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Classifying three-person-games

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Shenoy starts his fundamental paper on coalition formation by the following sentences:
"The theory of n-person cooperative games presented by von Neumann and Morgenstern is a mathematical theory of coalition behaviour. A fundamental problem posed in game theory is to determine what outcomes are likely to occur if a game is played by "rational players". Given an n-person cooperative game and assuming rational behaviour, it is natural to inquire (1) which of the possible coalitions can be expected to form and (2) what will be the final payoffs to each of the players."(*Shenoy* 1977)
In this paper I like to examine these questions simultaneously for all games: in this view a solution concept generates types of games, that are defined by

(1) a set of coalitions, that is "rational" resp. to the solution concept
(2) the form of set of payoffs being "rational" resp. to the solution concept or the kind of procedure leeding to the set of "rational" payoffs.

A set of solution concepts generates types by

(1) the vector of types resp. to the solution concepts
(2) the set-theoretic relation (inclusion, equality, etc.) between the "rational" coalitions or payoffs that correspond to the different solution concepts.

Explicit classification of games is done for 3-person side-payment games - as an example.
Classification of games by solution concepts is giving a better theoretical understanding of the solution concepts and the differences betwenn them, and it is for importance for experimental games as well. By game-types it is possible to compare the spontaneous acceptance and the acceptance of recommandations of the different types of "rational" arguments, that correspond to the solution concepts. We require for an experimental study on the "fitness" of solution concepts to represent all types generated by the set of solution concepts in question.

Games and Solution Concepts

1.Games

1.1 Definition

A **game** is a pair (N,v) with N={1,2,...,n}, a finite subset of the natural numbers **N**, called players, and a real-valued function v:**P**(N) → **R**, v(0)=0. The elements of **P**(N):={S;S⊂N} are called coalitions.
Sometimes it will be useful to identify **P**(N) with 2^N by S→1_S. 1_S is the **characteristic function** of S - in general a characteristic function 1_A is admitting the values 0 and 1, and is given by $1_A(y)=1$ iff y is an element of A. Furthermore it is useful to think of 1_S as a N-vector or as a measure on N.
A **measure** (N,m) is a game, that is additive, i.e. for all coalitions S the value m(S) is equal to $\Sigma\{m_i; i \in S\}$.

1.2 A money interpretation

For a money interpretation of the game (N,v), we argue that v(S) is the amount of money S can get by cooperation (without cooperation of somebody else). Hence, everybody can be sure to gain at less his v({i}), and can hope to get a suitable part of v(S) if he is member of S and S can be established. The difficulties to establish a coalition (and to decide on a suitable distribution) are born out of the concurrence of possible coalitions, and the payoffs they can offer for cooperation. The arguments to agree on a certain cooperation or to reject it are examined by various "solution concepts".

1.3 Preferences and utility

Sometimes games (N,v) are applied on a larger class of conflicts. In this setup a coalition S can cause some results R(S), and the players are supposed to have a preference relation on $\bigvee\{R(S); S\subseteq N\}$. Under special circumstances (see *Rosenmüller* 1981) it is possible to represent the individual preferences by utility functions, that assign an individual value to each result. The scale of the values or the utilities are defined up to a monotone transformation. In some classes of conflicts it can be reasonable to suppose, that the individuals can judge the results by a stronger scale. For example, in the money interpretation of the game an interval scale is obvious.

The main problem for a preference-interpretation is: how can the individuals compare their individual scales. In the money interpretation individuals all operate on the same scale, and by that there is no difficulty to judge the gains and losses that are made by an transfer. For a short intruduction how to handle transfer of utilities in the general case see *Kahan/Rapoport* 1984.

1.4 Strategic equivalence

Monatary payoffs are certainly no stronger than intervally scaled. If there are sidepayments, then there must be within the game something like money. Therefore, it should not matter for any theory of coalition forming based on strategic thinking what magnitudes of cash the payoffs represent. The strategic thinking about a game should not be affected by positive affine transformations, i.e. transformations that only alter the arbitrary unit and the zero-point of the payoffs.
Formally a positive affine transformation (a,m), a>0, m real-valued measure, transforms a given game v into w=av+m. Especially for a coalition S we have w(S)=av(S)+m(S).

1.5 Symmetry

Let us define the **symmetry group** $\Gamma=\Gamma(N,v)$ of a game. Permutations of N induce motions in coalitions and motions in games. The symmetry group Γ is the subgroup of permutations π satisfying $v=\pi v$.

Γ partitions the player set N into equivalence classes $\tilde{I}$ called **types**:
we write $i\sim j$ iff player i is in the orbit of j, i.e. $i\in\Gamma j$.
Let $\tilde{I} := \{j\in N, i\sim j\}$ and $\tilde{N}=N/\Gamma=\{\tilde{I};i\in N\}$.

We say players i and j are **symmetric** in a game (N,v) iff $v(S\cup\{i\})=v(S\cup\{j\})$ whenever S and $\{i,j\}$ are disjoint. Symmetric players are of the same type, but there are games with two players of the same type, that are not symmetric. For example: $N=\{1,2,3,4\}$, $v(S)=1$ iff $S=\{1,3\}$ or $S=\{2,4\}$ or S contains three or four elements, else $v(S)=0$. The symmetry group is the group of cyclic permutations. Hence, there is only one type, but 1 and 2 are not symmetric. If for all types all players of that type are symmetric, then the game is said to have the **symmetry property**.

1.6 Some special classes of games

The game (N,v) is called **superadditive** iff $v(S)+v(T)\le v(S\cup T)$ for all disjoint coalitions.
The game (N,v) has **constant-sum** iff $v(S)+v(N\setminus S)=v(N)$ for all coalitions S.
The game (N,v) is called **convex** iff $v(S)+v(T)\le v(S\cup T)+v(S\cap T)$ for all coalitions S and T.

By weakening the symmetry-comparison of two players we get the **desirability relation:** $i\succeq j$ (i is more desirable than j) iff $v(S\cup\{i\})\ge v(S\cup\{j\})$ whenever S and $\{i,j\}$ are disjoint.
This relation is transitive (*Maschler/Peleg* 1966, section 9).

If it is complete, then the relation induces an order on the set of types, and two players are of the same type iff they are symmetric i.e. if the relation is complete then the game has the symmetry property.
We say, the game (N,v) is an **ordered game** if its desirability relation is complete.

The game (N,v) is called **simple** iff the set of values of v is equal to {0,1}. (For a survey on simple games and for possible interpretations see *Shapley* 1962.)

2. The space of n-person games

2.1 The space of n-person games and the generating polytopes

Let N be fixed. The set of all games (N,v), i.e. of all n-person games can be structured as real vector-space of dimension 2^n-1, taking the values v(S), S non-empty, as coordinates.
The following basis of the vector-space is of special interest:
The game v_S is called **unanimity game** on S,
iff $v_S(T)=1$ for all S satisfying $S\subset T$ and
=0 otherwise. The set $\{v_S : S\subset N\}$ forms a basis.
In the next step we focus on the $2^n-(n+1)$-dimensional subspace defined by $v(\{1\})=\ldots=v(\{n\})=0$. The games of this subspace are called **0-normalized.**
For any game v the vector $(v(\{1\}),\ldots,v(\{n\}))$ is called **threat-point**, and the corresponding measure is called **threat-measure**; both are denoted by $\underline{x}$.
We define $v':=v-\underline{x}$ to be the **0-normalized version** of the game v.
Since v' is obtained by a positive affine transformation, namely $(1,\underline{x})$, both games v and v' are strategically equivalent.
By this argument we can focus our attention on the subspace above.

