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Discounted Two-Person Zero-Sum Games with Lack of Information on Both Sides

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0. Introduction

Subject of this paper is a model of a game with incomplete information that has already been considered by Mertens and Zamir (1971-72). They concentrated their interest on a finitely repeated game with an average payoff function, although they already claimed analogous results to hold for discounted payoffs. Here we shall only consider discounted payoffs which leads us to completely different arguments. By these methods one can circumvent the drawbacks appearing in the proof by Mertens and Zamir pointed out by Armbruster (1983). The so called "recursive structure" of the game resp. the recursive formula whose motivation was not very convincing (but whose validity was essential for the whole result) appears in this paper as an easy consequence of the main result.

1. The Model

An infinitely repeated discounted two-person zero-sum game with incomplete information on both sides is based on the following data:

- finite sets I and J
  (sets of actions for players 1 and 2)
- finite sets R and S
  (sets of types for player 1 and 2 resp. states of nature)
- for every $(r,s) \in R \times S$ and $I \times J$-matrix $A^{r,s}$
  (payoff matrices)
- two $I \times J$-matrices E and F with entries from alphabets $\mathcal{E}$ resp. $\mathcal{F}$
  (signalling matrices)
- a probability distribution $p \in \Delta(R{\times}S)$ ($\Delta(R{\times}S)$ denotes the set of probabilities on $R{\times}S$)
- a real number $\lambda \in (0,1)$
  (discount factor)

The game runs as follows:

- At stage 0 $(r,s) \in R{\times}S$ is selected according to p. Both players know p but player 1 is only informed about the choice of $r \in R$ and player 2 about the choice of $s \in S$. It is not assumed that p is the product of its marginal distributions. The knowledge of their own types enables the players to compute a posteriori probabilities on their opponent's types.

- At each stage $t \in \mathbb{N}$ both players choose independently parameters $i_t \in I$ resp. $j_t \in J$.

- Afterwards the letters $E(i_t,j_t)$ and $F(i_t,j_t)$ are announced to player 1 and player 2 respectively. They are not told the actual value $A^{r,s}(i_t,j_t)$ of the payoff matrix.

- Both players have perfect recall, i.e. they can use all information they get in the course of the game up to stage t for their decision at stage t+1.

- Player 1 receives from player 2 the amount $(1-\lambda) \sum_{t=1}^{\infty} \lambda^{t-1} A^{r,s}(i_t, j_t)$.

According to the description above player 1 can make his choice of parameters at stage t dependent on his type r, the sequence of letters $(e_1, \ldots, e_{t-1})$ from the alphabet $\mathcal{E}$ he has heard and of course his own actions $(i_1, \ldots, i_{t-1})$. In order to simplify notation let us assume that the sequence of letters already includes the information about his past actions, i.e. $E(i,j) \neq E(i',j')$ $\forall\, i \neq i' \in I;\ j,j' \in J$. Then a strategy of player 1 consists of an infinite sequence $\sigma = (\sigma_1, \sigma_2, \ldots)$ of mappings $S_t : R \times \mathcal{E}^{t-1} \to \Delta(I)$ resp. stochastic kernels $\sigma_t \mid R \times \mathcal{E}^{t-1} \Longrightarrow I$.

With an analogous assumption a strategy of player 2 is given by a sequence $\tau = (\tau_1, \tau_2, \ldots)$ of mappings $\tau_t : S \times \mathcal{F}^{t-1} \to \Delta(J)$ resp. stochastic kernels $\tau_t \mid S \times \mathcal{F}^{t-1} \Longrightarrow J$.

The sets of strategies are denoted by

$$\Sigma = \prod_{t=1}^{\infty} \Sigma_t \quad \text{and} \quad \Upsilon = \prod_{t=1}^{\infty} \Upsilon_t .$$

Let $H = I \times J$. In the following the identity $h = (i,j)$ resp. $h_t = (i_t, j_t)$ is always tacitly assumed. Using the strategies $\sigma = (\sigma_1, \sigma_2, \ldots)$ and $\tau = (\tau_1, \tau_2, \ldots)$ the players generate the probability distribution $\underline{P}^p_{(\sigma,\tau)}$ on $R \times S \times H^{\infty}$ (endowed with the topology of rectangular subsets):

$$\underline{P}^p_{(\sigma,\tau)}(r,s,h_1 \ldots h_T) =$$

$$p(r,s) \prod_{t=1}^{T} \sigma_t(r, E(h_1), \ldots E(h_{t-1}); i_t) \cdot \tau_t(s, F(h_1), \ldots F(h_{t-1}); j_t)$$

The payoff function (as a function on $\Sigma \times \Upsilon$) is naturally defined as the expectation of the discounted payoffs with respect to this distribution; explicitly:

$$\alpha_\lambda^p(\sigma,\tau) = \int_{R\times S\times H^\infty} (1-\lambda) \sum_{t=1}^{\infty} \lambda^{t-1} A^{r,s}(h_t)\, d\underline{P}^p_{(\sigma,\tau)}$$

$$= E^p_{(\sigma,\tau)} \left((1-\lambda) \sum_{t=1}^{\infty} \lambda^{t-1} \underline{A}_t\right)$$

($\underline{A}_t$ denotes a random variable on $R\times S\times H^\infty$, $\underline{A}_t(r,s,h_1,h_2,\ldots) = A^{r,s}(h_t)$)

Hence, we have a noncooperative two-person zero-sum game in normal form $\Gamma_\lambda(p) = (\Sigma,\Upsilon,\alpha_\lambda^p)$. The strategy sets are compact, the payoff function is continuous and affine in every component of σ and τ. Consequently it is quasiconcave (quasiconvex) in σ (τ). Thus the min-max theorem guarantees the existence of a value $v_\lambda(p)$.

