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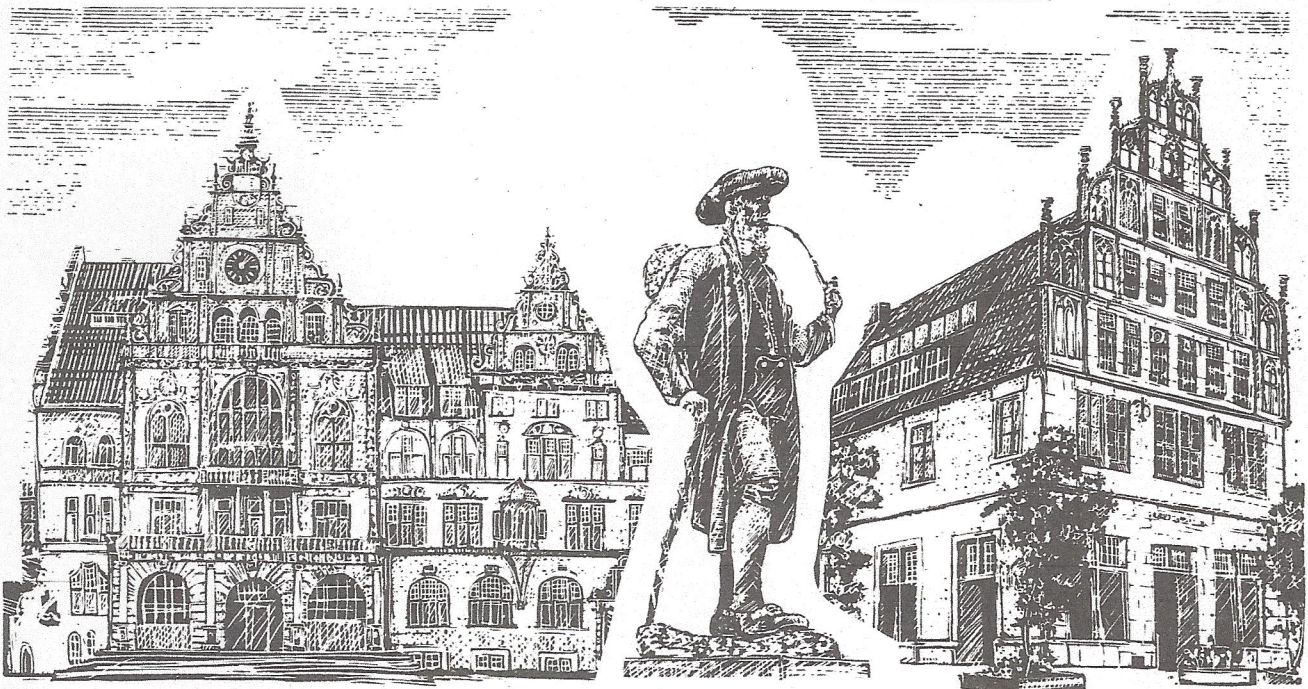
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An Invariance Theorem for
Preferences and Some Applications

Walter Trockel
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H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der

Universität Bielefeld

Adresse / Address:

Universitätsstraße

4800 Bielefeld 1

Bundesrepublik Deutschland

Federal Republic of Germany

0. Introduction

The present work originates from a problem arising in the analysis of aggregate consumer demand, when consumers are allowed to have non-convex preferences. For a thorough treatment of this topic, known under the heading "Smoothing Demand by Aggregation", and for further references the reader is referred to TROCKEL (1984).

A main difficulty in deriving a continuous or even continuously differentiable aggregate demand function in this context consists in finding an adequate formalization of suitable dispersion for distributions on a space \mathcal{P} of preferences. This problem was solved by E.DIERKER, H.DIERKER, and TROCKEL (1984) by making the space \mathcal{P} of preferences a G -space, the group G acting on \mathcal{P} being the finite-dimensional space of price-variations. The transportation of the Lebesgue measure - more precisely the Haar measure on G - to the different orbits in \mathcal{P} provides the basis for formalizing a suitable dispersion assumption for distributions on \mathcal{P} . Any potential smoothing effect on the demand aggregated over all preferences in an orbit relies, therefore, on the availability of a Lebesgue-continuous probability. Accordingly, orbits are not allowed to degenerate to singletons if this approach is expected to work. Hence, we are led to the question as to which preferences are invariant under the group actions, "price-invariant" for short.

This problem has been posed by GRANDMONT (1987), who also quoted the Cobb-Douglas representable preferences as an example for a class of monotone continuous price-invariant preferences which, however, are strictly convex.