A game is called **0-nonnegative** if its 0-normalized version v' is nonnegative. Let $v^+:=\max\{v,\underline{x}\}$. The game v^+ can be seen as representing the same demands of the players as v, in so far as an inferior coalition, lifted up by v^+, can be replaced without any loss by "going alone".
By this argument we can reduce our attention to the 0-nonnegative 0-normalized games, that represent all games by $v\to v'^+$, for all solution concepts that admit "concurrence of coalition structures".

The set of all 0-nonnegative 0-normalized games is a $2^n-(n+1)$-
-dimensional cone with vertex (0,...,0). It can be generated by a convex polytope. To this end we cut the cone by an adequate hyperplane of the subspace of 0-normalized games. As an example take the hyperplane $\Sigma\{v(S); S \text{ coalition}\}=1$. A convex polytope attained by this procedure is called **generating polytope.**
Every 0-nonnegative 0-normalized game is strategically equivalent to (0,...,0) or a game contained in the generating polytope.
By this argument we can reduce our attention to the generating polytope. The generating polytope is $2^n-(n+2)$-dimensional (for n=3 we get 3 dimensions, for n=4 we get 10 dimensions).

2.2 Reducing the generating polytope

Permuting the players we get at most n! versions of a game. These games only differ in the names of the players. So we can argue, that they represent the same game. We would prefer to reduce the generating polytope in such a way, that there would be exactly one game representing the permutation-versions. But in general there is no procedure to do this in such a way that the reduced generating polytope is a convex polytope too.
If a game is lacking in the symmetry property, then players of one type cannot be permuted freely.
In order to fix names of the players, we use the relation induced by the quotas and >.
The **quotas** are defined by: q(S)=v(S), S containes n-1 elements, q is measure. Now, we can reduce the generating polytope by permutations. The games of the reduced generating polytope can be given by the following coordinates:
$(v(12),\ldots,v(\{3,\ldots,n\}),q,v(N))$,
$q(N)+\Sigma\{v(S);\ S \text{ contains not } n-1 \text{ elements}\}=1,\ q_1 \geq q_2 \geq \ldots \geq q_n$.
The vertices of the reduced generating polytope are 1_S,
S contains not n-1 elements, and (0,...,0,1,...,1,0,...,0,0)/r with r being the number of quotas equal 1 (r=1,...,n).
The reduced generating polytope is convex.
If the symmetry group of a game is not the trivial one (equal to {identity}), then the game is on the boundary of the reduced generating polytope. Hence, only some games on the boundary are to identify.

3. Kinds of rationality

3.1 Solution concepts based on excesses

We look at a fixed game (N,v).
A measure (on N) is called **payoff-vector.**
A **coalition structure** $\underline{B}$ is a partition of N.
An **individually rational payoff configuration**, in short: i.r.p.c., is a pair $(x,\underline{B})$, with $\underline{B}$ being a coalition structure and a measure x, that satisfies:

1. $x(B)=v(B)$ for all $B\in\underline{B}$ and
2. $x_i \geq v(\{i\})$ for all players i.

Given a payoff-vector (or an i.r.p.c.) the **excess** $e(x,S)$ of a coalition S is defined by $e(x,S)=(v-x)(S)$.

The **core for coalition structure $\underline{B}$**, or short: $\underline{B}$-core, is the set of all payoff-vectors x that satisfy:

1. $(x,\underline{B})$ is an i.r.p.c. and
2. the excess $e(x,S)$ is nonpositive for all coalitions S.

The {N}-core was introduced by *Gillies* 1959.
If x is element of the $\underline{B}$-core, then no coalition can be better-off by itself.

We turn next to the bargaining set. This solution concept is defined in terms of "objections" and "counterobjections".
An **objection** of player i against player j at x is a pair (y,S), y a payoff-vector, S a coalition, such that:

1. i is element of S, and j is not,
2. i is strictly better-off in y than in x, i.e. $y_i > x_i$,
3. no member of of S is worse-off in y than in x, i.e. $y_k \geq x_k$ for all $k\in S$, and
4. S can guarantee y to its members, i.e. $v(S)\geq x(S)$.

A **counterobjection** (z,T), z payoff-vector, T coalition, (of player j) to such an objection (y,S) is defined to satisfy:

1. j is element of T, and i is not,
2. $z_k \geq x_k$ for all $k \in T \setminus S$
3. z is not worse than y for the players turned from S,
   i.e. $z_k \geq y_k$ for all $k \in T \cap S$.
4. T can guarantee z to its members, i.e. $v(T) \geq z(T)$.

For an i.r.p.c. to be "stable", every coalition of the coalition structure must be able to defend their payoffs against possible objections by counter-objections.

The **bargaining set** for coalition structure $\underline{B}$ is defined as the set of all i.r.p.c. $(x,\underline{B})$, that are stable in the sense, that every objection at x can be countered by a counterobjection.

Remark. The bargaining set and the core make no use of interpersonal comparisons of utility.

Remember the notion of the excesses. An excess $e(x,S)$ represents the collective gain or loss for S if the members of S withdraw from x to form the coalition S. In that sense the excess is a measure for the strengt of relative satisfaction or dissatisfaction respectively to a proposed payoff-vector x.

The following two solution concepts, namely the kernel and the nucleolus, are comparing payoff-vectors by their excesses, to find the "best" of them, i.e. that payoff-vectors that admit the highest degree of satisfaction of all single players (for the kernel) or of all coalitions (for the nucleolus). For another interpretation that does not use interpersonal comparisons of utility see *Maschler/Peleg/Shapley* 1979, p.330.

The comparison of the excesses by players uses the following definition: $s(i,j,x) := \max\{e(x,S);\ i \in S \text{ and } j \in N \setminus S\}$.

The **kernel** for coalition structure $\underline{B}$ is defined to be the set of all payoff-vectors x with $(x,\underline{B})$ being an i.r.p.c., that "equalize the maximal excesses if it is possible", formally:

$s(i,j,x) = s(j,i,x)$ or $x_i = v(\{i\})$ or $x_j = v(\{j\})$

for all coalitions B of coalition structure $\underline{B}$ and for all pairs $\{i,j\}$ of players contained in B.

For the comparison of the excesses by coalitions let L(x) be a 2^n-vector, the elements of which are the excesses e(x,S), $S \subset N$, arranged in nonincreasing order (i.e. $(L(x))_r \leq (L(x))_s$ whenever r>s).
The **nucleolus** for coalition structure $\underline{B}$ is the set of all payoff-vectors x that satisfy:

1. (x,$\underline{B}$) is an i.r.p.c.
2. L(x) is lexicographically minimal on the set of payoff-vectors y with (y,$\underline{B}$) i.r.p.c.

If the set of payoff-vectors y satisfying (y,$\underline{B}$) i.r.p.c. is nonempty, the nucleolus consists of a single element (*Schmeidler* 1969, *Kohlberg* 1971).

