as the self imposed capacity constraint at price p. All the customers are perfectly informed about the prices charged by the various firms and try to buy from the cheapest firm(s). If there are several cheapest firms they split up equally among these. First all customers place their orders with the cheapest firm(s). These orders are fullfilled, until the cheapest firm(s)' capacity is exhausted. The remaining (unserved) customers now place their orders with the next cheapest firm(s) (again splitting up equally, if there are several). This procedure is repeated until either all the customers are served or all firms are exhausted. The *rationing scheme*[1] induced by this mechanism is formalised as follows: Assume (w.l.o.g. by symmetry) that the firms $i = 1, \ldots, N$ are active $(0 \leq N \leq n)$ charging prices $p_1, \ldots, p_N$, and the remaining firms $N+1, \ldots, n$ are not active. We write the strategy n-tupel with N active firms as

$$s = (s_1, \ldots, s_n) = (p_1, \ldots, p_N, \underbrace{\text{n.a.}, \ldots, \text{n.a.}}_{n-N}) =: \vec{p}^{\,N}$$

Then the residual demand faced by firm i is[2]

$$D_i(\vec{p}) = \frac{\max\{1 - \sum_{j: p_j < p_i} v'^{-1}(p_j),\, 0\}}{\#\{j \in \{1, \ldots, N\} \;:\; p_j = p_i\}}\,. \tag{2.4}$$

The quantity actually sold by firm i is

$$q_i\left(\vec{p}^{\,N}\right) = \min\{D_i\left(\vec{p}^{\,N}\right), v'^{-1}(p_i)\}. \tag{2.5}$$

For an action vector $s = (s_1, \ldots, s_n)$ the profit of firm i is given by

$$\pi_i(s) = \begin{cases} s_i \cdot q_i\left(\vec{p}^{\,N}\right) - v\left(q_i\left(\vec{p}^{\,N}\right)\right) - F & \text{for } s_i \in [0, L] \\ 0 & \text{for } s_i = \text{n.a.} \end{cases} \tag{2.6}$$

This defines an n-player game G with strategy spaces $S_i = [0, L] \cup \{\text{n.a.}\}$ and payoff functions π_i given by (2.6). If we wish to single out player i, we write $s = (s_i, s_{-i})$ where $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. The defintion of a Nash equilibrium is obvious:

[1]By the simple demand structure this rationing scheme coincides with efficient rationing as well as with proportional rationing.

[2]$\#A$ is the cardinality of the set A.

Definition 2.1 *$s^* = (s_1^*, \ldots, s_n^*)$ is called a Nash equilibrium of G if $\forall\, i = 1, \ldots, n$, $\forall\, s_i \in [\,0\,, L] \cup \{n.a.\}$:*

$$\pi_i(s^*) \geq \pi_i(s_i, s_{-i}^*) \ .$$

Let $AC(q) = \frac{v(q)+F}{q}$ be the average cost of producing q units of the commodity. Since $v'' > 0, AC$ has a unique minimum and we can define the "competitive price" by

$$p_c := \min_q AC(q) \ , \tag{2.7}$$

(i.e. minimum average cost). The competitive output is defined by

$$q_c := AC^{-1}(p_c).$$

Define $m := \frac{1}{q_c}$. For technical simplicity we make

Assumption 1 $\quad m \in \mathbb{Z}$.

Assumption 2

$$L \cdot \frac{1}{n} - v\left(\frac{1}{n}\right) - F \geq 0$$

This assumption claims that the number of firms is small enough such that the firms can all profitably operate in the market at some price, possibly the monopoly price, which is equal to the reservation price in our model. We do not require the number of firms to be equal to m, the efficient number of firms, a social planner would install in order to minimize total costs.

It is sometimes argued that m would be the natural number of firms if free entry is admitted. This reasoning is actually flawed (see also Section 2.2 on the Salop–Stiglitz–model). In [14] we convert Assumption 2, that is, we try to endogenize the number of active firms in a suitable supergame, whereas here we take the number of firms as exogenously given.

Note that Assumption 2 implies $L \geq p_c$.

For simplicity we further assume

Assumption 3 $\quad v'^{-1}(L) \leq 1$.

Assumption 3 claims that the market is large enough such that a single firm's self imposed capacity constaint at the monopoly price is not greater than the whole market.[3]

Unless explicitly stated otherwise, Assumptions 1 – 3 will be maintained throughout this paper.

[3]This is not a serious restriction but avoids tedious distinctions of several cases in the repeated game.

Proposition 2.1 *If $n \cdot v'^{-1}(L) \leq 1$, the oligopoly game G has a unique equilibrium* $\vec{p}^* = (L, \ldots, L)$.

Proof: Obvious. Since each firm produces at its capacity bound, charging a lower price than L yields a lower profit. Q.E.D.

Proposition 2.2 *If $n \cdot v'^{-1}(L) > 1$, the oligopoly game G' has no equilibrium (in pure strategies).*

Proof: The proof is simple and well known. There can be no symmetric equilibrium. For, suppose that all the firms face excess capacity, it will pay for any firm to undercut its competitors. If all the firms exhaust their self imposed capacity constraints, a possible equilibrium price must be smaller than L by presupposition. Hence it will pay to charge a higher price, say L.

At an asymmetric price configuration, there must be at least one firm which exhausts its capacity by Assumption 3. Such a firm can always increase its profit by raising its price slightly. Q.E.D.

2.2 The Bertrand–Edgeworth–Problem in a Model with Imperfect Information (the Salop–Stiglitz–Model)

It is not difficult to extend the arguments of Proposition 2.2 to the Salop-Stiglitz-model [16] (see also [6,17]). In their model the customers are not automatically informed about the price–seller–correspondence. All they know is the distribution of the prices. There are two types of customers each of them having differently high cost to purchase the price–seller–correspondence. SALOP and STIGLITZ and also WILDE [17] get four cases, dependent on the ratio of "high–cost" and "low–cost–customers": no equilibrium at all, a single price equilibrium at the monopoly price and two kinds of *two–price–equilibria*. One of the two price equilibria is of the form: β percent of the firms charge p_c and $(1-\beta)$ percent of the firms charge L. In this case, the "high–cost–customers' " cost c_h is greater than $(1-\beta)(L-p_c)$.

The other two–price–equilibrium has the form: β percent of the firms charge p_c and $(1-\beta)$ percent of the firms charge a price $p_h < L$. In the latter case, the "high–cost–customers' " cost c_h is equal to $(1-\beta)(L-p_c)$. They are now indifferent between purchasing the price–seller–correspondence or selecting randomly. It is assumed that they will do the latter. In equilibrium SALOP & STIGLITZ and also WILDE require that all the firms make zero profits (neglecting integer problems). Due to the U–shaped average cost curve, at the competitive price, each firm produces a unique amount of output, the competitive output. But this means that each firm has monopolistic power on the customers it serves. Thus, any of the firms which charge the competitive price has an incentive to deviate by charging a higher price, exploiting also the perfectly informed customers who will not be served by the remaining firms at price p_c. In the first type of *two-price-equilibrium*, the deviating firm can charge $L - \varepsilon$, in the second type, it can charge $p_h - \varepsilon$, still capturing all of the unserved informed customers plus an equal share of the uninformed ones (an equal share with respect to the remaining firms which are not sold out). So, the equilibria SALOP & STIGLITZ claim to get, are not really Nash–equilibria.