In the present paper this problem is solved by the application of our main result (Proposition 1.1) which gives a classification of translation-invariant continuous preferences on a finite-dimensional vector space. This is done by exploiting the fact that price-invariant preferences on \mathbb{R}_{++}^1

correspond, by isomorphism, to translation-invariant preferences on \mathbb{R}^1 . But analysis of invariance-properties for preferences on \mathbb{R}^1 amounts to analysis of social welfare orderings. One might expect therefore to find an answer to our problem in the social welfare literature. Indeed, if we restrict our interest to monotone preferences, the problem can be solved by application of some versions of a utilitarianism theorem, for instance MASKIN (1978), GEVERS (1979), ROBERTS (1980) and the survey article of D'ASPROMONT (1985). In fact, the earliest and at the same time most general utilitarianism result on monotone social welfare orderings is Theorem 4.3.1 in BLACKWELL and GIRSHICK (1954) which was stated, however, in the framework of individual decision making under uncertainty. In the social welfare framework monotony assumptions are quite natural, since they represent the Pareto principle. In demand theory, however, there is no reason to a priori restrict attention to monotone preferences. Therefore, one needs a more general result. But such a result, once it is available, represents at the same time also quite a general version of a utilitarianism theorem.

In the first section we shall state and prove our main results: Propositions 1.1 and 1.2 and discuss some technical aspects. Section 2 consists of an application of Proposition 1.1 to the price-invariance problem in demand analysis. In section 3 we apply our results to social choice and discuss the relation to certain known results on utilitarianism. Section 4 contains some concluding remarks.

1. Translation-Invariant Preferences

Let R be a preference relation on \mathbb{R}^1 , $l \in \mathbb{N}$, i.e. a complete transitive binary relation on \mathbb{R}^1 . For given R we denote by I and P , respectively, the associated indifference and strict preference relation. Formally:

$$\forall x, y \in \mathbb{R}^1 : \begin{aligned} xIy &\iff xRy \text{ and } yRx \\ xPy &\iff xRy \text{ and not } yRx. \end{aligned}$$

For any $x \in \mathbb{R}^1$ the set $I_x := \{y \in \mathbb{R}^1 \mid yIx\}$ is the (R-) indifference set of x .

Moreover, we define

$$\begin{aligned} R^{-1}(x) &:= \{y \in \mathbb{R}^1 \mid yRx\}, & P^{-1}(x) &:= \{y \in \mathbb{R}^1 \mid yPx\} \\ R(x) &:= \{y \in \mathbb{R}^1 \mid xRy\}, & P(x) &:= \{y \in \mathbb{R}^1 \mid xPy\}. \end{aligned}$$

Definition 1.1: The preference relation R on \mathbb{R}^1 is upper (resp. lower) semi-continuous at $x \in \mathbb{R}^1$ iff $R^{-1}(x)$ (resp. $R(x)$) is closed. R is upper (resp. lower) semi-continuous if it is so at every $x \in \mathbb{R}^1$. R is continuous (at $x \in \mathbb{R}^1$) iff it is upper and lower semi-continuous (at $x \in \mathbb{R}^1$).

Recall that R is upper resp. lower semi-continuous resp. continuous iff it allows an upper resp. lower semi-continuous resp. continuous utility representation.

Definition 1.2: A preference relation R on \mathbb{R}^1 is called (translation-)invariant iff

$$\forall x, y, z \in \mathbb{R}^1 : xRy \iff x+z Ry+z$$

Our aim in this section is it to classify invariant preferences. We shall do this by distinguishing between the two cases with and without continuity assumptions on preferences. First, we shall state and prove a proposition

which is based on a continuity assumption on preferences. This is quite natural in demand theory, where our motivation originates from. A second proposition will be derived then without any continuity assumption. Both can be related then to existing results on social welfare orderings.

Proposition 1.1:² Let R be a translation-invariant preference on \mathbb{R}^1 which is (upper or lower) semi-continuous at some $x \in \mathbb{R}^1$. Then R admits a linear utility representation u which is zero iff R is trivial, i.e. $\forall x, y \in \mathbb{R}^1: x I y$.

Proof: By translation-invariance we have for all $x \in \mathbb{R}^1$, $y I x \iff y - x I 0$, hence $I_x = x + I_0$. For the same reason we have for all $x \in \mathbb{R}^1$

$$\begin{aligned} R(x) &= x + R(0), \\ R^{-1}(x) &= x + R^{-1}(0). \end{aligned}$$

Hence, semi-continuity at some x implies semi-continuity. Since $R^{-1}(0) = -R(0)$, semi-continuity of R implies continuity.

