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**A Multi-Person Approach Towards the
Maximization of the Average
Rate of Return**

by

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I. Introduction

Since the seventies theoretical effort on optimization of portfolios consisting of fixed-income securities continuously grew. To a large extent this is due to the fact that fixed-income securities represent the by large most important type of investment for investors like insurance companies and pension funds. Their attractiveness is due to the priori knowledge on intermediate and terminal payoffs. This, however, does not mean that the rate of return is known at the date of purchase, since, due to uncertain re-investment opportunities of coupons, the promised yield and the realized yield to maturity generally differ. But, of course the variation of prices at any date before maturity is smaller than the variation of stock prices.

In this paper we shall be confined to "active bond portfolio management" which presupposes the idea of receiving abnormal profits by means of the superior abilities of the portfolio manager.

An active portfolio manager makes decisions with respect to three major factors. They greatly consist of

performing interest-rate forecasts or estimates of future bond prices,

defining the maturity structure of the portfolio and

selecting the individual bonds with differing maturities and coupon rates for a diversified portfolio.

We shall confine ourselves to the last point. The preceding decisions (for example the scenario forecasting of the portfolio manager) are contained in the parameters of our model.

The uncertain data as given by the interest forecasts represent the only source of risk being dealt within this exposition.

In particular we are not concerned with the default risk and the risk related to investments in foreign currency bonds. Moreover we do not consider call options nor the uncertain gap between the real rate of return and the nominal rate of return.

As far as the assumptions on the goals of the investors are concerned we shall assume that they are going to maximize the average rate of return of the securities hold in their portfolio (up to their planning horizon). This assumption means to reduce the evaluation of a stream of cash inflows to a one-dimensional characteristic.

For the ease of presentation we exclude any form of imperfectness on the securities' market as for instance given by transaction costs, taxes and indivisibilities for the purchase of securities.

Within this exposition we shall analyze the interdependent optimization of portfolios of (at least two) investors. This type of analysis requires further motivation.

The common type of analysis of portfolio management assumes the economic environment to be fixed, in which the investor chooses the optimum mix of securities (given his characteristics). This Robinson Crusoe-like decision making presupposes implicitly certain assumptions to hold, if optimization along these lines can reasonably be called rational. These ideal assumptions however are far from being satisfied in reality (for large investors) and thus a model which bases decisions on those assumption must necessarily come to erroneous recommendations in as much optimum portfolio selection is concerned.

Three observations enforce us to go beyond the common models:

We presume that the market clearing price for bonds depends on demand (and supply) in an exogeneously given manner. Thus any identical choice of the portfolios by a relevant share of the participants of the bond market provides a price shock (for simplicity no lasting effects are assumed to exist).

The effect of such a change of prices however is not restricted on this particular investor, but has a kindred impact on the yields obtained by all of his competitors.

Consequently all investors' decisions are interrelated.

In fact, due to modern information processing and the efficiency of securities markets smaller investors have access to identical information almost simultaneously. To a large extent they use similar strategies for portfolio optimization. The similarity of optimization strategies is due to the fact that software packets for optimization are sold to large numbers of medium-size investors. Consequently, important groups of investors (importance measured by their accumulated budgets), following the recommendations of these optimization programs, come to identical conclusions for rearrangements of portfolios. Similar investors are grouped together, these groups of real investors appear as the individuals in the present model, whence we derive the relevance of the model as far as the description of the real world is concerned.

Secondly recall the fact that there is a time lag between the observation of current prices and the execution of a buy- or sell order based on these data. This lag is due to

time consuming information processing

and

the fact that the execution of the order by itself is time consuming, since queues are served according to the FIFO-principle.

The effect of this time lag is that the realized prices of bonds differ from those which were observed and provided the basis for the buy-order. Thus decision-making cannot be mapped into a Robinson Crusoe decision model, or, equivalently in a "multi-person" model in which investors can be ranked such that any investor sees the effect of the decisions of his predecessors on prices of bonds. On the contrary, decisions are to be found simultaneously or, without recurring to the physical flow of time, in ignorance from one another.

Thus we find simultaneity of decisions.

As a third point we note that future commitments differ implying that the cash inflows have to be adapted accordingly. Even in a simple model we may find that investors' objective functions differ from one another (we shall comment on this in a later section).