2. I - Concavity / II - Convexity

Suppose player 1 performs a "type-dependent lottery" represented by a stochastic kernel q from his set of types R into another finite set which won't be specified at this point, i.e. he chooses an element μ according to the probability distribution $q(r;\cdot)$ where r is his true type. An outsider knowing the probability p on $R \times S$ and the outcome of the lottery can compute the following posterior probability:

$$\mathrm{Prob}(r,s\,|\,\mu) = \frac{p(r,s)\; q(r;\mu)}{\sum_{r'\in R} p(r')\; q(r';\mu)} = c_\mu(r) \cdot p(r,s).$$

The conditional probability on $R \times S$ is obtained from p by multiplying with a number independent of s.

On the other hand if p is a convex scombination $\sum_\mu \bar{q}(\mu)\; p^\mu$ of probabilities p^μ fulfilling the above property there exists a lottery yielding the p^μ's as posteriors, namely

$$q(r;\mu) = \frac{\bar{q}(\mu)\; p^\mu(r,s)}{p(r,s)} = \bar{q}(\mu) \cdot c_\mu(r) \qquad \text{(independent of s!)}$$

$$\mathrm{Prob}(r,s\,|\,\mu) = \frac{p(r,s)\; \dfrac{\bar{q}(\mu)\; p^\mu(rs)}{p(r,s)}}{\sum_{r',s'} p(r',s')\; \dfrac{\bar{q}(\mu)\; p^\mu(r',s')}{p(r',s')}} = p^\mu(r,s)$$

The set of possible conditional probabilities after a type dependent lottery of player 1 is denoted by

$$\Delta_I(p) = \{p' \in \Delta(R\times S) : \exists \in \mathbb{R}^R : p'(r,s) = c(r)\; p(r,s)\ \forall s \in S\}$$

Analogously:

$$\Delta_{II}(p) = \{p' \in \Delta(R\times S) : \exists\, d \in \mathbb{R}^S : p'(r,s) = d(s)\; p(r,s)\ \forall r \in R\}$$

Both sets are compact and convex. The extreme points are given by those functions c,d which take the value zero for all but one argument $r \in R$ resp. $s \in S$.

It is now convenient to think of player 1 using such a type dependent lottery in order to select a strategy $\sigma^{\mu} \in \Sigma$ which he is going to employ during the whole game.

From the point of view of the outsider (or player 2 neglecting his own type) this strategy effects that with probability $\tilde{q}(\mu)$ the game $\Gamma_{\lambda}(p^{\mu})$ is played where player 1 employs strategy σ^{μ}. Of course it is impossible to express this conduct in terms of behaviour strategies. Usually the restriction to behaviour strategies is justified by quoting AUMANN (1964) who establishes an equivalence between behaviour and mixed strategies in infinite extensive games with perfect recall. But as the coming arguments heavily depend on the feasibility of imitating the above strategy by behaviour strategies the construction will be given explicitly.

Definition 1:

A function $f : \Delta(R\times S) \to \mathbb{R}$ is called (strictly) concave w.r.t. I (I-concave) iff $f|_{\Delta_I(p)}$ is (strictly) concave for all $p \in \Delta(R\times S)$.

A function $f : \Delta(R\times S) \to \mathbb{R}$ is called (strictly) convex w.r.t. II (II-convex) iff $f|_{\Delta_{II}(p)}$ is (strictly) convex for all $p \in \Delta(R\times S)$.

$\text{cav}_I f$ is the smallest function g that satisfies

- $g(p) \geq f(p) \quad \forall p \in \Delta(R\times S)$
- g is I-concave

$\text{vex}_{II} f$ is the largest function g that satisfies

- $g(p) \leq f(p) \quad \forall p \in \Delta(R\times S)$
- g is II-convex.

Proposition 1:

Let $\tau \in \Upsilon$ be an arbitrary strategy of player 2. The function $f(p) = \max_{\sigma\in\Sigma} \alpha^{p}_{\lambda}(\sigma,\tau)$ is I-concave.

Proof:

Let $p \in \Delta(R\times S)$, p^1, $p^2 \in \Delta_I(p)$ such that $\bar{q}\, p^1 + (1-\bar{q})\, p^2 = p$ for some $\bar{q} \in (0,1)$.

We have to show that

$$f(p) \geq \bar{q}\, f(p^1) + (1-\bar{q})\, f(p^2).$$

This is done by giving strategy σ that achieves exactly the payoff $\bar{q}\, f(p^1) + (1-\bar{q})\, f(p^2)$.