3 The Repeated Market Game

Since there is no pure strategy equilibrium in the one shot game apart from special examples, we are now interested in the question whether there is a stationary equilibrium in pure strategies in the repeated game with an infinite time horizon.

3.1 The Model

All the assumptions about firms and consumers hold in each period: each customer demands exactly one unit of the homogenous commodity in every period up to the reservation price L. The commodity is *not* durable. The customers are always perfectly informed about the prices in each period, they try to place their orders always with the cheapest firms and split up equally if there are several and they do not become biased towards certain firms.

We denote by $s(t)$ the action vector of period t. If $s^\infty = \{s(t)\}_{t=0}^{\infty} \in \Omega := S^\infty$ is an action path, the whole payoff of firm i over all infinitely many periods is given by

$$v_i(s^\infty) := \sum_{t=1}^{\infty} \delta^t \pi_i(s(t)) \ . \tag{3.1}$$

Some more notations are needed: if p^* is a price, we write $\overline{p^*} = \underbrace{(p^*, \ldots, p^*)}_{n}$ and $(p_i, \overline{p^*_{-i}}) = (\underbrace{p^*, \ldots, p^*}_{i-1}, p_i, \underbrace{p^*, \ldots, p^*}_{n-i})$.

The highest profit that can be earned at price p (that is, at which price equals marginal cost) is denoted by

$$\pi^*(p) = p \cdot v'^{-1}(p) - v(v'^{-1}(p)) - F \ .$$

Note that under Assumption 3

$$\sup_{p_i < p^*} \pi(p_i, \overline{p^*_{-i}}) = \pi^*(p) ,$$

this is the upper bound for profits a firm can earn by undercutting a symmetric price outcome $\overline{p^*}$.

The following lemmata and definitions are needed.

Lemma 3.1 *$\pi^*(p)$ is strictly increasing in p and $\frac{d\pi}{dp}(p) = v'^{-1}(p)$.*

Lemma 3.2 $\frac{d^2\pi}{(dp)^2}(p) = \frac{1}{v''(v'^{-1}(p))} > 0.$

Lemma 3.3 *Let $S'_i = [0, L]$ $\forall i \in I$ (there is no option to be inactive for a moment). If $L > v'\left(\frac{1}{n}\right)$ (which is satisfied under Assumption 3),*

i) there is a unique $p^0 \in S'_i$ such that

$$\pi_i\left(L, \overline{p^0_{-i}}\right) = \sup_{p_i < p^0} \pi_i\left(p_i, \overline{p^0_{-i}}\right) \tag{3.2}$$

ii)

$$\max_{p_i \in S'_i} \pi_i\left(p_i, \overline{p^0_{-i}}\right) \qquad \textit{exists}$$

iii)

$$v'^{-1}\left(\frac{1}{n}\right) < p^0 < v'^{-1}\left(\frac{1}{n-1}\right)$$

Proofs: See the appendix.

Lemma 3.3 claims that there is a price p^0 such that the best response against $\overline{p^0_{-i}}$ is to charge the monopoly price. Actually player i is (almost) indifferent between charging the monopoly price and undercutting p^0 slightly.

Definition 3.1 *Define the* punishment action p^{pun} *by the solution of (3.2) in p^0.*

Lemma 3.4 *For all $p_{-i} \in S'_{-i} = [0, L]^{n-1}$ we have*

$$\sup_{p_i \in S'_i} \pi_i\left(p_i, \overline{p^{pun}_{-i}}\right) \leq \sup_{p_i \in S'_i} \pi_i\left(p_i, p_{-i}\right)$$

Lemma 3.4 claims that a firm cannot be held down lower than $\sup_{p_i \in S'_i}(p_i, \overline{p^{pun}_{-i}})$ by any other $(n-1)$-vector of prices p_{-i}.

Definition 3.2 *A symmetric vector* $\overline{p} = (\underbrace{p, \ldots, p}_{n})$ *is called a* (symmetric) stationary equilibrium outcome (st.e.o.) *if each firm charges p in every period of the game, and there is a subgame perfect equilibrium strategy σ supporting this outcome.*

We can now give a characterization of the set of all stationary symmetric equilibrium outcomes for the nontrivial case $L > p_c$. Notice that in this case $\frac{1}{v'^{-1}(L)} < m$.

Theorem 3.1 *Assume* $L > p_c$.
a) *If* $n > m$, $\overline{p}$ *is a st.e.o. iff*

$$\pi^*(p) \leq \frac{1}{1-\delta}\pi(\overline{p}) \ . \tag{3.3}$$

b) *If* $\frac{1}{v'^{-1}(L)} < n \leq m$, $\overline{p}$ *is a st.e.o. iff*

$$\pi^*(p) + \frac{\delta}{1-\delta}\pi^*\left(p^{\text{pun}}\right) \leq \frac{1}{1-\delta}\pi(\overline{p}) \ . \tag{3.4}$$

c) *If* $n \leq \frac{1}{v'^{-1}(L)}$, *there is a unique st.e.o.* $\overline{p} = (L, \ldots, L)$.

We will sketch the proof below. A formal proof is given in the appendix.

The theorem gives us a full characterization for all stationary equilibrium prices which can be supported by a subgame perfect equilibrium in pure strategies for an arbitrary discount factor $0 < \delta < 1$. In case c), the trivial case, we have a kind of monopolistic oligopoly without tacit collusion. The monopolistic outcome is merely due to the self imposed capacity constraints, thus there is no difference to the one shot game (cf. Prop. 2.1).

For the other cases we would like to get an upper and a lower bound for symmetric stationary equilibrium prices (st.e.p.'s). By the non–linearity of the equations (3.3) and (3.4) in p, however, we cannot in general solve for p. Indeed, (3.3) and (3.4) each *do* yield an upper and a lower bound for the equilibrium prices, or there is no solution at all. To see this, observe that π^* is convex in p by Lemma 3.2, whereas the R.H.S.'s of (3.3) and (3.4) are linear in p. The second term on the L.H.S. of (3.4) is independent of p. Moreover, the derivative of π^* is not bounded. This implies that there must be an upper bound for the st.e.p.'s, if there is a price p satisfying (3.3) or (3.4), respectively. On the other hand $\pi^*(p_c) = 0$, and $\pi^*\left(p^{\text{pun}}\right) > 0$ for $n \leq m$ by Lemma 3.3 iii). But $\pi(\overline{p_c}) \leq 0$. This implies a lower bound for st.e.p.'s which is not less than p_c.

Unequality (3.3) states simply that the profit earned by undercutting once must be smaller than the sum of profits earned when never deviating from p. In particular, this means that after deviating a firm can be always held down to zero by a subgame perfect equilibrium strategy. But this is only so if n is greater than m. For in this case, $n-1$ firms have enough capacity to serve the whole market even at prices slightly below the minimum average cost p_c.