Summarizing:

we assume simultaneous decisions of at least two investors, who are characterized by different objective functions such that one investor's decision has an impact on his and the other investors' profits.

In short, we formulated a game-theory problem.

As a consequence, every investor has to think of the possible decisions of his competitors inasmuch as his expected payoff is concerned.

Since the investor is interested in maximizing the average yield of the securities within his portfolio as a rational subject he has to impute the same maximization-efforts being provided by his competitors. This gives rise to some minimum requirement on joint, but independent and individuum--centered behavior. It amounts especially to require that a recommendation of rational behavior in a multi-person decision situation satisfies stability in as much that any deviation from the recommendation proposed by theory provides less profit. In fact the (Nash-) equilibrium is characterized by the fact that any person's strategy is a best reply to the opponents' strategies. Any recommendation which does not satisfy this requirement would be self-destroying and bear the incentive to deviate from the recommendation for at least one decision-maker. Such a recommendation never would be followed by rational decision-makers. Of course, by joint deviation, which presumes coalition-formation the payoff could prove augmentable simultaneously, however the absence of self-binding power hampers the decision-makers from getting together (see T. Schelling [60]). It should be noted that the (Nash-) equilibrium condition is only a necessary but by no means a sufficient condition for rational behavior, since in many multi--person decision situations a lot of equilibrium strategy vectors exist and thus a normative theory of rational behavior has to select one of these equilibria (see J. Harsanyi and R. Selten [88]).

Let us conclude the present paragraph with a comment on the range of implications of our observation, it deals with the restriction on the bond market.

Due to the different trading volumes the degree of interdependence of decision is even greater on stock markets than on bond markets. However for the ease of demonstration we decided to present our ideas in the bond market context. The effect of demand shocks on prices, which is basic to the switch from a "Robinson Crusoe" decision model to a multi-person decision model quantitatively has to be derived from real data.

II. The Model

Within the present chapter we shall deal with the structural assumptions of the model, starting with the economic environment of the investors.

1. The economic environment

First of all there is given a finite number of well defined *investment-alternatives*, enumerated by $k = 1, \dots, K$.

These investment-alternatives are assumed to be bonds and notes, respectively, characterized by their principal, their (annual) coupons and their term to maturity, formally we define them by a vector of payoffs $(c_1, c_2, \dots, c_\tau)$, where τ denotes the term to maturity. The principal which is paid at the term to maturity is incorporated in the payoffs c_τ , thus

$$c_t(k) = \begin{cases} \text{coupon}(k,t) & , t < \tau(k) \\ \text{coupon}(k,t) + \text{principal} (=100) & , t = \tau(k) \\ 0 & , \text{otherwise.} \end{cases}$$

Secondly, there are given *interest rate curves* $(m_t(s))_{t=1, \dots, T}$, $s=0, \dots, S$ where, given s , for any date $t = 1, \dots, T$, $m_t(s)$ defines the rate of return for an investment spanning the period $[t, T]$. T denotes the *planning period*, which is defined to be the maximum term to maturity of the available securities.

For simplicity assume $m_t(s) = m(s)$. The parameters s (mnemotechnical for scenario) expresses the investors' ignorance on future. A realization of future thus is reduced to the choice of one element of a number of interest curves.

For convenience we assume that the interest rate forecasts of all investors coincide.

Let μ denote a probability distribution on the set of scenarios $\{1, \dots, S\}$. μ is the *numerical valuation* on the number of possible "futures".

We assume that μ is common to all investors.

We may also consider differing subjective probabilities μ_1, \dots, μ_N for the realization of future scenarios (if being common knowledge of the investors).

Next there is a vector $(\gamma_1, \dots, \gamma_K)$ called *sensitivity*. It is used to describe the reaction of current prices to demand. Also for simplicity we assume the components of the vector to coincide. The formal definition and use of this parameter will be given later on.

The set of data describing the economic environment is completed with an interest rate i^0 which - to a certain extent - defines the price of the bonds at time 0. The number i^0 is called *reference interest rate*, we shall comment on the role of this parameter afterwards.

The primary characteristics of a fixed income-security are given by its return and price (disregarding risk and liquidity).