Let σ^1, $\sigma^2 \in \Sigma$ be optimal strategies giving the payoff $f(p^1)$ resp. $f(p^2)$. The idea is that player 1 performs a type dependent lottery yielding the posterior p^1 with total probability $\bar{q}$, p^2 with probability $(1-\bar{q})$ and then makes use of the corresponding optimal strategies. (In order to prove I-concavity two outcomes of the lottery are enough. The construction of optimal strategies would need $|R|$ outcomes.)

In terms of behaviour strategies his conduct can be described as follows: Define $\sigma \in \Sigma$ by

$\sigma_T(r,e_1,\ldots,e_{T-1};i) =$

$$\frac{\bar{q}\,\frac{p^1(r,s)}{p(r,s)}\prod_{t=1}^{T}\sigma^1_t(r,e_1\ldots e_{t-1};i(e_t)) + (1-\bar{q})\,\frac{p^2(r,s)}{p(r,s)}\prod_{t=1}^{T}\sigma^2_t(r,e_1\ldots e_{t-1};i(e_t))}{\bar{q}\,\frac{p^1(r,s)}{p(r,s)}\prod_{t=1}^{T-1}\sigma^1_t(r,e_1\ldots e_{t-1};i(e_t)) + (1-\bar{q})\,\frac{p^2(r,s)}{p(r,s)}\prod_{t=1}^{T-1}\sigma^2_t(r,e_1\ldots e_{t-1};i(e_t))}$$

Observe first that the condition p^1, $p^2 \in \Delta_I(p)$ implies that the quotients $\frac{p^1(r,s)}{p(r,s)}$ and $\frac{p^2(r,s)}{p(r,s)}$ are independent of s, thus the strategy is well defined and admissible for player 1. Secondly remember that the letters announced to the players include their own choice of action. $i(e_t)$ denotes the t-th stage parameter of player 1.

Straightforward computation shows that

$$\underline{P}^{p}_{(\sigma,\tau)} = \bar{q}\ \underline{P}^{p^1}_{(\sigma^1,\tau)} + (1-\bar{q})\ \underline{P}^{p^2}_{(\sigma^2,\tau)} .$$

Consequently σ produces the payoff

$$\begin{aligned}\alpha^{p}_{\lambda}(\sigma,\tau) &= \bar{q}\ \alpha^{p^1}_{\lambda}(\sigma^1,\tau) + (1-\bar{q})\ \alpha^{p^2}_{\lambda}(\sigma^2,\tau) \\ &= \bar{q}\ f(p^1) + (1-\bar{q})\ f(p^2)\end{aligned}$$ □

Corollary 2:

$v_\lambda(p)$ is I-concave and II-convex.

Proof:

v_λ is a minimum of I-concave functions.

II-convexity follows for duality reasons. □

3. The NR-Game

In equilibrium it can be assumed that both players know their opponent's strategy. In this case they also know the distribution $\underline{P}^p_{(\sigma,\tau)}$ so that they are able to compute a posteriori probabilities on their opponent's types after each stage of the game. Let us now state explicitly, what player 2 can compute after stage 1 if he receives the signal $f \in \mathcal{F}$ his true type being $s \in S$.

$$\underline{P}^p_{(\sigma,\tau)}\,(r\,|\,s,f)$$

$$= \frac{p(r,s) \sum_{i,j:F(i,j)=f} \sigma_1(r;i)\ \tau_1(s;j)}{\sum_{r'} p(r',s) \sum_{i,j:F(i,j)=f} \sigma_1(r';i)\ \tau_1(s;j)}$$

According to the assumption that player 2's signal f includes his own choice of action j, there is only one $j(f) \in J$ that may lead to signal f.

$$= \frac{p(r,s) \sum_{i:F(i,j(f))=f} \sigma_1(r;i)}{\sum_{r'} p(r',s) \sum_{i:F(i,j(f))=f} \sigma_1(r';i)}$$

If player 1 doesn't want to give away any private information, i.e. if he doesn't want to enable player 2 to update his posteriors, this expression must be equal to $p(r|s)$ for all $r \in R$, which is equivalent to

$$\sum_{i:F(i,j(f))=f} \sigma_1(r;i) = \sum_{r'} p(r'|s) \sum_{i:F(i,j(f))=f} \sigma_1(r';i) \quad \forall\, r \in R$$

$$\Leftrightarrow \quad \sum_{i:F(i,j(f))=f} \sigma_1(r;i) = \sum_{i:F(i,j(f))=f} \sigma_1(r';i) \qquad \forall\, r,\ r' \in R$$

Definition 2:

A first stage strategy $\sigma \in \Sigma_1$ of player 1 is called non-revealing (NR), if

$$\sum_{i:F(i,j(f))=f} \sigma_1(r;i) = \sum_{i:F(i,j(f))=f} \sigma(r';i) \qquad \forall\, r, r' \in R, \forall\, f \in \mathcal{F}$$

A first stage strategy $\tau \in \Upsilon_1$ of player 2 is called non-revealing (NR), if

$$\sum_{j:E(e(i),j)=e} \tau(s;j) = \sum_{j:E(e(i),j)=e} \tau(s';j) \qquad \forall\, s, s' \in S, \forall\, e \in \mathcal{E}$$

The sets of NR first stage strategies are denoted by NR(I) resp. NR(II). They are always non-empty.