Let $x, y \in I_0$, i.e. $x I 0 I y$. By invariance we get $x - y I -y I 0$, i.e. $x - y, -y \in I_0$. So I_0 is a subgroup of \mathbb{R}^1 . Moreover, $I_0 = R^{-1}(0) \cap R(0)$ is closed. Hence, there are some integers $p, r \leq 1$ such that I_0 is isomorphic as a topological group to $\mathbb{R}^p \times \mathbb{Z}^{r-p}$, while the quotient group \mathbb{R}^1/I_0 is topologically isomorphic to $T^{r-p} \times \mathbb{R}^{n-r}$ ($T \cong \mathbb{R}^{n-1}/\mathbb{Z}$ denotes the circle or torus group), (cf. Theorem 6 in MORRIS (1977)).

Now, for any $x, y \in I_0$ and any $n, m \in \mathbb{Z}$ we have $nx + my \in I_0$.

Clearly, the same holds true then for any $n, m \in \mathbb{Q}$.

²I thank Martin Heuer for an idea which led to a simplification of the original proof.

Therefore the discrete factor z^{r-p} of I_0 must degenerate to 0. Hence, $r = p$ and $I_0 \cong \mathbb{R}^p$, $\mathbb{R}^1 \setminus I_0 \cong \mathbb{R}^{1-p}$.

Now, if $p = 1$, then R is trivial and representable by the zero-functional. Assume therefore that T is non-trivial and let $z \in P - z$. Hence, $P^{-1}(0)$ and $P(0)$ are both open and non-empty. Therefore, $\mathbb{R}^1 \setminus I_0 = P^{-1}(0) \cup P(0)$ cannot be connected, hence $\dim I_0 \geq 1-1$. Since R is non-trivial, we have $\dim I_0 = 1-1$. Now any $p \in I_0^\perp$ defines by the inner product $x \mapsto \langle p, x \rangle$ a linear utility representation u for R .

q.e.d.

The continuity of R has been used in a twofold way in the proof of the above proposition.

First, the closedness of indifference sets I_x , $x \in \mathbb{R}^1$ was caused by continuity. Secondly, the ordering of the closed indifference manifolds in such a way that "above" each indifference set are only better elements and "below" are only worse ones is also due to continuity.

Without continuity of R we are left with cosets I_x , $x \in \mathbb{R}^1$ of a subgroup I_0 of \mathbb{R}^1 which, however, need not be closed, and the indifference manifolds could be ordered in a quite arbitrary way. This would be the case even if the indifference manifolds would be assumed to be closed.

This is possible since the continuity of R which is equivalent to the continuity of its representation $u : \mathbb{R}^1 \rightarrow \mathbb{R}$, amounts to the simultaneous continuity of both of the two maps pr and \tilde{u} into which u can be decomposed as follows:

$$\mathbb{R}^1 \xrightarrow{pr} \mathbb{R}^1 / R \xrightarrow{\tilde{u}} \mathbb{R} : x \mapsto I_x \mapsto u(x)$$

where \mathbb{R}^1 / R is endowed with the identification topology.

Translation-invariance, although it allows to derive continuity of R from semi-continuity at some point, is far from being strong enough to replace continuity or monotony assumptions.

If we want a result in the same spirit as Proposition 1.1 without a continuity assumption on R , we have to use some monotony, or, more generally, some assumption which determines some "better"- and "worse"- directions in \mathbb{R}^1 . Moreover, the identity of R with a preference represented by a linear functional will be weakened to an inclusion.

We shall describe "better"-directions in the following by a proper cone, the linear representations being elements of its dual cone. We have to recall, therefore, these notions.

Definition 1.3 (cf. SCHAEFER (1966)): The set $C \subset \mathbb{R}^1$ is a proper cone iff

i) $C + C \subset C$ ii) $\forall \lambda > 0 : \lambda C \subset C$ iii) $C \cap (-C) = \{0\}$.

The dual cone of a cone $C \subset \mathbb{R}^1$ is the set

$C' := \{p \in \mathbb{R}^1 \mid \forall x \in C: \langle p, x \rangle \geq 0\}$.

Clearly, any proper cone is convex. Moreover, it is well-known that $C'' \equiv C$ if C is closed.

Definition 1.4 (cf. GRODAL et al. (1984)): Let $C \subset \mathbb{R}^1$ be a proper cone. A preference relation R on \mathbb{R}^1 is weakly C-monotone at $x \in \mathbb{R}^1$ iff $x + C R x$. It is weakly C-monotone iff it is so at every $x \in \mathbb{R}^1$.