For $n \leq m$, this is different. Here a firm can only be held down to $\pi^*(p^{\text{pun}}) > 0$ rather than to zero. Hence, on the L.H.S. of (3.4), we have the profit earned by undercutting p, the equilibrium price, plus the discounted sum of profits by undercutting p^{pun} forever (more precisely we have suprema of those profits).

It follows from Theorem 3.1 that for all equilibrium prices p, $p \geq AC\left(\frac{1}{n}\right)$ must hold, otherwise a firm could leave the market, which guarantees it a profit of zero. Indeed, to show $p \geq AC\left(\frac{1}{n}\right)$ is part of the proof.

We will write for short:

$$\pi^L(p) := \pi_i^L(p) := \pi_i(L, \overline{p_{-i}})$$
$$= L \cdot \max\{0\ ,\ 1-(n-1)v'^{-1}(p)\} - v\left(\max\{0\ ,\ 1-(n-1)v'^{-1}(p)\}\right) - F$$

That is, $\pi^L(p)$ is the profit earned by exploiting the residual demand, if there is any, not being satisfied by the $n-1$ firms which charge p. Theorem 3.1 also implies that for any stationary equilibrium price p we have:

$$\pi^L(p) < \pi^*(p) \tag{3.5}$$

This implies that, if undercutting does not pay, charging a higher price does not pay, either. This is a consequence of the following Lemma:

Lemma 3.5 *Prices not greater than p^{pun} cannot be stationary equilibrium prices.*

Proof: See the appendix.

Corollary 3.1 *For any st.e.p. we have*

$$\pi^L(p) < \pi^*(p)\ .$$

Proof: If p is a st.e.p., it follows from Lemma 3.5 that $p > p^{\text{pun}}$. By definition of p^{pun} we have $\pi^L(p^{\text{pun}}) = \pi^*(p^{\text{pun}})$. By its definition π^L is not increasing in p, whereas π^* is increasing in p by Lemma 3.1. Hence, $\pi^L(p) < \pi^L(p^{\text{pun}}) = \pi^*(p^{\text{pun}}) < \pi^*(p)$. Q.E.D.

The following result guarantees nonnegative profits for prices satisfying (3.3).

Lemma 3.6 *Let $n > m$. If p satisfies (3.3), then $p > AC\left(\frac{1}{n}\right)$.*

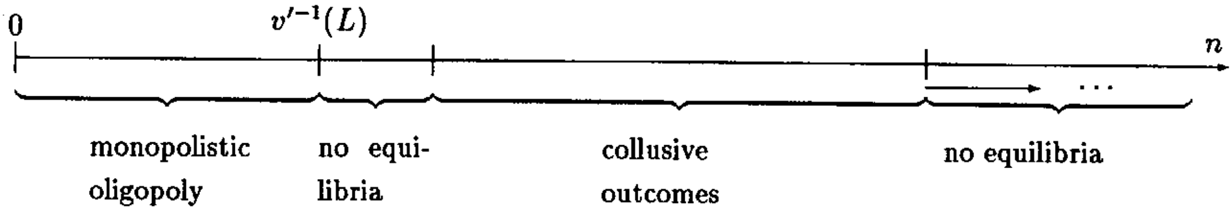

Figure 1: *Existence of equilibria depending on oligopoly size.*

Proof: Assume $p \leq AC\left(\frac{1}{n}\right)$. This is the same as $\pi(\overline{p}) \leq 0$. **Case a)** $p > p_c$, then $\pi^*(p) > 0 \geq \pi(\overline{p}) \geq \frac{1}{1-\delta}\pi(\overline{p})$, contradicting (3.3). **Case b)** $p = p_c$, $\Rightarrow \pi^*(p) = 0$. But since $n > m$, $\pi(\overline{p_c}) < 0$, contradicting (3.3). **Case c)** $p < p_c$, $\Rightarrow \pi^*(p) \geq \underbrace{\pi(\overline{p})}_{<0} > \frac{1}{1-\delta}\pi(\overline{p})$, a contradiction. Q.E.D.

Lemma 3.7 *Let $n \leq m$. If p satisfies (3.4), then $p > AC\left(\frac{1}{n}\right)$.*

Proof: By Lemma 3.5, $p > p^{pun}$. By Lemma 3.3 iii), $p^{pun} > v'\left(\frac{1}{n}\right)$. But for $n \leq m$, $v'\left(\frac{1}{n}\right) \geq AC\left(\frac{1}{n}\right)$. Hence $p > AC\left(\frac{1}{n}\right)$. Q.E.D.

As we saw, the equilibrium conditions (3.3) and (3.4) give upper and lower bounds for the st.e.p.'s. The motivation to establish the theory of supergames mainly was to explain cooperative outcomes, better called tacit collusion, as equilibrium outcomes of noncooperative games. Hence, we should above all be interested in the upper bounds for st.e.p.'s, that is, the outcomes of tacit collusion. These upper bounds are given by

$$\min\left\{L\,,\,\max\left\{p \mid \pi^*(p) \leq \frac{1}{1-\delta}\pi(\overline{p})\right\}\right\} \qquad \text{for } n > m$$

and

$$\min\left\{L\,,\,\max\left\{p \mid \pi^*(p) \leq \frac{1}{1-\delta}\left[\pi(\overline{p}) - \delta\pi^*(p^{pun})\right]\right\}\right\} \qquad \text{for } n \leq m\ .$$

But also the lower bounds for the st.e.p.'s are of some interest. If we take the lower bound as a function of n (neglecting that n is an integer, for a moment), its intersection with the upper bound for st.e.p.'s determines the region of the oligopoly size n for which proper collusive stationary equilibrium outcomes are possible. This is illustrated in Figure 1) and in the example below (see also figures 2 and 3).

Notice that there may be even a gap for oligopoly sizes having a st.e.p.. This gap may arize between the monopolistic oligopoly size and the collusive size, since a st.e.p. p has to be strictly greater than $v'\left(\frac{1}{n}\right)$.

To prove Theorem 3.1 we employ simple optimal penal codes developed by ABREU [1,2] and extended to price setting games with (constant) capacity constraints by LAMBSON

[11]. The idea is to charge the punishment price p^{pun} (defined in Def. 3.1) during a number of periods at the beginning of the penal code. The penal code "ends" by returning to the highest possible equilibrium price p^u forever. An intermediate price p^l, with $p^{\text{pun}} \leq p^l < p^u$, will be charged for exactly one period after charging p^{pun}, and before returning to p^u in order to exactly achieve the level a player can be held down to.