Define a payoff function on NR(I) $\times$ NR(II) by

$$\beta^p(\sigma,\tau) = \sum_{r,s} p(r,s) \sum_{i,j} \sigma(r;i)\ \tau(s;j)\ A^{r,s}(i,j)$$

which is exactly the first undiscounted stage payoff these strategies would induce in $\Gamma_\lambda(p)$. Let u(p) be the value of the game (NR(I), NR(II), β^p) (the NR-game).

4. The Limiting Behaviour of v_λ

From the first the existence of a limit function $\lim_{\lambda \to 1} v_\lambda$ is not clear. But we have

Lemma 3:
v_λ is Lipschitz-continuous with the same constant for all $\lambda \in (0,1)$.

Proof:

Let $p^1, p^2 \in \Delta(R\times S)$ and let (σ^1,τ^1), (σ^2,τ^2) be equilibrium strategies in $\Gamma_\lambda(p^1)$ resp. $\Gamma_\lambda(p^2)$. W.l.o.g. we can assume that $v_\lambda(p^1) > v_\lambda(p^2)$.

$$\begin{aligned}
& v_\lambda(p^1) - v_\lambda(p^2) \\
= \quad & \alpha_\lambda^p(\sigma^1,\tau^1) - \alpha_\lambda^p(\sigma^2,\tau^2) \\
\leq \quad & \alpha_\lambda^p(\sigma^1,\tau^2) - \alpha_\lambda^p(\sigma^1,\tau^2) \\
= \quad & \sum_{r,s} (p^1(r,s) - p^2(r,s)) \int_{H^\infty} (1-\lambda) \sum_{t=1}^{\infty} \lambda^{t-1} A^{r,s}(h_t)\, d\underline{P}^{r,s}_{(\sigma^1,\tau^2)} \\
\leq \quad & \|p^1 - p^2\| \cdot M \\
& \quad M = \max_{i,j,r,s} |A^{r,s}(i,j)|
\end{aligned}$$

□

The set of functions $\{v_\lambda : \lambda \in (0,1)\}$ is equicontinuous and uniformly bounded, so we can assume the existence of a uniformly convergent subsequence. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence with $\lambda_n \in (0,1)$ $\forall n$ $\lambda_{n'} \to 1$ and the corresponding sequence $(v_{\lambda_n})_{n\in\mathbb{N}}$ of value functions may converge to v.

We can also presume the existence of converging sequences $\sigma_1^n \to \sigma \in \Sigma_1$, $\tau_1^n \to \tau \in \Upsilon_1$ of optimal first stage strategies in $\Gamma_{\lambda_n}(p)$.

Proposition 4:

If v is strictly I-concave at p, then σ is NR.

If v is strictly II-convex at p, then τ is NR.

Proof:

It suffices to prove the first statement.

We need a representation of the payoff functions in which the first stage payoffs are treated separately:

$$\alpha_\lambda^p\,(\sigma,\tau)$$

$$= \quad E^p_{(\sigma,\tau)}\,((1-\lambda)\sum_{t=1}^{\infty}\lambda^{t-1}\,\underline{A}_t)$$

$$= \quad E^p_{(\sigma,\tau)}\,((1-\lambda)\underline{A}_1) + E^p_{(\sigma,\tau)}\,((1-\lambda)\sum_{t=2}^{\infty}\lambda^{t-1}\,\underline{A}_t)$$

$$= \quad (1-\lambda)\ \beta^p(\sigma_1,\tau_1) + E^p_{(\sigma,\tau)}\,((1-\lambda)\sum_{t=1}^{\infty}\lambda^{t-1}\,\underline{A}_t \mid h_1))$$

$$= \quad (1-\lambda)\ \beta^p(\sigma_1,\tau_1) + E^p_{(\sigma,\tau)}\,(E^{p(\cdot|h_1)}_{(\sigma(h_1),\tau(h_1))}\,((1-\lambda)\sum_{t=1}^{\infty}\lambda^{t}\,\underline{A}_t))$$

$$= \quad (1-\lambda)\ \beta^p(\sigma_1,\tau_1) + \lambda\sum_{h_1} P^p_{(\sigma,\tau)}(h_1)\ \alpha_\lambda^{p(\cdot|h_1)}\,(\sigma(h_1),\ \tau(h_1))$$

with

$$p(\cdot|h_1) = \underline{P}^p_{(\sigma,\tau)}\,(\cdot|h_1)$$

conditional probability on $R \times S$ given h_1

$$\sigma(h_1)_t = \sigma_{t+1}\,(\cdot,E(h_1),\ldots)$$

$$\tau(h_1)_t = \tau_{t+1}\,(\cdot,F(h_1),\ldots)$$

i.e. $\sigma(h_1)$ is obtained from σ by inserting $E(h_1)$ as first stage signal and then shifting the stages.

$$
\begin{aligned}
& v_\lambda(p) \\
= \quad & \min_{\tau\in\Upsilon} \max_{\sigma\in\Sigma} \alpha_\lambda^p(\sigma,\tau) \\
= \quad & \min_{\tau_1} \max_{\sigma_1} \min_{(\tau_2,\tau_3,\ldots)} \max_{(\sigma_2,\sigma_3,\ldots)} \alpha_\lambda^p(\sigma,\tau) \\
\leq \quad & \min_{\tau_1} \max_{\sigma_1} \min_{(\tau_2,\tau_3,\ldots)} \max_{(\sigma_2,\sigma_3,\ldots)} \alpha_\lambda^p(\sigma,\tau) \\
\leq \quad & \max_{\sigma} \min_{\tau} \alpha_\lambda^p(\sigma,\tau) \\
= \quad & v_\lambda(p)
\end{aligned}
$$

Due to the min-max theorem both inequalities can be replaced by equalities.