Proposition 1.2: Let C be a proper cone with non-empty interior $\overset{\circ}{C}$. Let R be a translation-invariant preference relation which is weakly C -monotone at some $\bar{x} \in \mathbb{R}^1$. Then there exists an element p of the dual cone C' such that:

$$\forall x, y \in \mathbb{R}^1 : \langle p, x \rangle > \langle p, y \rangle \implies x P y.$$

Proof: First we show that $c \in P 0$ for all $c \in \overset{\circ}{C}$, briefly $\overset{\circ}{C} \subset P 0$, unless R is trivial. Assume there is $c \in \overset{\circ}{C}$ such that $0 \not R c$. As $\bar{z} + \overset{\circ}{C} \subset R \bar{z}$ by assumption, and thus by invariance $\overset{\circ}{C} \subset R 0$ we get $c \in I 0$. Then, by invariance, $qc \in I 0$ for all $q \in \mathbb{Q}$. Now, for any $x \in \mathbb{R}^1$ there are $q, q' \in \mathbb{Q}$ such that $qc \in x + \overset{\circ}{C}$ and $q'c \in x - \overset{\circ}{C}$. Therefore $0 \in I qc \subset R x \subset R q'c \subset I 0$ for all $x \in \mathbb{R}^1$. So R is trivial.

Now let $M := R^{-1}(0) + \overset{\circ}{C}$. This implies $\overset{\circ}{C} \subset M$. As for any $x \in \mathbb{R}^1$ the set $x + \overset{\circ}{C}$ is open, also $M = \bigcup_{x \in R^{-1}(0)} x + \overset{\circ}{C}$ must be open.

Next we show that M is convex. Let $y = x + c$, $y' = x' + c'$ with $x, x' \in R^{-1}(0)$ and $c, c' \in \overset{\circ}{C}$. Choose $\lambda \in (0, 1)$ and let

$\bar{c} := \lambda c + (1-\lambda) c'$ and $\bar{x} := \lambda x + (1-\lambda) x'$. Now consider $\bar{y} := \bar{x} + \bar{c}$.

As C is a proper cone we have $\bar{c} \in \overset{\circ}{C}$. For $q, q' \in \mathbb{Q}$ close enough to λ and $1-\lambda$, respectively, we have

$\bar{y} = \bar{x} + \bar{c} = qx + q'x' + (\lambda - q)x + (1 - \lambda - q')x' + \bar{c}$ with

$(\lambda - q)x + (1 - \lambda - q')x' + \bar{c} \in \overset{\circ}{C}$. Since $x, x' \in R^{-1}(0)$, by invariance also $qx, q'x' \in R^{-1}(0)$. Thus, $\bar{y} \in M$.

So 0 is on the boundary of the convex open set M . Therefore, there exists a supporting hyperplane to M through 0 , i.e., $\exists p \in \mathbb{R}^1$, $p \neq 0$ such that $\langle p, y \rangle \geq 0$ for any $y \in M$ and, by $\overset{\circ}{C} \subset M$, in particular, for any $y \in \overset{\circ}{C}$. Hence we get $\langle p, y \rangle \geq 0$ for all $y \in C$. Also $\langle p, x \rangle \geq 0$ for any $x \in R^{-1}(0)$. Hence, $\langle p, y - x \rangle < 0$ implies $0 \in P y - x$, and thus, by invariance, $x \in P y$.

q.e.d.

An immediate consequence of our two propositions is the following

Corollary 1.1: A non-trivial continuous preference relation R on \mathbb{R}^1 is translation-invariant if and only if $\exists p \in \mathbb{R}^1, p \neq 0$
 $\forall C \subset \mathbb{R}^1, C$ proper cone with $p \in C$ the preference relation R is C '-monotone.

Our Proposition 1.1 is obviously related to a well-known theorem on the functional equation

$$(1.1) \quad f(x+y) = f(x) + f(y),$$

which states that a solution $f : \mathbb{R} \rightarrow \mathbb{R}$ for this equation is continuous iff it is continuous at some point which is the case if and only if f is linear.

This result does not imply Proposition 1.1 because the invariance of R does not imply that a utility representation u of R fulfills the functional equation

$$(1.2) \quad u(x+y) = u(x) + u(y).$$

For instance,

$$u : \mathbb{R}^2 \rightarrow \mathbb{R} : x \mapsto (x_1+x_2)^2$$

does not solve the functional equation, but represents an invariant preference. However, every additive utility function defines, indeed, an invariant preference relation.

On the other hand Proposition 1.1 could be used for an alternative proof of the result quoted above.