4 An Example (Quadratic Cost Functions)

Let

$$C(q) = \frac{\alpha}{2}q^2 - F$$

F.o.c.'s yield

$$v'^{-1}(p) = \frac{p}{\alpha}\,, \qquad \text{hence} \qquad p_c = \sqrt{2F\alpha}\,, \quad q_c = \sqrt{\frac{2F}{\alpha}}$$

$$\pi^*(p) = \frac{p^2}{2\alpha} - F\,, \quad \pi(\overline{p}) = p\frac{1}{n} - \frac{\alpha}{2n^2} - F$$

Consider first the case $n > m$. Substituting these expressions in (3.3) yields

$$p \leq \frac{\alpha}{(1-\delta)n}\left[1 + \sqrt{\delta\left[1 - (1-\delta)(q_c n)^2\right]}\right]$$

and

$$p \geq \frac{\alpha}{(1-\delta)n}\left[1 - \sqrt{\delta\left[1 - (1-\delta)(q_c n)^2\right]}\right]$$

The argument of the square root is not negative iff

$$n \leq \frac{1}{q_c\sqrt{1-\delta}} = \frac{1}{\sqrt{1-\delta}}m$$

The smallest upper bound for the oligopoly size which still has an equilibrium depends additionally on L. This bound is given by

$$n \leq \min\left\{\frac{\alpha L}{L^2(1-\delta) + \delta\alpha^2 q_c^2}\left[1 + \sqrt{\delta\left[1 - \left(\frac{\alpha q_c}{L}\right)^2\right]}\right]\,,\ \frac{1}{\sqrt{1-\delta}}m\right\}\,.$$

The first term in brackets is achieved by the intersection of $p(n) = \frac{\alpha}{(1-\delta)n}\left[1 - \sqrt{\delta\left[1-(1-\delta)(q_c n)^2\right]}\right]$ and $p = L$. Notice that for $\delta \to 1$, $\frac{1}{\sqrt{1-\delta}}m$ goes to infinity, whereas the first term in brackets converges to $\frac{L}{2F}\left[1 + \sqrt{1 - \frac{2F\alpha}{L^2}}\right]$. But $n \leq \frac{L}{2F}\left[1 + \sqrt{1 - \frac{2F\alpha}{L^2}}\right]$ is equivalent to $L \geq AC\left(\frac{1}{n}\right)$. Similarly, the lower bound for p converges to $AC\left(\frac{1}{n}\right)$. In other words, all "strictly" individually rational prices

($p > AC\left(\frac{1}{n}\right)$) of a symmetric outcome are stationary equilibrium prices, if δ is sufficiently close to one, which is consistent with the folk theorem (cf. FUDENBERG, MASKIN:86 [10]).

Next let $n \leq m$. For the punishment price we get:

$$p^{pun} = (1/(n-1)^2+1) \cdot \left[(\alpha - L)(n-1) + \sqrt{L^2(n-1)^2 + \alpha(2L-\alpha)}\right]$$

Observe that for large values of L

$$p^{pun} \approx \frac{1}{(n-1)^2+1} \cdot [(\alpha - L)(n-1) + L \cdot (n-1)] = \frac{\alpha(n-1)}{(n-1)^2+1} \approx \frac{\alpha}{n-1} = v'\left(\frac{1}{n-1}\right).$$

(3.4) yields

$$p \leq \frac{\alpha}{(1-\delta)n}\left[1 + \sqrt{\delta\left[1-(1-\delta)\left(\frac{n \cdot p^{pun}}{\alpha}\right)^2\right]}\right]$$

and

$$p \geq \frac{\alpha}{(1-\delta)n}\left[1 - \sqrt{\delta\left[1-(1-\delta)\left(\frac{n \cdot p^{pun}}{\alpha}\right)^2\right]}\right]$$

Since $p^{pun} > v'\left(\frac{1}{n}\right)$ we have $p^{pun} > \frac{\alpha}{n}$. Hence the lower bond for prices is greater than $\frac{\alpha}{(1-\delta)n}\left[1 - \sqrt{\delta\left[1-(1-\delta)\left(\frac{n \cdot (\alpha/n)}{\alpha}\right)^2\right]}\right] = \frac{\alpha}{n} = v'\left(\frac{1}{n}\right)$.

Figures 2 and 3 show several shapes of the sets of st.e.p.'s depending on the oligopoly size.

5 Asymmetric and Nonstationary Equilibrium Outcomes

There are also asymmetric stationary equilibria. But there is no stationary equilibrium with some firms being inactive as the following proposition shows.

Proposition 5.1 *If $s = (s_1, \ldots, s_n)$ is a st.e.o., then $\forall i \in I$ $s_i \neq$ n.a..*

Proof: Suppose $s = (s_1, \ldots, s_n)$ is a st.e.o. with $s_i =$ n.a. for some i.

Case a) $s_j = p_c$ or $s_j =$ n.a. $\forall j \neq i$. This means, that no firm makes positive profits. Since firms have to make nonnegative profits in equilibrium $\#\{i \mid s_i \neq \text{n.a.}\} = m$. But then it pays for any firm k with $s_k \neq$ n.a. to raise the price for at least one period and to leave the market afterwards. This yields positive profits, a contradiction.

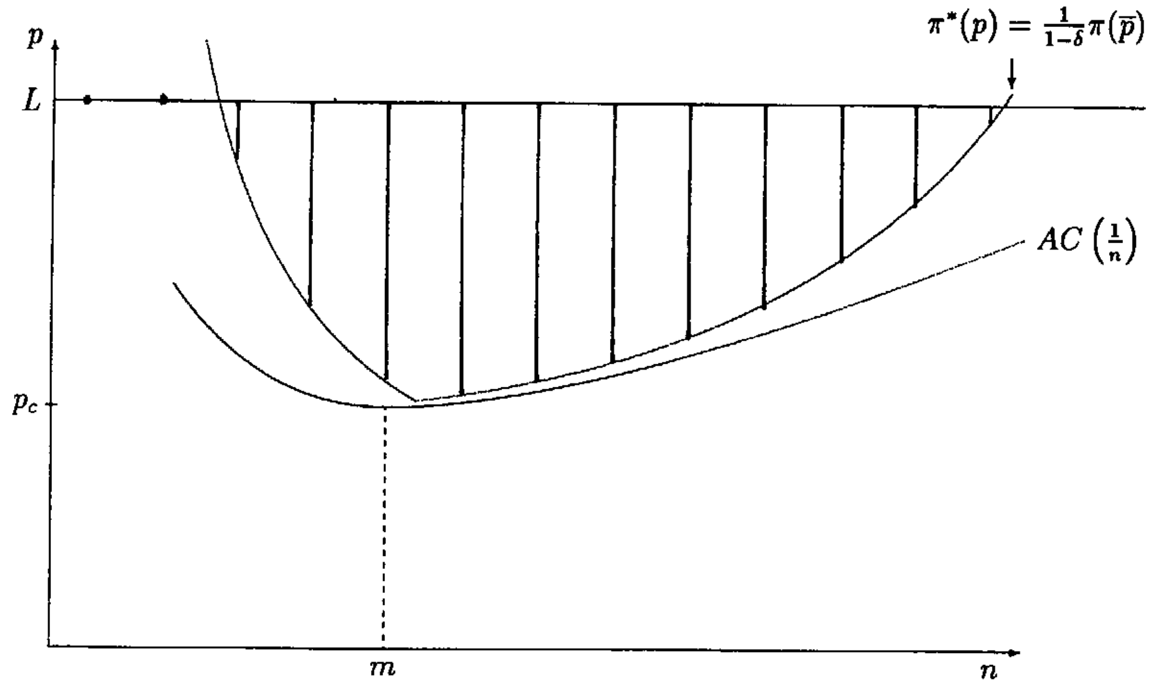

Figure 2: *The vertical lines represent the equilibrium region for a qudratic cost function. In this example, where relatively high marginal costs are taken, the upper bound for prices is given by the reservation price.*

