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**REMARKS ON COOPERATIVE GAMES
WITH INCOMPLETE INFORMATION**

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§ 0 Introduction

This paper is dealing with a version of the characteristic function of an NTU game in the case of incomplete information. The model is based essentially on the work of HARSANYI-SELTEN and MYERSON. However, we argue that the question of a suitable extension of a bargaining solution (a suitably modified NASH-value) should be raised again. On one hand, MYERSON's objection against the HARSANYI-SELTEN are worth a discussion, on the other hand his axiomatic approach yields a strong discontinuity. We discuss at length a slightly modified version ("ex post individual rationality") of the HARSANYI-SELTEN value which seems to be more satisfying in view of the family of examples under consideration.

§ 1 The Model

Let $I = \{1, \dots, n\}$ represent the *players* or individuals involved in the game. For each $i \in I$ let there be given a finite set T^i representing the possible "*types*" player i can find himself to be in.

p denotes a probability on $T = T^1 \times \dots \times T^n$, called the "*joint distribution of types*".

Next, let $\underline{X} \subseteq \mathbb{R}^n$ be a compact convex set representing the possible "*alternatives*"; $\underline{x} \in \underline{X}$ is a single specified alternative, called the *threatpoint*.

Finally, for $i \in I$ a function

$$U^i : T \times \underline{X} \rightarrow \mathbb{R}$$

(continuous and concave) reflects the *utility* of player $i \in I$ if the players find themselves in the position of types $(t_1, \dots, t_n) \in T$ and x is the alternative chosen. We write

$$U : T \times \underline{X} \rightarrow \mathbb{R}^n, U_t(x) = (U_t^1(x), \dots, U_t^n(x)).$$

Definition 1.1: $\Gamma = (I, T, p, \underline{X}, \underline{x}, U)$

is a cooperative game with incomplete information.

A (tentative) interpretation of how this game is actually being played is provided as follows:

Let $(\Omega, \underline{F}, P)$ be a probability space and

$$\tau : \Omega \rightarrow T$$

be a random variable such that $p = \tau P = P \circ \tau^{-1}$ is the distribution of τ . By this random device an n -tuple of types $\tau(\omega)$ is drawn by chance and player $i \in N$ observes *his own* type $\tau^i(\omega)$. Given this state of affairs, the player may communicate and, via a binding (and enforcable) contract, agree upon jointly choosing some $x \in \underline{X}$. If so, player i is awarded the utility

$$U_{\tau(\omega)}^i(x).$$

If players fail to agree upon such a contract, they will receive the utility

$$U_{\tau(\omega)}^i(\underline{x}).$$

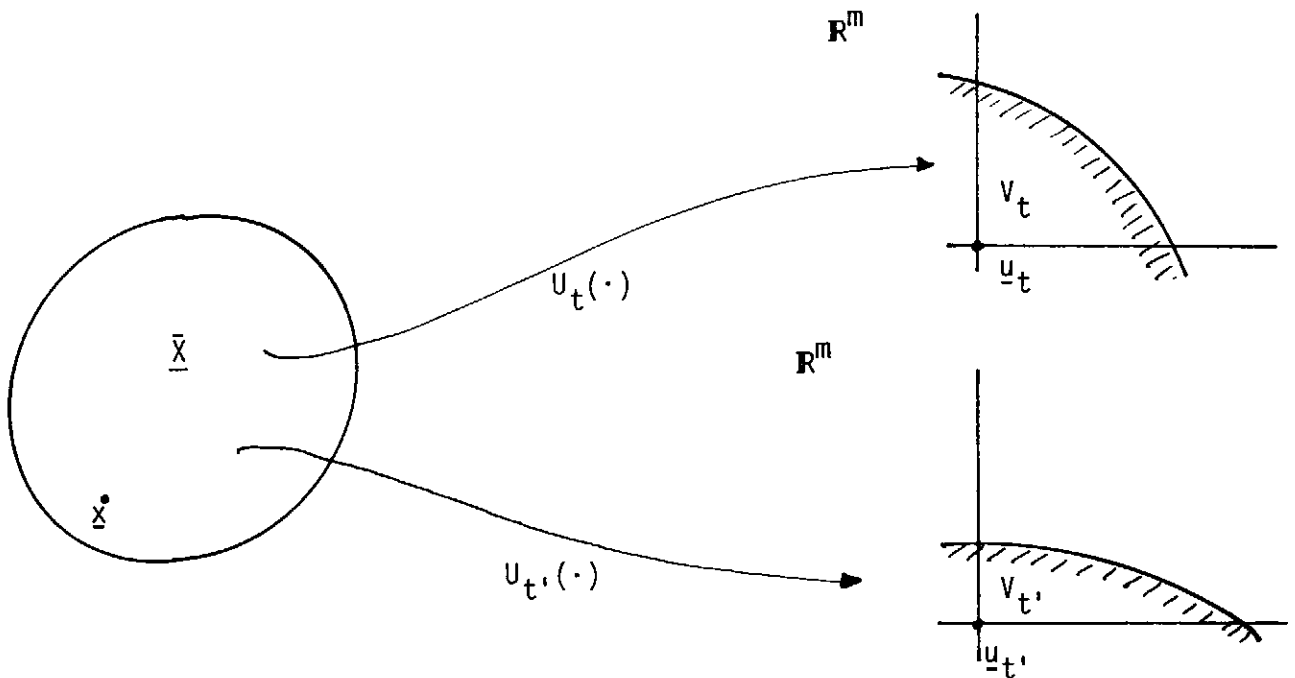
Remark 1.2: For $t \in T$ consider the set

$$V_t = \text{CPH} \{U_t(x) \mid x \in \underline{X}\}$$

(where CPH denotes the "comprehensive hull") and the utility vector

$$\underline{u}_t = U_t(\underline{x}) \in V_t$$

If chance chooses $\tau(\omega) = t$, then players actually are engaged in a unanimous game (a Nash Bargaining problem) (V_t, \underline{u}_t) – but they are unaware of the real t apart from each player's own coordinate.



(Fig.1)

Hence we may visualize the fact that the players simultaneously consider bargaining problems $(V_t, \underline{u}_t)_{t \in T}$ – each of them being assessed with a certain conditional probability by each player according to his observation.

Note that the requirements imposed upon U and the fact that V_t is taken to be a comprehensive hull imply that V_t is a convex set which is compactly generated (i.e., the convex comprehensive hull of a compact set).

Now, by agreeing on a contract which states that $x \in \underline{X}$ is the common choice, the players simultaneously choose $u_t = U_t(x) \in V_t$ for each t - again, each player only being aware of his coordinate.

Remark 1.3: HARSANYI and SELTEN ([2], 1972) in a different framework consider the "generalized Nash bargaining problem" which, in our present set-up may be defined as follows.

Introduce player i 's conditional expectation of payoff given that t_i is known to him, i.e.

$$A_{t_i}^i(x) = E_P\{U_{\tau}^i(x) \mid \tau^i = t_i\}$$

and let the (comprehensive hull) of all feasible payoffs for all types of all players be

$$V := \text{CPH} \left\{ (A_{t_1}^1(x))_{t_1 \in T^1}, \dots, (A_{t_n}^n(x))_{t_n \in T^n} \right\} \subseteq \mathbb{R}^{|T^1| + \dots + |T^n|};$$

also introduce the corresponding *threat point*

$$\tilde{u} := ((A_{t_1}^1(\underline{x}))_{t_1 \in T^1}, \dots, (A_{t_n}^n(\underline{x}))_{t_n \in T^n}) \in \mathbb{R}^{|T^1| + \dots + |T^n|}$$

By a system of seven axioms these authors arrive at the "generalized Nash solution" which is the (unique) utility vector \tilde{u} obtained by maximizing the "generalized Nash Product"

$$\prod_{i \in I} \prod_{t_i \in T^i} (u_{t_i}^i - \tilde{u}_{t_i}^i)^{p_{t_i}^i}$$

with

$$p_{t_i}^i := P(\tau_i = t_i) = \sum_{\substack{t' \in T \\ t'_i = t_i}} P(\tau = t')$$

More precisely, the contract suggested is an $\tilde{x} \in \underline{X}$ such that

$$\tilde{u} = ((A_{t_1}^1(\tilde{x}))_{t_1 \in T^1}, \dots, (A_{t_n}^n(\tilde{x}))_{t_n \in T^n})$$

maximizes the generalized Nash product.

Later on MYERSON in a series of papers [6] [7] introduced the concept of a "Bayesian incentive compatible mechanism" (BIC) based on similar devices employed by HURWICZ [4] and D'ASPREMONT/GERARD-VARET [1]. In his first approach he just restricts the SELTEN-HARSANYI solution to the set of BIC's while in his second approach he axiomatizes a further solution concept. In a further paper MYERSON also considers a generalized NTU-Shapley value [8].

Note that in our approach the set \underline{X} is already assumed to be compact and convex (and the utility concave and continuous). Thus, "joint randomizing" is generally not necessary. The above mentioned papers, however, deal with finite \underline{X} upon which players may jointly randomize.

Despite the fact that these concepts may be "imbedded" in our present one, the introduction of the present model bears a slightly different flavour.

Definition 1.4: A mechanism is a mapping $\mu : T \rightarrow \underline{X}$.

Remark 1.5: Mechanisms have been introduced by MYERSON into the context of the Nash bargaining situation; of course the concept is known in various contexts. A mechanism represents the idea of the following type of agreement between the players : each player $i \in I$ will report a type $t_i \in T^i$ and then $\mu(t) \in X$ is the alternative agreed upon.

It may be observed, however, that this interpretation has a certain influence concerning the story about the characteristic or "coalitional" function. "Binding agreements" should now be possible with respect to mechanisms - which is more to ask for than was included in our model so far.

For the "judicial" procedure which so far made it possible to (bindingly) "agree upon x " (an enforceable contract or a referee accepting contracts) should be capable of extending its functioning towards mechanisms.

Thus "law enforcement" is a new aspect which enters the picture. For, obviously players may get vital information by the announcements being made (which trigger the mechanism) and, accordingly, they might very well regret having agreed to this mechanism in the face of chance's choice of t . Thus, as x cannot be registered unless t is chosen, the duties of the "referee" must be greatly extended.

At the same time we should make up our mind concerning the temporal relationship between bargaining about the mechanism μ and the chance move that discloses t . For, if bargaining takes place after players know their type and mechanisms can be enforced, then players may disclose information by insisting on mechanisms favorable to them under certain conditions and extremely unfavorable under different ones. (This is particularly the case if a mechanism is "Bayesian incentive" – a term to be discussed later.) However, we basically want to insist on the notion that players inherently cannot disclose information concerning their type.

Thus we come up with a modified story as to how "the game is being played":

First stage: players bargain about a mechanism μ . If they agree upon some such μ , they may register it with a referee ("the court"), who is able to enforce it. If they do not agree, \underline{x} will be enforced later on.

Given μ , a new game in strategic form, called Γ^μ , arises which is verbally described as follows:

- ★) Chance chooses $t \in T$.
- ★★) Player $i \in I$, his information set being described by $t_i \in T^i$, announces, that he is type $s_i \in T^i$.
- ★★★) Player i receives $U_t^i(\mu(s))$.

Remark 1.6: Consider the game Γ^μ . A strategy of player i in this game is given by a mapping

$$\sigma^i : T^i \rightarrow T^i$$

Meaning "announce $s_i = \sigma^i(t_i)$ if t_i is your type").

In view of this incomplete information concerning the other players types, player i 's expected payoff in the subgame $\Gamma_{\bar{t}}^\mu$ determined by chance's choice of \bar{t} depends on \bar{t}_i only

and is given by

$$(1) \quad \bar{A}_{\bar{t}}^i(\sigma^1, \dots, \sigma^n) = E_P(U_{\tau}^i(\mu \circ \sigma \circ \tau) \mid \tau^i = \bar{t}_i) = \bar{A}_{\bar{t}_i}^i(\sigma^1, \dots, \sigma^n)$$

where $\sigma \circ \tau$ means $(\sigma_1 \circ \tau_1, \dots, \sigma_n \circ \tau_n)$.

Clearly, his payoff in the game is

$$\begin{aligned} & \bar{A}^i(\sigma^1, \dots, \sigma^n) \\ &= E_P U_{\tau}^i(\mu \circ \sigma \circ \tau) \\ (2) \quad &= \int E_P U_{\tau}^i(\mu \circ \sigma \circ \tau) \mid \tau_i = t_i) P(\tau^i \in dt_i) \\ &= \int \bar{A}_{t_i}^i(\sigma_1, \dots, \sigma_n) p^i(dt_i) \end{aligned}$$

where p^i denotes the distribution of τ^i which can be computed by means of p via

$$p^i(F_i) = \int_{(T^1 \times \dots \times F_i \times \dots \times T^n)} dp(t_1, \dots, t_i, \dots, t_n) = p(T^1 \times \dots \times F_i \times \dots \times T^n)$$

Remark 1.7: A particular strategy for player i is "telling the truth", represented by the identity map

$$\begin{aligned} \bar{\sigma}_i &: T^i \rightarrow T^i \\ \bar{\sigma}_i(t_i) &= t_i \end{aligned}$$

Definition 1.8: μ is *Bayesian incentive compatible (BIC)* if "telling the truth" is a Nash equilibrium in Γ^μ .