3.2 Solution concepts based on surplus by single players

The following solution concepts focus the value added by a single player entering a coalition. Let us define for $S \subset N$, $i \in N \setminus S$ the **marginal value** m(i,S) of player i at S by
$m(i,S) := v(S \cup \{i\}) - v(S)$.
The following solution concepts differ in treating the marginal value. The Shapley-value (*Shapley* 1953) is distributing v(N) according to the expectation of the marginal value. Hence, the marginal values are seen as measuring the worth of a player. Equal share of surplus is distributing positive marginal values among the members of the newly formed coalition $S \cup \{i\}$: here the marginal value is seen as gained by means of the extension of the coalition.
Both concepts can be understood as based on the following procedure: Given an ordering on the players (represented by numbers E1 up to En), an attempt is made to extend every coalition reached {E1,E2,...,Er} to {E1,...,E(r+1)}. For the Shapley-value coalition N={E1,...,En} is reached, whereas for the equal share of surplus the process can stop before.

Let us fix an order of entry E. Then the vector g(E) defined by $(g(E))_i := v(\{E1,...,Ei\}) - v(\{E1,...,E(i-1)\})$ assigns the gains of entry to the entering player until N is reached.
The gains of entry are marginal values: $(g(E))_i = m(i,\{E1,...,E(i-1)\})$.
The **Shapley-value** $\Phi(v)$ is defined by
$\Phi(v) = \Sigma\{g(E);\ E \text{ is an order on } N\}/n!$.

It can be shown that the Shapley-value is the only function on the space of games that satisfies the following axioms:

1. $\Phi(v)(N) = v(N)$,
   i.e. the Shapley-value distributes v(N) among the players
2. $\pi(\Phi(v)) = \Phi(\pi v)$,
   i.e. the Shapley-value does not depend on names
3. $\Phi(v+w) = \Phi(v) + \Phi(w)$,
   i.e. adding games allows to add the values
4. If $m(i,S)=0$ for all S, $i \in N\setminus S$, then $(\Phi(v))_i = 0$,
   i.e. the dummies are assigned a worth of zero.

Remark. Also the Shapley-value can be extended to coalition structures (see *Aumann/Drèze* 1974). But this extension exhibits some strangeness (see *Shenoy* 1977).
The additivity in axiom 3. is a very efficient tool, since the unanimity games form a basis.

Let us fix an order of entry E. Let us imagine player E1 up to E(i-1) are already together in a room. Now, when player Ei is called to present a possible gain, some players had already formed a coalition, say B(E,i-1), and some players with a number less than i - let us denote this set by C(E,i-1) - are already waiting. If player Ei can offer $v(\{E1,...,Ei\})$ greater than the sum of all claims presented - that is $v(B(E,i-1)) + \underline{x}(C(E,i-1) \cup \{Ei\})$ - than the new coalition $\{E1,...,Ei\} = B(E,i)$ is formed and C(E,i) is empty, else $B(E,i) = B(E,i-1)$ and $C(E,i) = \{Ei\} \cup C(E,i-1)$.
Formally: The **surplus** of the i-th player according to E is defined by

1. $s(E,i) = v(\{E1,...,Ei\}) - v(B(E,i-1)) - \underline{x}(C(E,i-1) \cup \{Ei\})$,
2. if $s(E,i) > 0$: $B(E,i) = \{E1,...,Ei\}$ and C(E,i) empty,
3. if $s(E,i) \leq 0$: $B(E,i) = B(E,i-1)$ and $C(E,i) = C(E,i-1) \cup \{Ei\}$.

The surplus will be distributed equally. Hence we define the **equal share of surplus** for order of entry E to be the i.r.p.c. $(e(E),\underline{B}(E))$ satisfying

1. $\underline{B}=\{\{B(E,n)\}\cup\{\{i\};i\in N\backslash B(E,n)\}$,
2. $(e(E))_i=v(\{i\})+\Sigma\{s(E,j)/j;\ j\geq i \text{ and } s(E,j)>0\}$.

Remark. The equal share of surplus is well defined, i.e. is an i.r.p.c., since only gains, that result from transitions from $B(E,i)$ to $B(E,i+1)$, are distributed.

3.3 Other solution concepts

The following solution concepts are not taken into consideration in this paper. Nevertheless they are worth mentioning in order to show, that there are other reasonable concepts to examine bargaining behaviour.

The definition of stable sets (or von Neumann-Morgenstern solutions) uses the notion of "domination". Let x and y be payoff-vectors. Then **x dominates y with respect to coalition** S iff

1. $v(S)\geq x(S)$ and
2. $x_i>y_i$ for all $i\in S$.

Next, **x dominates y** - we write x dom y - iff there exists a coalition S such that x dominates y with respect to S.
A **stable set** for coalition structure $\underline{B}$ is a subset Z of the i.r.p.c. with coalition structure $\underline{B}$ such that

1. there do not exist $x,y\in Z$ with x dom y and
2. for every i.r.p.c. $(y,\underline{B})$, $y\notin Z$ there exists an $x\in Z$ such that x dom y.

The first property is called internal stability or internal consistency, and the second one is called external stability or external domination.
For a survey on stable sets see *Lucas* 1970.

Remember the definition of the {N}-core. The **least-core** for the game v is the core of that game w, $w(S)=v(S)$ for all $S\neq N$, $w(N)=v(N)+t$, that has the minimal t with respect to non-empty cores.

For the equity core the members of a coalition S are supposed to expect the equity distribution y(S)=(v(S),...,v(S))/number of members of S. Hence, they will reject any payoff-vector x such that $(y(S))_i > x_i$ for all $i \in S$.

The **equity core** is defined to be the set of all payoff-vectors x satisfying

1. x(N)=v(N) and
2. there exist no coalition S with $(y(S))_i > x_i$ for all $i \in S$.

The equity core is a weaker solution concept than the {N}-core in a sense that it always contains the {N}-core, if the {N}-core exists.

For a discussion of this concept see *Selten* 1972 and 1976, *Okada/Funaki* 1982.

The goal to take the equity distribution into consideration has the equity core in common with concepts like the equal division kernel (*Crott/Albers* 1981 and 1982) and the equal excess model (*Komorita* 1979).

4. The rational coalition structure

For solution concepts, that are defined for coalition structures, it is natural, to ask for the transitions between coalition structures, that are induced by the individual interests.
First let us compare i.r.p.c. $(x,\underline{B})$ and i.r.p.c. $(y,\underline{C})$:
$(x,\underline{B})>(y,\underline{C})$ iff there exist a coalition $B \in \underline{B}$ with $x_i > y_i$ for all its members $i \in B$.
We argue that $(y,\underline{C})$ is dominated by $(x,\underline{B})$ because all members of B are better-off in x, and in a consequence coalition B withdraws from $(y,\underline{C})$.
Let a solution concept L assign to each coalition structure $\underline{B}$ a set of "rational" payoff-vectors, then
$\underline{B}>\underline{C}$ iff there exists a coalition $B \in \underline{B}$, such that for every $y \in L(\underline{C})$ exists an $x \in L(\underline{B})$ satisfying $x_i > y_i$ for all members $i \in S$.

