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Nash–Implementation with Renegotiation in the Case of Two Agents

by

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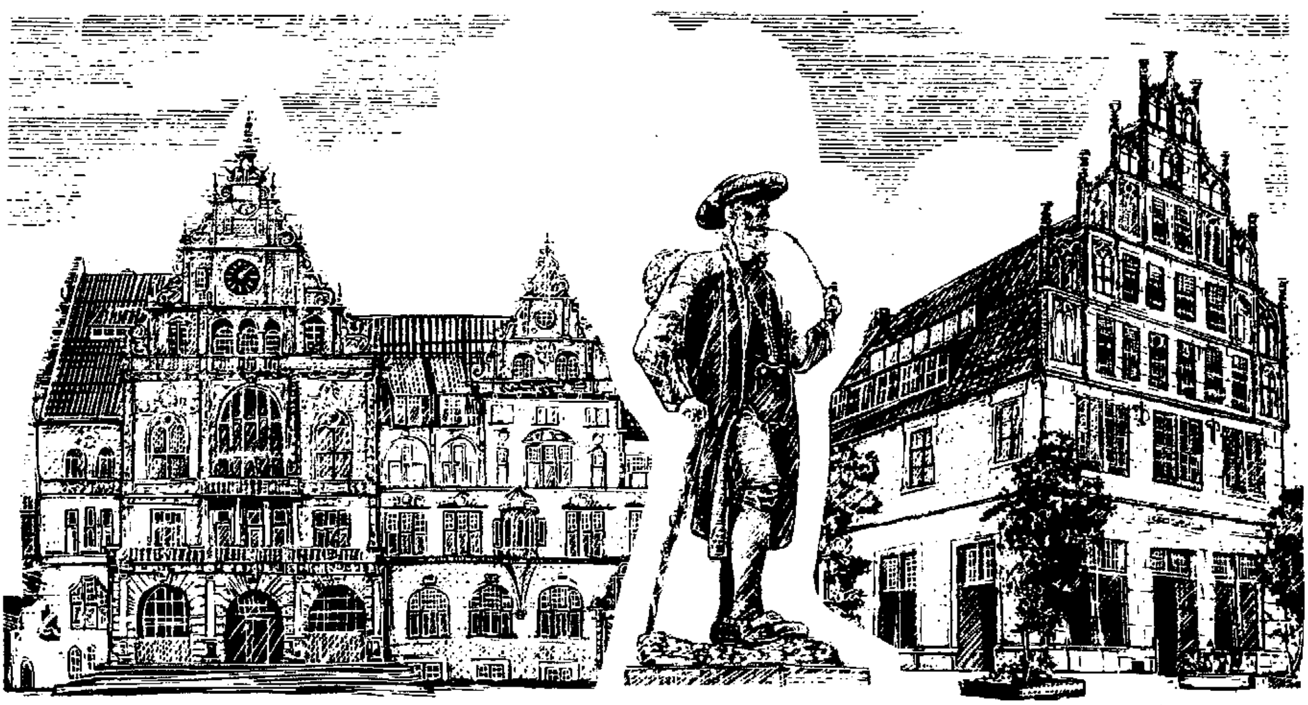

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The present paper supplements Homann (1989) on which it relies heavily. It is not self–contained because the description of the motivating contract model and some results in Homann (1989) are presupposed.

A revelation principle for Nash–implementation with renegotiation is formulated. Furthermore necessary conditions for implementability of a social choice correspondence according to this concept of implementation are specified. At last sufficient conditions for this kind of implementability are derived and they are compared with the sufficient conditions in Homann (1989) which are stronger than the conditions analysed in the present paper.

The following notation is used in this paper.
Let $N = \{1,2\}$ be the set of agents and let X denote the set of possible outcomes. Corresponding to each agent $i \in N$ there is defined a subset $\mathcal{R}_i$ of the set $\mathcal{R}$ of complete, reflexive, transitive binary relations on X. It is assumed that there exists an outcome $x_0 \in X$ with $x\, R_i\, x_0$ for all $x \in X$, all $R_i \in \mathcal{R}_i$ and all $i \in N$. For each element R_i of $\mathcal{R}_i$ relations P_i and I_i on X are defined by $x\, P_i y \Leftrightarrow \neg\, y\, R_i\, x$, $x\, I_i y \Leftrightarrow x\, R_i y \wedge y R_i x$ for all $x,y \in X$ and all $i \in N$. A finite set Θ of states of the world with $|\Theta| > 1$ is given. Each state $\vartheta \in \Theta$ and each $i \in N$ determine a preference relation $R_i(\vartheta) \in \mathcal{R}_i$ of agent i in state ϑ. A mechanism g with sets S_1, S_2 is a function $g : S_1 \times S_2 \rightarrow X$ whereby S_i is the set of actions of agent $i \in N$. A mechanism g determines an outcome for both agents which depends on the choice of an action by each agent. A mechanism g and a state $\vartheta \in \Theta$ induce a normal–form–game (g,ϑ) with complete information whereby the agent $i \in N$ evaluates the outcome determined by g and the chosen actions according to his preference relation $R_i(\vartheta)$. A renegotiation function is a function $h : X \times \Theta \rightarrow X$ with the following properties:

1) $h(x,\vartheta)\, R_i(\vartheta) x$ for all $x \in N$, all $\vartheta \in \Theta$ and all $i \in N$ (Individually rational),

2) $h(x,\vartheta) \in SP(\vartheta)$ for all $\vartheta \in \Theta$ and all $x \in X$ with
$SP(\vartheta) = \{z \in X \mid \text{for all } y \in X : z\, I_i(\vartheta) y \text{ for all } i \in N \text{ or it exists an } i \in N \text{ with } z\, P_i(\vartheta) y\}$
for all $\vartheta \in \Theta$ (Pareto–efficient),

3) $h(x,\vartheta) = x$ for all $x \in SP(\vartheta)$ and all $\vartheta \in \Theta$.

A mechanism g, a renegotiation function h and a state $\vartheta \in \Theta$ induce a normal–form–game $(h(g(\cdot),\vartheta),\vartheta)$ with complete information.

In the following the term $(\bar{s}_i, s_{-i})$ with $i \in N$, $\bar{s}_i \in S_i$, $s \in S_1 \times S_2$ denotes the element $\tilde{s} \in S_1 \times S_2$ with $\tilde{s}_i = \bar{s}_i$ and $\tilde{s}_j = s_j$ for $j \in N$, $j \neq i$.

The set of Nash–equilibria (in pure strategies) of a game (g,ϑ) with $g : S_1 \times S_2 \to X$ and $\vartheta \in \Theta$ is defined by

$$NE(g,\vartheta) = \{(\bar{s}_1,\bar{s}_2) \in S_1 \times S_2 \mid \text{for all } i \in N : g(\bar{s}_1,\bar{s}_2)\, R_i(\vartheta)\, g(s_i,\bar{s}_{-i}) \text{ for all } s_i \in S_i\}.$$

The set of Nash–outcomes of a game (g,ϑ) is denoted by
$NO(g,\vartheta) = \{x \in X \mid \text{it exists } (s_1,s_2) \in NE(g,\vartheta) \text{ with } x = g(s_1,s_2)\}$.

The set of Nash–equilibria of a game $(h(g(\cdot),\vartheta),\vartheta)$ and the set of Nash–outcomes of such a game are defined analogously.

A social choice function (SCF) is a function $f : \Theta \to X$. An SCF f induces a social choice correspondence (SCC) $F : \Theta \twoheadrightarrow X$ such that $F(\vartheta) = \{x \in X \mid x\, I_i(\vartheta)\, f(\vartheta) \text{ for all } i \in N\}$ for all $\vartheta \in \Theta$. In the following it is assumed that $f(\vartheta) \in SP(\vartheta)$ for all $\vartheta \in \Theta$ and that the considered SCC F is induced by the considered SCF f.
A mechanism g implements an SCC F in Nash–equilibria with an a priori given renegotiation function h if it holds:

4) for all $\vartheta \in \Theta$: $NO(h(g(\cdot),\vartheta),\vartheta) \neq \phi$ and $NO(h(g(\cdot),\vartheta),\vartheta) \subset F(\vartheta)$.

A mechanism g implements an SCC F in Nash–equilibria with renegotiation if 4) is satisfied for all renegotiation functions h (i.d. with properties 1), 2), 3)).

In the following necessary conditions and a revelation principle for Nash–implementation with renegotiation are derived.
For this purpose it is necessary to modificate the following condition 5) which is identical with 7) in Homann (1989) (ib. p.10). This condition requires that a mechanism g and a renegotiation function h satisfy:

5) for all $\vartheta \in \Theta$ it exists an $s \in NE(h(g(\cdot),\vartheta),\vartheta)$ with $h(g(s),\vartheta) = g(s)$.