Now, given some trivial positivity conditions on p , which we adopt for the remaining exposition, this means that $\bar{\sigma}$ is a Nash equilibrium in each $\Gamma_{\bar{t}}^\mu$. If we focus our interest

on player 1, then, in view of $\bar{\sigma} \circ \tau = \tau$, the equilibrium condition writes

$$\begin{aligned} & E_P U_\tau^1 (\mu \circ \tau \mid \tau^1 = \bar{t}_1) \\ & \geq E_P U_\tau^1 (\mu(\sigma^1 \circ \tau^1, \tau^2, \dots, \tau^n) \mid \tau^1 = \bar{t}_1) \end{aligned}$$

and if $p(\cdot \mid \bar{t}_1)$ denotes the distribution of τ_2, \dots, τ_n under $P(\cdot \mid \tau^1 = \bar{t}_1)$, this inequality is

$$\begin{aligned} & \int_{\bar{t}_1} U_{t_2 \dots t_n}^1 (\mu(\bar{t}_1, t_2, \dots, t_n)) dp(t_2, \dots, t_n \mid \bar{t}_1) \\ & \geq \int_{\bar{t}_1} U_{t_2 \dots t_n}^1 (\mu(\sigma_1(\bar{t}_1), t_2, \dots, t_n)) dp(t_2, \dots, t_n \mid \bar{t}_1) \end{aligned}$$

As $\sigma_1(\bar{t}_1)$ is running through T^1 while σ_1 is running through all possible strategies of player 1, we come up with a trivial

Lemma 1.9: μ is BIC if and only if for any $(\bar{t}_1, \bar{s}_1) \in T^1 \times T^1$ we have

$$\begin{aligned} & \int_{T^2 \times \dots \times T^n} U_{\bar{t}_1 \dots}^1 (\mu(\bar{t}_1, \dots)) dp(\dots \mid \bar{t}_1) \\ (3) \quad & \geq \int_{T_2 \times \dots \times T_n} U_{\bar{t}_1 \dots}^1 \mu(\bar{s}_1, \dots) dp(\dots \mid \bar{t}_1) \end{aligned}$$

Note that, of course, $p(\dots | t_1)$ can be computed by means of p , clearly

$$p(t_2, \dots, t_n | t_1) = P(\tau^2 = t_2, \dots, \tau^n = t_n | \tau^1 = t_1)$$

$$= \frac{P_{t_1, \dots, t_n}}{\sum_{s_2, \dots, s_n} P_{t_1, s_2, \dots, s_n}}$$

At this stage we should again discuss the temporary relationship between bargaining about the mechanism and the chance move that assigns the type t_i to player i . Following HOLMSTRÖM-MYERSON [3] we distinguish 3 categories in which players decisions (and bargaining) can be viewed.

1. "*ex ante*" - before chance moves players will discuss the decisions to be made and, if possible, agree upon a BIC-mechanism. At this stage an important criterium that matters for the decision is the expected payoff

$$(4) \quad \bar{A}^{i\mu}(\bar{\sigma}) = E_P(U_\tau^i(\mu \circ \tau))$$

hence, if we adopt the notion that players basically agree upon BIC-mechanisms then at this first stage we might consider the idea that they are essentially bargaining concerning payoffs from the feasible set

$$(5) \quad \text{CPH} \{ \bar{A}^{1\mu}(\bar{\sigma}), \dots, \bar{A}^{n\mu}(\bar{\sigma}) \mid \mu \text{ BIC} \}.$$

2. "*in mediis*" (or *interim*) - after the chance move, that is, when every player knows his type t_i and before any announcements have been made concerning every other players type. We could also say that the *in mediis* situation occurs after chance has selected one of the subgames Γ_t^μ that were mentioned earlier. Given this information, player i 's payoff is given by

$$(6) \quad \bar{A}_{t_i}^{i\mu} := \bar{A}_{t_i}^{i\mu}(\bar{\sigma}) = E_P(U_\tau^i(\mu \circ \tau) \mid \tau_i = t_i).$$

Even, as we have constantly stressed to be the case, bargaining takes place *ex ante* then nevertheless players might argue about their situation *in mediis* and considerations concerning what is "fair", "equitable", or what

kind of power a player has during the bargaining process, will also touch the possible situation in mediis. In this case they are actually considering the following "feasible set" of possible payoffs:

Definition 1.9: Let

$$\nabla := \text{CPH} \left\{ (A_{t_1}^1(\mu))_{t_1 \in T^1}, \dots, (\bar{A}_{t_n}^n(\mu))_{t_n \in T^n} \mid \mu \text{ BIC} \right\}$$

then the following is a trivial statement (cf. also MYERSON [5]):

Lemma 1.10: $\nabla \subseteq V$ and ∇ is a compact, convex, and comprehensive subset of

$$\mathbb{R}^{|T^1| + \dots + |T^n|}.$$

This is seen by essentially using the linearity of $\bar{A}_{t_i}^i(\cdot)$ in μ as well as the fact that BIC's, being defined by linear inequalities of type (2), form a compact convex polyhedron (in \underline{X}^T).

In fact, in [5] MYERSON evaluates the HARSANYI-SELTEN value essentially with respect to ∇ .

Our discussion concerning the temporal aspects of our model should continue with

3. "ex post" – that is after the announcements have been made that were agreed upon according to the mechanism μ . This notion is particularly intriguing because, if the players had agreed in advance upon a BIC mechanism then, after these announcements, every player is completely informed about the prevailing types. This fact was not initially included in the model, rather it is a consequence of the adoption of BIC mechanisms. Hence, every player is now aware of the bargaining situation $(V_t, \underline{u}_t)_{t \in T}$ – and he might very well regret being in a situation like this in view of what the mechanism has in store for him. That is, even if bargaining takes place *ex ante* players, while arguing about mechanisms, will point out that they could end up with a certain probability $p(\{t\})$ in this bargaining situation and that the mechanism μ – if considered to be fair or equitable – should

somehow reflect its properties within this situation. Of course, player i 's payoff *ex post* is described by $U_t^i(\mu(t))$ – but while the discussion is taking place *ex ante*, the distribution of these payoffs which is given by the law p matters as well.

Actually it would seem that consideration of the *ex post* situation is also induced by the application of BIC mechanisms. For, such mechanisms are devised in order to render a transfer of information between the players feasible. The use of such information can be to the advantage of all players – or of a few ones. However, this advantage materializes only when the additional knowledge in some sense enters the communication process which takes place *ex ante*. Perhaps one can argue that the introduction of BIC mechanisms makes no sense unless we assume that players want to use arguments from the *ex post* situation within the *ex ante* bargaining procedure.

The notion of "probability invariance" ([6]) should also be discussed in this context. It would seem that this idea more or less presses the fact that V is the only and essential quantity a solution concept should depend upon. If so, then a probability invariant solution concept depends only on the "*in mediis*" stage of the whole game and does not reflect the concerns of the players with respect to the *ex post* situations.

§ 2 Examples

Example 2.1:

Two players divide 100 dollars. While player 1 has a utility function linear in money, player 2 may be one of two types having either linear or logarithmic utility accordingly. Thus we have

$$\underline{X} = \{x \in \mathbb{R}_+^2 \mid x_1 + x_2 \leq 100\}, \quad \underline{x} = (0,0)$$

$$T^1 = \{*\}, \quad T^2 = \{a, r\}$$

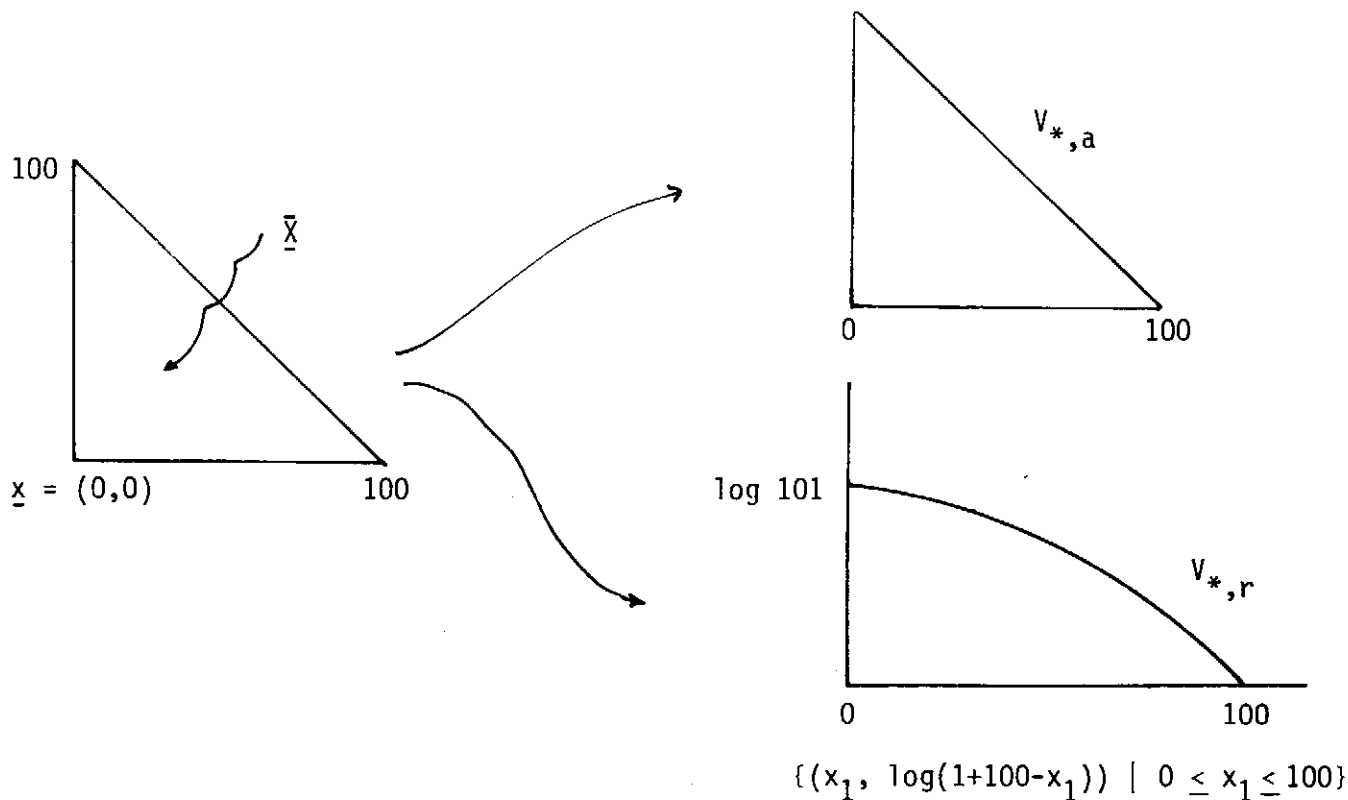
$$p = (p_a, p_r) \geq 0, \quad p_a + p_r = 1$$

$$(1) \quad U_{*,a}^1(\cdot) = U_{*,r}^1(\cdot) = U^1, \quad U^1(x) = x_1$$

$$U_{*,a}^2(x) = U_a^2(x) = x_2$$

$$U_{*,r}^2(x) = U_r^2(x) = \log(1+x_2)$$

The "cooperative situation" $(\underline{X}, V_{t(t \in T)})$ is obviously represented as follows



(Fig.2)

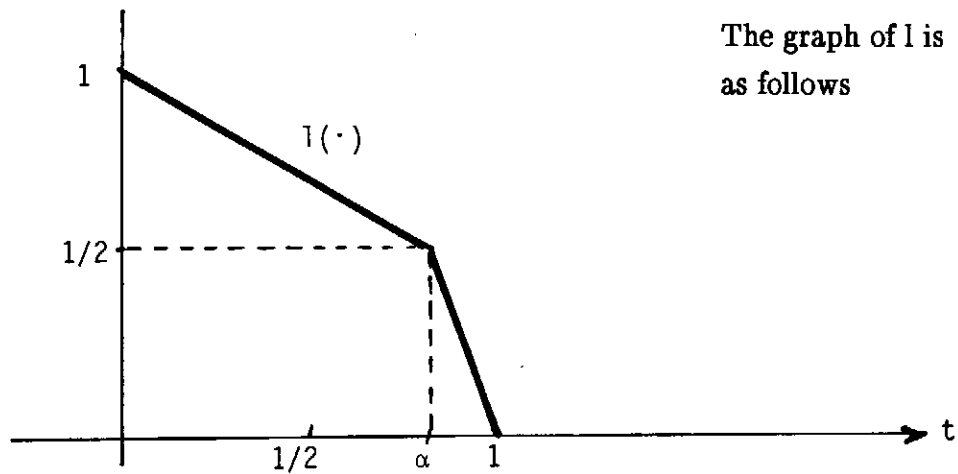
Example 2.2: Similar to Example 2.1., let players divide just 1 dollar and player 2 in situation of being type "r" has the utility

$$U_{\star,r}^2(x) = U_r^2(x) = x_2 \frac{1(x_1)}{1-x_1}$$

where

$$(2) \quad l(t) = \begin{cases} \frac{2\alpha-t}{2\alpha} & 0 \leq t \leq \alpha \\ \frac{1-t}{2(1-\alpha)} & \alpha \leq t \leq 1 \end{cases}$$

and $\frac{1}{2} < \alpha < 1$ denotes a parameter.