It is assumed that the coupons are immediately and fully reinvested at the interest rate defined by the scenario. Thus generally the promised yield and the realized yield, which depends on the actual scenario, differ.

For simplicity we further assume that the bonds and notes are purchased only on the coupon dates, thereby we eliminate the problem of accrued interest.

Given some fixed scenario (interest rate m) then the *return* received at some time H is defined as the discounted sum of payoffs by

$$r(H) = \sum_{t=1}^T c_t \cdot (1+m)^{H-t}.$$

The return splits into the payoff resulting from coupons and their re-investment up to the planning horizon and to the payoff resulting from selling the security before maturity, where its price is assumed to be subject to the prevailing market interest (at H). The latter is equal to

$$(1+m)^{r-H} \cdot \sum_{t=H+1}^{\tau} c_t \cdot (1+m)^{(r-H)-t},$$

discounted to the present time by the factor $(1+m)^H$. Thereby the yield from H to τ is equal to the prevailing interest rate (at time H).

Given the price p and the return (at time H) then the *realized yield* (up to H) for this security is equal to

$$y(H) = {}^H\sqrt{r(H) / p} - 1.$$

Some "prices" p^0 - which provide only some rough orientation - are defined based on the reference interest rate i^0 , using a distinguished interest rate curve $m(s^0)$.

For reasons which shall become clear later $m^0 = m(s^0)$ is called *status quo curve*.

Given the status quo curve m^0 it is assumed that for all securities the promise yield up to T coincides with the reference interest rate, i.e.

$$\bigwedge_k i^0 = {}^T\sqrt{r(T)(k) / p^0(k)} - 1.$$

This defines the current "calculatory" prices

$$p^0 = (p_k^0)_{k=1, \dots, K}.$$

Another way of defining the value of securities is to presume the coincidence of the yields up to the individual terms of maturity. Both assumptions are not crucial but only simplify the exposition. They extricate us from introducing one of the theories relating liquidity and yield, for example the liquidity premium theory (see R. Fuller and J. Farrell [88], pp 412-413).

These theories provide explanations for the historical fact that upward sloping yield curves tend to occur more frequently. The liquidity premium theory for example regards a yield premium as compensation for investing longer term to be the reason for the above phenomenon. This yield premium is motivated by the assumption of risk-averse investors and the observation that longer term bonds have greater price variations for a given change in interest rate and therefore are associated with more risk than short term securities.

2. The investors

The economic environment having been defined the investors $n = 1, \dots, N$ as decision makers enter the scene. We assume that the investors may vary according to two criteria.

Any investor is assumed to have some individual *planning horizon* H . For simplicity we assume that he is bound to dispose of his capital only at H , up to this point there exists neither any cash inflow nor do there exist future commitments before the end of his planning period. We recall that the investors are assumed to maximize the average yield of the securities being held in their portfolio, given their planning horizon.

We shall now comment on the role of the planning horizon in as much the utility functions of the investors - and, derived thereupon, the optimum portfolio - is concerned.

Observe that, presupposing uncertain future the ranking of investment-alternatives with respect to their yield depends on the planning horizon of the investor. To see this effect we give a numerical example.

Example:

Let investment-alternative 1 be a 10% coupon bond with 11 years to maturity, alternative 2 be a 10% coupon bond with 7 years to maturity. Assume the capital to be invested to be 100,000.

Then, assuming scenarios to be defined by (flat) interest curves $m_1 = 12$ and $m_2 = 8$ we find

$$\begin{aligned} 1)(i) \quad r_{m_1}^1(7) &= 194,815 & r_{m_2}^1(7) &= 195,852 \\ r_{m_1}^2(7) &= 200,890 & r_{m_2}^2(7) &= 189,228, \end{aligned}$$

whence

$$\begin{aligned} (ii) \quad y_{m_1}^1(7) &= 9.99 & y_{m_2}^1(7) &= 10.07 \\ y_{m_1}^2(7) &= 10.47 & y_{m_2}^2(7) &= 9.53 \end{aligned}$$