This means that for all $\vartheta \in \Theta$ at least one Nash–equilibrium of the game with renegotiation $(h(g(\cdot),\vartheta),\vartheta)$ already yields the same outcome in the game without renegotiation (g,ϑ). It is obvious that this Nash–equilibrium is also a Nash–equilibrium of (g,ϑ). Therefore a kind of revelation principle can be formulated in the case of Nash–implementation with an a priori given renegotiation function h by a mechanism g whereby g and h satisfy 5) (cf. Homann (1989), p.11).

A modification of 5) yields the condition that a mechanism g and a renegotiation function h satisfy:

6) for all $\vartheta \in \Theta$ it exists an $s \in NE(h(g(\cdot),\vartheta),\vartheta)$ with

$g(s) \in SP(\vartheta)$ ($\Leftrightarrow h(g(s),\vartheta) = g(s)$) or $g(s) \in A(\vartheta)$

whereby $A(\vartheta) = \{x \in X \backslash SP(\vartheta) \mid$ it exists $z \in SP(\vartheta)$ such that for all $i \in N$: $z\ R_i(\vartheta) x$ and for all $\tilde{z} \in SP(\vartheta) \backslash \{z' \in X \mid$ for all $i \in N$: $z'\ I_i(\vartheta)\ z\}$ it exists a $j \in N$ with $x\ P_j(\vartheta)\ \tilde{z}\}$ for all $\vartheta \in \Theta$.

For $x \in SP(\vartheta)$ the element $h(x,\vartheta)$ is uniquely determined by the property 3). For $x \in A(\vartheta)$ any different renegotiation functions h, $\tilde{h}$ with 1), 2), 3) yield $h(x,\vartheta) I_i(\vartheta) \tilde{h}(x,\vartheta)$ for all $i \in N$. For $x \notin SP(\vartheta) \cup A(\vartheta)$ there exist renegotiation functions h, $\tilde{h}$ with 1), 2), 3) such that it exists a $j \in N$ with $h(x,\vartheta)\ P_j(\vartheta)\ \tilde{h}(x,\vartheta)$.

In many considered models it holds $A(\vartheta) = \phi$ for all $\vartheta \in \Theta$ such that 6) is identical with 5). For example this is implied by the assumption that for all $\vartheta \in \Theta$ all $i \in N$ and all $z \in X \backslash SP(\vartheta)$ there exists an $x \in SP(\vartheta)$ with $x\ I_i(\vartheta) z$.

Condition 6) is reasonable because thus the agents can be sure that a certain Nash–equilibrium of the game with renegotiation can be played without knowledge of the applied renegotiation function. Now a kind of revelation principle can be shown which is a generalization of the result in Homann (1989) (ib. p.11).

Proposition 1:

If an SCC F is implemented in Nash–equilibria with an a priori given renegotiation function h by the mechanism g and if g and h satisfy 6) then there exists a revelation mechanism $\tilde{g} : \Theta \times \Theta \rightarrow X$ with $\tilde{g}(\vartheta,\vartheta) = f(\vartheta)$ for all $\vartheta \in \Theta$ and $\tilde{g}(\vartheta,\phi) = g(s_1(\vartheta),s_2(\phi))$ for all $\vartheta,\phi \in \Theta$, $\vartheta \neq \phi$ whereby $s(\vartheta) = (s_1(\vartheta),\ s_2(\vartheta))$ is a Nash–equilibrium of $(h(g(\cdot),\vartheta),\vartheta)$

considered in condition 6). This mechanism $\tilde{g}$ implements f (and F) truthfully in Nash–equilibria (i.d. $\forall \vartheta \in \Theta : (\vartheta,\vartheta) \in NE(\tilde{g},\vartheta)$).

Proof:

An element $s \in NE(h(g(\cdot),\vartheta),\vartheta)$ with $g(s) \in SP(\vartheta) \cup A(\vartheta)$ is considered (namely: $s = s(\vartheta)$). Because of the assumption it yields $h(g(s),\vartheta) \in F(\vartheta)$.

Two cases are distinguished:

a) $g(s) \in SP(\vartheta)$: Because of 1) it is obvious that $s \in NE(g,\vartheta)$ and therefore it results $(\vartheta,\vartheta) \in NE(\tilde{g},\vartheta)$.

b) $g(s) \in A(\vartheta)$: For all $i \in N$ and all $\tilde{s}_i \in S_i$ it holds $h(g(s),\vartheta)\ R_i(\vartheta)\ h(g(\tilde{s}_i,s_{-i}),\vartheta)$.