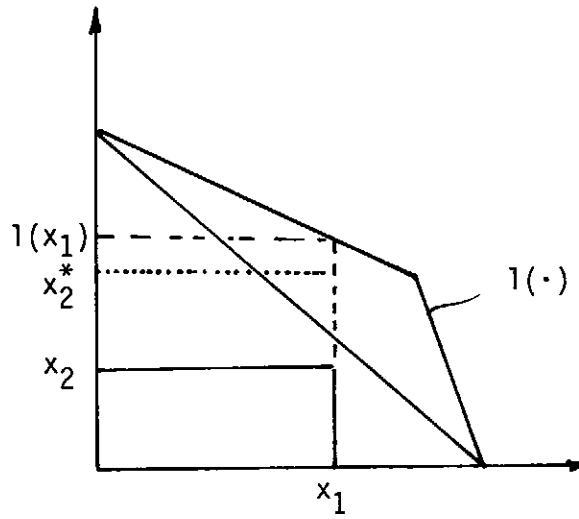
(Fig.3)

and in case of $t = (\star,r)$, an agreement $(x_1, x_2) \in \underline{X}$ results in utilities

$$(x_1, x_2^\star) = (x_1, U_r^2(x))$$

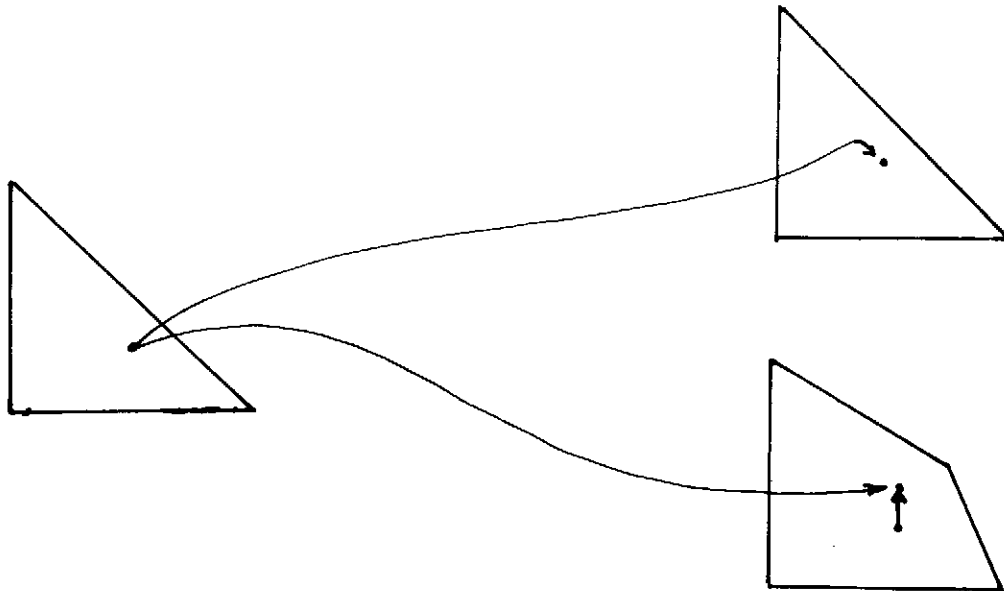
for the players such that

$$(3) \quad \frac{x_2^\star}{x_2} = \frac{1(x_1)}{1-x_1}$$



(Fig.4)

That is, player 2's utility in this case is increasing proportionally to $\frac{l(x_1)}{1-x_1}$; this leads to the following situation



(Fig.5)

Example 2.3: Consider two Bimatrix-Games, called "TOP" and "BOTTOM", each being represented by two matrices as follows

$$(4) \quad \begin{array}{l} \text{TOP:} \\ \text{BOTTOM:} \end{array} \quad \begin{array}{cc} \left[\begin{array}{cc} 1 & 4 \\ 0 & 6 \end{array} \right] & \left[\begin{array}{cc} 6 & 4 \\ 0 & 0 \end{array} \right] \\ \left[\begin{array}{cc} 5 & 6 \\ 0 & 6 \end{array} \right] & \left[\begin{array}{cc} 6 & 2 \\ 0 & 0 \end{array} \right] \end{array}$$

Both games have a Nash equilibrium in pure strategies as indicated.

These games may be used to construct another one as follows:

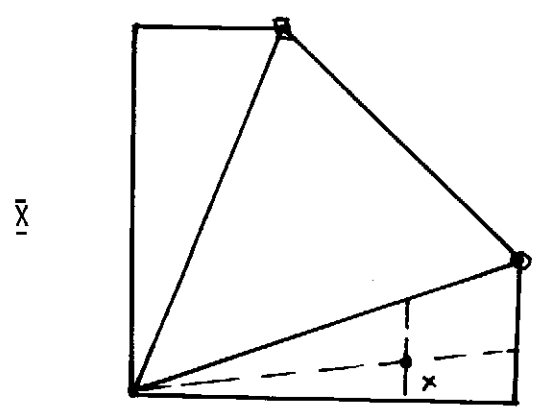
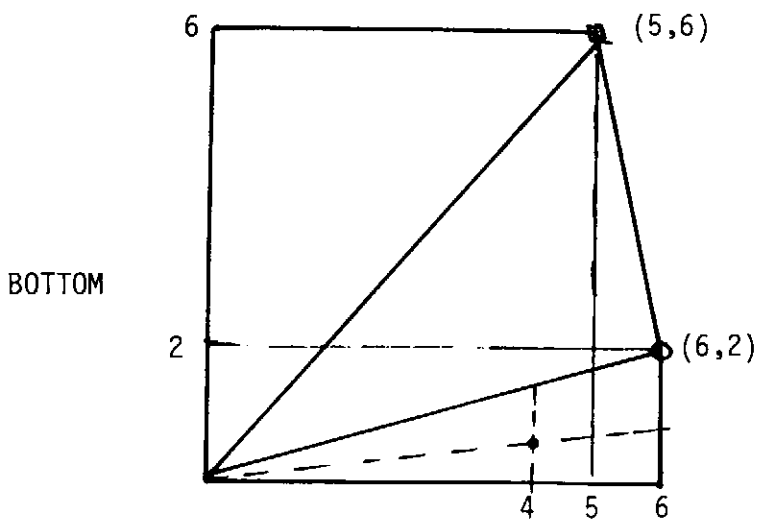
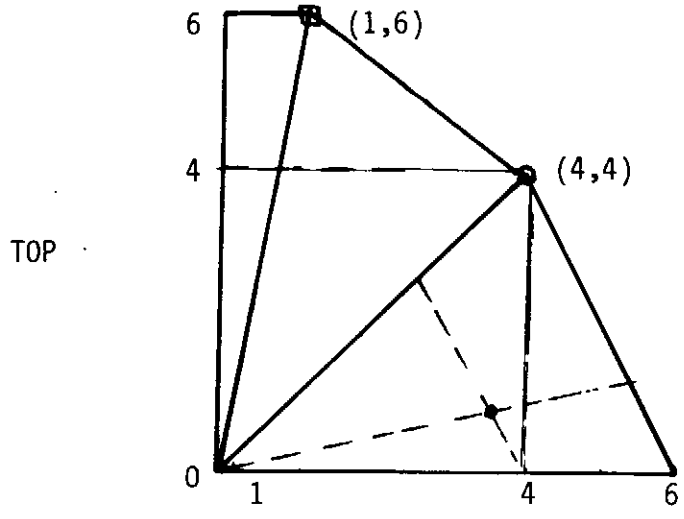
Player one is given the information TOP or BOTTOM. Both players may then choose jointly randomized (correlated) strategies and, after agreeing to what kind of strategy they want to apply, receive their corresponding expected value.

This model, which is constructed rather in the spirit of HARSANYI and SELTEN [2] as well as MYERSON [5] may be presented in our present framework. Note, however, that we have to agree upon a "threatpoint" – which could be generated by adding a row and a column of zeros in each matrix – or by having each player play "max-min".

Also note that the parametrisation via the joint random strategies is not 1-1.

Hence by choosing a different parametrisation we certainly change the game – and it is a matter of discussion whether 1-1 parametrisations are relevant.

Our parametrisation, as indicated is chosen such that the Pareto curve is parametrized linearly in an obviously way and any non Pareto efficient point is connected to zero, thus defining a Pareto efficient point – the properties of which determine the parametrization of the non Pareto efficient point.



(Fig.6)

MYERSON considers an example in [5], [6], this is the only one so far existent in the literatur, for which this solution and the HARSANYI-SELTEN solution were computed. Again, as we used "joint randomized" mixed strategies, this example is presented more in a "Bimatrix set up" spirit. However, it could be argued that it fits much more in our present framework (which allows for a continuous share of money).

This example is presented somewhat more formal as follows.

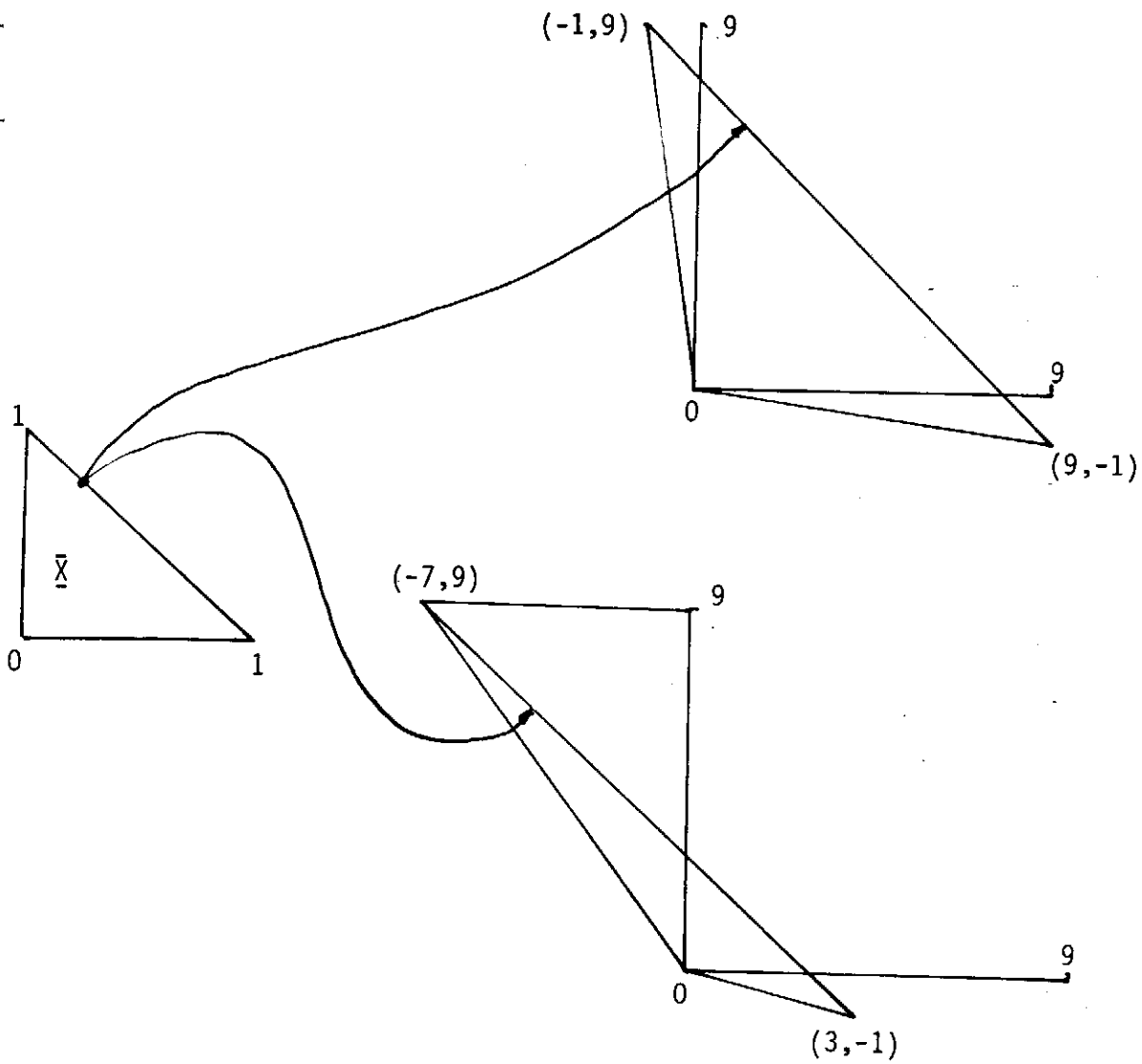
Example 2.4: (Two players, $I = \{1,2\}$). $T^1 = \{T,B\}$, $T^2 = \{\star\}$

i.e., player 1 is TOP or BOTTOM, while player 2's type is fixed. Let

$$\underline{X} = \{x \in \mathbb{R}_+^2 \mid x_1 + x_2 \leq 1\}$$

Possible states of nature are (T,*) and (B,*), let us omit the *. Then the utilities are given by

$$(5) \quad \begin{aligned} U_T^1(x) &= 10 x_1 - (x_1 + x_2) \\ U_B^1(x) &= 10 x_1 - 7(x_1 + x_2) \\ U^2(x) &= 10 x_2 - (x_1 + x_2) \end{aligned}$$



(Fig.7)

Note that the utilities can also be written

$$\begin{aligned} U_T^1(x) &= 9(x_1 + x_2) - 10x_2 \\ (6) \quad U_B^1(x) &= 3(x_1 + x_2) - 10x_2 \\ U^2(x) &= 9(x_1 + x_2) - 10x_1 \end{aligned}$$

and in particular, on the Pareto surface ($x_1 + x_2 = 1$), we have

$$\begin{aligned} U_T^1(x) &= 9 - 10 x_2 \\ (7) \quad U_B^1(x) &= 3 - 10 x_2 \\ U^2(x) &= 9 - 10 x_1 \end{aligned}$$

Thus the interpretation (cf. [5]) is as follows: The players may engage in some joint venture which generates costs of 10 units. For player two, its worth is 9 units while for player 1 it is 9 or three units, according to whether he is type TOP or BOTTOM. x_i denotes the amount player $3-i$ has to pay (and \underline{X} suggests, that the venture can also be obtained on a lower "activity level" $x_1 + x_2 < 1$ - in which case the agreement, however, would not be Pareto efficient).

§ 3 BIC-mechanisms visualized

Consider the case (as suggested by the examples in the previous section) that $I = \{1,2\}$ and player 2 has only one type available, say $T^2 = \{\star\}$. We have then $p(\cdot | t_1) = \delta_{\star}$ and hence

$$\int_{T_2} U_{(t_1, \cdot)}^1(\mu(t_1, \cdot)) dp(\cdot | t_1) = U_{(t_1, \star)}^1(\mu(t_1, \star))$$

such that the inequality in Lemma 1.7. reads for all $t_1 \in T_1$:

$$(1) \quad U_{(t_1, \star)}^1(\mu(t_1, \star)) \geq U_{(t_1, \star)}^1(\mu(s_1, \star))$$

whatever $s_1 \in T^1$.