Define a first stage strategy τ_{NR} for player 2 by

$\tau_{NR}(s;j) = \frac{1}{|J|} \quad \forall\ s;j.$ $\tau_{NR} \in NR(II)$ regardless of the signalling matrix of player 1.

Let us now consider the coverging subsequence v_{λ_n} together with the optimal first stage strategies σ^n resp. τ^n:

$$
\begin{aligned}
& v_{\lambda_n}(p) \\
& = \min_{\tau_1} \max_{\sigma_1} \min_{(\tau_2,\tau_3,\ldots)} \max_{(\sigma_2,\sigma_3,\ldots)} \alpha_\lambda^p(\sigma,\tau) \\
& = \min_{\tau_1} \max_{\sigma_1} ((1-\lambda_n)\ \beta^p(\sigma_1,\tau_1) + \\
& \qquad \min_{(\tau_2,\tau_3,\ldots)} \max_{(\sigma_2,\sigma_3,\ldots)} \lambda_n \sum_{h_1} \underline{P}^p_{(\sigma_1,\tau_1)}(h_1)\ \alpha_{\lambda_n}^{p(\cdot|h_1)}(\sigma(h_1),\tau(h_1))
\end{aligned}
$$

$$(1) \quad \leq \min_{(\tau_2,\tau_3,\ldots)} \max_{(\sigma_2,\sigma_3,\ldots)} \lambda_n \sum_{h_1} \underline{P}^p_{(\sigma_1,\tau_{NR})}(h_1)\, \alpha_{\lambda_n}^{p(\cdot|h_1)}(\sigma(h_1),\tau(h_1)) + (1-\lambda_n)\, M$$

$$(2) \quad \leq \min_{(\tau_2,\tau_3,\ldots)} \lambda_n \sum_{h_1} \underline{P}^p_{(\sigma^n,\tau_{NR})}(h_1) \max_{\sigma\in\Sigma} \alpha_{\lambda_n}^{p(\cdot|h_1)}(\sigma,\tau(h_1)) + (1-\lambda_n)\, M$$

$$(3) \quad \leq \min_{(\tau_2,\tau_3,\ldots)} \lambda_n \sum_{f} \underline{P}^p_{(\sigma^n,\tau_{NR})}(f) \max_{\sigma\in\Sigma} \alpha_{\lambda_n}^{p(\cdot|f)}(\sigma,\tau(h_1)) + (1-\lambda_n)\, M$$

$$(4) \quad = \lambda_n \sum_{f} \underline{P}^p_{(\sigma^n,\tau_{NR})}(f) \min_{\tau} \max_{\sigma} \alpha_{\lambda_n}^{p(\cdot|f)}(\sigma,\tau) + (1-\lambda_n)\, M$$

$$= \lambda_n \sum_{f} \underline{P}^p_{(\sigma^n,\tau_{NR})}(f)\, v_{\lambda_n}(p(\cdot|f)) + (1-\lambda_n)\, M$$

Inequality (2) is valid because in the preceding term player 1's strategy from stage 2 on can only depend on the value of his signalling matrix $E(h_1)$ while the second expression implies that he can refer directly to the first stage history h_1, thus enlarging the set of strategies he is maximizing above.

Inequality (3) follows from proposition 1 and the fact that

$\underline{P}^p_{(\sigma^n,\tau_{NR})}(\cdot|f)$ is a convex combination of the probabilities

$\{\underline{P}^p_{(\sigma^n,\tau_{NR})}(\cdot|h) : F(h) = f\}$ and

$\underline{P}^p_{(\sigma^n,\tau_{NR})}(\cdot|h) \in \Delta_I\, (\underline{P}^p_{(\sigma^n,\tau_{NR})}(\cdot|f)) \quad \forall\, h : F(h) = f.$

The explanation of equality (4) is similar to the argument given for inequality (2). But in this case the strategic potentiality of player 2 remains the same.

Letting λ_n go to 1 we have

$$v(p) \leq \sum_f \underline{P}^p_{(\sigma,\tau_{NR})}(f)\ v(p(\cdot(f))$$

τ_{NR} is non-revealing, that implies $p(\cdot|f) \in \Delta_I(p)\ \forall f \in \mathcal{F}$.
Since v is assumed to be strictly I-concave at p no posteriors different from p can occur with positive probability. Player 2 chooses each of his actions with positive probability, consequently player 1's strategy has to be NR. □

Lemma 5:
If σ is NR, then $v(p) \leq u(p)$.
If τ is NR, then $v(p) \geq u(p)$.