We argue that $\underline{C}$ is dominated by $\underline{B}$ because of the coalition $B \in \underline{B}$, that will counter any proposal of $\underline{C}$ by withdrawal.
A discussion of both relations is found in *Shenoy* 1977, 1979.
A coalition structure $\underline{B}$ is called **rational** w.r.t. solution concept L iff "it is not dominated", i.e. there exists no coalition structure $\underline{C}$ satisfying $\underline{C}>\underline{B}$.
The set of rational coalition structures are in some sense the "core of the game reduced by the solution concept and played on coalition structures".

A coalition structure is called **maximal** iff the sum of the values of its coalitions is maximal resp. to all coalition structures.
For the core, the rational coalition structures are the maximal ones, and all cores for a rational coalition structure coincide (*Shenoy* 1977, 46-7).
For the bargaining set family no general existence theorem for rational coalitions is known at this time.

5. Main theorems on solution concepts

5.1 Solution concepts and operations on games

By easy calculations one can show that:

- core, the bargaining set family and the Shapley-value covary with positive affine transformations ("stategic equivalence"), and permutations on N ("symmetry")
- core and the bargaining set family give the same solution for a 0-normalized game and its 0-nonnegative hull, if there exists an i.r.p.c. for the considered coalition structure (else it is empty.

Kernel, nucleolus and Shapley-value preserve the desirability relation and assign zero-weight to dummies (cf. *Maschler/Peleg* 1966).

5.2 Balanced games and the core

Let us imagine the players can decide on the intensity b_S by that coalition S is operated. If the intensity to run a coalition is measured by a number between zero and one, the players are "fully employed" if $\Sigma\{b_S 1_S ; S \subset N\} = 1_N$.

Let us call an 2^n-vector b a **balanced system** for N iff $\Sigma\{b_S 1_S ; S \subset N\} = 1_N$.

For a game (N,v) let $b(v) := \sup\{\Sigma\{b_S v(S) ; S \subset N\}$; b is a balanced system for [N]

The number b(v) is the maximal gain in (N,v) players can get by splitting their engagement.

The game (N,v) is called **balanced** iff $b(v) \leq v(N)$.

The main theorem on the core (*Bondareva* 1963, *Shapley* 1967) is: {N}-core is not empty iff the game is balanced.

This theorem is proven by the duality theorem of linear programing. The primal programm is: find $\min\{x(N) ; x(S) \geq v(S)\}$, the dual: find $\max\{\Sigma\{c_S 1_S ; S \subset N\}$; $c \geq 0$ and $\Sigma\{c_S 1_S ; S \subset N\} = 1_N\}$.

So c is a balanced system. For a detailed proof see *Rosenmüller* 1981.

5.3 Bargaining set, kernel and nucleolus

By definitions the core is a subset of the bargaining set. For 0-nonnegative games the bargaining set is not empty (*Peleg* 1963 and 1967). The kernel is a subset of the bargaining set and it is not empty iff there exists an i.r.p.c. of the concidered coalition structure (*Davis/Maschler* 1965, sec.5). The nucleolus is a unique point within the kernel and as a function it is continuous and piecewise affine (*Schmeidler* 1969).

Let us fix some point x within the least-core. Then for every pair of players {i,j} we get a closed line segment of points within the least-core that are constructed by transfers between i and j; we call the line segment R({i,j}). The kernel has the following bisection property: x is element of the kernel iff for each pair {i,j} either x bisects R({i,j}) or x_i=v({i}) or x_j=v({j}) (*Maschler/Peleg/Shapley* 1979). For the nucleolus *Maschler/Peleg/Shapley* 1979 extend the procedure leading to least-core. The extended (lexicographic) procedure is shown to lead to the nucleolus. The authors write: "Finally, the intuitive interpretations of the two solution concepts are clarified: the kernel as kind of multi-bilateral bargaining equilibrium without interpersonal utility comparisons, in which each pair of players bisects an interval which is either the battleground over which they can push each other aided by their best allies (if they are strong) or the no-man's-land that lies between them (if they are weak); the nucleolus as the result of an arbitrator's desire to minimize the dissatisfaction of the most dissatisfied coalition."

By a paper of *Megiddo* 1974 we know, that the bargaining set family lacks of monotonicity on games, i.e. if the game is changed only by increasing the value for N, then the solutions do not guarantee not to decrease the payoff to some player.

5.4 Convex games and the Shapley-value

Recall the definition of a convex game and the definition of the gain g(E) according to a given order of entry. The following theorem holds for convex games (*Shapley* 1972):
g(E) is vertex of the {N}-core, and all vertices of the {N}-core are of such kind. In this sense the Shapley-value is the gravity center of the {N}-core.

Three-Person Games

1. Coordinates (cf.fig.1)

- The **quotas** are defined by: q(S)=v(S), S containes n-1 elements, q is measure. Three-person games are ordered games, and the desirability relation coincides with the relation induced by the quotas. Thus, we can reduce the generating polytope by permutations. The games of the reduced generating polytope can be given by the following coordinates: (q,v(N)), q(N)+v(N)=1, $q_1 \geq q_2 \geq q_3$. The vertices of the reduced generating polytope are (0,0,0,1), (1,1,1,0)/3, (1,0,0,0), (1,1,-1,0). Let ij:={i,j}.
- The reduced generating tetraeder contains four games equivalent to a **simple game**: (0,0,0,1), (1,1,1,2)/5, (1,0,0,1)/2, (1,1,-1,2)/3.
- A game has the **symmetry group** of all permutations iff it is element of the closed line segment with vertices (0,0,0,1) and (1,1,1,0)/3. The only other symmetry groups of three-person games are keeping fixed (1) the "big" player, (2) the "small" player, (3) all players. Games of case (1) are exactly the games within the convex hull of (1,0,0,0) and the above line segment minus this line segment. Games of case (2) are exactly the games within the convex hull of (1,1,-1,0) and the above line segment minus this line segment.
- There is only one **constant sum game** within the generating tetraeder: (1,1,1,2)/5.
- All three-person games are ordered. The **desirability relation** is the same as that one induced by the quotas.
- **Measures** are represented by the game (0,0,0,0). This game is the vertex of the generating cone.
- A game v is **superadditive** iff $v(N) \geq q(12)$; q(12)=v(N) is a hyperplane separating the reduced generating tetraeder into the closed tetraeder of superadditive games and the half-open polytop of the non-superadditive games. The vertices of the superadditive games are: (0,0,0,1), (1,1,1,2)/5, (1,0,0,1)/2, (1,1,-1,2)/3.
- A game is **convex** iff $v(N) \geq v(12)+v(13)=q(N)+q(1)$; by the separating hyperplane we get the closed tetraeder of convex games. The vertices are: (0,0,0,1), (1,1,1,4)/7, (1,0,0,2)/3, (1,1,-1,2)/3. The convex games are a subset of the superadditive games. Both tetraeders have two vertices in common.