The second inequality is trivially satisfied, as $\bar{t}_2 = \bar{s}_2 = \star$ is the unique element in T_2 .

Omitting the star \star , we have a BIC mechanism represented by a mapping

$$\mu : T^1 \rightarrow \underline{X}$$

such that

$$U_{t_1}^1(\mu(t_1)) \geq U_{t_1}^1(\mu(s_1))$$

for all $s_1, t_1 \in T_1$.

Assume now, in addition, that $T^1 = \{T, B\}$ consists of just two elements. In accordance with Remark 1.2., let us write

$$u_T := U_T(\mu(T)) \in V_T$$

$$u_B := U_B(\mu(B)) \in V_B$$

Within this simple set up the conditions for μ to be BIC are

$$u_T^1 \geq U_T^1(\mu(B))$$

$$u_B^1 \geq U_B^1(\mu(T))$$

Consider the first condition which we write

$$U_T(\mu(B)) \in \{u \mid u_T^1 \geq u_1\}$$

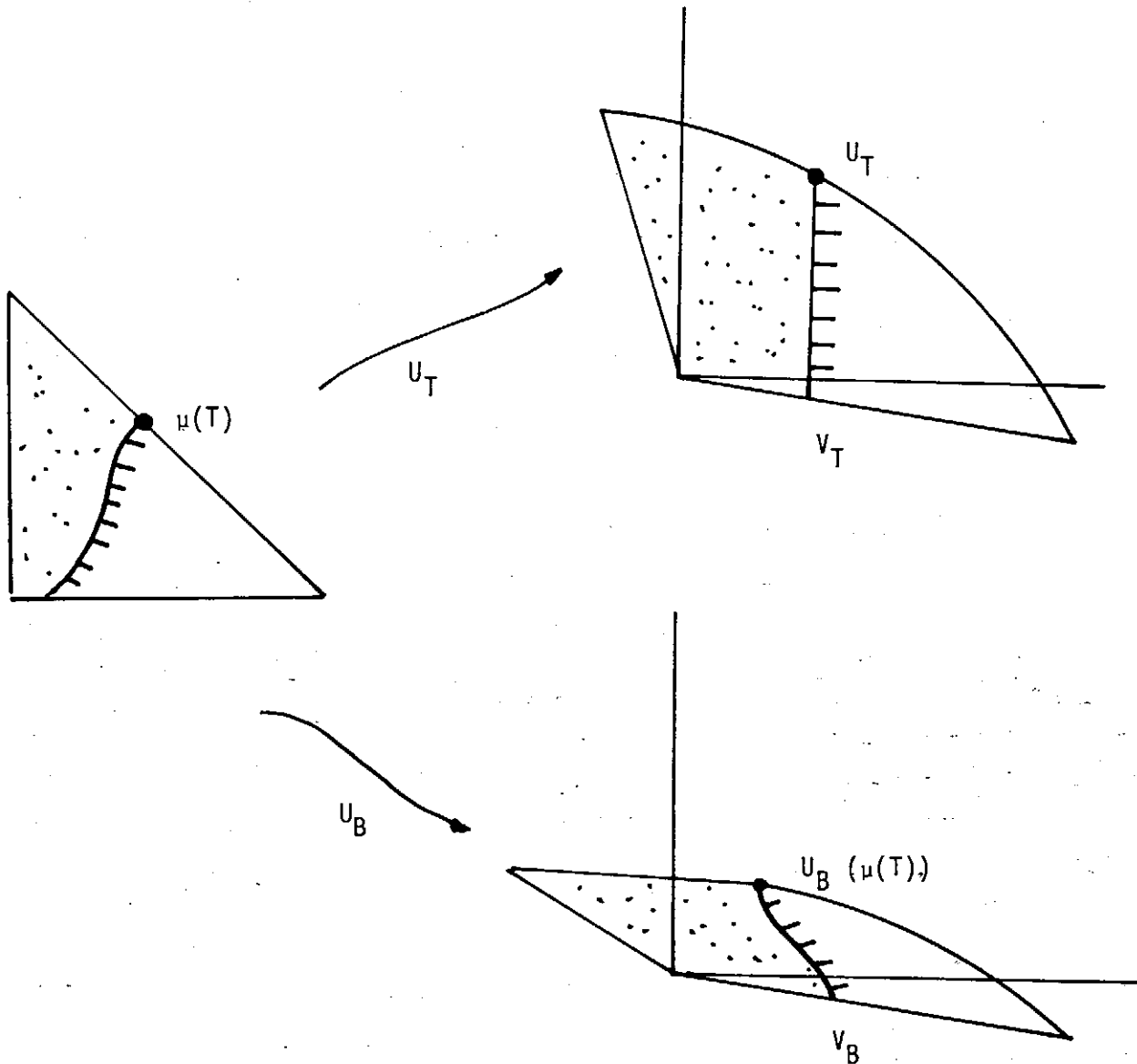
or

$$\mu(B) \in U_T^{-1}(\{u \mid u_T^1 \geq u_1\}).$$

Applying U_B we come up with

$$u_B \in U_B(U_T^{-1}(\{u \mid u_T^1 \geq u_1\})).$$

Assume that U_B and U_T are 1-1 and that U^1 is monotone in the first coordinate, i.e., the situation has been modelled in a way that player 1's utility increases with the 1st coordinate of $x \in \underline{X}$.



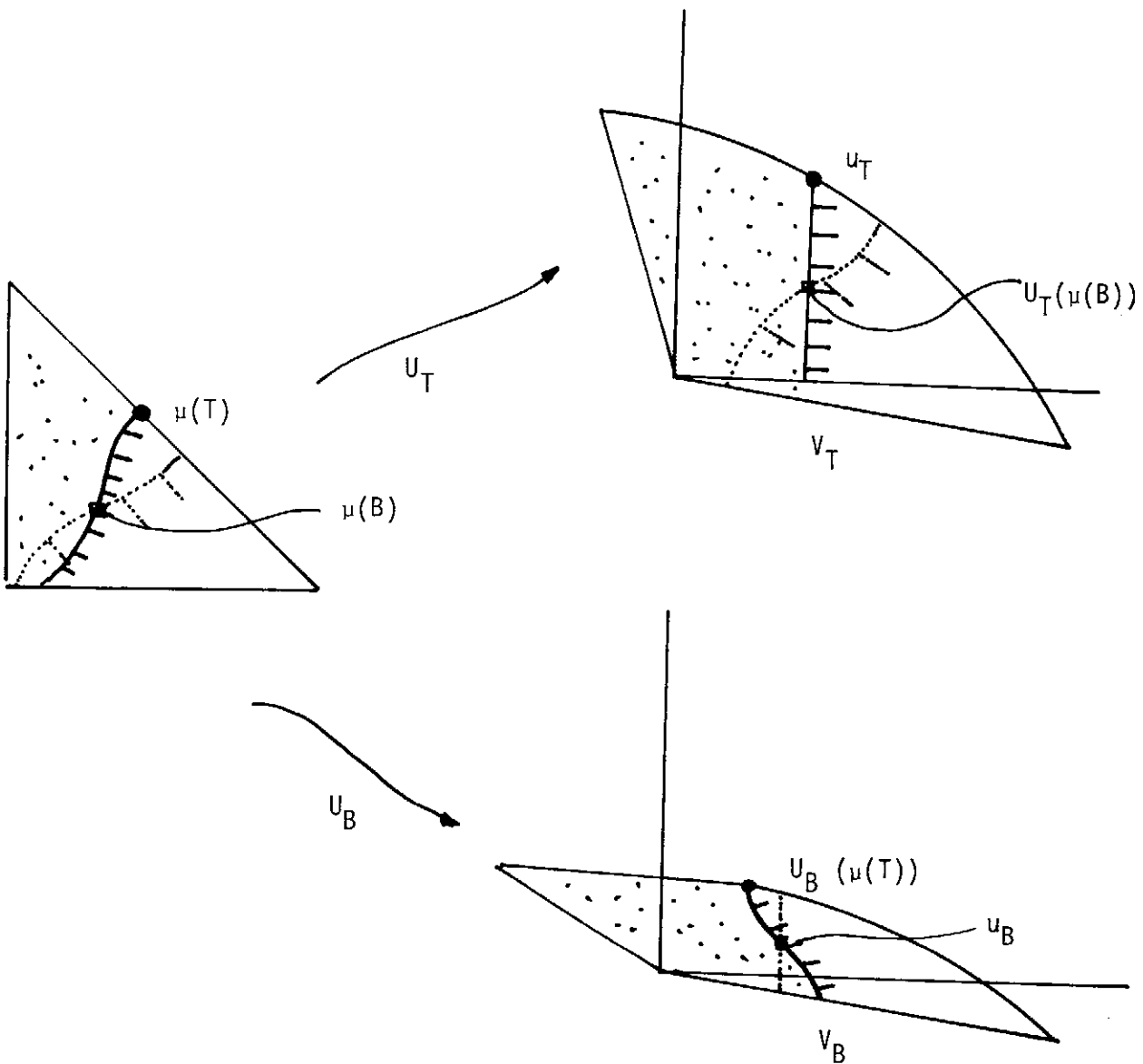
(Fig.8)

Then, intuitively, $U_T^{-1} (\{u \mid u_T^1 \geq u_1\})$ is a set in \underline{X} to the right of the curve $U_T^{-1} (\{u \mid u_T^1 = u_1\})$ and similarly $U_B (U_T^{-1} \{...\})$ can be visualized in V_B .

Similarly, the second equation in (*) will lead to a requirement

$$u_T \in U_T (U_B^{-1} (\{u \mid u_B^1 \geq u_1\}))$$

and the pair $(\mu(T), \mu(B))$ is BIC only if both commodities are satisfied which typically might be the case if the following picture is reflecting the true situation.



(Fig.9)

Of course, as U_B and U_T are 1-1 mappings the situation is completely depicted, say, by the 1-1 mapping

$$U_0 := U_T^{-1} \circ U_B : V_T \rightarrow V_B$$

and the two vectors $u^T \in V_T$, $u^B \in V_B$ which should satisfy

$$u_B \in U_0(\{u \in V_T \mid u_1 \leq u_T^1\})$$

$$u_T \in U_0^{-1}(\{u \in V_B \mid u_1 \leq u_B^1\})$$

Remark 3.1: Note that $\mu(T) = \mu(B)$ or

$$U_0(u_T) = u_B$$

always implies BIC; this is the situation where the mechanism doesn't react to the announcements of the players.

Next, let us adapt *Definition 1.9.* to the present situation. We have to consider typical elements of \mathbb{V} of the form

$$(\bar{A}_T^1(\mu), \bar{A}_B^1(\mu), \bar{A}_*^2(\mu))$$

where μ is BIC. Assuming that

$$P(\tau = T) = q$$

$$P(\tau = B) = 1-q$$

we have, e.g.

$$\bar{A}_T^1(\mu) = E_P(U_\tau^1(\mu \circ \tau) \mid \tau = T) = U_T^1(\mu(T)) = u_T^1$$

$$\bar{A}_B^1(\mu) = u_B^1$$

and

$$\begin{aligned} \bar{A}_*^2(\mu) &= E_P(U_\tau^2(\mu \circ \tau)) \\ &= q U_T^2(\mu(T)) + (1-q) U_B^2(\mu(B)) \\ &= q u_T^2 + (1-q) u_B^2 \end{aligned}$$

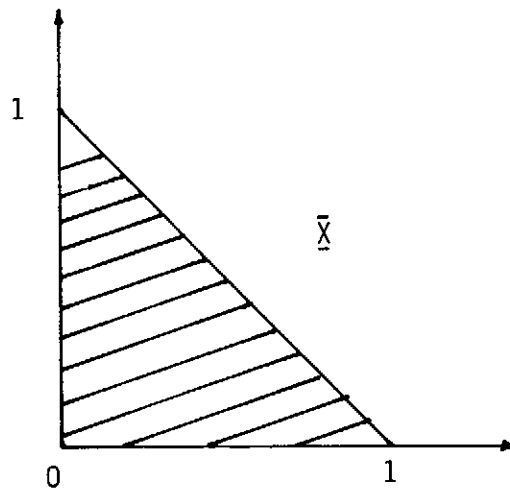
that is, if u_T, u_B are the utilities induced by BIC mechanism, then

$$(u_T, u_B) \rightarrow (u_T^1, u_B^1, q u_T^2 + (1-q) u_B^2)$$

generates the set ∇ as (u_T, u_B) ranges through the BIC utilities.