Proof:
Again we prove the first statement. Let $\tau_{NR} \in NR(II)$ be an optimal strategy for player 2 in the NR-game.

$$v_{\lambda_n}(p)$$

$$\leq \quad (1-\lambda_n)\ \beta^p(\sigma^n,\tau_{NR}) + \lambda_n \sum_f \underline{P}^p_{(\sigma^n,\tau_{NR})}(f)\ v_{\lambda_n}(p(\cdot(f))$$

$$\leq \quad (1-\lambda_n)\ \beta^p(\sigma^n,\tau_{NR}) + \lambda_n\ v_{\lambda_n}(p)$$

(because $p(\cdot|f) \in \Sigma_I(p)$ and I -concavity of v_{λ_n})

$$\Longrightarrow \quad v_{\lambda_n}(p) \leq \beta^p(\sigma^n,\tau_{NR})$$

$$\Longrightarrow \quad v(p) \leq \beta^p(\sigma,\tau_{NR}) \leq u(p)$$ □

Proposition 4 and Lemma 5 together yield

Lemma 6:
v strictly I-concave at p $\Longrightarrow v(p) \leq u(p)$
v strictly II-convex at p $\Longrightarrow v(p) \geq u(p)$.

Lemma 7:

$$v(p) = \underset{I}{\text{cav}}\ \min\{u(p), v(p)\}$$

$$v(p) = \underset{II}{\text{vex}}\ \max\{u(p), v(p)\}$$

Proof:

We prove the first statement.

Obviously we have

$$v(p) \geq \underset{I}{\text{cav}} \min\{u(p), v(p)\}$$

Let us now assume that for some $p \in \Delta(R\times S)$ the inequality is strict. Let the maximal difference be equal to ϵ.

$$\Delta_\epsilon = \{p \in \Delta(R\times S) :$$

$$v(p) - \underset{I}{\text{cav}} \min\{u(p), v(p)\} = \epsilon\}$$

Let p_o be an extreme point of $Co(\Delta_\epsilon)$. From

$$v(p_o) > \underset{I}{\text{cav}} \min\{u(p_o), v(p_o)\}$$

it follows that

$$v(p_o) > u(p_o).$$

Lemma 6 implies that v is not strictly I-concave at p_o. If p_o is no extreme point of $\Delta(R\times S)$ there is a line segment $[p_1,p_2] = Co\{p_1,p_2\}$ with $p_1,p_2 \in \Delta_I(p)$ containing p_o in its relative interior on which v is affine. Consequently

$$v - \underset{I}{\text{cav}} \min\{u,v\} \text{ is convex on } [p_1,p_2]$$

$$v - \underset{I}{\text{cav}} \min\{u,v\} \leq \epsilon$$

$$(v - \underset{I}{\text{cav}} \min\{u,v\})(p_o) = \epsilon$$

so that

$$v - \underset{I}{\text{cav}} \min\{u,v\} \equiv \epsilon \text{ on } [p_1,p_2].$$

At least one of the points p_1, p_2 is situated outside of $Co(\Delta_\epsilon)$. Thus p_o has to be an extreme point of $\Delta(R\times S)$. But in this case $v(p_o) = u(p_o)$ is trivially satisfied. Contradiction! □

Theorem 8:

$v_1 = \lim_{\lambda\to 1} v_\lambda$ exists.

It is uniquely determined by the functional equations

(1) $v(p) = \underset{I}{\text{cav}} \min \{u(p), v(p)\}$

(2) $v(p) = \underset{I}{\text{vex}} \max \{u(p), v(p)\}$.

Proof:

Since the limit of every convergent subsequence v_{λ_n} satisfies these functional equations we must show that their simultaneous solution is unique.

Now let $\underline{v}$ be any II-convex solution of (1) and $\bar{v}$ any I-concave solution of (2). We show that $\underline{v} \leq \bar{v}$.

(The other inequality follows analogously)

Assume that

$$\max_{p\in\Delta(R\times S)} \underline{v}(p) - \bar{v}(p) = \epsilon > 0$$

and let p_o be an extreme point of

$$Co\ \{p \in \Delta(R\times S) \cdot \underline{v}(p) - \bar{v}(p) = \epsilon\}$$

Consequently at least one of the functions $\underline{v}$, $\bar{v}$ is not equal to u at p_o. Let us consider the case $\underline{v}(p_o) \neq u(p_o)$. (If $\bar{v}(p_o) \neq u(p_o)$ similar arguments can be applied.)

It follows that $\underline{v}(p_o) > u(p_o)$ and that $\underline{v}$ cannot be strictly I-concave at p_o. Again there exists a line segment $[p_1,p_2]$ with $p_1,p_2 \in \Delta(p_o)$ containing p_o in its relative interior on which $\underline{v}$ is affine. Like in the proof of Lemma 7 we have

$$\underline{v} - \bar{v} \text{ is convex on } [p_1,p_2]$$

$$\underline{v} - \bar{v} \leq \epsilon$$

$$\underline{v}(p_o) - \bar{v}(p_o) = \epsilon$$

so that

$$\underline{v} - \bar{v} \equiv \epsilon \text{ on } [p_1,p_2].$$

Again a contradiction follows. □

Remarks on the interpretation of the foregoing results:
It is evident from theorem 8 that v_1 is I-concave and II-convex. Lemma 6 already shows that strict I-concavity and strict II-convexity at some point p together imply the identity of the limiting value of $\Gamma_\lambda(p)$ and the value u(p) of the corresponding NR-Game. Indeed the functional equations show that for all $p \in \Delta(R\times S)$ where v_1 and u don't coincide v_1 is either locally affine on $\Delta_I(p)$ or locally affine on $\Delta_{II}(p)$. Strict I-concavity (II-convexity) stands for the necessity to conceal one's private information from the opponent (Proposition 4), while affine pieces of the value function indicate the opportunity of using the private information in order to improve the payoff in comparison with the NR-game. $v_1(p) > u(p)$ implies local affinity on $\Delta_I(p)$ while $v_1(p) < u(p)$ implies local affinity on $\Delta_{II}(p)$. Mostly only one of the players is allowed to make use of his private information. In order to understand the reason for this phenomenon let us briefly mention the results for infinitely repeated games with limiting average payoff function, cf. Mertens-Zamir (1980). The payoff one has in mind is $\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} A^{r,s}(h_t)$ but in order to avoid convergence