A concluding observation is important for the later application of our results to social choice:

Translation-invariance does not imply the following invariance requirement:

$$(1.3) \quad \forall x, y \in \mathbb{R}^1 \quad \forall \lambda > 0: xRy \iff \lambda x R \lambda y$$

This has been observed already in BLACKWELL and GIRSHICK (1954). To see this one has to consider the set of (not necessarily continuous) solutions of the functional equation (1.1). Indeed, (1.1) implies

$f(mx) = mf(x)$ for all integers m , and, thus for all rational numbers m . Hence, (1.1) is equivalent to

$$(1.4) \quad f(qx+ry) = qf(x) + rf(y) \quad \forall x, y \in \mathbb{R} \quad \forall q, r \in \mathbb{Q}.$$

This means that every solution respects the vector space structure of \mathbb{R} over the field \mathbb{Q} . After choice of a basis B of this vector space, all solutions of (1.4) have the form

$$(1.5) \quad f(x) = \sum_{i=1}^n r_i f(b_i) \text{ whenever } x = \sum_{i=1}^n r_i b_i, \quad b_1, \dots, b_n \in B.$$

Here for any non-continuous solution f of (1.4) or, equivalently, of (1.1), any $\alpha \in \mathbb{R}$ can be represented as

$$\alpha = \lim_{n \rightarrow \infty} f(t_n) \text{ for a suitable sequence } (t_n)_{n \in \mathbb{N}}$$

converging to zero.

Now let R on \mathbb{R} be represented by $u : \mathbb{R} \mapsto \mathbb{R}$ with $u(x+y) = u(x) + u(y)$, where u is not continuous. Clearly, R is translation-invariant. Let $x \neq 0$, and, hence, $0 \neq x$. Take $\alpha > u(x)$. There must be a sequence $(x_n)_{n \in \mathbb{N}}$ converging to zero with $\lim_{n \in \mathbb{N}} u(x_n) = \alpha$. Therefore, there is a small $\lambda > 0$ such that

$$u(\lambda x) > u(x) > 0 > u(-x) > u(-\lambda x).$$

For $y := -x$ and $\mu := 1-\lambda$ we get

$$u(\mu y) = u(\lambda x - x) = u(\lambda x) - u(x) > 0 > u(x) - u(\lambda x) = u(x - \lambda x) = u(\mu x),$$

although $u(y) < u(x)$. Hence, R does not satisfy (1.3).

2. Application to Demand Analysis

Let \succsim be a preference relation on the strictly positive cone \mathbb{R}_{++}^1 of the commodity space \mathbb{R}^1 , $1 \in \mathbb{N}$. The space of (non-normalized) price systems is also modelled by \mathbb{R}_{++}^1 . By η, ξ, ζ we denote commodity bundles by π, ρ, τ price systems. A change of a price system $\pi = (\pi_1, \dots, \pi_1)$ to $\pi' = (\pi'_1, \dots, \pi'_1)$ can be effected by an operator ρ in the following way,

$$\rho * (\pi_1, \dots, \pi_1) = (\pi'_1, \dots, \pi'_1).$$

Obviously, ρ may be identified with the vector $(\pi_1^{-1} \pi'_1, \dots, \pi_1^{-1} \pi'_1)$. Then the operation on \mathbb{R}_{++}^1 represents coordinatewise multiplication. Therefore, $(\mathbb{R}_{++}^1, *)$ can be looked at as the group of price-variations, or, equivalently, of prices, acting on the price space \mathbb{R}_{++}^1 . Similarly, $(\mathbb{R}_{++}^1, *)$ acts on the commodity space \mathbb{R}^1 . This latter action, moreover, induces an action of $(\mathbb{R}_{++}^1, *)$ on a space \mathcal{P} of preferences on \mathbb{R}_{++}^1 in the following way:

$$\alpha: \mathbb{R}_{++}^1 \times \mathcal{P} \rightarrow \mathcal{P} : (\rho, \succsim) \mapsto \succsim_\rho$$

where \succsim_ρ is defined by

$$\rho * \xi \succsim_\rho \rho * \eta : \iff \xi * \eta.$$

We abbreviate $\rho * \xi$ by ξ^ρ .

Definition 2.1: A preference relation \succsim on \mathbb{R}_{++}^1 is called price-invariant iff

$$\forall \rho \in \mathbb{R}_{++}^1 : \succsim \equiv \succsim_{\rho}.$$

Now we shall exploit the fact that price-invariant preferences on \mathbb{R}_{++}^1 correspond in a one-one way to translation-invariant preferences on \mathbb{R}^1 .

Indeed, look at the topological group isomorphism

$$L: (\mathbb{R}_{++}^1, *) \rightarrow (\mathbb{R}^1, +) : \xi = (\xi_1, \dots, \xi_1) \mapsto x = (x_1, \dots, x_1)$$

where $\forall i \in \{1, \dots, 1\} : x_i = \ln \xi_i$.