For $H = 11$ on the other hand we find

$$\begin{aligned} 1)(i) \quad r_{m_1}^1(11) &= 306,545 & r_{m_2}^1(11) &= 266,454 \\ r_{m_1}^2(11) &= 316,104 & r_{m_2}^2(11) &= 257,442, \end{aligned}$$

whence

$$\begin{aligned} (ii) \quad y_{m_1}^1(11) &= 10.72 & y_{m_2}^1(11) &= 9.32 \\ y_{m_1}^2(11) &= 11.03 & y_{m_2}^2(11) &= 9.98 \end{aligned}$$

An easy calculation shows that for probabilities $\mu \in [0.52\dots, 0.63\dots]$ an investor with planning horizon equal to 7 years prefers alternative 1 to alternative 2, whereas an investor with planning horizon equal to 11 prefers alternative 2 to alternative 1. This fact shows the qualitative impact of the planning horizon on the investor's optimization problem.

The second characteristic of any investor is his *weight as a participant* of the market. It is assumed that his weight can be given as a number between zero and one such that the weight vector (w^1, \dots, w^N) , summing up to unity, describes the shares of demand of the investors at time 0 and thereby the impact of the investors demand to the prices of bonds. This point is dealt with in the next section.

3. Prices and demand

The yield of a security depends on the return being received at the planning horizon and also on the current price which has to be paid for the security. This price certainly depends on the demand for this security and thus has to be taken into consideration from those investors which are large enough to induce price changes on the market.

The present model represents only a partial analysis of the market for fixed income securities. Here prices are not deduced as being the result of the equality of demand and supply. Such an approach would imply some premises concerning the (utility) maximizing behavior on suppliers (and issuers) of securities (which are not present in our model).

We assume that the price reaction to the aggregated demand is given exogeneously. Thus, on the demand side of the market there remains to perform the (micro-) analysis of the individual demand of the investors in order to find the structure of the individual demand. The equilibrium being found as the solution to the (basically) yield-maximizing behavior of the investors may hopefully be identified as a substructure of rational behavior in an appropriate general model, which also comprizes issuers of securities. The solution to this problem however has to be left for further research.

The present approach on defining prices comprizes two of the three different schools of thought on the matter of securities' values (see D. Fisher and R. Jordan [79], pp 78).

i^0 , the common yield of all securities reflects the "fundamentalists" approach stating that the price of a security is equal to the discounted value of the stream of income from that security. Price changes therefore only come from anticipations' changes and the major source for the latter is new information, which in our model is parametrized by i^0 .

The second ingredient to prices comes from the "random walk theorists" school. They claim, thereby being diametrically opposed to the "technical analysts" - who try to predict prices from historical record - that securities markets are almost perfect and thzus securities' prices

should reflect all the information available to the market participants. In our model this means that they anticipate decisions of their competitors in as much as these decisions have an impact on bonds' prices.

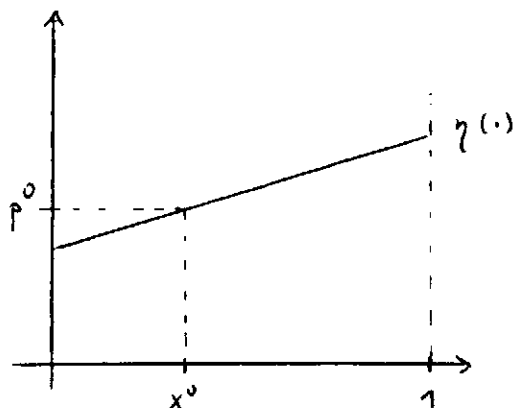
Summarizing, we shall restrict ourselves to generating prices as follows: We assume that the main influence on securities prices is given by the assumption of an (exogeneously given) common yield to the planning period for all securities. Formally this common yield is defined by the reference interest rate i^0 at time 0.

Formally it is assumed that p^0 are the bonds' prices if the current demand (share) is equal to x^0 . However, deviations of demand from this exogeneously given structure provide deviations from the bonds' prices p^0 .

We may interpret this assumption by saying that the additional demand for securities meets the supply provided by the issuers and thus no price reaction emerges.

For our purposes, namely proving the compulsion of a game-theoretical analysis of the behavior of investors on the fixed-income securities market, the standard concept of economic theory on the relation of variation of prices and demand is sufficient.