2.**The core** (cf.fig.2)

- A **coalition structure** is called **maximal** iff the sum of the values of its coalitions is maximal (resp. to all coalition structures). If a coalition structure is not maximal then its core is empty. The cores of maximal coalition structures coincide.
- Usually the notion "core" means "core for the coalition structure of the grand coalition", i.e. {N}. Then we get a non-empty core iff the game is balanced. The game is **balanced** iff $v(N) \geq q(12)$ and $v(N) \geq q(N)$. By considering the separating hyperplanes we get the closed polytope of balanced games as the convex hull of the five games (0,0,0,1), (1,1,1,3)/6, (1,0,0,1)/2, (1,1,0,2)/4 and (1,1,-1,2)/3. A convex game is balanced and a balanced game is superadditive.
- If the grand coalition structure is not maximal, then {12,3} is maximal. The core for {12,3} is empty iff $v(12) < v(13)+v(23)$, or in terms of quotas: iff $q_3 > 0$. The coalition structure {12,3} is maximal iff the game is not superadditive or $v(12)=v(N)$. The core for {12,3} is not empty for exactly the games in the closed convex polytope generated by the following six games: (1,0,0,0), (1,0,0,1)/2, (1,1,0,0)/2, (1,1,0,2)/4, (1,1,-1,0) and (1,1,-1,2)/3.
- The non-empty core for {N} and for {12,3} coincide on the closed triangle with vertices (1,0,0,1)/2, (1,1,0,2)/4, (1,1,-1,2)/3. The non-empty core for {12,3} and for {13,2} coincide on the closed line segment between (1,0,0,0) and (1,0,0;1)/2. The core for {23,1} is empty.
  The non-empty cores of {N}, {12,3} and {13,2} coincide in exactly one point: the game representing the simple game (1,0,0,1). The core of this game contains only one element, that gives zero to the smaller players, inspite of the fact that the strong player needs the cooperation of the small ones.
- Rational coalition structures are all maximal coalition structures with non-empty cores.
- We can distinguish the games by the dimension of the core (or -finer- by the number of vertices of the core). The core is one-dimensional iff it is the {12,3}-core and the game is not element of the closed quadrangle with vertices (1,0,0,0), (1,0,0,1)/2, (1,1,0,0)/2, (1,1,0,2)/4. The core is a single point iff the game is element of the above quadrangle.

- We got ten **types** of games:

0.	(0,0,0,0)	single point for all coalition structures
1.	"above"	two-dimensional core for {N}
2.	"left"	one-dimensional core for {12,3}
3.	"right"	empty core
4.	"left wing"	one-dimensional core for {N} and {12,3}
5.	"right wing"	single point for {N}
6.	"vertical wing"	single point for {12,3}
7.	"horizontal line"	single point for {N} and {12,3}
8.	"vertical line"	single point for {12,3} and {13,2}
9.	(1,0,0,1)/2	single point for {N}, {12,3}, {13,2}

Remark: The coalition of the two "smaller" players is never effective.

3.**The Kernel** (cf.fig.2,3)

- The kernel for three-person games is a unique point for any coalition structure. It coincides with the nucleolus. For the coalition structure {N} we get: If the core is empty, the kernel coincides with the bargaining set; if the core is not empty, the bargaining set coincides with the core.
  For the other coalition structures bargaining set and kernel and -if not empty: the core- coincide.
  (*Davis/Maschler* 1965)
- The kernel for {N} is central in the least core
  (*Maschler/Peleg/Shapley* 1972)
- The kernel is a continuous piecewise affine function on the three-person-games. The regions of linearity can be identified with a set of coalitions, relevant for the maximal excesses. The regions corresponds to a special kind of dynamics for the **bargaining procedure proposed by the kernel**. The regions are defined by inequalities of excesses up to an equality. By that reason, the regions are closed convex polytopes. They can intersect on their boundaries.
- For the coalition structure {1,2,3} there is only one i.r.p.c. and the kernel is trivial.

- For the pair structures {13,2} and {23,1} there are two regions of linearity, depending on whether the quota of the smallest player is nonnegative or not. The regions intersect at the zero quota. If there is no negative quota, the pair-coalition distributes the quotas; if there is a negative quota, the third player can demand more, namely $v(3)=0$.
- The pair structure {12,3} distributes the quotas to player 1 and 2 - in any game.
- For the grand coalition there are six regions of linearity. We follow the notation and the result of *Davis/Maschler* 1965:
  A. There is no relevant pair coalition. This is the case for $3v(12) \leqslant v(N)$. The value of the grand coalition is divided equally.
  B. There is one relevant pair coalition, namely 12. This is the case for $\max\{q(N), q(N)+3q_3\} \leqslant v(N) \leqslant 3v(12)$.
     By dividing $v(N)-v(12)$ into two equal parts the pair coalition separates out the smallest player. There are three regions of this kind depending on the set of relevant one-person-coalitions:
     B_1. Both 1 and 2 cannot demand more than zero from the remaining part of $v(N)$, not yet distributed among the players.
     B_2. Player 1 can demand more than zero, but not player 2.
     B_3. Both can demand more than zero.
     In any case there will be an equal division of that part of $v(N)$ that remains when all demands are satisfied.
  C. All pair coalitions are relevant. This is the case for all quotas are nonnegative and $q(N)-3q_3 \leqslant v(N) \leqslant q(N)+3q_3$. The kernel assigns the quota to any player plus an equal part of $v(N)-q(N)$; that part can be negative.
  D. In all remaining games the smallest player cannot demand for more than zero. There are two regions:
     D_1: Player 2 can demand zero or more.
     D_2: Only player 3 can demand for more than zero.
     If a game is of type D, then it is not superadditive.

- The following list contains the vertices of the regions according to the kernel for {N}:

game	kernel	vertex of region						
(0,0,0,1)	(1,1,1)/3	A						
(1,1,-1,6)/7	(2,2,2)/7	A	B_3					
(1,0,0,3)/4	(1,1,1)/3	A	B_3	B_2				
(1,1,1,6)/9	(2,2,2)/9	A	B_3	B_2	B_1	C		
(1,1,-1,2)/3	(1,1,0)/3		B_3	B_2	B_1			
(1,1,0,2)/4	(1,1,0)/4				B_1	C	D_1	
(1,0,0,1)/2	(1,0,0)/2			B_2	B_1	C	D_1	D_2
(1,1,-1,1)/2	(1,1,0)/4				B_1		D_1	
(1,1,1,0)/3	(0,0,0)					C	D_1	D_2
(1,0,0,0)	(0,0,0)							D_2
(1,1,-1,0)	(0,0,0)						D_1	D_2

- The rational coalition structures for the kernel are that coalition structures rational for the bargaining set given in *Shenoy* 1977, theorems 3.28-30. {N} is rational iff the {N}-core is not empty. All pair-structures are rational iff quotas are nonnegative and {N}-core is empty or a single point. {12,3} is rational iff one quota is negative and {N}-core is empty or not of full dimension. We got six **types** and can list them by using the core-types:

0. (0,0,0,0) as core-type 0. : any coalition structure
1. "above" as core-type 1. : {N} is rational
2. "left" as core-type 2. : {12,3}
3. "right" or "vertical wing" or "vertical line" as core-types 3.,6.,8. : all pair structures
4. "left wing" as core-type 4. : {N} and {12,3}
5. "right wing" or "horizontal line" or (1,0,0,1)/2 as core-types 5.,7.,9. : {N} and all pair structures