Its inverse is

$$E : (\mathbb{R}^1, +) \rightarrow (\mathbb{R}_{++}^1, *) : x \mapsto \xi = (\exp x_1, \dots, \exp x_1).$$

The isomorphism L preserves the vector ordering and induces a bijection from \mathcal{P} onto some space of preferences on \mathbb{R}^1 , i.e. $\succsim \mapsto L^*(\succsim)$ via

$$x = L(\xi) \quad L^*(\succsim) \quad L(\eta) = y \iff \xi \succsim \eta.$$

Obviously, $R := L^*(\succsim)$ is translation-invariant if and only if \succsim is price-invariant. To apply Proposition 1.1 we just have to look what happens to hyperplanes in \mathbb{R}^1 under the map E .

Consider a typical hyperplane

$$H := H_{pc} := \{x \in \mathbb{R}^1 \mid \langle p, x \rangle = c\}, \quad p \in \mathbb{R}^1 \setminus \{0\}, \quad c \in \mathbb{R}.$$

We get

$$\begin{aligned} E(H_{pc}) &= \langle \xi \in \mathbb{R}_{++}^1 \mid \langle p, (\ln \xi_1, \dots, \ln \xi_1) \rangle = c \rangle \\ &= \langle \xi \in \mathbb{R}_{++}^1 \mid \ln(\xi_1^{p_1} \dots \xi_1^{p_1}) = c \rangle. \end{aligned}$$

Denoting $\exp c$ by \bar{c} we have

$$E(H_{pc}) = \langle \xi \in \mathbb{R}_{++}^1 \mid \xi_1^{p_1} \dots \xi_1^{p_1} = \bar{c} \rangle.$$

This is a typical indifference class of a non-trivial continuous price-invariant preference on \mathbb{R}_{++}^1 . It is immediate that such a preference is locally non-satiated. Since E and L are preserving the vector ordering of \mathbb{R}^1 , monotony properties are preserved when going from R to \succeq or vice versa.

In general, there are price-invariant preferences which do not generate demand correspondences. Take $l = 2$, $p_1 = 1/3$, $p_2 = -2/3$. Then a typical indifference curve of a price-invariant preference in \mathbb{R}_{++}^1 has the form

$$\langle \xi \in \mathbb{R}_{++}^2 \mid \xi_1 = \bar{c} \xi_2^2 \rangle, \bar{c} \in \mathbb{R}_+.$$

If the representing utility function increases with \bar{c} then the preference is strictly convex, but it fails to have a maximal element in the budget set $\langle \xi \in \mathbb{R}_{++}^1 \mid \langle \pi, \xi \rangle \leq w \rangle$ for any $\pi \in \mathbb{R}_{++}^1$ and $w \in \mathbb{R}_{++}$.

Considering the formula for $E(H_{pc})$ we see that it is a Cobb-Douglas indifference set if and only if $p \in \mathbb{R}_{++}^1$. But this is the case if and only if the hyperplane H_{pc} has its normal p in the strictly positive cone \mathbb{R}_{++}^1 . Accordingly, under an additional monotony assumption the continuous price-invariant preferences are exactly the Cobb-Douglas representable ones.

Proposition 2.1: Let \succeq be a non-trivial continuous price-invariant preference relation on \mathbb{R}_{++}^1 : Then

$\exists p \in \mathbb{R}^1 \setminus \{0\} \forall \xi, \eta \in \mathbb{R}_{++}^1$:

$$[\xi > \eta] \iff [\xi_1^{p_1} \dots \xi_n^{p_1} > \eta_1^{p_1} \dots \eta_n^{p_1}].$$

Whenever $[\xi - \eta \in \mathbb{R}_{++}^1]$ implies $[\xi \succeq \eta]$ then $p \in \mathbb{R}_+^1 \setminus \{0\}$.

If $p \in \mathbb{R}_+^1 \setminus \{0\}$ then $[\xi - \eta \in \mathbb{R}_{++}^1]$ implies $[\xi > \eta]$, i.e. \succeq

is monotone. $p \in \mathbb{R}_+^1$ if and only if $[\xi - \eta \in \mathbb{R}_+^1 \setminus \{0\}]$

implies $[\xi > \eta]$, i.e. \succeq is strongly monotone.

The corresponding result for $R = L^*(\succeq)$ is an immediate consequence of Proposition 1.1. The present version then follows from isomorphism.

Proposition 2.1. asserts, in particular, that the non-trivial continuous weakly monotone ($[\xi - \eta \in \mathbb{R}_+^1] \implies [\xi \succeq \eta]$) preference relations which are price-invariant are exactly the Cobb-Douglas representable ones.

This result overlaps with one due to H.DIERKER (1986). She states that among all homogeneous demand functions which satisfy Walras Law the only ones which are invariant under the action of the group of price-variations are those generated by Cobb-Douglas utility functions. The specific context of demand functions covers even those which are not derivable from utility maximization. On the other hand it does not provide an answer for nonconvex preferences, convex ones which do not satisfy Walras Law, and those for which there does not even exist a demand correspondence.