The simplest form of this relation (see J. Hirshleifer [88]) is given by supposing the variation of prices to be derived from a linear price reaction function. Here x^0 serves as the origin for the demand-price function, such that prices p^0 clear the market for demand x^0 . The price reaction function is assumed to be (affin-) linear, its grade expresses the sensitivity γ of prices with respect to demand.



Since the aggregated demand x , which is responsible for the price, is combined from the individual demands (weighted by the individual trading volumes) each investor may, supposing the demands of his competitor being given, compute the bonds prices and thereby the yield of his securities as depending on his own demand.

Let the vector $w = (w^1, \dots, w^N)$ denote the weights of the investors. Then, given the vector $x = (x^1, \dots, x^N)$ of individual demands, the actual price of bonds becomes

$$p(x) = p^0 \cdot \eta(x), \text{ where}$$

$$\eta(x) = 1 + \gamma \cdot (w^N \cdot x^N - x^0),$$

($w^N \cdot x^N$ denotes the scalar product of w^N and x^N).

III. Existence of Equilibria

We argued that the exclusion of gains by deviation from a commonly announced behavior is a minimum requirement on rational behavior in a multi-person decision problem. The solutions of this extension of the von Neumann minmax-principle are called equilibria. Their existence in a decision situation with multi-linear utility functions was first established by J. Nash [50]. Formally equilibria are to be defined as follows:

Let x_n denote the set of available strategies of player n and denote, for $x^N = x_1 x_2 \dots x_N$, the utility- (or payoff-) functions by

$$f_n : x^N \longrightarrow \mathbb{R}.$$

A vector of strategies $x^{*N} \in x^N$ is called (*Nash-*) *equilibrium*, if for all n and all $x_n \in x_n$:

$$f_n(x^{*N}) \geq f_n(x^{*N-1}, x_n).$$

The existence of equilibria may be derived under various assumptions on the set of strategies and properties of the payoff-functions. For our purposes the following formulation, provided by H. Nikaido and K. Isoda [55] is sufficient:

Theorem:

Let $\Gamma = (\alpha_1, \dots, \alpha_N; u_1, \dots, u_N)$ be an N-person game satisfying

α_n is a compact and convex subset of some \mathbb{R}^m , $n = 1, \dots, N$

$u_n : \alpha_1 \times \dots \times \alpha_N \longrightarrow \mathbb{R}$ is continuous, $n = 1, \dots, N$ and for all $n \in \{1, \dots, N\}$ and fixed strategies $x_k \in \alpha_k$, $k \neq n$ the function

$u_n(x_1, \dots, x_{n-1}, x_{n+1}, \dots, x_N) : \alpha_n \longrightarrow \mathbb{R}$ is concave.

Then Γ has at least one equilibrium point.

We shall now verify for our particular situation the requirements stated in the theorem. In our case the available strategies are given by the set of all portfolios consisting of securities 1, ..., K. This set is a compact and convex subset of \mathbb{R}^K , a simplex Δ^K , and thus satisfies the requirements. Thus there remains to prove the concavity of the average yield as a function defined on Δ^K , given any combination of portfolios held by the competitors.

The rest of this section is devoted to the investigation of the functional relationship of portfolios and the average yield of the securities contained in it. Here we fix (for the moment) the portfolios of the competitors and thus assume a Robinson Crusoe decision situation. The isolation of an investor will provide the basis for deriving the existence of an equilibrium portfolio vector and the proves the efficiency of the optimization and equilibrium search algorithm. The main observation is that the yield-function is concave over the set of portfolios.

It is sufficient to assume a fixed scenario to be given since the expected mean yield which is the objective function of the investor - a weighted average of the scenario's yield - then becomes a concave function too.

Recall that in the previous section we provided a formula for the reaction function of the bonds' prices to demand, based on what we called the market structure x^0 and the related bonds' prices p^0 .

Since we assume the portfolios of the competitors to be fixed, say z , we may regard the price reaction function as a function of the investor's portfolio x , i.e.