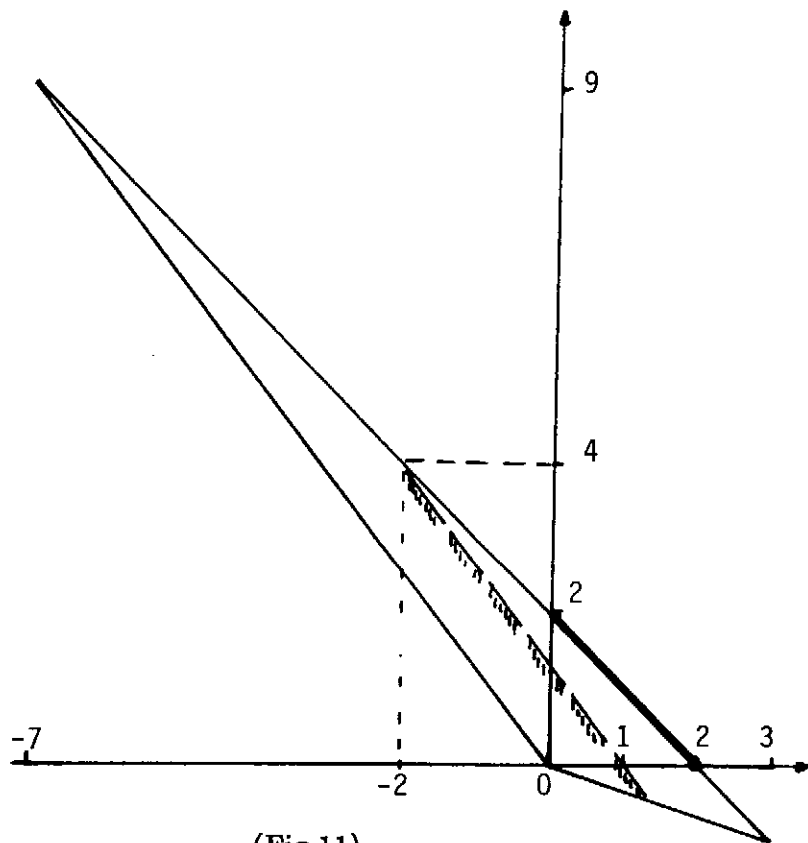
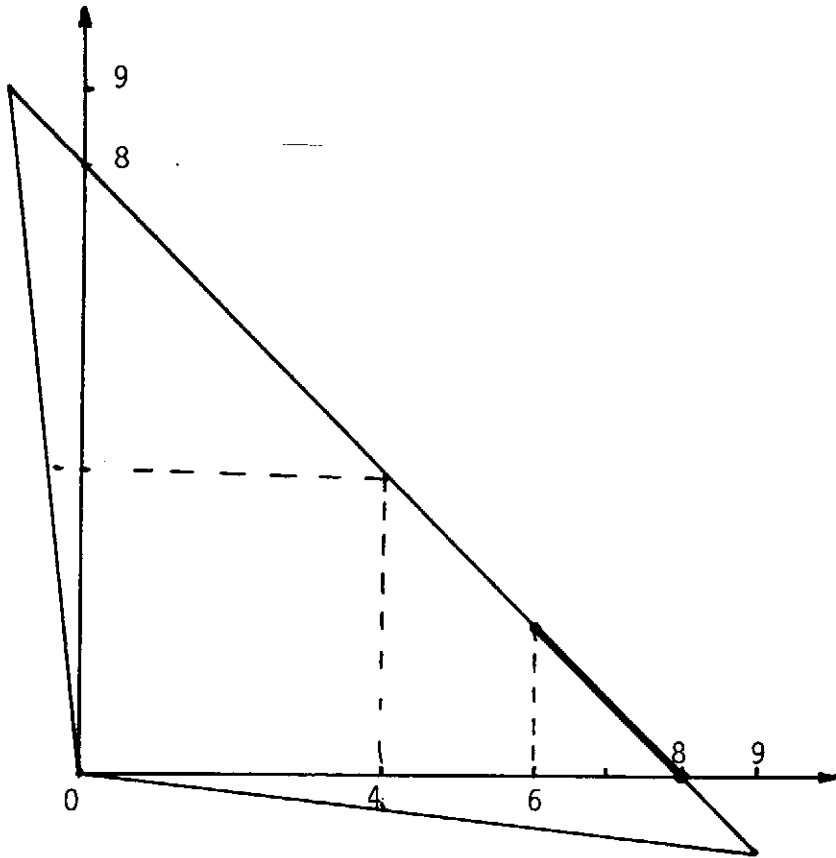
Example 3.2: (i.e., Example 2.4.)

In Example 2.4., V_T and V_B can easily be visualized, we omit the parameter set \underline{X} which is



(Fig.10)

For instance $u_T = (4,4)$ and $u_B = (-2,4)$ constitute a BIC mechanism.



(Fig.11)

Let us consider the case that $q = 9/10$. The following pairs of utilities constitute BIC's with the property that

$$(u_T^1, u_B^1, q u_T^2 + (1-q) u_B^2)$$

is an *extreme point* in ∇ . These are (for $q = 9/10$):

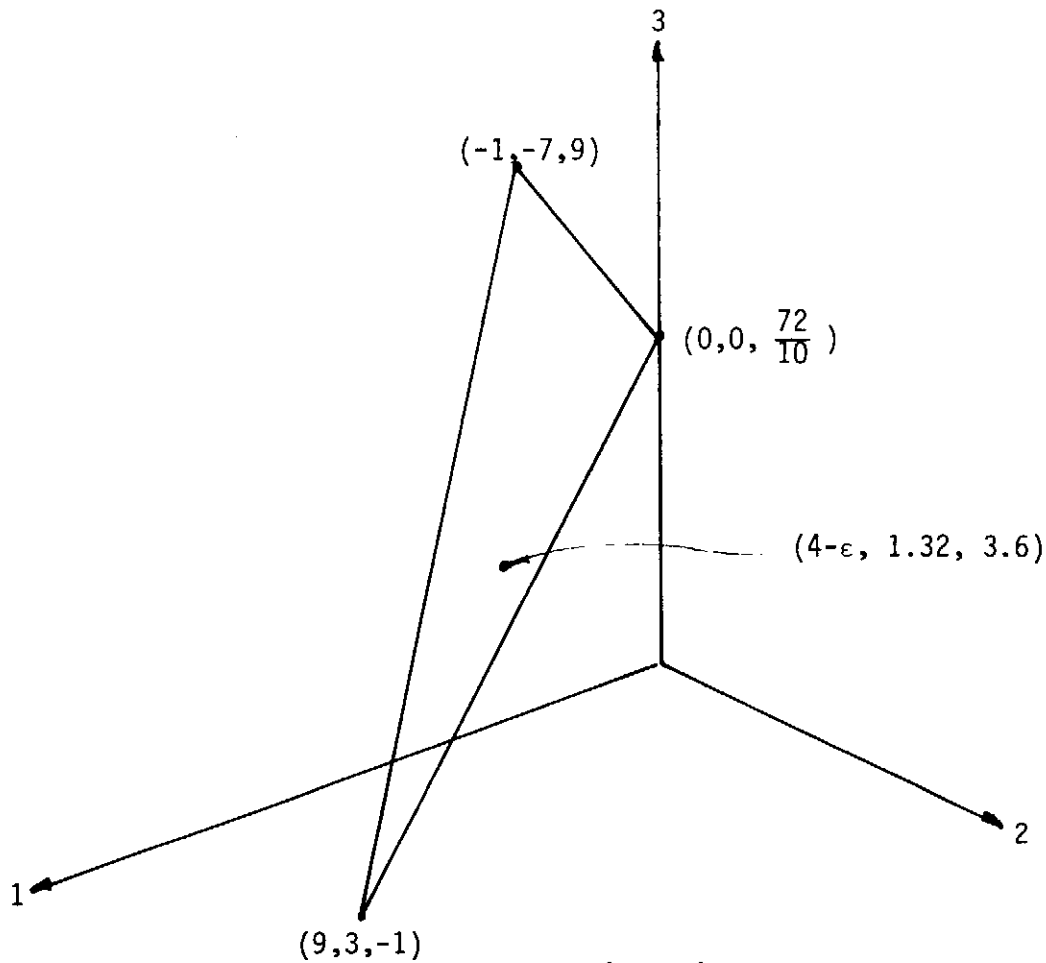
$$(-1,9), (-7,9) \rightarrow (-1,-7,9)$$

$$(0,8), (0,0) \rightarrow (0,0, \frac{72}{10})$$

$$(9,-1), (3,-1) \rightarrow (9,3,-1)$$

It may be verified that these are all the BIC's yielding extreme points in ∇ . Thus ∇ is the convex hull of

$$(-1,-7,9), (0,0, \frac{72}{10}), (9,3,-1)$$



(Fig.12)

The maximizer of the generalized Nash Product

$$u_1^{9/10} u_2^{1/10} u_3$$

is approximately of the shape

$$(4-\epsilon, 1.32, 3.6)$$

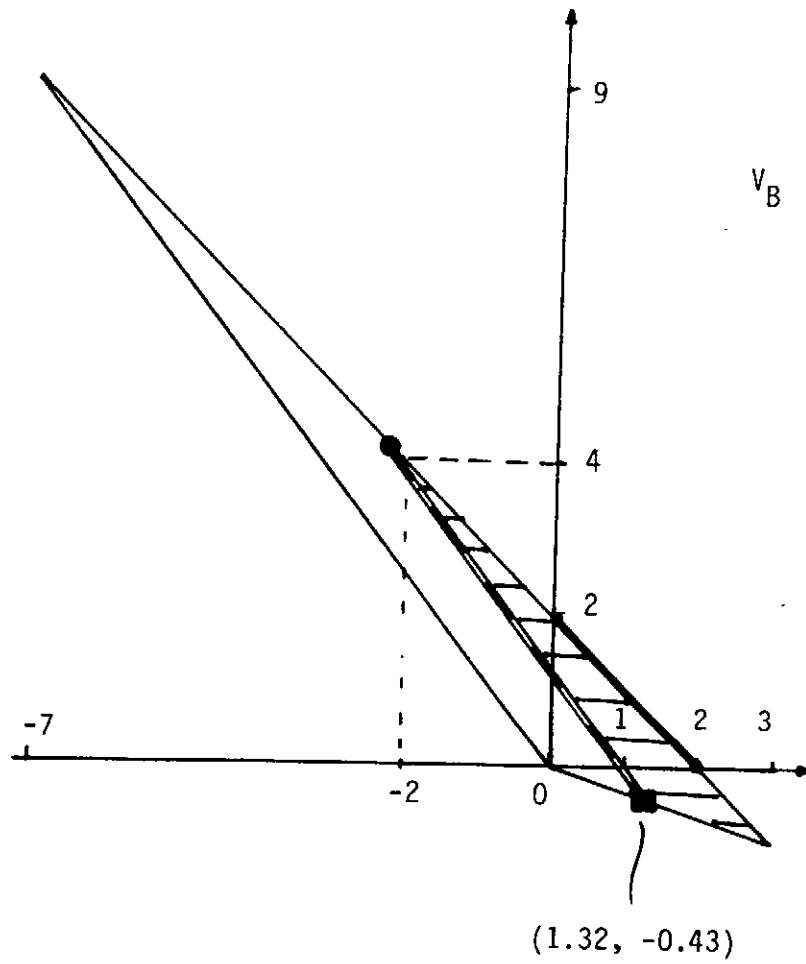
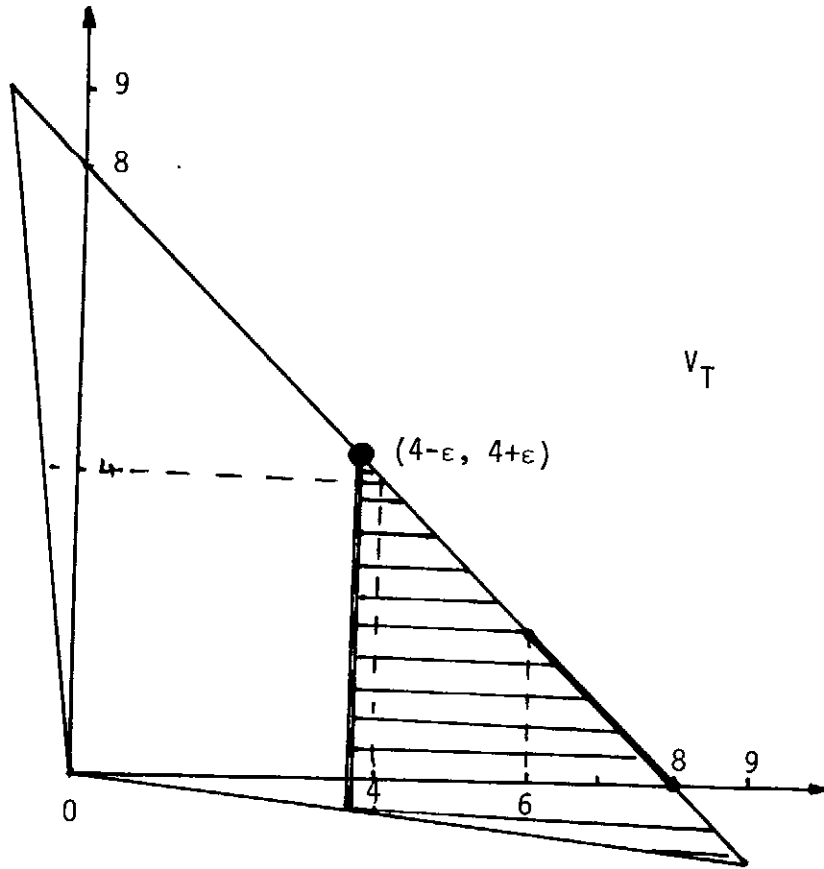
and is "implemented" by the BIC utilities

$$\bar{u}_T = (4-\epsilon, 4+\epsilon)$$

$$\bar{u}_B = (1.32, -0.43)$$

$$\rightarrow (4-\epsilon, 1.32, 3.6 + \frac{9}{10} \epsilon - \frac{0.43}{10})$$

which means $\epsilon = \frac{0.43}{9}$. This example corresponds to MYERSON's computation in [5] .



(Fig.13)

§ 4 Ex post arguments

We have previously argued our basic assumptions, that is, we assume that bargaining concerning μ takes place *ex ante* – so as to be sure that there is a well defined game Γ^μ such that "telling the truth" is a Nash equilibrium.

Also we are assuming that "announcing types" takes place *in mediis*. However, player i , when discussing μ is mentally passing by all possible states $\bar{t}_i \in T^i$ he may find himself to be in, that is, that player i is concerned with what happens if chance has moved.