3. Application to Social Choice²

Our results of section 1 can be applied to the social choice context. We have to interpret then \mathbb{R}^1 as the utility space, the elements of which represent utility allocations to the l individuals of a society resulting from certain social alternatives. The relation R on \mathbb{R}^1 has to be looked at then as a planner's preference on utility allocations. This takes us into the welfarism framework, where social welfare functionals can be perfectly represented by induced social welfare orderings (cf. D'ASPROMONT (1985)).

There are some versions of a utilitarianism theorem all of which are related to our results presented in section 1. Theorem 3 of D'ASPROMONT and GEVERS (1977), Theorem 2 of GEVERS (1979), Theorem 3.3.3 in D'ASPROMONT (1985), and Theorem 2 of ROBERTS (1980) are modifications or reinterpretations of Theorem 4.3.1 due to BLACKWELL and GIRSHICK (1954). They do not rely on any continuity assumption on R . The theorem of MASKIN (1978) makes a stronger assertion under different assumptions and requires continuity of R .

MASKIN'S result is related to our Proposition 1.1, while the others are related to Proposition 1.2.

Let us first sketch the framework in which we can compare those results.

Let Z be a set of at least three social alternatives. Let $N = \{1, \dots, l\}$ be the set of individuals constituting the society. Let \mathcal{P} be the set of preferences on Z and \mathcal{U} the set of bounded real-valued functions on $N \times Z$. For any $u \in \mathcal{U}$ and $i \in N$ the function $u(\cdot, i)$ represents the preference of individual i on Z . A social welfare functional (SWFL) is a mapping from some subset of \mathcal{U} into \mathcal{P} . The "welfarism theorem" (cf. Theorems 2.1 and 2.3 in D'ASPROMONT (1985)) asserts that a SWFL satisfying assumptions of *unrestricted domain* and *strong neutrality* (the latter one being equivalent

²For notational details of this section we refer to MASKIN (1978), ROBERTS (1980), D'ASPROMONT (1985), and BLACKWELL and GIRSHICK (1954).

then to *Pareto-indifference* together with *independence of irrelevant alternatives for pairs*) induces a social welfare ordering (SWO) as follows (by $u(z)$ we mean the vector $(u(z,1), \dots, u(z,1))$):

$$\forall x, x' \in \mathbb{R}^1 : x R x' \iff$$

$$[\exists u \in \mathcal{U} \exists z, z' \in Z: \quad \text{i) } u(z) = x \quad \text{ii) } u(z') = x' \quad \text{iii) } z f(u)z']$$

$$\text{i.e. } R = \{(u(z), u(z')) \in \mathbb{R}^{21} \mid (u, z, z') \in \text{graph } f\}.$$

These versions of the utilitarianism theorem not assuming continuity of R which are quoted above, are implied by Theorem 4.3.1 in BLACKWELL and GIRSHICK (1954) except for Theorem 2 by ROBERTS (1980). In fact, Roberts claims that his result is "slightly stronger" than that of BLACKWELL and GIRSHICK. This is true only in the sense that Robert's result is not restricted to the welfarism framework, since he does not assume neutrality. But otherwise, i.e. as a result on SWO's it is not stronger than that of BLACKWELL and GIRSHICK.

First, Robert's assumption C U C (CC* in GEVERS (1977)) modelling *unit comparability* implies translation-invariance (L3 in Blackwell's and Girshick's book), but not vice versa, as was shown on the last page of section 1. Secondly, as the trivial preference shows, Blackwell's and Girshick's *weak monotony assumption* (L2) does not imply Robert's *weak Pareto-criterion* (P). Therefore, since (without continuity) P and L2 are different. Robert's result is, in spite of the fact that in his proof he makes only use of L3 rather than of C U C, not stronger than the Blackwell and Girshick result.

Anyway, both results are immediate consequences (as far as SWO's are concerned) of our Proposition 1.2.

To get the Blackwell and Girshick theorem one has to observe that their monotony condition L2 is just weak C-monotony for $C = \mathbb{R}_+^1$.

Robert's weak Pareto-criterion (P) implies C-monotony for $C := \mathbb{R}_{++}^1$. In both cases one has $C' = \mathbb{R}_+^1$ and the assertions on the social welfare ordering R follow from the assertion in our Proposition 1.2.

Let us compare now Maskin's result with our Proposition 1.1. Maskin asserts that *if a SWFL satisfies independence, the strong Pareto-criterion, anonymity, elimination of indifferent individuals, continuity, and full comparability then it is the utilitarianism principle*

i.e.

$$\forall x, y \in \mathbb{R}^1 \quad \forall u \in \mathcal{U}$$

$$x \succ f(u) \succ y \iff \sum_{i=1}^1 u(x, i) \geq \sum_{i=1}^1 u(y, i)$$

The elimination of indifferent individuals is formally a restatement of the *strong separability condition* used by DEBREU (1960) to establish additive separability of utility functions. Maskin derives separability of the representation of the SWO R, using this assumption.