4. The Shapley-Value (cf. fig. 4,5)

- If a function is linear on the games of the reduced generating polytope, preserves symmetry and assigns an individually rational payoff vector to each game, then it is the equal distribution of v(N). This fact can be seen at the four vertices of the reduced generating polytope: (0,0,0) is assigned to three of them, and (1,1,1)/3 to the fourth one. Therefore the Shapley-value is **not individually rational** for all games. The region of individual rationality and that one without individual rationality are separated by the triangle of (1,1,1,0)/3, (2,0,0,1)/3, (1,1,-1,2)/3 generated by the equation: the value of the smallest player is zero. The first game has a value with three zeros, the second one with two zeros, and the last one exhibits one zero.
- **Shapley-value and kernel for {N}** coincide:
  0. For the game (0,0,0,0).
  1. On the closed line segment (1,1,1,6)/9 to (0,0,0,1) (within kernel-region A.).
  2. On kernel-region B. the payment for the player separated out demands the equality v(N)=2q(N). This equality yields type B_2. Together with the evaluation of the other payments we get the triangle with the vertices (1,1,1,6)/9, (1,0,0,2)/3, (1,1,-1,2)/3 (the corresponding values are (2,2,2)/9, (2,1,1)/6, (1,1,0)/3).
  3. On kernel-region C. we get q(N)=3q(i) and equal quotas. Hence, the region of coincidence is the closed line segment (1,1,1,6)/9 to (1,1,1,0)/3.
  4. On kernel-region D. coincidence is reached on the closed line-segment between (2,0,0,1)/3 and (1,1,1,0)/3.

  Coincidence in regions 0.,1.,3. is due to symmetry.
- The relation between **Shapley-value and core** can be described as follows: the Shapley-value is element of the {N}-core for three of the five vertices of the polytope of the balanced games ({N}-core is not empty), namely (0,0,0,1), (1,1,1,3)/6, (1,1,-1,1)/2. For the other two vertices the Shapley-value is not 12-rational. By the corresponding inequality we get the other two vertices of the region "Shapley-value is element of the {N}-core": (4,0,0,5)/9 and (2,2,0,5)/9.

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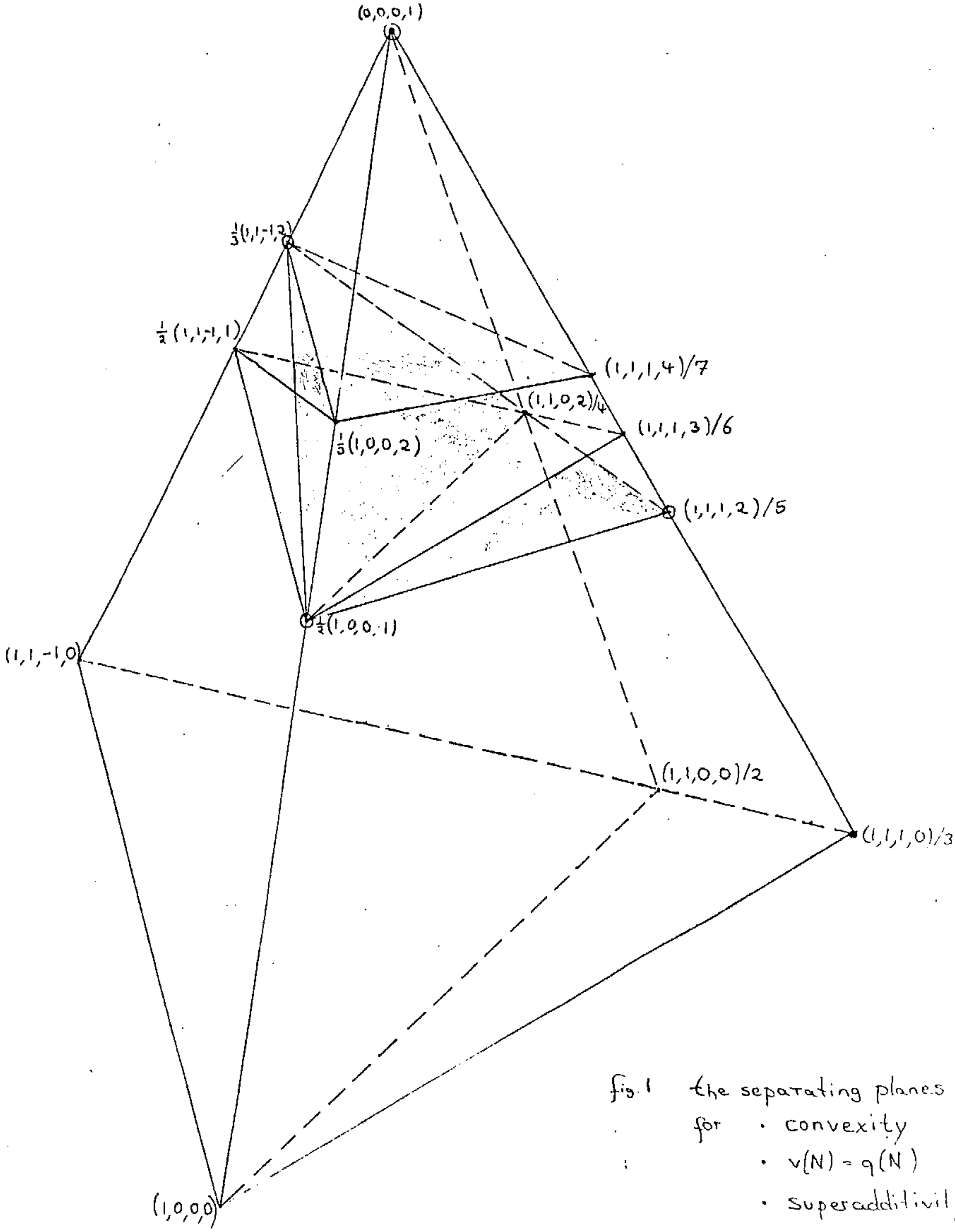

fig. 1 the separating planes
for · convexity
· $v(N) = q(N)$
· superadditivity
· $q_3 = 0$