With fixed $i \in N$, $\tilde{s}_i \in S_i$ either it holds $f(\vartheta)\ I_j(\vartheta)\ h(g(\tilde{s}_i,s_{-i}),\vartheta)$ for all $j \in N$ and therefore $f(\vartheta)\ R_j(\vartheta)\ g(\tilde{s}_i,s_{-i})$ for all $j \in N$ or it holds $h(g(s),\vartheta)\ P_i(\vartheta)\ h(g(\tilde{s}_i,s_{-i}),\vartheta)$ and $h(g(\tilde{s}_i,s_{-i}),\vartheta)\ P_j(\vartheta)\ h(g(s),\vartheta)$ for $j \in N$, $j \neq i$. This implies $h(g(\tilde{s}_i,s_{-i}),\vartheta)\ P_j(\vartheta)\ g(s)$ because of 1) and therefore $g(s)\ P_i(\vartheta)\ h(g(\tilde{s}_i,s_{-i}),\vartheta)$ because $g(s) \in A(\vartheta)$. Thus it results $g(s)\ P_i(\vartheta)\ g(\tilde{s}_i,s_{-i})$ because of 1). This yields $(\vartheta,\vartheta) \in NE(\tilde{g},\vartheta)$.

□

From the proof it becomes evident that only because of the case (it exist $\vartheta \in \Theta$, $s \in S_1 \times S_2$ with: $g(s) \in A(\vartheta)$, $h(g(s),\vartheta) \in F(\vartheta)$ and it exist $i \in N$, $\tilde{s}_i \in S_i$ such that $h(g(s),\vartheta)\ I_j(\vartheta)\ h(g(\tilde{s}_i,s_{-i}),\vartheta)$ for all $j \in N$) the revelation mechanism $\bar{g}(\vartheta,\phi) = g(s_1(\vartheta),s_2(\phi))$ for all $\vartheta,\phi \in \Theta$ instead of $\tilde{g}$ generally does not provide the result of proposition 1.

Proposition 1 is a first step in formulating a revelation principle. The following classification serves for the derivation of a revelation principle and of necessary conditions for implementability. It modificates the classification 8)a) – 8)d) in Homann (1989) (ib. p.12) in such a way that 8)d) is divided in three subcases.

A mechanism $g : S_1 \times S_2 \rightarrow X$ is given and an arbitrary $h : X \times \Theta \rightarrow X$ with 1), 2), 3) is considered. If there exist $\vartheta \in \Theta$, $s \in S_1 \times S_2$, $i \in N$, $\tilde{s}_i \in S_i$ with $g(s) = f(\vartheta)$ and $f(\vartheta)\ R_i(\vartheta)\ g(\tilde{s}_i,s_{-i})$ then six cases can be distinguished for $y = g(\tilde{s}_i,s_{-i})$:

7)a) It holds: $f(\vartheta)\ P_i(\vartheta)\ y$ and for $j \in N,\ j \neq i : y\ P_j(\vartheta)\ f(\vartheta)$.
This implies: $h(y,\vartheta)\ P_j(\vartheta)\ f(\vartheta)$ and $f(\vartheta)\ P_i(\vartheta)\ h(y,\vartheta)$
because of $h(y,\vartheta) \in SP(\vartheta)$ and $|N| = 2$.

7)b) It holds: $f(\vartheta)\ P_i(\vartheta)\ y$ and for $j \in N,\ j \neq i : y\ I_j(\vartheta)\ f(\vartheta)$.
This implies: $f(\vartheta)\ R_i(\vartheta)\ h(y,\vartheta)$ because otherwise $f(\vartheta) \notin SP(\vartheta)$.

7)c) It holds: $f(\vartheta)\ I_i(\vartheta)\ y$ and for $j \in N,\ j \neq i : f(\vartheta)\ I_j(\vartheta)\ y$.
This implies: $h(y,\vartheta) = y$ and $y \in F(\vartheta)$.

7)d_1) It holds: $f(\vartheta)\ R_i(\vartheta)\ y$ and for $j \in N,\ j \neq i : f(\vartheta)\ P_j(\vartheta)\ y$.
Also for all $z \in SP(\vartheta)$ with $z\ P_i(\vartheta)\ f(\vartheta)$ it holds: $y\ P_j(\vartheta)\ z$.
This implies: $f(\vartheta)\ R_i(\vartheta)\ h(y,\vartheta)$ because of 1), 2).