Linearity of this representation follows from *full comparability*.

Full Comparability: $\forall u, u' \in \mathcal{U}$, if $b > 0$, $a \in \mathbb{R}$ such that

$$\forall i \in N \quad \forall z \in Z$$

$$u(z, i) = b u'(z, i) + a$$

then $f(u) = f(u')$

Full Comparability is weaker than C U C which looks the same except that in C U C the constant $a \in \mathbb{R}$ may depend on $i \in N$. As already shown in section 1, full comparability does not imply, nor is it implied by translation-invariance.

Yet, if we replace full comparability by translation-invariance in Maskin's theorem, we get the separability together with the linearity of the representation of R from Proposition 1.1. This allows us to drop the requirement of elimination of indifferent individuals. Also, continuity can be weakened

to semi-continuity at some point, and the strong Pareto-principle employed by Maskin, can be replaced by weak \mathbb{R}_+^1 -monotony. Clearly, like Maskin we have to add the assumption of anonymity. Noticing that Maskin's independence assumption puts his result in the welfarism framework, we can use his proof to get the following result on social welfare orderings in which $v, w \in \mathbb{R}^1$ represent utility allocations $u(z), u(z')$ for certain social alternatives, $z, z' \in Z$.

Proposition 3.1: If a translation-invariant SWO R on \mathbb{R}^1 satisfies semi-continuity at some point and weak \mathbb{R}_+^1 -monotony then:

$$\exists p \in \mathbb{R}_+^1, p \neq 0 \forall v, w \in \mathbb{R}^1 : v R w \iff \sum_{i=1}^1 p_i v_i \geq \sum_{i=1}^1 p_i w_i.$$

Under the additional assumption of anonymity p can be chosen as $p = (1, \dots, 1)$.

4. Concluding Remarks

It has been the aim of the present paper to underpin the usefulness of the concept of a group action for aggregation of consumer's demand relations. We could establish that in all "relevant" cases orbits of "similar" consumers induced by the group action, are non-degenerated and hence large enough to allow for suitable diversification of tastes. The only monotone continuous preferences with degenerate orbits are the Cobb-Douglas representable ones. But those are unimportant for aggregation purposes because the derived demand functions have already the nice structure one would like to get from aggregation for other preferences. Useful as Cobb-Douglas utility functions may be for textbooks, they certainly do not build the basis for individual demand behavior nor are they of fundamental importance for demand theory. Yet, given their omnipresence in the economic literature, it seems worthwhile to draw attention to the fact that Proposition 2.1 can be looked at as a classification result for Cobb-Douglas representable preferences. It provides necessary and sufficient conditions for preferences to be Cobb-Douglas representable. There has been a different statement in SAMUELSON (1965) classifying Cobb-Douglas representable preferences as exactly those allowing simultaneous additive separability of a utility representation and its indirect utility function. As HICKS (1969) and then, in a thorough analysis, SAMUELSON (1969) (cf. also LAU (1969)) observed, this statement relying on a wrong statement in HOUTHAKKER (1960) is false. So simultaneous additivity does not single out only the Cobb-Douglas representable preferences.

The group action on the basis of which preference diversification was modelled, has, in fact, been used also in applied demand analysis and economics. The concept of household equivalent scales (cf. BARTEN (1964), DEATON and MUELLBAUER (1980), and JORGENSON and SLESNICK (1984)) just expresses the idea that all households' preferences in a society belong to one orbit derived from a certain preference by stretching the axes of the commodity space. It seems promising to analyze whether interpersonal comparisons could be based on the concept of a group action on preferences.

While in the context of aggregation of demand the difference of preferences in the same orbit is of fundamental importance in creating diversification of taste, and, therefore, different demand behavior at the same prices, it is just the dual aspect which is crucial in our application in social choice. There the planner's preference ignores differences within the orbits, i.e., all individuals in the same orbit have exactly the same influence on the social welfare ordering. This informational aspect of invariance plays an important role in social choice theory and is reflected in many of the requirements made there (cf. SEN (1986)).

It seems on the first sight surprising that the same mathematical result can be usefully applied in so different parts of economics as demand theory, empirical demand analysis, social choice, and individual decision making under uncertainty, where normative and positive, ordinal and cardinal standpoints are involved. But in all those cases aggregation is involved in one or the other way, and weighting systems, probabilities, and price systems are only different names for the same mathematical tool, a normalized positive linear functional.

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