problems one could define a payoff function (and thus a two-person zero-sum game in normal form) by means of a Banach Limit. These games generally have no value. The formal reason is the impossibility to fulfil the requirements of the minmax theorem. Choosing a topology for the strategy spaces one has to give up either compactness or continuity of the payoff function. An intuitive reason why minmax and maxmin don't coincide may be explained in the following way:
Suppose we are in minmax situation, player 1 knows player 2' strategy and is able to compute posteriors on player 2's types. If player 2's strategy involves the use of private information he reveals it to player 1. Player 1 may wait an arbitrary number of steps in order to exhaust a maximal amount of information because this does not affect the average payoff of an infinite number of stages. After he has learned everything there is to learn he decides how he is going to apply his own private information. Of course the maxmin strategies provide a reversed order of information release. If in the discounted game the discount factor λ is large, i.e. if the total weight of the first stages is small and the equilibrium strategies of both players comprise the use of private information, both players could deviate from equilibrium by switching to something close to minmax resp. maxmin strategies of the undiscounted game. If only one of the players is supposed to make use of his private information, this problem does not occur.

5. Standard Signalling Case

We obtain the standard signalling case from the preceding model by defining E(i,j) = F(i,j) = (i,j). The players are informed about their opponent's choice of action. Both players have the same information on the past moves. Note that this effect cannot be produced by simply giving both players equal information matrices. We could no longer assume that the signalling matrix contains the player's choice of action. Comprising them in the same matrix would indeed mean standard signalling. On the other hand if the player's actions are not reproduced by the information matrix even the same signal will lead the players to different conclusions if they combine it with the knowledge of their own action. The common history both players can refer to is crucial for the following theorem:

Theorem 9: (Recursive Formula)

$$v_\lambda(p) = \min_{\tau\in\Upsilon_1} \max_{\sigma\in\Sigma_1} \left((1-\lambda)\ \beta^p(\sigma,\tau) + \lambda \sum_{h_1} \underline{P}^p_{(\sigma,\tau)}(h_1)\ v_\lambda(p(\cdot|h_1))\right)$$

Proof:

We need a second look at the proof of proposition 4:

Inequality (1) is omitted, we don't estimate the first stage payoff and don't instert τ_{NR}. Inequality (2) becomes an equality like (4) (due to standard signalling) and inequality (3) becomes unnecessary. Remark that the NR-property of τ_{NR} is only used in (3).

□

Theorem 9 can be interpreted as follows:

The value of the game $\Gamma_\lambda(p)$ remains the same if the rules are modified like this: After stage one the game is cut off and the players begin a new discounted game whose payoffs are weighted by λ. In the new game the players don't refer to what happened at stage 1, i.e., they forget both choices of actions but instead they learn the a-posteriori-probability on R×S they could compute if they knew both first stage strategies. (In fact the value wouldn't change either if the players could further remember the first stage history, see (2) and (4) in the proof of proposition 4.)

It is crucial for the validity of theorem 9 that in the standard signalling case both players can compute the same a-posteriori-probability. The above property is usually referred to as the recursive structure of repeated games with incomplete information.

Define a mapping $\phi_\lambda : \mathcal{C}^o(\Delta(R\times S),\mathbb{R}) \to \mathcal{C}^o(\Delta(R\times S),\mathbb{R})$ by

$$\phi_\lambda\, v(p) = \min_{\tau\in\Upsilon_1} \max_{\sigma\in\Sigma_1} \left((1-\lambda)\, \beta^p(\sigma,\tau) + \lambda \sum_h \underline{P}^p_{(\sigma,\tau)}(h)\; v_\lambda(p(\cdot|h))\right)$$

Lemma 10:

ϕ_λ is contracting: $\| \phi_\lambda v - \phi_\lambda w \| \leq \lambda \| v - w \|$

Proof:

Let (σ^o,τ^o) and (σ^1,τ^1) be first stage strategies achieving the minmax for v resp. w.