Moreover, we favor the idea that the players settle for some BIC mechanism and hence they will eventually know the other fellows types; this fact is also known to the players in advance. Of course this idea is based upon the basic assumption of the possibility of "law enforcement".

If bargaining at the *ex ante* stage allows for arguments which refer to the stages *ex ante* and *in mediis*, then we should also allow for arguments derived from the situation *ex post*. Player i knows that, given a BIC mechanism, he will eventually know the truth that is the realization of the type selecting random variable. Therefore he might as well be worried *ex ante* what will happen to him *ex post* – that is, he considers what he will be awarded by μ not only w.r.t. his expectations but also w.r.t. all possible $t \in T$. It is likely that players in such situation will argue that they cannot accept a mechanism which leaves them with a possible payoff which is not even individually rational – even if this should happen with small probability. This means we want to introduce the notion of *ex post* individual rationality.

Definition 4.1: $\mu \in \text{BIC}$ is called *ex post i.r.* if, for all $i \in I$ and all $t \in T$

$$(1) \quad U_t^i(\mu(t)) \geq U_t^i(\underline{x})$$

or, for short, if we write this as a vector inequality

$$(2) \quad U_t(\mu(t)) \geq U_t(\underline{x}) \quad (t \in T)$$

or

$$(3) \quad u_t \geq \underline{u}_t \quad (t \in T)$$

We shall use the notation " $\mu \in \text{BICIR}$ " to indicate that $\mu \in \text{BIC}$ is i.r..

Let us shortly discuss the notion of a "solution concept". Such concept should also in some sense depend on all three stages of the game. In particular, it is possible that the concept is influenced by *ex post* arguments (as the players in bargaining about mechanisms will use such arguments), a fact that might violate MYERSON's probability invariance axiom. First of all the consideration of the *ex post* stages of the game lead to the introduction of the set

$$(4) \quad \begin{aligned} V_t^+ &:= \text{CPH} \{U_t(x) \mid x \in \underline{X}, U_t(x) \geq U_t(\underline{x})\} \\ &= \text{CPH} \{u_t \mid u_t \in V_t, u_t \geq \underline{u}_t\} \end{aligned}$$

and, of course, the fact that a mechanism μ is *ex post* individually rational and BIC induces that

$$(5) \quad U_t(\mu(t)) \in V_t^+ \quad (t \in T)$$

Note that we are still entertaining the idea of a 1-1 parametrization of the sets V_t ; hence a mechanism is essentially described by indicating all vectors $U_t(\mu(t)) \in V_t^+$ ($t \in T$). Roughly speaking this means that μ can be regarded as an element of $(\mathbb{R}^n)^T$.

Of course we are also interested in the set

$$(6) \quad \nabla^+ = \text{CPH} \{(\bar{A}_{t_i}^i(\mu))_{i \in I, t_i \in T^i} \mid \mu \in \text{BICIR}\} \subseteq \mathbb{R}^{|T^1| + \dots + |T^n|}$$

Vectors in this set are generally denoted by $y = (y_{t_i}^i)_{i \in I, t_i \in T^i}$. In particular the case

that the bargaining process fails to reach an agreement corresponds to a certain mechanism $\underline{\mu}$, $\underline{\mu}(t) = \underline{x}$ ($t \in T$); this way the threatpoint is imbedded into the set of mechanisms, more precisely, the threatpoint is regarded as an element of BICIR. Of course we have $u(\underline{\mu}(t)) = \underline{u}_t$ for all $t \in T$ and hence, as the strategic behavior of the players within the game $\Gamma^{\underline{\mu}}$ does not influence the payoff, we find that failure of reaching an agreement corresponds to the *in mediis* expectation of a payoff of

$$\begin{aligned}
 (7) \quad \bar{A}_{t_i}^i(\mu) &= E_P(U^i(\mu \circ \tau) \mid \tau_i = t_i) \\
 &= E_P(\underline{u}_{\tau}^i \mid \tau_i = t_i) \\
 &=: \underline{y}_{t_i}^i.
 \end{aligned}$$

Next, the generalized HARSANYI-SELTEN value is the mapping σ that assigns to every Γ the unique element $\bar{y} \in V^+$ that maximizes the (weighted) Nash coordinate product

$$(8) \quad \prod_{\substack{i \in I \\ t_i \in T^i}} (y_{t_i}^i - \underline{y}_{t_i}^i)^{p_i(t_i)}$$

where y ranges over the set V^+ ; we write $\sigma(\Gamma) = \bar{u}$. Compare the discussion of the HARSANYI-SELTEN value in Section 1 and the definition of p_i for this purpose.

Given Γ , we shall say that $\bar{\mu}$ implements $\sigma(\Gamma) = \bar{y}$ if

$$(9) \quad \bar{A}_{t_i}^i(\bar{\mu}) = \bar{y}_{t_i}^i$$

holds true. *A priori* it is not clear whether the implementation of the value is unique.

Most likely it is true that σ can be axiomatized by means of the axioms which were presented by HARSANYI and SELTEN [2] - provided the IIA-axiom is suitably formulated within the framework of the sets V_t . We will offer no formal proofs for these facts.

Rather the remainder of this section will be devoted to an extensive discussion of an example where σ is computed for all probabilities p .

This example has already been discussed in Section 2 (Example 2.4) and Section 3 (Example 3.1). The results obtained by our computations for various probabilities P does not seem all together unconvincing. In this context it is worth noting that

MYERSON's axiomatic approach [6], when studied for all probabilities P results in a discontinuity for $q = P(\tau = T) = \frac{1}{4}$ (WEIDNER [8]). It is an open question whether this is a structural problem, however, on first sight it is not unlikely that the probability invariance axiom when imposed upon a value may cause continuity problems.

Example 4.2: (= Example 3.1. = Example 2.4.)

Recall that, as in SECTION 3 a pair of utilities u_T, u_B resulting from a mechanism μ , i.e.

$$u_T = U_T(\mu(T)), u_B = U_B(\mu(B))$$

gives rise to an element of V^+ via

$$(10) \quad (u_T, u_B) \rightarrow (u_T^1, u_B^1, qu_T^2 + (1-q)u_B^2) = (y_T^1, y_B^1, y_{\star}^2);$$

recall that $P(\tau = T) = q$.

It is not hard to see that in order for the right side of (10) to be an extreme point of V^+ , it is necessarily true that either u_T or u_B is an extreme point of V_T^+ or V_B^+ . Moreover if, say u_T is extreme in V_T^+ , then in the framework of this example u_B is extreme in

$$U_B \circ U_T^{-1} (\{u \in V_T^+ \mid u_T^1 \geq u_1\})$$

Therefore, the following procedure is adopted in order to obtain the extreme points of V^+ :

(*) List extreme points $\hat{u}_T \in V_T^+, \hat{u}_B \in V_B^+$.

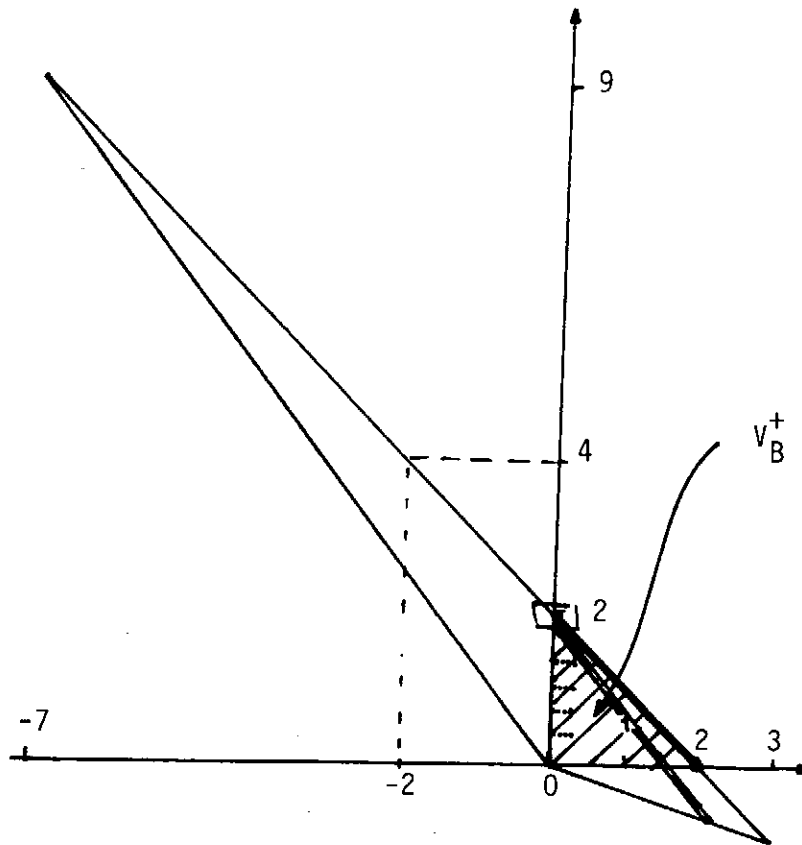
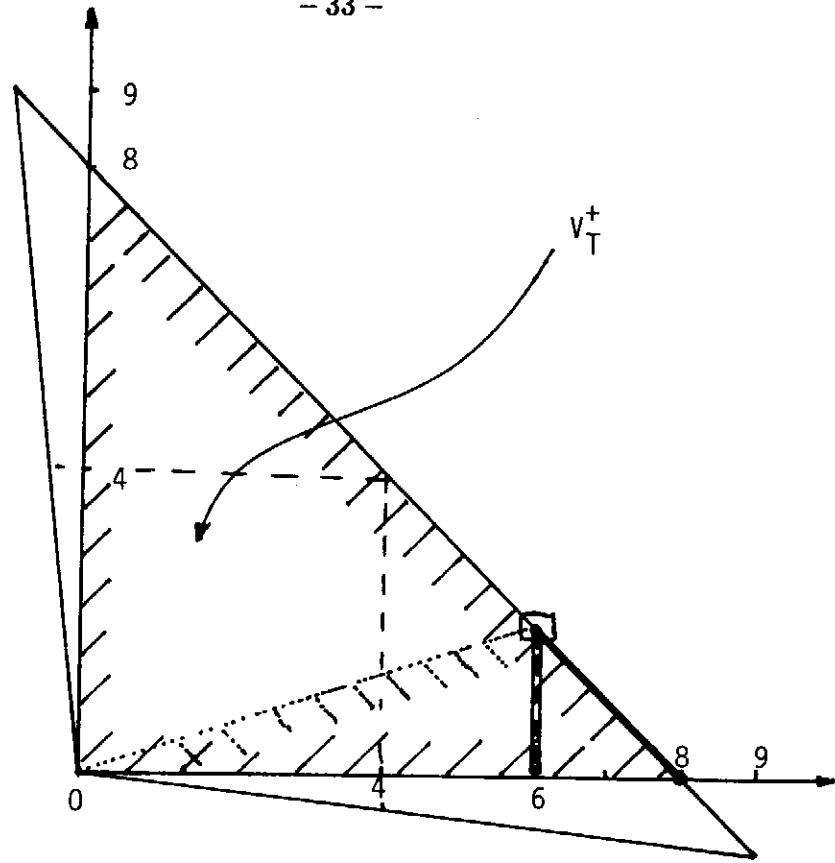
(**) Suppose \hat{u}_T is extreme in V_T^+ , combine with some $\hat{u}_B \in V_B^+, \hat{u}_B$ extreme in $U_B \circ U_T^{-1} (\{u \mid u_T^1 \geq u_1\})$.

According to this procedure we find

$$(6,2) (0,2) \rightarrow (6,0,2)$$

$$(0,8) (0,0) \rightarrow (0,0,8q)$$

$$(8,0) (2,0) \rightarrow (8,2,0)$$



(Fig.14)

Fig. 14 shows the situation for (6,2) (0,2). E.g. (0,2) is extreme in V_B^+ and (6,2) is extreme in

$$U_T \circ U_B^{-1} (\{u \in V_B^+ \mid u_B^1 \geq u_1\})$$

The set V^+ , resulting as the convex hull of (6,0,2), (0,0,8q), and (8,2,0) from this procedure is sketched in Fig. 15. The triangle Δ as depicted represents the Pareto surface of V^+ (which is its comprehensive hull) only if $q > \frac{1}{4}$.

The hyperplane including Δ is

$$\{y \mid ay = c\}$$

with $a = a^q = (q - \frac{1}{4}, 1-q, \frac{3}{4})$, $c = 6q$.