7)d_2) It holds: $f(\vartheta)\ R_i(\vartheta)\ y$ and for $j \in N,\ j \neq i : f(\vartheta)\ P_j(\vartheta)\ y$.
Also for all $z \in SP(\vartheta)$ it holds $f(\vartheta)\ R_i(\vartheta)\ z$.
This implies: $f(\vartheta)\ R_i(\vartheta)\ h(y,\vartheta)$ because of 1), 2).

7)d_3) It holds: $f(\vartheta)\ R_i(\vartheta)\ y$ and for $j \in N,\ j \neq i : f(\vartheta)\ P_j(\vartheta)\ y$.
Also it exists a $z \in SP(\vartheta)$ with $z\ P_i(\vartheta)\ f(\vartheta)$ and $z\ R_j(\vartheta)\ y$.
Then there exist renegotiation functions $h, \tilde{h}$ with 1), 2), 3)
such that $f(\vartheta)\ R_i(\vartheta)\ h(y,\vartheta)$, $\tilde{h}(y,\vartheta)\ P_i(\vartheta)\ f(\vartheta)$.

Only in the case 7)d_3) the properties 1), 2), 3) do not determine the relation of agent i in state ϑ between $h(y,\vartheta)$ and $f(\vartheta)$ for any renegotiation function h.

In the case 7)d_2) the form of the game $(h(g(\cdot),\vartheta),\vartheta)$ with any renegotiation function h can be specified more precisely. If $s \in S_1 \times S_2$ is a Nash–equilibrium of the game $(h(g(\cdot),\vartheta),\vartheta)$ with a fixed renegotiation function h and if $h(g(s),\vartheta) \in F(\vartheta)$ then 7)d_2) implies $h(g(\bar{s}_j,s_{-j}),\vartheta) \in F(\vartheta)$ for all $\bar{s}_j \in S_j$ and therefore $f(\vartheta)\ R_k(\vartheta)\ g(\bar{s}_j,s_{-j})$ for all $k \in N$ and all $\bar{s}_j \in S_j$. Because of $f(\vartheta)\ R_i(\vartheta)\ h(g(\tilde{s}),\vartheta)$ for all $\tilde{s} \in S_1 \times S_2$ it results $(\bar{s}_j,s_{-j}) \in NE(h(g(\cdot),\vartheta),\vartheta)$ for all $\bar{s}_j \in S_j$.

The classification 7) essentially relies on the assumption $|N| = 2$. If $|N| > 2$ is presumed then there are more cases in which renegotiation functions $h, \tilde{h}$ with 1), 2), 3) exist such that $f(\vartheta)\ R_i(\vartheta)\ h(y,\vartheta)$ and $\tilde{h}(y,\vartheta)\ P_i(\vartheta)\ f(\vartheta)$.

The case $(f(\vartheta)\ R_i(\vartheta)y$ and for all $k \in N,\ k \neq i : f(\vartheta)\ R_k(\vartheta)y$ and it exists $j \in N,\ j \neq i$ with $f(\vartheta)\ P_j(\vartheta)y)$ corresponds to the case $7)d_1) \vee 7)d_2) \vee 7)d_3)$ and the subcase corresponding to $7)d_3)$ implies the existence of such renegotiation functions.

But in addition the case $(f(\vartheta)\ R_i(\vartheta)y$ and it exists $\{j,k\} \subset N\backslash\{i\}$ with $yP_j(\vartheta)\ f(\vartheta)$ and $f(\vartheta)\ P_k(\vartheta)y)$ also has this property.

In the following it is said that an element $y \in X$ satisfies 7)a) w.r.t. $(i,\vartheta) \in N \times \Theta$ if the subcase 7)a) of the case $f(\vartheta)\ R_i(\vartheta)\ y$ holds (analogously this definition is applied to the other cases of 7).

This classification is used in the proof of the following proposition.

Proposition 2:

An SCC F and a mechanism $g : S_1 \times S_2 \to X$ are given.
If it holds: $NO(h(g(\cdot),\vartheta),\vartheta) \neq \phi$ and $NO(h(g(\cdot),\vartheta),\vartheta) \subset F(\vartheta)$ for all $h : X \times \Theta \to X$ with 1), 2), 3) then g and each h with 1), 2), 3) satisfy 6).

Proof:

It is assumed that it exists an $\bar{h} : X \times \Theta \to X$ with 1), 2), 3) such that 6) is not fulfilled i.d. it exists $\vartheta \in \Theta$ such that for all $s \in NE(\bar{h}(g(\cdot),\vartheta),\vartheta) : g(s) \notin SP(\vartheta) \cup A(\vartheta)$. This ϑ is considered in the following.