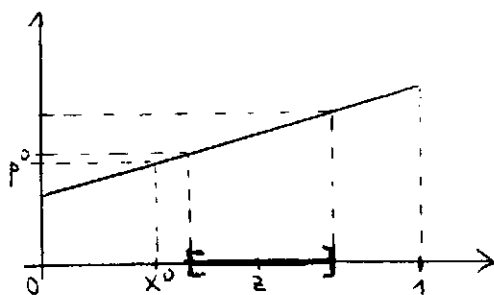
$$p(k;x|z) = p^0(k) \cdot \eta(k;x|z),$$

where

$$\eta(k;x|z) = 1 + \gamma \cdot ((w \cdot x_k + (1-w) \cdot z_k),$$

w the weight of the investor.

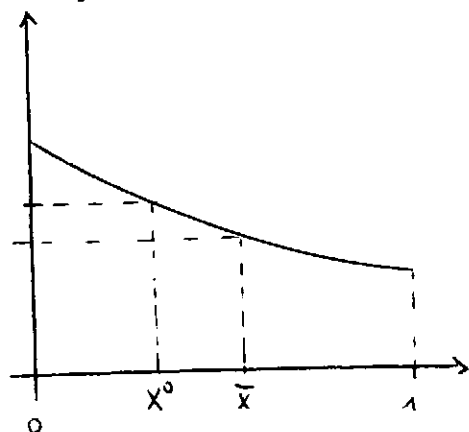
Given the aggregate portfolio z of the competitors the investor's choice of shares among his investment alternatives only gives rise to minor variations, depending on his weight w (and the sensitivity γ). We have



The aggregate portfolio z (including the portfolio of the distinguished investor and his competitors) gives rise to a yield of the individual securities (from the view of an investor with planning horizon H) by

$$\begin{aligned} y^H(k;x|z) &= \sqrt[H]{r^H(k;x|z)/p(k;x|z)} - 1 \\ &= \sqrt[H]{r^H(k;x|z)/p^0(k) \cdot \eta(k;x|z)} - 1. \end{aligned}$$

The following shape for the mean yield of the individual securities contained in the portfolio (as a function of their share) will be found:



Remark:

$y^H(k; \cdot | z) : [0, 1] \mapsto \mathbb{R}_+$ is convex for all k, H and z .

Proof: Easy calculation.

Despite the yield of a specific security being a *convex* function of the securities share within the portfolio the mean yield of the securities contained in the portfolio comes out to be a *concave* function of the bonds' shares.

We shall now use the same symbol y to denote the yield of the security, given the aggregate portfolio or the combination of the investor's portfolio x and the aggregate portfolio z of all his competitors, this will cause no confusion.

The average yield from the portfolio now becomes

$$y^H(x|z) = \sum x(k) \cdot y^H(k; x|z).$$

We claim

Proposition:

$y^H(\cdot | z)$ is concave for all z (and $H \geq 2$).

First observe that a sufficient condition for concavity of a function

$$\Psi : [0, 1] \mapsto \mathbb{R},$$

$$\Psi(u) = u \cdot \psi(u), \quad \psi : [0, 1] \mapsto \mathbb{R} \text{ antitonic,}$$

is given by

$$2 \cdot \psi'(u) < \psi''(u).$$

In fact,

$$\Psi'(u) = u \cdot \psi'(u) + \psi(u)$$

$$\Psi''(u) = u \cdot \psi''(u) + \psi'(u) + \psi'(u)$$

$$= u \cdot \psi''(u) + 2 \cdot \psi'(u)$$

$$< 0$$

if (presupposing antitonicity of ψ and $u \in [0, 1]$)

$$\psi''(u) < -2 \cdot \psi'(u).$$

Now observe that, given k , H and z ,

$$r^H(k; \cdot | z) : [0, 1] \mapsto \mathbb{R}$$

is antitonic, thus we may use the above condition to ensure concavity of

$$x(k) \cdot y^H(k; x(k) | z),$$

where $x(k)$ varies in the interval $[0, 1]$.