If we solve

$$(11) \quad \max \{y_1^q \cdot y_2^{1-q} \cdot y_3 \mid ay = c\}$$

then the maximizer is

$$\begin{aligned} \bar{y}^q &= \left(\frac{12}{4q-1} \frac{q^2}{4q-1}, 3q, 4q \right) \\ &= q \left(\frac{12q}{4q-1}, 3, 4 \right) \end{aligned}$$

If, in Fig. 15, we consider the projection on the 2-3-plane, then $\bar{y}^q \in \Delta$ requires two additional conditions, namely

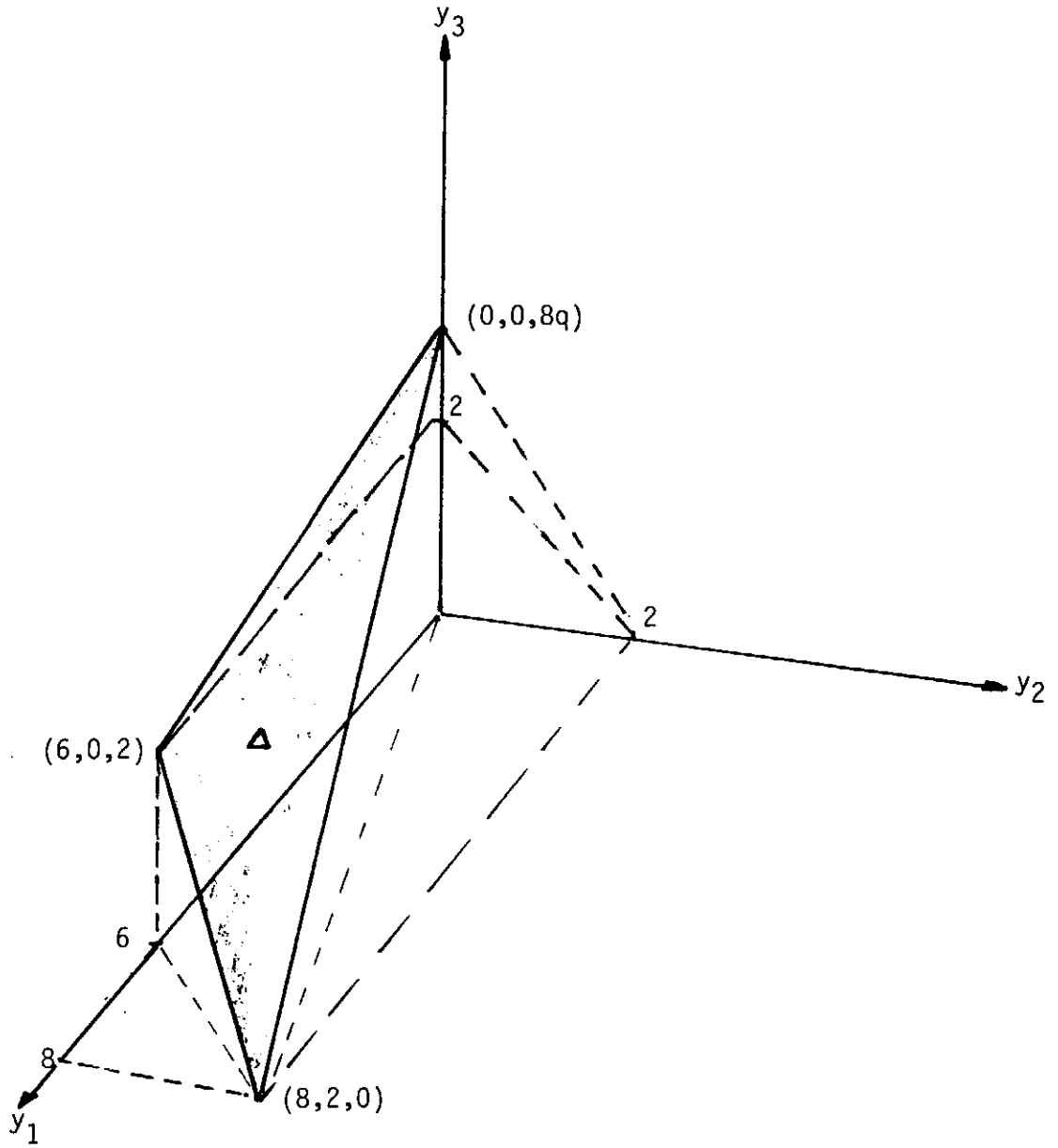
$$3q + 4q \geq 2$$

$$3q + 1 \leq 2$$

i.e.

$$(12) \quad \frac{2}{7} \leq q \leq \frac{1}{3}$$

Thus, for $\frac{2}{7} \leq q \leq \frac{1}{3}$, \bar{y}^q is the solution of the generalized Nash procedure (the generalized HARSANYI-SELTEN value).



(Fig.15)

We now want to "implement" \bar{y}^q , i.e., look for \bar{u}_T^q, \bar{u}_B^q such that

$$\begin{aligned}(\bar{u}_T^q, \bar{u}_B^q) &\rightarrow (\bar{u}_T^{q1}, \bar{u}_B^{q1}, q \bar{u}_T^{q2} + (1-q) \bar{u}_B^{q2}) \\ &= \bar{y}^q \\ &= \left(\frac{12-q^2}{4q-1}, 3q, 4q\right)\end{aligned}$$

Clearly, this gives three equations for the four coordinates of \bar{u}_T^q, \bar{u}_B^q ,

$$(13) \quad \begin{aligned} u_T^1 &= \frac{12q^2}{4q-1} \\ u_B^1 &= 3q \\ q u_T^2 + (1-q) u_B^2 &= 4q \end{aligned}$$

Assuming (by inspection of the situation as depicted in Fig.14) that \bar{u}_T is Pareto efficient in V_T^+ , the fourth equation is

$$u_T^1 + u_T^2 = 8$$

and the unique solution is

$$(14) \quad \begin{aligned} u_T^q &= \left[\frac{12q^2}{4q-1}, \frac{32q-8-12q^2}{4q-1} \right] \\ u_B^q &= \left[3q, \frac{12q^3 - 28q^2 + 7q}{4(1-q)(4q-1)} \right] \end{aligned} \quad \left(\frac{2}{7} \leq q \leq \frac{1}{3} \right)$$

The two boundary cases, $q = \frac{2}{7}$ and $q = \frac{1}{3}$ admit for the following implementations

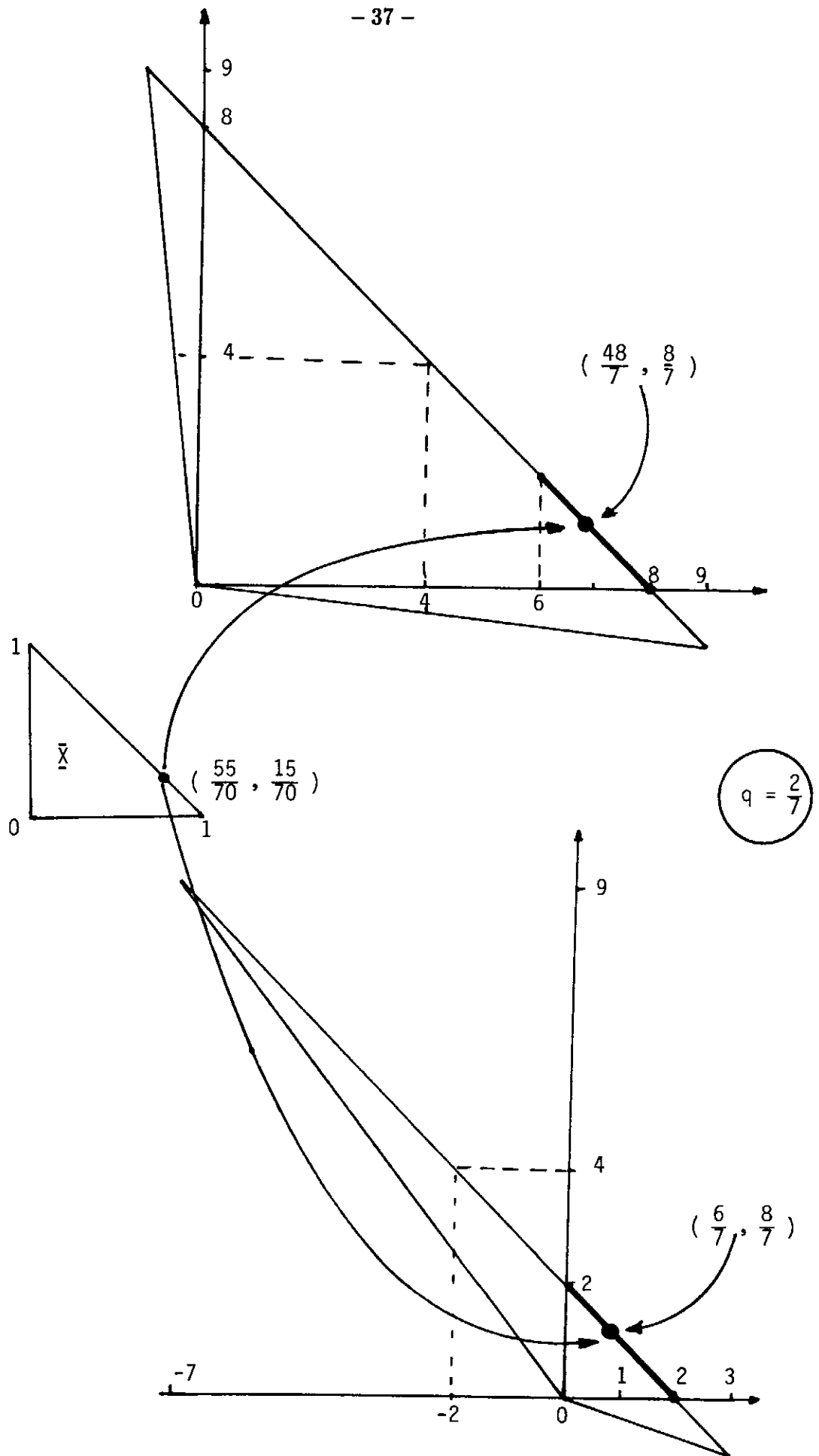
$$(15) \quad q = \frac{2}{7} : \left(\frac{48}{7}, \frac{8}{7} \right), \left(\frac{6}{7}, \frac{8}{7} \right) \rightarrow \left(\frac{48}{7}, \frac{6}{7}, \frac{8}{7} \right)$$

$$(16) \quad q = \frac{1}{3} : (4,4), (1,0) \rightarrow \left(4, 1, \frac{4}{3} \right)$$

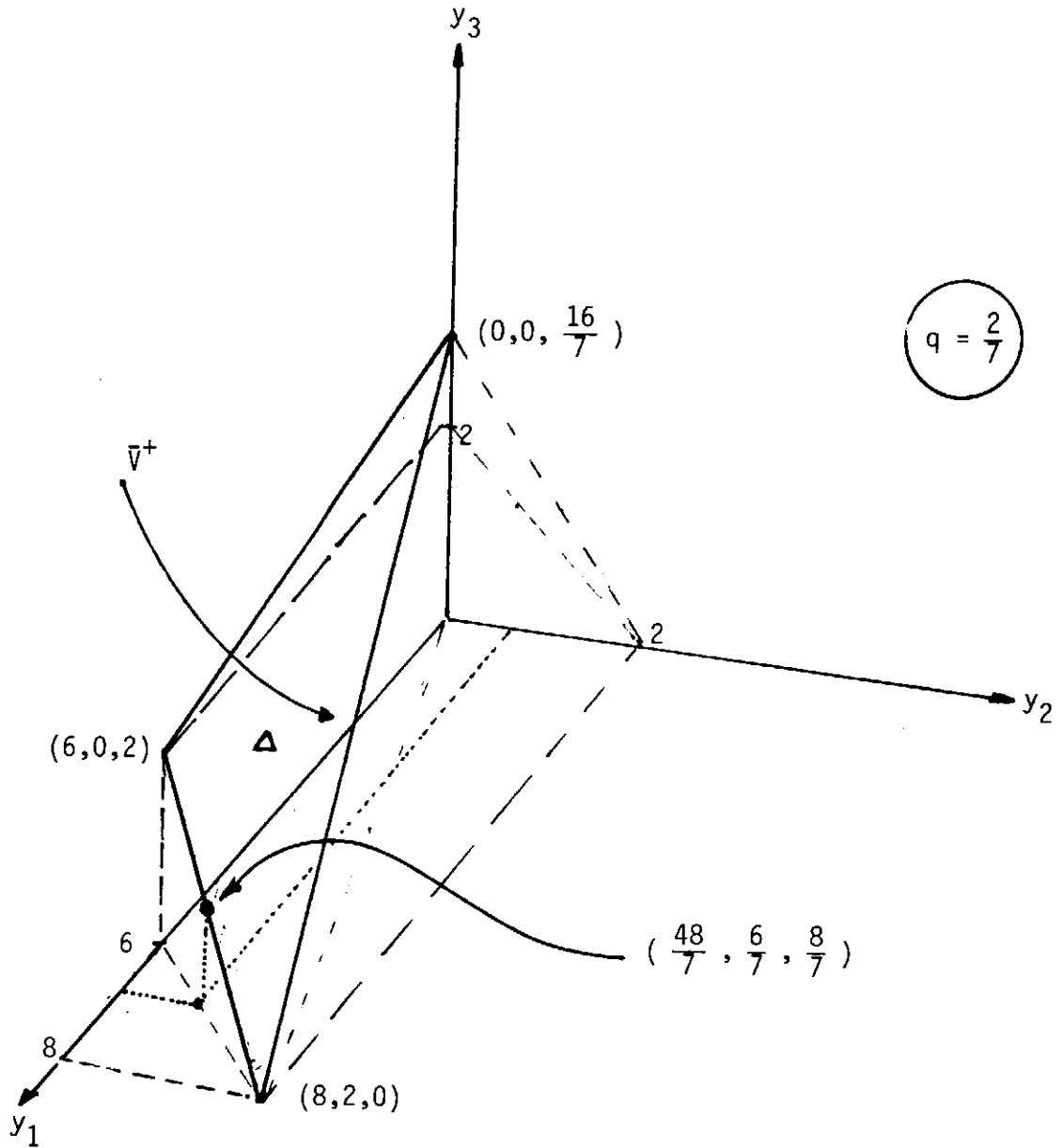
Note, that for $q = \frac{2}{7}$ we observe that \bar{u}^B is Pareto-efficient in V_B . Owing to this observation we conjecture – and verify – that $\mu(T) = \mu(B)$, that is $\bar{u}_T^{2/7}$ and $u_B^{2/7}$ are parametrized by the same $\bar{x} \in \underline{X} (= \mu(T) = \mu(B))$. Indeed, it turns out that this is $\bar{x} = \left(\frac{55}{70}, \frac{15}{70} \right)$, for in this case (cf. Example 2.4.)

$$U_T^1(\bar{x}) = 10x_1 - 1 = \frac{48}{7}, U_B^1(\bar{x}) = 10x_1 - 7 = \frac{6}{7}, U^2(\bar{x}) = 10x_2 - 1 = \frac{8}{7}$$

Clearly, it is Remark 3.1., that applies in this situation.