For each $s \in \bar{S}$ with
$\bar{S} = \{s \in S_1 \times S_2 |\ g(s) \in SP(\vartheta) \cup A(\vartheta) \wedge h(g(s),\vartheta) \in F(\vartheta)$ for all renegotiation functions h$\}$
the sets $M_i(s) = \{g(\tilde{s}_i, s_{-i}) |\ \tilde{s}_i \in S_i,\ \tilde{s}_i \neq s_i\}$ with $i \in N$ are considered. If there exists an $s \in \bar{S}$ such that for all $i \in N$ each element of $M_i(s)$ satisfies $7)a) \vee 7)b) \vee 7)c) \vee 7)d_1) \vee 7)d_2)$ w.r.t. (i,ϑ) then for this $s \in \bar{S}$ it holds:
$s \in NE(h(g(\cdot),\vartheta),\vartheta)$ for all $h : X \times \Theta \to X$ with 1), 2), 3).

This is in contradiction with the assumption for $\bar{h}$ because for this $s \in \bar{S}$:

$s \in NE(\bar{h}(g(\cdot),\vartheta),\vartheta)$ and $g(s) \in SP(\vartheta) \cup A(\vartheta)$.

Therefore it must hold: for all $s \in \bar{S}$ it exist an $i \in N$ and a $y \in M_i(s)$ such that y does not satisfy 7)a) v 7)b) v 7)c) v 7)d_1) v 7)d_2) w.r.t. (i,ϑ) (because $\bar{h}(y,\vartheta)\ \ P_i(\vartheta)\ f(\vartheta)$).

Now a function $\tilde{h} : X \times \Theta \to X$ with 1), 2), 3) can be constructed in such a way that $\tilde{h}(y,\vartheta) = \bar{h}(y,\vartheta)$ for all $y \in M$ with

$$M = \bigcup_{i \in N} \{y \in \bigcup_{s \in \bar{S}} M_i(s) \mid \bar{h}(y,\vartheta) \notin F(\vartheta) \text{ and}$$
$$y \text{ does not satisfy 7)a) v 7)b) v 7)c) v 7)}d_1\text{) v 7)}d_2\text{) w.r.t. } (i,\vartheta)\}.$$

and $\tilde{h}(g(s),\vartheta) \notin F(\vartheta)$ for all $s \notin \bar{S}$.

These both definitions do not contradict each other because $\bar{h}(y,\vartheta) \notin F(\vartheta)$ for all $y \in M$. The identity of $\tilde{h}$ with $\bar{h}$ on $M \times \{\vartheta\}$ guarantees $s \notin NE(\tilde{h}(g(\cdot),\vartheta),\vartheta)$ for $s \in \bar{S}$. Therefore it results $NO(\tilde{h}(g(\cdot),\vartheta),\vartheta) \cap F(\vartheta) = \phi$.

But this result is in contradiction with the assumption that g implements F in Nash–equilibria with renegotiation. □

Thus in the case of implementation in Nash–equilibria with renegotiation condition 6) does not cause a restriction of the set of all renegotiation functions.

Propositions 1 and 2 lead to a revelation principle for Nash–implementation with renegotiation.

<u>Proposition 3:</u>

If an SCC F is implemented in Nash–equilibria with renegotiation by the mechanism g then there exists a revelation mechanism $\tilde{g} : \Theta \times \Theta \to X$ which implements f (and F) truthfully in Nash–equilibria.

<u>Proof:</u>

Proposition 2 yields that g and each renegotiation function h with 1), 2), 3) satisfy 6). Therefore any h with 1), 2), 3) can be chosen and the revelation mechanism $\tilde{g}$ induced by this h according to the definition in proposition 1 can be considered. Then proposition 1 provides the result. □

The next proposition serves to determine necessary conditions for implementability of an SCC F in Nash–equilibria with renegotiation.