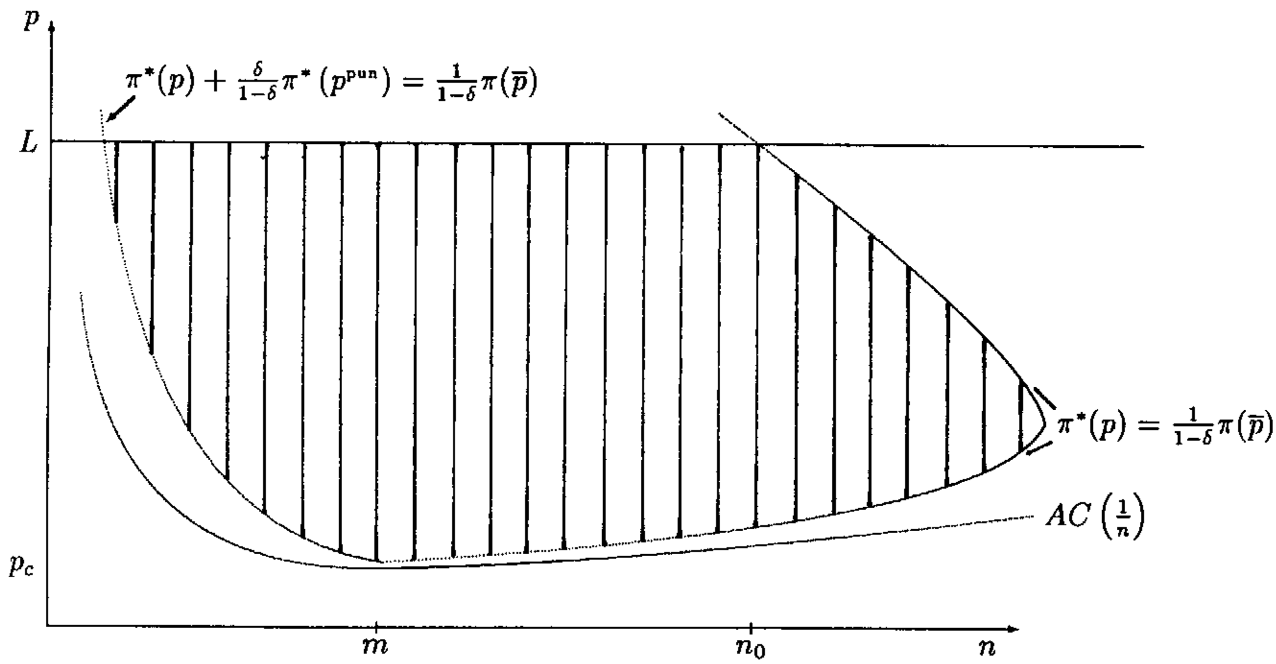

Figure 3: *The equilibrium region for lower marginal costs. In this case, the upper bound for prices decreases in n from n_0 on.*

Case b) There is exactly one firm j with $s_j = p_j > p_c$. If this firms makes positive profits, it will pay for firm i to undercut p_j and to leave the market thereafter (hit–and–run). If firm j makes zero profit, some other firms must be active by Assumption 3. For one of these it pays to raise the price up to $p_j - \varepsilon$ for one period. Keeping its own market share and getting at least some of j's demand it makes a positive profit for at least one period. Leaving the market after this period, firm j cannot get punished, a contradiction.

Case c) There are at least two firms j, k with $p_j, p_k > p_c$. If $p_j = p_k$, it pays for firm i to enter the market and to undercut $p_j = p_k$. If, say $p_j < p_k$, firm j must exhaust its capacity, if firm k does not make losses. But then it pays for firm i to hit–and–run by undercutting p_j, a contradiction again. Q.E.D.

The intuition behind this proposition is simply that collusive outcomes are jeopardized by hit–and–run of inactive forms. Competitive outcomes, on the other hand, cannot last since some firms may exploit their monopoly power on a market share which cannot be served by the remaining firms. Notice also that for stationary outcomes with some firms being inactive the most important qualifier of FUDENBERG and MASKIN's folk theorem [10] on repeated games with discounting is violated: this outcome is not in the interior of the set of individually rational outcomes.

There *are* asymmetric stationary equilibria with all the firms being active. If $\vec{p} = (p_1, \ldots, p_n)$, with (w.l.o.g.) $p_1 \leq \ldots \leq p_n$ and $p_i < p_{i+1}$ for some $i < n$, is a st.e.o., only the firms with the highest profits can have excess capacity. Such an equilibrium outcome is Pareto optimal (with respect to the firms) if and only if the highest price p_n equals L and there are no unserved consumers left. For, raising the price of firm i with $p_i < p_n$ lowers demand and profits for firm n.

Pareto optimal *stationary* equilibrium outcomes may not be Pareto optimal among *all* equilibrium outcomes if $n > m$. In this case it would be optimal to let exactly m firms to be active in each period. To avoid hit–and–run, however, each firm has to become active infinitely often.

We call (p, m) a *quasi–symmetric, quasi–stationary equilibrium outcome* if and only if in each period exactly m firms are active and charge the same price p.

Proposition 5.2 *Let $n = k \cdot m$, $k \in N$, $k \geq 2$, and assume there is a price p, satisfying (3.3). Then (L, m) is a quasi–symmetric, quasi–stationary equilibrium outcome only if*

$$\pi^*(L) \leq \frac{\delta^{k-1}}{1-\delta}\pi(L) \tag{5.6}$$

Proof: If in each period exactly m firms are active, at least one firm has to wait for at least $k-1$ period until it is its turn to become active. If a firm deviates, the same penal code as used in the proof of Theorem 3.1 will be employed. This means in particular, all the firms will become active after any deviation. Q.E.D.

If there is no integer k with $n = k \cdot m$, the factor $\frac{\delta^{k-1}}{1-\delta}$ has to be substituted by a more complicated one requiring some combinatorial considerations. Since one can doubt the empirical relevance of this kind of collusive paths, where firms become active periodically, we think it is not worth to elaborate this point further.

6 Concluding Remarks

We analysed a repeated price game with firms facing a U–shaped average cost curve, and gave a complete characterization of all stationary equilibrium outcomes in pure strategies. It is well known that in many games players can be held down to a lower level when mixed strategies are allowed. Few has been done to construct optimal penal codes in mixed strategies. Also ABREU [1,2] and LAMBSON [11] only employ pure strategies. A good reason for not considering optimal penal codes in mixed strategies is that one had to assume that not only outcomes but also the distribution functions of mixed strategies can be perfectly monitored by the other players, which seems to be not very plausible. Notice, however, that in our model for $n > m$, mixed strategies cannot help to punish a player more severely, since a firm can guarantee itself zero by leaving the maket forever, regardless of what the other players do. For the case $n \leq m$ an optimal penal code in mixed strategies could possibly yield a lower payoff than $\frac{1}{1-\delta}\pi^*(p^{pun})$ after deviation.