Recalling the definition of $y^H(\cdot)$ we infer

$$\frac{d}{dx} r^H(k; x | z) = \text{const} \cdot \frac{d}{dx} (\eta(k; x | z))^{-1/H}.$$

In view of the condition on the first and second derivative of η it is sufficient to consider (for $\eta(k; x | z) = 1 + \gamma \cdot ((w \cdot x(k) + (1-w) \cdot z(k)) - x^0(k))$)

$$\begin{aligned} & \frac{d}{dx} (\eta(k; x | z))^{-1/H} \\ &= (-1/H) \cdot (1 + \gamma \cdot ((w \cdot x + (1-w) \cdot z) - x^0))^{-\frac{1}{H}+1} \cdot \gamma \cdot w \end{aligned}$$

and

$$\begin{aligned} & \frac{d^2}{dx^2} (\eta(k; x | z))^{-\frac{1}{H}} \\ &= \frac{1}{H} \cdot \left(\frac{1}{H}+1\right) \cdot (1 + \gamma \cdot ((w \cdot x + (1-w) \cdot z) - x^0))^{-\frac{1}{H}+2} \cdot \gamma^2 \cdot w^2. \end{aligned}$$

The condition

$$\frac{d^2 \eta}{dx^2} < -2 \cdot \frac{d\eta}{dx}$$

holds if

$$\begin{aligned} & \frac{2 \cdot \gamma \cdot w}{H} \cdot (1 + \gamma \cdot ((w \cdot x + (1-w) \cdot z) - x^0))^{-\frac{1}{H}+1} \\ & < \frac{\gamma^2 \cdot w^2}{H} \cdot \left(\frac{1}{H}+1\right) \cdot \frac{(1 + \gamma \cdot ((w \cdot x + (1-w) \cdot z) - x^0))^{-\frac{1}{H}+1}}{1 + \gamma \cdot ((w \cdot x + (1-w) \cdot z) - x^0)} \end{aligned}$$

i.e. if

$$2 > \gamma \cdot w \cdot \left(\frac{1}{H}+1\right) \cdot \frac{1}{1 + \gamma \cdot ((w \cdot x + (1-w) \cdot z) - x^0)}.$$

The last inequality surely holds for $H \geq 2$ and $0 \leq \gamma < \frac{1}{2}$.

As a sum of concave functions

$$x(k) \cdot y^H(k; x(k), z), \quad k \in K$$

$y^H(x|z)$ is immediately observed to be concave in x , which proves the proposition.

IV. The Technical Process of Equilibrium Search:

The equilibrium-condition has been realized to represent a minimum requirement for rational behavior of investors, however striving for an equilibrium is not an easy task to be performed. The decisive obstacle comes from the fact that even if the existence of an equilibrium is established by (usually) a fixed-point argument, almost no information is available on the number of equilibria

the payoff vectors related to them

and

analytically safe procedures, algorithms to find them (other than the exhaustive search).

For N -person games with a multi-linear structure of payoff functions the Lemke-Howson algorithm (in case of non-degeneracy) provides some equilibria (see J. Rosenmüller [71], [82]). Recently I. Krohn, S. Moltzahn, J. Rosenmüller, P. Sudhölter and H.-M. Wallmeier [89] provided an efficient method for the computation of at least one equilibrium in the 2-person case. It is based on the Lemke-Howson algorithm and is similar to the simplex method.

In our particular context there is no multi-linear structure available. Consequently, the existence, structure and variability of equilibrium portfolio combinations can only be studied "empirically". In the sequel we shall describe a procedure for finding equilibrium portfolio strategies.

It is the specific structure of the model, in particular the form of the price-reaction to aggregated demand which allows for finding equilibria different to the "exhaustive search"-approach. The algorithm to be sketched below works on a random basis. By changing the portfolios of the investors sequentially in line of a higher yield finally an equilibrium is attained.

The statement of a successful search for an equilibrium however could not be proven rigorously yet, the statement thus has to be viewed rather as an empirical than a mathematical one.

The numerical results suggest uniqueness of the equilibrium since varying the initial portfolios always leads to the same equilibrium.

The equilibrium-search algorithm:

For a given combination of portfolios the individual incentive for deviation is studied by looking for a potential improvement of yields. When performing this payoff optimization procedure the competitors' portfolios are assumed to be fixed. If an improvement is found, the combination of portfolios does not satisfy the equilibrium condition. In this case the investor changes his portfolio by defining his new strategy as to be an appropriate convex combination of the former candidate and the newly found improvement. After revision of his portfolio the next investor (they are ordered along a circle) tries to find an improvement. The procedure stops if no investor finds an improvement given a certain combination of portfolios. The combination being found is a (Nash-) equilibrium portfolio since there is no improvement by unilateral deviation.

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