Proposition 4:

If a mechanism g implements an SCC F in Nash–equilibria with renegotiation then for all $\vartheta \in \Theta$ one of the following cases a), b) holds:

a) it exists an $s \in S_1 \times S_2$ with $h(g(s),\vartheta) \in F(\vartheta)$ for all h with 1), 2), 3) and for all $i \in N$ and all $\tilde{s}_i \in S_i$ the element $g(\tilde{s}_i,s_{-i})$ satisfies 7)a) ∨ 7)b) ∨ 7)c) ∨ 7)d_1) ∨ 7)d_2) w.r.t. (i,ϑ) (i.d. $s \in NE(h(g(\cdot),\vartheta),\vartheta)$ for all renegotiation functions h),

b) it exist $s,\bar{s} \in S_1 \times S_2$ with $\{h(g(s),\vartheta),h(g(\bar{s}),\vartheta)\} \subset F(\vartheta)$ for all h with 1), 2), 3) and it exist $y \in X$, $i \in N$, $\tilde{s}_i \in S_i$, $\hat{s}_j \in S_j$ with $j \in N$, $j \neq i$ such that: y satisfies 7)d_3) w.r.t. (i,ϑ), (j,ϑ) ($\Rightarrow f(\vartheta)\ P_i(\vartheta)y,\ f(\vartheta)\ P_j(\vartheta)y$) and $g(\tilde{s}_i,s_{-i}) = y, g(\hat{s}_j,\bar{s}_{-j}) = y$.

Proof:

For fixed $\vartheta \in \Theta$ the sets

$S = \{s \in S_1 \times S_2 \mid g(s) \in SP(\vartheta) \cup A(\vartheta),\ h(g(s),\vartheta) \in F(\vartheta)$ for all h with 1), 2), 3), it exists an $\tilde{h}$ with 1), 2), 3) such that $s \in NE(\tilde{h}(g(\cdot),\vartheta),\vartheta)\}$

and $\bar{S} = \{s \in S \mid$ it exist $i \in N$, $\tilde{s}_i \in S_i$ with: $g(\tilde{s}_i,s_{-i})$ satisfies 7)d_3) w.r.t. $(i,\vartheta)\}$

are considered.

Because of proposition 2 the mechanism g and each h with 1), 2), 3) satisfy 6). Condition 6) implies that S is not empty. In the proof of proposition 1 it is shown that for all $s \in S$, all $i \in N$ and all $\tilde{s}_i \in S_i$ it holds:

$f(\vartheta)\ R_i(\vartheta)\ g(\tilde{s}_i, s_{-i})$ (i.d. $g(\tilde{s}_i, s_{-i})$ satisfies 7)a) ∨ 7)b) ∨ 7)c) ∨ 7)d_1) ∨ 7)d_2) ∨ 7)d_3) w.r.t. (i,ϑ)).

If a renegotiation function $\tilde{h}$ with 1), 2), 3) can be constructed in such a way that

(*) $\tilde{h}(g(\tilde{s}_i,s_{-i}),\vartheta)\ P_i(\vartheta)\ f(\vartheta)$

for all $s \in \bar{S}$, $i \in N$, $\tilde{s}_i \in S_i$ with: $g(\tilde{s}_i,s_{-i})$ satisfies 7)d_3) w.r.t. (i,ϑ)

then it holds:

$s \notin NE(\tilde{h}(g(\cdot),\vartheta),\vartheta)$ for all $s \in \bar{S}$ because
$s \in NE(\tilde{h}(g(\cdot),\vartheta),\vartheta)$ implies $\tilde{h}(g(s),\vartheta) \in F(\vartheta)$ which is in contradiction with (*).

Thus for all $s \in NE(\tilde{h}(g(\cdot),\vartheta),\vartheta) \cap S$ it holds: $s \notin \bar{S}$
and therefore: for all $i \in N$ and all $\tilde{s}_i \in S_i$:
$g(\tilde{s}_i,s_{-i})$ satisfies 7)a) ∨ 7)b) ∨ 7)c) ∨ 7)d_1) ∨ 7)d_2) w.r.t. (i,ϑ).
Because of 6) it exists an $s \in NE(\tilde{h}(g(\cdot),\vartheta),\vartheta) \cap S$.

If such a renegotiation function $\tilde{h}$ cannot be constructed then it must exist $s \in \bar{S}$, $i \in N$, $\tilde{s}_i \in S_i$ with: $g(\tilde{s}_i,s_{-i})$ satisfies 7)d_3) w.r.t. (i,ϑ) such that (*) cannot be chosen. Because $y = g(\tilde{s}_i,s_{-i})$ satisfies 7)d_3) w.r.t. (i,ϑ) there exists a $z \in SP(\vartheta)$ with $zP_i(\vartheta)\ f(\vartheta)$ and $zR_j(\vartheta)y$. The only conditions for $\tilde{h}$ are the properties 1), 2), 3) and (*). 1), 2), 3) cannot prevent to choose $\tilde{h}(y,\vartheta) = z$. Only (*) can be in contradiction with the choice $\tilde{h}(y,\vartheta) = z$. Thus it must exist $\bar{s} \in \bar{S}$, $\hat{s}_j \in S_j$ for $j \in N$, $j \neq i$ such that $y = g(\hat{s}_j,\bar{s}_{-j})$ and such that y satisfies
7)d_3) w.r.t. (j,ϑ) (i.d. $f(\vartheta)\ P_i(\vartheta)\ \tilde{h}(y,\vartheta)$, $\tilde{h}(y,\vartheta)\ P_j(\vartheta)\ f(\vartheta)$ because of (*) for $\bar{s}$).