W.l.o.g. let $\phi_\lambda v(p) > \phi_\lambda w(p)$.

$$\begin{aligned}
&\phi_\lambda v(p) - \phi_\lambda w(p)\\
= \quad & (1-\lambda)\, \beta^p\, (\sigma^o,\tau^o) + \lambda \sum_h \underline{P}^p_{oo}(h)\; v(p_{oo}(\cdot|h))\\
& - (1-\lambda)\, \beta^p\, (\sigma^1,\tau^1) + \lambda \sum_h \underline{P}^p_{11}(h)\; v(p_{11}(\cdot|h))\\
= \quad & (1-\lambda)\, \beta^p\, (\sigma^o,\tau^1) + \lambda \sum_h \underline{P}^p_{o1}(h)\; v(p_{o1}(\cdot|h))\\
& - (1-\lambda)\, \beta^p\, (\sigma^o,\tau^1) + \lambda \sum_h \underline{P}^p_{o1}(h)\; v(p_{o1}(\cdot|h))\\
\leq \quad & \lambda \sum_h \underline{P}^p_{o1}(h)\; \| v - w \|\\
= \quad & \lambda \| v - w \|
\end{aligned}$$

□

Corollary 11:

v_λ is determined uniquely by the recursion formula.

In the case of incomplete information on one side, i.e. $|S|=1$, v_λ has a special property:

Proposition 12:

$$v_\lambda \le v_\mu \quad \text{if} \quad \lambda \le \mu < 1.$$

Proof:

First of all we have to show a preliminary statement: The undiscoundted first stage equilibrium payoff is not smaller than the value.

Let σ_1, τ_1 be optimal first stage strategies in $\Gamma_\lambda(p)$,

$\bar{\sigma}_\lambda(i)$ the total probability action i,

$\bar{\sigma}_\lambda(i) = \sum_r p(r)\ \sigma_\lambda(r;i)$

$$v_\lambda(p)$$

$$= \quad (1-\lambda)\ \beta^p\ (\sigma_\lambda,\tau_\lambda) + \lambda \sum_i \bar{\sigma}_\lambda(i)\ v_\lambda(p_\lambda(\cdot\,|i))$$

$$\le \quad (1-\lambda)\ \beta^p\ (\sigma_\lambda,\tau_\lambda) + \lambda\ v_\lambda(p)$$

$$\Longrightarrow \quad v_\lambda(p) \le \beta^p\ (\sigma_\lambda,\tau_\lambda)$$

$$\Longrightarrow \quad \beta^p\ (\sigma_\lambda,\tau_\lambda) \le \sum_i \bar{\sigma}_\lambda(i)\ v_\lambda(p_\lambda(\cdot\,|i))$$

Let us now assume that $\max_{p \in \Delta(R)} (v_\lambda(p) - v_\mu(p)) = \epsilon > 0$

$$
\begin{aligned}
& v_\lambda(p) - v_\mu(p) \\
= \quad & (1-\lambda)\ \beta^p\ (\sigma_\lambda, \tau_\lambda) + \lambda \sum_i \bar{\sigma}_\lambda(i)\ v_\lambda(p_\lambda(\cdot|i)) \\
& - (1-\mu)\ \beta^p\ (\sigma_\mu, \tau_\mu) - \mu \sum_i \bar{\sigma}_\mu(i)\ v_\mu(p_\mu(\cdot|i)) \\
\leq \quad & (1-\mu)\ \beta^p\ (\sigma_\lambda, \tau_\mu) + \mu \sum_i \bar{\sigma}_\lambda(i)\ v_\lambda(p_\lambda(\cdot|i)) \\
& - (\lambda-\mu)\ \beta^p\ (\sigma_\lambda, \tau_\lambda) + (\lambda-\mu) \sum_i \bar{\sigma}_\lambda(i)\ v_\lambda(p_\lambda(\cdot|i)) \\
& - (1-\mu)\ \beta^p\ (\sigma_\lambda, \tau_\mu) - \mu \sum_i \bar{\sigma}_\lambda(i)\ v_\mu(p_\lambda(\cdot|i)) \\
\leq \quad & \mu \sum_i \bar{\sigma}_\lambda(i)\ (v_\lambda(p_\lambda(\cdot|i) - v_\mu(p_\lambda(\cdot|i))) \\
\leq \quad & \mu\,\epsilon \qquad \forall\ p \in \Delta(R) \qquad \square
\end{aligned}
$$

which yields a contradiction.

The proof of proposition 12 shows that the (undiscounted) equilibrium payoff decreases from stage to stage. The formal reason is the concavity of the value function, intuitively one could argue that the amount of information the uninformed player gathers can only increase in the course of the game and thus his capability of reducing player 1's payoffs increases as well. Consequently the value of the game becomes smaller as the weight of the initial stages decreases.

References:

Armbruster, W. (1983): A Formal Proof of the Recursive Formula for the Value of Finitely Repeated Two-Person Zero-Sum Games with Lack of Information on Both Sides. mimeo.

Aumann, R.J. (1964): Mixed and Behaviour Strategies in Infinite Extensive Games. Annals of Mathematics Studies 52, Princeton University Press, 627-650.

Mertens, J.-F. and S.Zamir (1971/72): The Value of Two-Person Zero-Sum Repeated Games with Lack of Information on Both Sides, International Journal of Game Theory 1, 39-64.

Mertens, J.-F. (1971/72): The Value of Two-Person Zero-Sum Repeated Games - The Extensive Case. IJGT 1, 217-225.

Mertens, J.-F. and S.Zamir (1980): Min Max and Max Min of Repeated Games with Incomplete Information. IJGT 9, 201-215.

Sorin, S. (1980): An Introduction to Two-Person Zero-Sum Repeated Games with Incomplete Information. IMSSS-Economics, Stanford University, TR 312