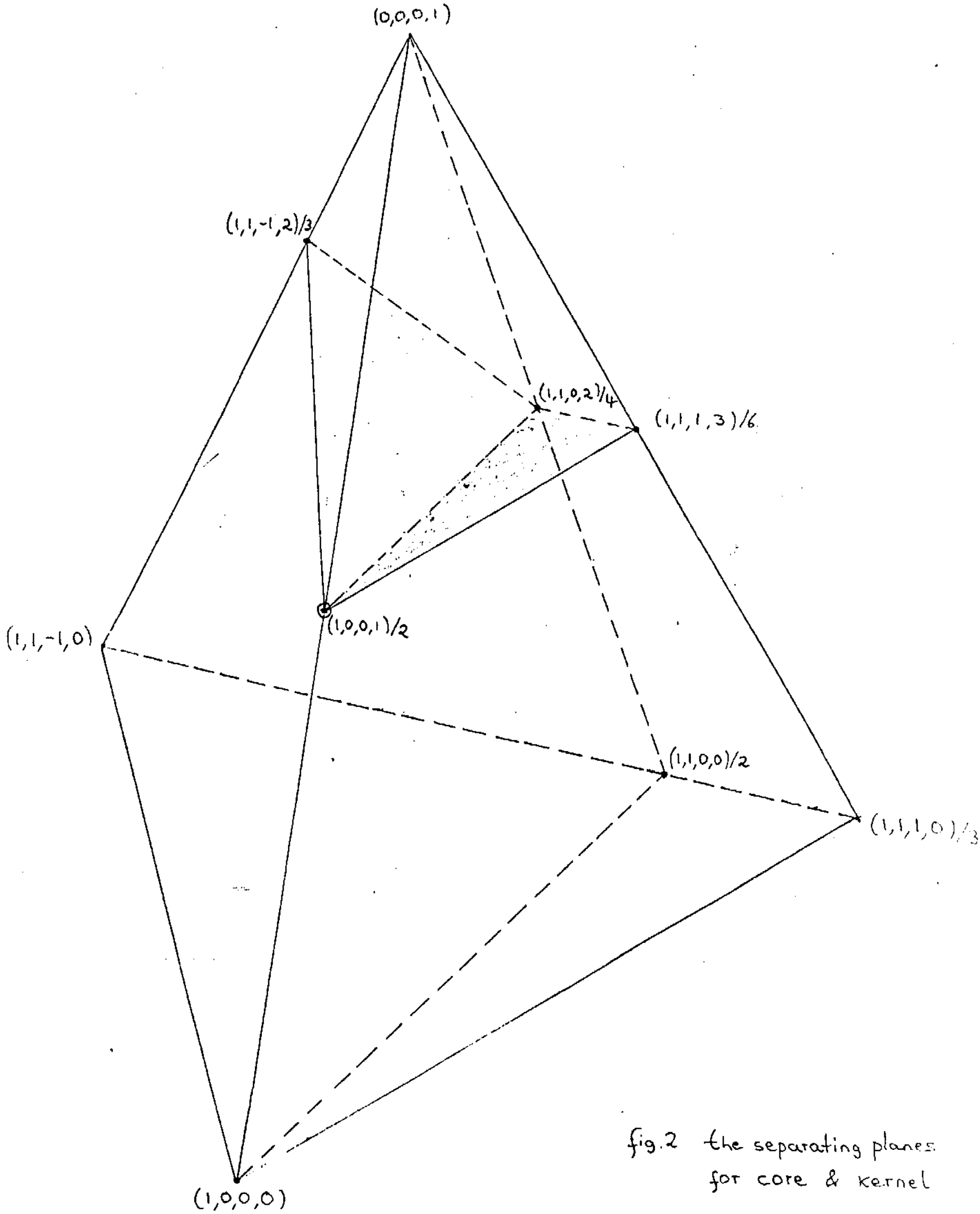

fig.2 the separating planes for core & kernel

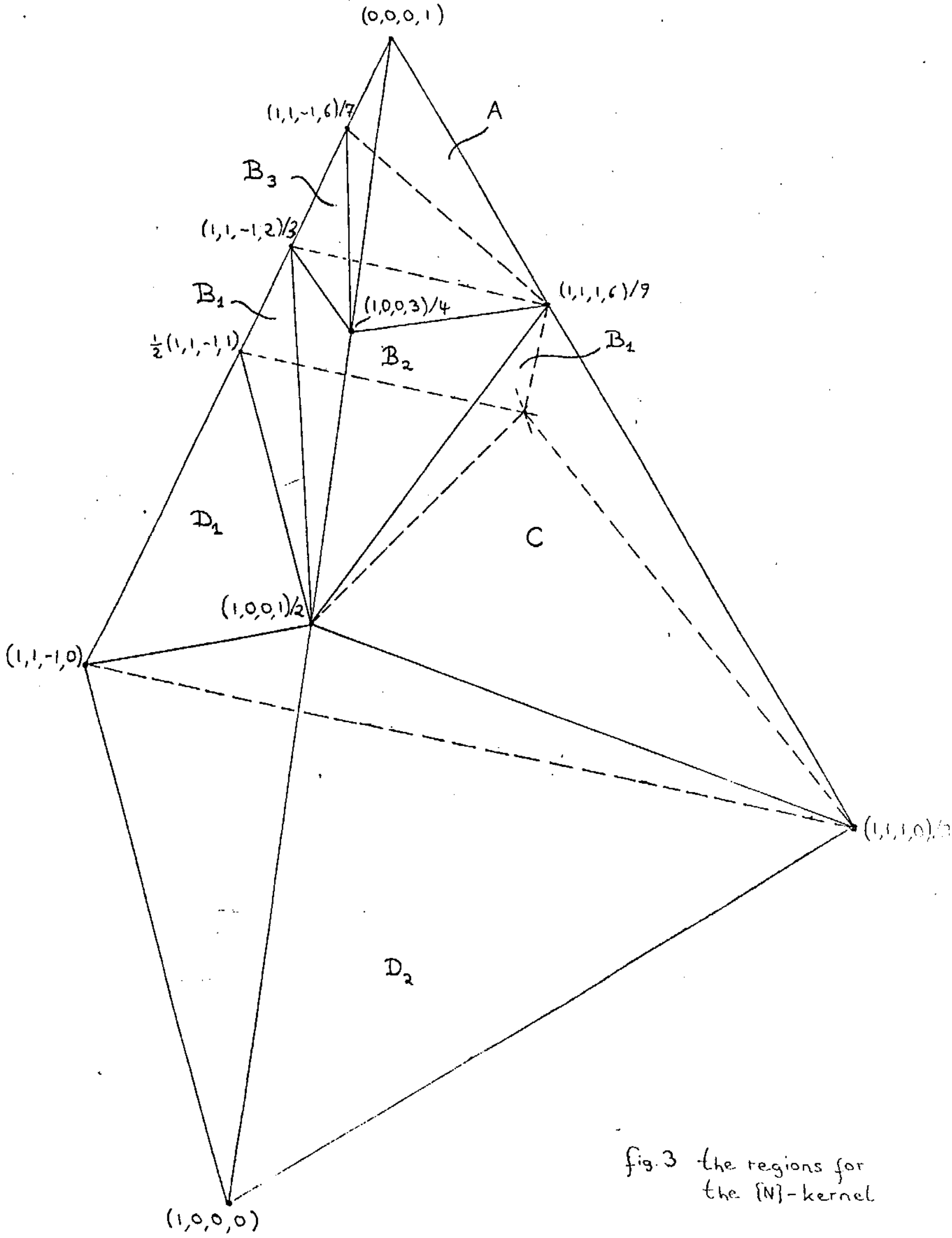

fig. 3 the regions for the {N}-kernel

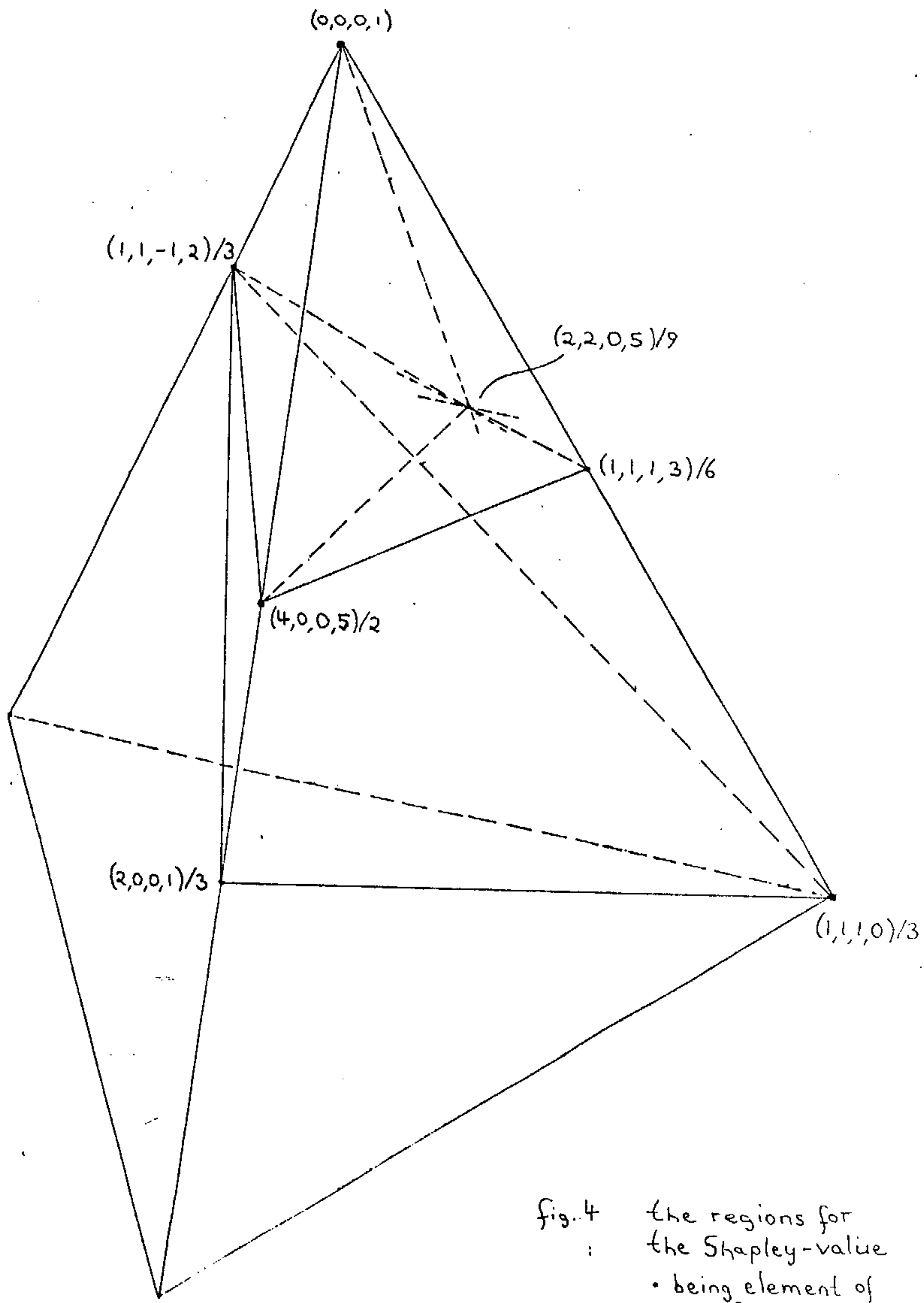