□

Proposition 4 also shows that such a revelation mechanism $\tilde{g}$ as considered in proposition 3 need not implement F in Nash–equilibria with renegotiation because case b) of proposition 4 can occur.

Sufficient conditions for implementability of an SCC according to the considered concept of implementation can be derived from the necessary conditions stated in proposition 4.

If a mechanism g exists such that case a) of proposition 4 is satisfied for all $\vartheta \in \Theta$ then a revelation mechanism $\bar{g} : \Theta \times \Theta \to X$ is induced by g in the following manner:
it is specified $\bar{g}(\vartheta,\vartheta) = f(\vartheta)$ for all $\vartheta \in \Theta$
and if $s(\vartheta) = (s_1(\vartheta), s_2(\vartheta))$ denotes a fixed Nash–equilibrium of $(h(g(\cdot),\vartheta),\vartheta)$ for all renegotiation functions h which is considered in case a) of proposition 4 then it is defined: $\bar{g}(\vartheta,\phi) = g(s_1(\vartheta),s_2(\phi))$ for all $\vartheta,\phi \in \Theta$, $\vartheta \neq \phi$.

Whereas the revelation mechanism used in proposition 3 depends on the chosen renegotiation function the mechanism $\bar{g}$ is the same for all renegotiation functions. The revelation mechanism $\bar{g}$ implements the SCC F in Nash–equilibria with renegotiation. This results from the following.

It is sufficient for an SCC F to be implementable in Nash–equilibria with renegotiation that for each pair $(\vartheta,\phi) \in \Theta \times \Theta$ with $\vartheta \neq \phi$ there exists an $x(\vartheta,\phi) \in X$ which satisfies

8)i) 7)a) $\vee$ 7)b) $\vee$ 7)c) $\vee$ 7)d_1) $\vee$ 7)d_2) w.r.t. $(2,\vartheta)$

and

8)ii) 7)a) $\vee$ 7)b) $\vee$ 7)c) $\vee$ 7)d_1) $\vee$ 7)d_2) w.r.t. $(1,\phi)$.

If 8) is satisfied then the revelation mechanism $\hat{g} : \Theta \times \Theta \to X$ with $\hat{g}(\vartheta,\vartheta) = f(\vartheta)$ for $\vartheta \in \Theta$ and $\hat{g}(\vartheta,\phi) = x(\vartheta,\phi)$ for all $\vartheta,\phi \in \Theta$ with $\vartheta \neq \phi$ implements F in Nash–equilibria with renegotiation because 3) and the classification 7) imply that $\hat{g}$ satisfies for all $h : X \times \Theta \to X$ with 1), 2), 3): $(\vartheta,\vartheta) \in NE(h(\hat{g}(\cdot),\vartheta),\vartheta)$ and $h(\hat{g}(\vartheta,\vartheta),\vartheta) \in F(\vartheta)$ for all $\vartheta \in \Theta$.

The sufficient conditions for an SCC F to be implementable in Nash–equilibria with renegotiation which are used in Homann (1989) (ib. 9)i) – 9)iv), pp. 12–13) are almost identical with the conditions 8). They are different in so far as in Homann (1989) the cases 7)d_1), 7)d_2) are not considered in the conditions. Thus the sufficient conditions in Homann (1989) are stronger than the conditions in this paper.

But often the considered model is constructed in such a manner that the cases 7)d_1), 7)d_2) do not occur. This holds for example if for all $\vartheta \in \Theta$ all $i \in N$ and all $z \in X \backslash SP(\vartheta)$ there exists an $x \in SP(\vartheta)$ with $x I_i(\vartheta) z$.

Reference

Homann, D. (1989), Incomplete Contracts and Nash–Implementation in the Case of Two Agents, Working Paper No.170, Institute of Mathematical Economics, University of Bielefeld.