Since our attention was mainly directed on the issue of increasing marginal costs and since we use the same demand structure a companion paper [14] we assumed "box–demand" for simplicity in this paper. Price supergames with increasing marginal costs and a more general demand function and differnt rationing rules also deserve attention.

Although we charactrized the equilibrium region for different numbers of firms, we took this number as exogenously given in the current paper. In [14] we relax Assumption 2 and assume the contrary. That is, we assume that there are always some firms which threaten to enter the market, making the market contestable. Assuming this, unfortunately there will no stationary equilibrium in pure strategies in the purely repeated game. If we introduce an entry cost, however, we get stationary equilibria. This makes the game a time dependent supergame rather than a purly repeated game. In such a game penal codes turn out to be more complicated and dependent on the history, in the sense that for the same player different penal codes my be started depending on what kind of deviation has happened.

A further assumption that can be called into question is that firms produce to order after prices have been set. Relaxing this assumption one had to enlarge the strategy space by also introducing quantities as a further dimension. This has been done for instance by MASKIN [13] when he investigated mixed strategies in the one shot game. Doing this also in a repeated game raises similar conceptual problems as when allowing for mixed strategies. To get optimal penal codes firms had to produce something in excess on a punishment path in order to prevent a firm to deviate to a higher price. However, there is a temptation to deviate by producing not in excess since excess production will not be sold if no one deviates during the punishment. To assume perfect monitoring of stocks does not seem to be very plausible here. We have to relegate these problems to further research.

A Appendix

In the following we often write π instead of π_i if statements hold for all i.

A.1 Proofs of the Lemmata

Proof of Lemma 3.1: Since $\pi^*(p) = p \cdot v'^{-1}(p) - v\left(v'^{-1}(p)\right) - F$, we get

$$\begin{aligned} \frac{d\,\pi^*}{dp}(p) &= v'^{-1}(p) + p \cdot \left[v'^{-1}\right]'(p) - v'\left(v'^{-1}\right)(p) \cdot \left[v'^{-1}\right]'(p) \\ &= v'^{-1}(p) + p \cdot \left[v'^{-1}\right]'(p) - p \cdot \left[v'^{-1}\right]'(p) \\ &= v'^{-1}(p) \; > 0 \qquad \text{since } v' > 0\,. \qquad \text{Q.E.D.} \end{aligned}$$

Proof of Lemma 3.2:

$$\frac{d^2\pi^*}{(dp)^2}(p) = \left(v'^{-1}\right)'(p) = \frac{1}{v''\left(v'^{-1}(p)\right)} > 0 \qquad \text{since } v'^{-1} > 0, \quad v'' > 0. \qquad \text{Q.E.D.}$$

Proof of Lemma 3.3: Observe that $\sup_{p_i'>p} \pi_i(p_i', \overline{p_{-i}}) = \pi_i(L, \overline{p_{-i}}) =: \pi^L(p)$. On the other hand $\sup_{p_i'<p} \pi_i(p_i', \overline{p_{-i}}) = \pi^*(p)$. Now

$$\pi^L(p) = -F \qquad \forall p \geq v'\left(\frac{1}{n-1}\right) \;,$$

since in this case $n-1$ firms can serve the whole market and there is no residual demand. On the other hand:

$$\begin{aligned} \pi^L(p) &> \pi(\overline{p}) & \forall p \leq v'\left(\tfrac{1}{n}\right)\,, \\ \pi^*(p) &= \pi(\overline{p}) & \forall p \leq v'\left(\tfrac{1}{n}\right)\,, \\ \text{and} \qquad \pi^*(p) &\geq \pi(\overline{p}) > -F & \forall p \geq v'\left(\tfrac{1}{n}\right)\,. \end{aligned}$$

It follows that

$$\pi^L\left(v'\left(\frac{1}{n-1}\right)\right) = -F \quad < \quad \pi^*\left(v'\left(\frac{1}{n-1}\right)\right) \tag{A.7}$$

$$\text{and} \qquad \pi^L\left(v'\left(\frac{1}{n}\right)\right) \quad > \quad \pi^*\left(v'\left(\frac{1}{n}\right)\right) \ . \tag{A.8}$$

Since π^L and π^* are continuous and strictly decreasing and strictly increasing, respectively, on $[v'\left(\frac{1}{n}\right), v'\left(\frac{1}{n-1}\right)]$, there is a unique price p^0 with $\pi^L(p^0) = \pi^*(p^0)$. This proves i). Since $\max_{p_i'<p} \pi_i(p_i', \overline{p_{-i}})$ does not exist, $\arg\max_{p_i' \in S_i'} \pi_i(p_i', \overline{p_{-i}^0}) = L$, proving ii). iii) follows from (A.7) and (A.8). Q.E.D.

Proof of Lemma 3.4: Since π^L and π^* are continuous and strictly decreasing and strictly increasing, respectively, on $[v'\left(\frac{1}{n}\right), v'\left(\frac{1}{n-1}\right)]$, there is obviously no *symmetric* punishment vector $\overline{s_{-i}}$ which is worse for player i than $\overline{p_{-i}^{\text{pun}}}$, if player i plays the best response.

It remains to show that a player cannot be held down to a lower level by using asymmetric punishment tupels. W.l.o.g. assume player N is the deviator. Suppose $\tilde{p}_{-i} = (p_1, \ldots, p_{N-1})$ with $p_1 \le p_2 \le \ldots \le p_{N-1}$ and $p_i < p_{i+1}$ for some $i \in \{1, \ldots, N-2\}$ is a more severe punishment than $\overline{p_{-i}^{\text{pun}}}$. Let $d_i^{N-1}(p_1, \ldots, p_{N-1})$ be the demand of firm $i = 1, \ldots, N-1$, if only $N-1$ firms are in the market and charge prices $p_1, \ldots, p_{N-1}$. And let $q_i^{N-1}(p_1, \ldots, p_{N-1}) = \min\{v'^{-1}(p_i)\ , \ d_i^{N-1}(p_1, \ldots, p_{N-1})\}$. Let $k := \max\{j \mid q_j = v'^{-1}(p_j)\}$ be the greatest index of a firm that produces at full capacity. Further let $g := \max\{j \in \{1, \ldots, N-1\} \mid d_j^{N-1}(p_1, \ldots, p_{N-1}) > 0\}$, be the greatest index of a firm that has still positive demand.

Case a): $k = g$, $p_g \le p^{\text{pun}}$. Then $q_N^D(L, \tilde{p}_{-i}) \ge q_N^D(L, \overline{p_{-i}^{\text{pun}}})$, and hence, $\pi_i(L, \tilde{p}_{-i}) \ge \pi_i(L, \overline{p_{-i}^{\text{pun}}})$, a contradiction.

Case b): $k = g$, $p_g > p^{\text{pun}}$. Then $\pi^*(p_g) > \pi^*(p^{\text{pun}})$ and firm N can get almost $\pi^*(p_g)$ by untercutting firm g, a contradiction.

Case c): $k < g$, $p_g \le p^{\text{pun}}$. Then

$$1 = \sum_{j=1}^{g} q_j^{N-1}(p_1, \ldots, p_{N-1}) \le \sum_{j=1}^{N-1} q_j^{N-1}(\overline{p_{-i}^{\text{pun}}}) = (N-1) \cdot v'^{-1}(p^{\text{pun}}) < 1$$

by Lemma 3.3.iii), which is not possible.