(Fig.16)



(Fig.17)

Continuing our analysis for $q > 1/3$ the solution of the maximizing problem suggested by (11) moves out of the triangle Δ as the roof of the feasible set (cf. Fig.15) becomes very "steep". Thus, the maximal generalized Nash Product is attained on the line $(8,2,0), (0,0,8q)$. It is not hard to verify that the maximizer is

$$(17) \quad \bar{y}^q = (4, 1, 4q) \quad (1/3 \leq q \leq 1),$$

and this is implemented by

$$(18) \quad \bar{u}_T = (4, 4) \text{ and } \bar{u}_B = (1, 0).$$

As for $\frac{1}{4} < q \leq \frac{2}{7}$, the solution of (11) has to be found on the line (6,0,2), (8,2,0) (cf. Fig.15).

However, for $q \leq \frac{1}{4}$ this line constitutes all of the Pareto-efficient surface of V^+ and hence we have to solve our maximization problem with respect to this line for the remaining q , that is, for $0 \leq q \leq \frac{2}{7}$. The constraints defining this line are given by

$$y_2 + y_3 = 2$$

$$y_1 - y_2 = 6$$

and the Lagrange procedure yields for the second coordinate of \bar{y}^q an equation

$$\frac{q}{6 + y_2} + \frac{1 - q}{y_2} - \frac{1}{2 - y_2} = 0$$

Hence we come up with

$$\bar{y}^q = \frac{1}{2} (\sqrt{49 - 54q + 9q^2} + (7 + 3q), \sqrt{49 - 54q + 9q^2} - (5 - 3q), -\sqrt{49 - 54q + 9q^2} + (9 - 3q)) \quad (0 \leq q \leq \frac{2}{7})$$

where $\sqrt{\quad}$ stands for $\sqrt{49 - 54q + 9q^2}$.

Now, for $q = \frac{2}{7}$ this yields nicely $(\frac{48}{7}, \frac{6}{7}, \frac{8}{7})$, and hence we have a continuous extension of the value. For $q = 0$ we obtain indeed

$$\bar{y}^0 = (7, 1, 1)$$

which is of course implemented by

$$\bar{u}_T = (7, 1), \bar{u}_B = (1, 1)$$

Thus we remark that for the extreme cases $q = 1$ and $q = 0$ the value coincides with the Nash value of the TOP and BOTTOM versions of the characteristic function.

The following table presents an overview:

$$1 \geq q \geq \frac{1}{3} :$$

$$(4,4), (1,0) \rightarrow (4,1,4q)$$

$$q = \frac{1}{3} :$$

$$(4,4), (1,0) \rightarrow (4,1, \frac{4}{3})$$

$$\frac{1}{3} \geq q \geq \frac{2}{7} :$$

$$\left[\frac{12q^2}{4q-1}, \dots \right], (3q, \dots) \rightarrow \left[\frac{12q^2}{4q-1}, 3q, 4q \right]$$

$$q = \frac{2}{7} :$$

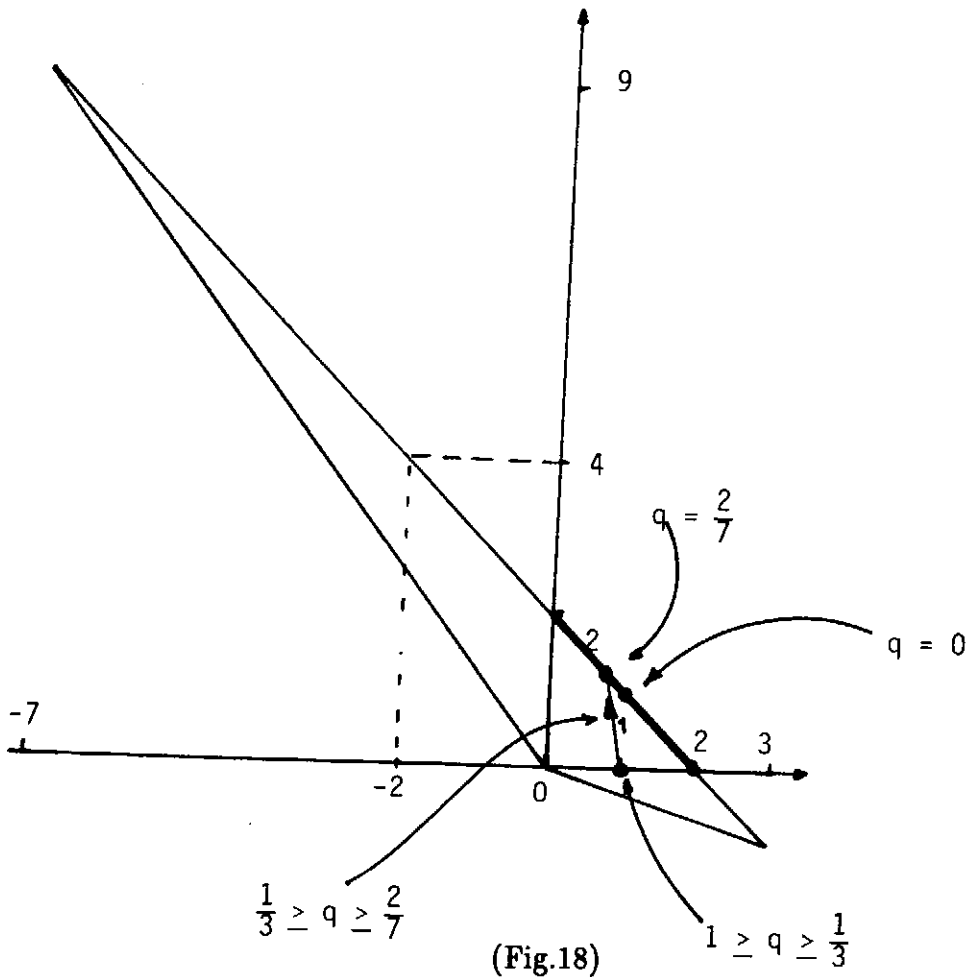
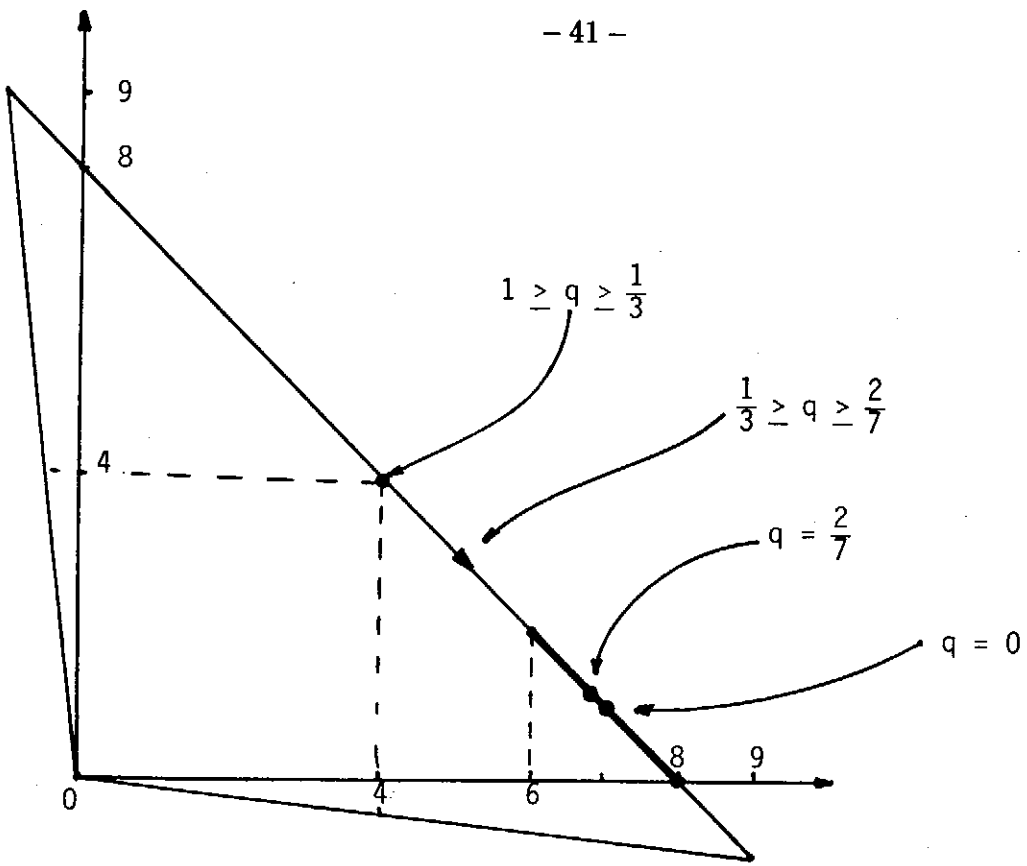
$$\left(\frac{48}{7}, \frac{8}{7} \right), \left(\frac{6}{7}, \frac{8}{8} \right) \rightarrow \left(\frac{48}{7}, \frac{6}{7}, \frac{8}{7} \right)$$

$$\frac{2}{7} \geq q \geq 0 :$$

$$\dots \rightarrow \frac{1}{2} (\sqrt{} + (7+3q), \dots, \dots)$$

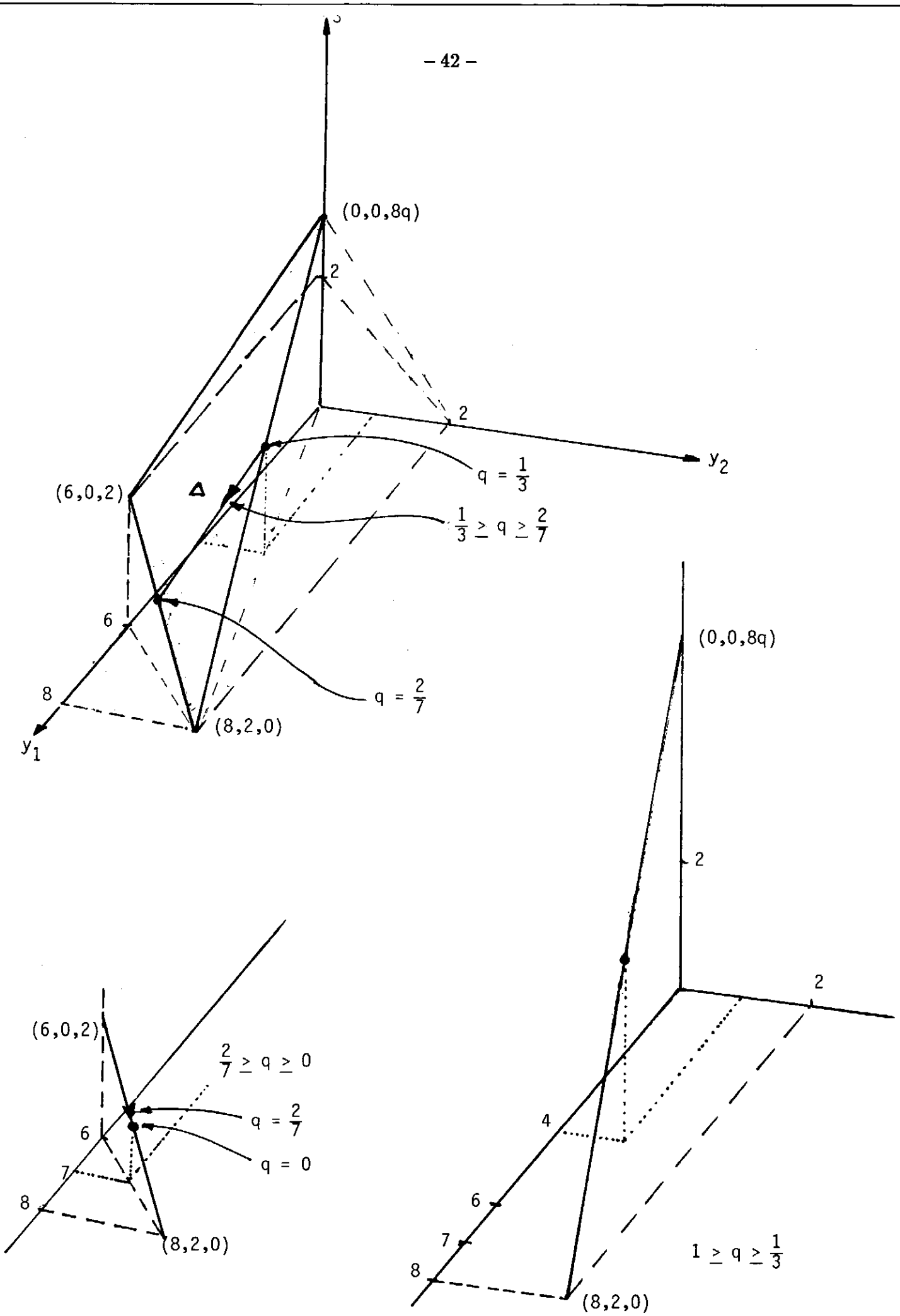
$$q = 0 :$$

$$(7,1), (1,1) \rightarrow (7,1,1)$$



(Fig.18)

Fig. 18 indicates the implementations for varying q and Fig. 19 shows the movement of \bar{y}^q in V^+ .



(Fig.19)

The reader should judge for himself how far he is satisfied with the value suggested by the procedure presented. It seems to be an advantage that the value for small or large q coincides with the Nash value in the *ex post* situations. Whether this coincidence is already justified for $q \geq \frac{1}{3}$ is of course questionable. Another slightly puzzling property is the lacking *ex post* monotonicity w.r.t. the "bottom"-state of nature, which is exhibited by studying the situation at $q = \frac{1}{3}$, $q = \frac{2}{7}$, $q = 0$.

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