fig. 4 : the regions for the Shapley-value
- being element of the {N}-core
- being individually rational

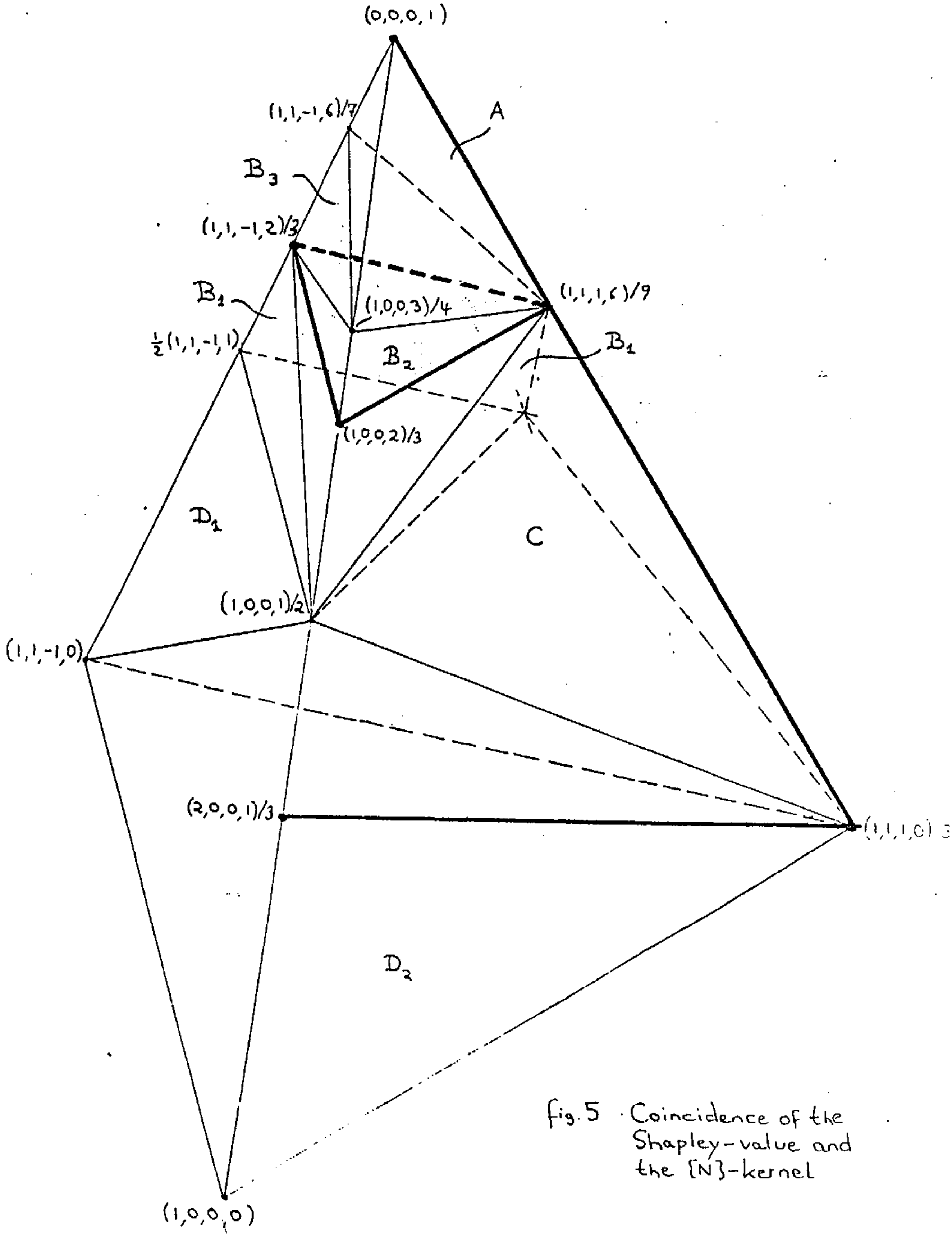

fig. 5 Coincidence of the Shapley-value and the {N}-kernel