Case d): $k < g$, $p_k \ge p^{\text{pun}}$. Undercutting p_k would yield $\pi^*(p_k) \ge \pi^*(p^{\text{pun}})$, a contradiction.

Case e): $k < g$, $p_k < p^{\text{pun}} < p_g$. In this case we get

$$\sum_{j:p_j=p_g} d_j^{N-1}(p_1, \ldots, p_{N-1}) < v'^{-1}(p^{\text{pun}}) \ ,$$

otherwise firm N could earn a higher profit than $\pi^*(p^{pun})$ by undercutting p_g slightly. But this leads to

$$1 = \sum_{i=1}^{k} v'^{-1}(p_i) + \sum_{j:p_j=p_g} d_j^{N-1}(p_1,\ldots,p_{N-1}) \; < \; \sum_{i=1}^{k} v'^{-1}(p^{pun}) + v'^{-1}(p^{pun})$$

$$\leq \; (N-1)v'^{-1}(p^{pun}) < 1 \; ,$$

a contradiction.

Proof of Lemma 3.5: For $n > m$ we have $p^{pun} < v'^{-1}\left(\frac{1}{n-1}\right) \leq v'^{-1}\left(\frac{1}{m}\right) = p_c$. Hence for all $p \leq p^{pun}$ we get $\pi(\overline{p}) \leq \pi(\overline{p^{pun}}) < 0$. Therefore, exit is more "profitable" than charging p forever.

For $n \leq m$ we show that

$$\pi^L(p) + \frac{\delta}{1-\delta}\pi^*(p^{pun}) > \frac{1}{1-\delta}\pi(\overline{p}) \; .$$

For $p \leq p^{pun}$ we know that $\pi^L(p) \geq \pi^L(p^{pun}) = \pi^*(p^{pun})$. Hence, for $p \leq p^{pun}$

$$\pi^L(p) + \frac{\delta}{1-\delta}\pi^*(p^{pun}) \geq \frac{1}{1-\delta}\pi^*(p^{pun}) > \frac{1}{1-\delta}\pi\left(\overline{p^{pun}}\right) \geq \frac{1}{1-\delta}\pi\left(\overline{p}\right) \; .$$

Q.E.D.

A.2 Proof of Theorem 3.1

We call a strategy generated by $n+1$ paths $q^0, q^1, \ldots, q^n$ with $q^i \in \Omega = S^\infty$ a *simple strategy profile* if q^0 is the initial path and q^i is the path that will be started if player i has deviated singly in the last period of some path q^j, $j \in \{0, 1, \ldots, n\}$.[4] If $\pi_i : S \to \mathbb{R}$ is the single period payoff function for player i and if $c = \{c(\tau)\}_{\tau=0}^{\infty} \in \Omega$ we write

$$\begin{aligned} v_i(c,t) &:= \sum_{\tau=0}^{\infty} \delta^\tau \pi_i(c(t+\tau)) \\ v_i(c) &:= v_i(c,0) \end{aligned}$$

ABREU demonstrated in [2] that a simple strategy profile is subgame perfect if and only if

$$\forall i \in I \quad \forall j \in \{0,1,\ldots,n\} \quad \forall \tau \geq 0 \quad \forall s_i \in S_i \setminus \{q_i^j(\tau)\} \quad :$$
$$\pi_i(s_i, q_{-i}^j(\tau)) + \delta v_i(q^i) \leq v_i(q^j, \tau) \tag{A.9}$$

In words, a single deviation from any path at any period must not pay.

[4]If the reader is not familiar with strategies defined by simple penal codes, she/he is relegated to Abreu:86,88 or Requate'90a,b.

Sufficiency

Before constructing an optimal penal code some preparations are needed. For each $n > m$ for which there is p' which satisfies (3.3) we define

$$p^u := p^u(n) = \min\{L\ ,\ \max\{p \mid p \text{ satisfies } (3.3)\}\} \qquad (A.10)$$

For each $n \leq m$ for which there is p'' which satisfies (3.4) we define

$$p^u := p^u(n) = \min\{L\ ,\ \max\{p \mid p \text{ satisfies } (3.4)\}\} \qquad (A.11)$$

Claim 1: If $n > m$, $\forall\, 0 < \delta < 1\ \exists\, T_0' \in \mathbb{N}$ such that

$$\sum_{t=0}^{T_0'-1} \delta^t \pi(\overline{p^{pun}}) + \sum_{t=T_0'}^{\infty} \delta^t \pi(\overline{p^u}) \leq 0 \qquad (A.12)$$

Notice that p^{pun} and p^u depend on n. If we want to emphasize this, we will write $p^{pun}(n)$ and $p^u(n)$.

Proof: For $n > m$ we have $n - 1 \geq m = \frac{1}{q_c}$. Hence $\frac{1}{n-1} \leq q_c \Rightarrow v'\left(\frac{1}{n-1}\right) \leq v'(q_c) = p_c$. By Lemma 3.3.iii) $p^{pun} < v'\left(\frac{1}{n-1}\right) \leq p_c$ leading to $\pi(\overline{p^{pun}}) < 0$. Since the second term of (A.12) becomes arbitrarily small for T_0' sufficiently large, the claim obviously holds.

Claim 2: If $n \leq m$, $\forall\, 0 < \delta < 1\ \exists\, T_0' \in \mathbb{N}$ such that

$$\sum_{t=0}^{T_0'-1} \delta^t \pi(\overline{p^{pun}}) + \sum_{t=T_0'}^{\infty} \delta^t \pi(\overline{p^u}) \leq \frac{1}{1-\delta}\pi^*(p^{pun}) \qquad (A.13)$$

Proof: Since $v'\left(\frac{1}{n}\right) < p^{pun}(n)$ we get $v'^{-1}(p^{pun}) > \frac{1}{n}$. Hence $\pi(\overline{p^{pun}}) < \pi^*(p^{pun})$. The rest is obvious.

Next define for each n

$$T_0 := \min\{T_0' \mid T_0' \text{ satisfies (A.12) if } n > m \text{ or (A.13) if } n \leq m\ \}. \qquad (A.14)$$

By Lemma 3.6 and 3.7, $p^u > AC\left(\frac{1}{n}\right)$ holds for all n for which p^u exists. Moreover $\pi\left(\overline{p^u}\right) > \pi^*(p^{pun})$ since p^u is an equilibrium price. Hence $T_0 \geq 1$. Further define the "last" punishment action (price) $p^l = p^l(n)$ by

$$\sum_{t=0}^{T_0-2} \delta^t \pi(\overline{p^{pun}}) + \delta^{T_0-1}\pi\left(\overline{p^l}\right) + \sum_{t=T_0}^{\infty} \delta^t \pi\left(\overline{p^u}\right) \leq \max\{0\ ,\ \frac{1}{1-\delta}\pi^*(p^{pun})\} \qquad (A.15)$$

Since $\pi(\overline{p})$ is continuous and increasing in p, there exists a price $p^l(n)$ with $p^{pun}(n) \leq p^l(n) < p^u(n)$ satifying (A.14).

We can now define the strategy, that is, the initial path and the penal code.

The *initial path* is defined by

$$q^0(\tau) = (\underbrace{p, \dots, p}_{n}) \qquad \forall \tau \geq 0 \tag{A.16}$$

For all $i \in I$, we define

$$q^i(\tau) = \begin{cases} \overline{p^{pun}} & \text{for} \quad 0 \leq \tau \leq T_0 - 1 \\ \overline{p^l} & \text{for} \quad \tau = T_0 \\ \overline{p^u} & \text{for} \quad \tau \geq T_0 + 1 \end{cases} \tag{A.17}$$

where p^u, T_0 and p^l are define by (A.10), (A.11), (A.14) and (A.15).

The values of the paths are $\forall i \in I \; \forall \tau$:

$$v_i(q^0, \tau) = \frac{1}{1-\delta}\pi(\overline{p}) > 0 \qquad \text{by Lemmata 3.6, 3.7} \quad ,$$

and additionally $\forall j \in \{1, \dots, n\}$:

$$v_i(q^j, \tau) \geq v_i(q^j) = \begin{cases} 0 & \text{if} \quad n > m \\ \frac{1}{1-\delta}\pi^*(p^{pun}) & \text{if} \quad n \leq m \end{cases}$$

The inequality is due to the fact that the actions (prices) of the punishment paths are nondecreasing.

To show that the strategy defined by (A.16) and (A.17) is a subgame perfect equilibrium, we have to check that (A.9) holds for all i, j, τ. To see that it does not pay to deviate from the initial path is easy. (A.9) takes the form of (3.3) and (3.4) for undercutting. That it does not pay to deviate to a higher price follows from Corollary 3.1.

It remains to show that (A.9) also holds for the punishment paths:

case a) $n > m$.

subcase i): $0 \leq \tau < T_0$: In this case we have $q^i(\tau) = \overline{p^{pun}}$ and (A.9) takes the form

$$\pi_i(s_i, \overline{p^{pun}_{-i}}) \leq v_i(q^i, \tau) \qquad \forall \, s_i \in S_i \tag{A.18}$$

Since for $n > m$ we have $p^{pun} < p_c$, we get $\pi_i^*(p^{pun}) < 0$. Hence the best response against $\overline{p^{pun}_{-i}}$ is s_i = n.a. which yields a profit of zero. On the other hand, the RHS of (A.18) is not smaller than zero by construction.

subcase ii): $\tau > T_0$: In this case, (A.9) holds if

$$\pi_i^*(p^u) \leq \frac{1}{1-\delta}\pi_i\left(\overline{p^u}\right) \tag{A.19}$$

which holds by definition of p^u and (3.3).

subcase iii): $\tau = T_0$: Then (A.9) holds if

$$\pi_i^*\left(p^l\right) \leq \pi_i\left(\overline{p^l}\right) + \frac{\delta}{1-\delta}\pi_i\left(\overline{p^u}\right) \tag{A.20}$$

To show this observe first that for $p = p^u = p^u(n)$ we get by (A.19):

$$\pi_i^*\left(p^u\right) - \pi_i\left(\overline{p^u}\right) \leq \frac{\delta}{1-\delta}\pi_i\left(\overline{p^u}\right)$$

Hence it suffices to show that

$$\pi_i^*\left(p^l\right) - \pi_i\left(\overline{p^l}\right) \leq \pi_i^*\left(p^u\right) - \pi_i\left(\overline{p^u}\right) \tag{A.21}$$

Now set $\phi(p) := \pi_i^*(p) - \pi_i(\overline{p})$. Then $\phi'(p) = v'^{-1}(p) - \frac{1}{n}$. The derivative is not negative iff $v'^{-1}(p) \geq \frac{1}{n} \Leftrightarrow p \geq v'\left(\frac{1}{n}\right)$. Now $p^l(n) \geq p^{pun}(n) > v'\left(\frac{1}{n}\right)$ for $n > m$. Hence ϕ is strictly increasing for $p \geq p^l$ and we get $\phi(p^l) < \phi(p^u)$, which establishes (A.21).

case b) $n \leq m$.

subcase i): $0 \leq \tau < T_0$: In this case we have $q^i(\tau) = \overline{p^{pun}}$. Since $\pi^*(p^{pun}) > 0$ for $n \leq m$, (A.9) holds if

$$\pi_i^*(p^{pun}) + \frac{\delta}{1-\delta}\pi_i^*(p^{pun}) \leq$$

$$\sum_{t=\tau}^{T_0-2} \delta^t \pi_i\left(\overline{p^{pun}}\right) + \delta^{T_0-1}\pi_i\left(\overline{p^l}\right) + \sum_{t=T_0}^{\infty} \delta^{T_0-1}\pi_i\left(\overline{p^u}\right) \tag{A.22}$$

The LHS of (A.22) is equal to and the RHS is not smaller than $\frac{1}{1-\delta}\pi^*(p^{pun})$ by construction.

subcase ii): $\tau > T_0$: In this case we have $q^i(\tau) = \overline{p^u}$. By Lemma 3.5, we know that $p^u > p^{pun}$. Therefore the best response against $(\overline{p^u_{-i}})$ is undercutting. Hence (A.9) takes the form

$$\pi_i^*(p^u) + \frac{\delta}{1-\delta}\pi_i^*(p^{pun}) \leq \frac{1}{1-\delta}\pi_i\left(\overline{p^u}\right) . \tag{A.23}$$

But this holds by (3.4) and the definition of p^u.

subcase iii): $\tau = T_0$: In this case we have $q^i(\tau) = \overline{p^l}$. But then (A.9) holds if

$$\pi_i^*(p^l(n)) + \frac{\delta}{1-\delta}\pi_i^*(p^{pun}) \leq \pi\left(\overline{p^l}\right) + \frac{\delta}{1-\delta}\pi_i\left(\overline{p^u}\right) \tag{A.24}$$

This holds by the same arguments as in subcase iii) of case a).

Necessity

For $n > m$ a firm can clearly not be held down to lower value than zero, since a firm can guarantee itself always zero by leaving the market.

For $n \leq m$, by the Lemmata 3.3 and 3.4, a firm cannot be held down to a lower value than $\pi^*(p^{pun})$ (which is greater than zero for $n \leq m$) in each period. Hence, if $\pi^*(p) + \frac{\delta}{1-\delta}\pi^*(p^{pun}) > \frac{1}{1-\delta}\pi(\overline{p})$ it always pays for any firm to undercut p once and to undercut p^{pun} afterwards forever, that is, to put up with reimposement of punishment in each period. Q.E.D.

Remark A.1 *Notice that the penal code employed in the proof of Theorem 3.1 is in general not unique. For $n > m$ any price not greater than p_c could be charged in the first period of punishment. What matters is that the value of the punishment paths equals zero and the profit of the punishment path is not greater than zero if the deviator plays the best response. For $n \leq m$, at least the punishment action in the first period is unique. Afterwards any continuation path with value equal to $[1/(1-\delta)]\pi^*(p^{pun}) - \pi\left(\overline{p^{pun}}\right)$ would do the job.*

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