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Incentive-Compatible Cost-Allocation Schemes
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Incentive-compatible cost-allocation schemes

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1 Introduction

The need for incentive compatible cost-allocation schemes, or more generally incentive compatible allocation schemes, stems from the premise that there is no omniscient, omnipotent and benevolent central, planner. Implementation of efficient allocation is obtained via Nash equilibria (or some of their refinements). The allocation mechanism depends on the whole domain of agents' conceivable characteristics and not on the actual characteristics. The common knowledge assumption implicit in the Nash equilibrium concept implies that when the mechanism is employed the agents know, in addition to their own characteristics, the actual characteristics of others.

The research in this area began with a quest for revelation mechanismes in which truth is unconditionally a best-response strategy. The fundamental result obtained in this area is the Gibbard-Satterthwaite impossibility theorem. Gibbard (1973) extended this result to more general allocation schemes (in which truth is meaningless as a strategy). To capture the idea that the actions of an agent are based on his characteristics only (and not on his beliefs about the others' characteristics) one is led to consider mechanisms that yield dominant strategies for every agent. Gibbard (1973) obtained the impossibility result for general schemes by reducing the problem to revelation mechanisms using (and introducing) the "revelation principle". The introduction of the Bayesian framework essentially did not change the conclusion. (In Bayesian (Harsanyi-Nash) equilibria, common knowledge of the prior over the agents' characteristics is assumed).

One can question the impossibility result when allocation schemes are planned for a restricted family of agents' characteristics. For instance, "median" is a non-manipulable and efficient voting rule for single peaked preferences, or truthful revelation is a dominant strategy in second-price private-value auctions. On the other hand, the Clark-Groves mechanism for the financing of public projects has been shown by Green and Laffont (1979) to be either non-feasible or inefficient. Moreover, these latter authors showed that there exists no efficient revelation mechanism for this problem.

In the simple public project problem it is assumed that every agent assigns a monetary value to the project. The project is built if and only if the sum of the values exceeds its cost, and in this case the question is how to allocate the cost among the agents. The Green-Laffont result asserts that any scheme that succeeds in eliciting the true valuations generically results in collective revenues that are either short of cost (non-feasibility) or exceed cost (inefficiency). If the truth is not elicited, then inefficiency may arise, in the sense that either the project will be built when it should not or vice versa. Alternatively, the revelation principle implies that it suffices to restrict attention to revelation mechanisms.

In the cost-allocation problems it is assumed that a set of alternatives is given together with a cost function which assigns a real number (the cost) to each alternative. The cost-allocation scheme assigns to each alternative and every agent his share of the cost of the alternative so that the total charges exactly cover the cost of the alternative. Each agent is characterized by a true signal not known to the center (or anyone else). The payoff to every agent depends on the selected alternative, his true signal and his share in the total cost. A cost-allocation scheme is incentive compatible if and only if, for every agent, sending the true signal is a dominant strategy, i.e., it is his best strategy independently of the others' signals.

This note was motivated by a problem regarding a group of public utilities located in relatively close proximity to each other. In a certain period of time some of these utilities are confronted by a shortage of capacity (or by an expensive production cost) while others have excess capacity. The potential buyers cooperate to purchase their needs from the potential sellers in the most efficient way. The minimization of total cost involves the solution of transportation problems.

The optimal solution is an outcome of an overall minimazation of costs which takes into account the whole system. Thus it may yield distortions, whereby some agents subsidize others (see Samet, Tauman and Zang (1984)). Hence transfer payments should be used to provide the agents with the right incentive to cooperate. In other words, the total (minimum) cost should be allocated "properly". But this is not sufficient. Since the needs of the potential buyers are private information, a prespecified cost-allocation rule may be manipulable in the sense that some buyers may find it profitable not to reveal their true characteristics. For instance, some may report demands lower than their true ones and self-produce (at high cost) the remainder, while others may ask to buy more than they actually need and dispose of the excess if by doing so they will sufficiently reduce their per-unit charge. Since misrepresentation of the true characteristics yields,

in general, non-efficient outcomes it is important to design non-manipulable cost-allocation schemes which yields individually rational outcomes which are efficient, if such schemes exist.

In this note it is shown that for a large class of problems there are cost-allocation schemes which are incentive compatible and yield individually rational and efficient outcomes. For example, cost-allocation schemes which extract from every agent an average of his marginal contribution to the cost of a random coalition of agents are shown to be incentive compatible and individually rational. In contrast, it is shown that the scheme which allocates the cost via prices which are proportional to the marginal cost prices is not incentive compatible.

2 The Model

The special features of our model are: a set $N = \{1, 2, \ldots, n\}$ of agents who operate as monopolists in n locations $A_1, A_2, \ldots, A_n$ (e.g., public utilities). Every $i \in N$ can produce any quantity x_i for a cost of $f_i(x_i)$, where $f_i : \mathbb{R}_+ \to \mathbb{R}_+$. In addition, there is another production possibility located at A_0 (the center). This may be either an established plant (or a set of plants owned by agents not in N) or a plant jointly built by the agents in N to cut their local costs. Let $f_0(y_1, \ldots, y_n)$ be the minimum cost of producing $\sum_{i \in N} y_i$ units at A_0 and transporting y_1 units to A_1, y_2 units to A_2, etc., using the most efficient configuration. It is assumed that for any $i \in N$, f_i is nondecreasing on $\mathbb{R}_+$, $f_i(0) = 0$, and f_i is continuous on $\mathbb{R}_{++}$. This allows for fixed cost components to the local technologies. As for the joint technology, it is assumed that f_0 is nondecreasing on $\mathbb{R}^n_+$, $f_0(0) = 0$, and $f_0(y) \leq \underline{\lim}_{m\to\infty} f_0(y^m)$ whenever $y^m \to y$. For example, suppose that $f_0(y) = s(\sum_{i\in N} y_i) + \sum_{i \in N} r_i(y_i)$, where $s(\sum_{i\in N} y_i)$ is the joint production cost of $\sum_{i\in N} y_i$ units in A_0 and $r_i(y_i)$ is the transportation cost of shipping y_i units from A_0 to A_i. Suppose that s and r_i, $i \in N$, are nondecreasing on $\mathbb{R}_+$, continuous on $\mathbb{R}_{++}$, and $s(0) = r_i(0) = 0$. In this case $f_0(y) \leq \underline{\lim}_{m\to\infty} f_0(y^m)$ whenever $y^m \to y$.

Let $F : \mathbb{R}^n_+ \to \mathbb{R}$ be the cost function of providing each agent with his consumption level at his location in the most efficient way. That is,

$$\begin{aligned} F(x_1, \ldots, x_n) \;\; = \;\; & \min_\delta \left[f_0(x-\delta) + \textstyle\sum_{i\in N} f_i(\delta_i)\right] \\ & \text{s.t.} \quad \delta \leq x, \\ & \qquad\;\; \delta \in \mathbb{R}^n_+. \end{aligned} \tag{1}$$

Let $\mathcal{F}_x : \prod_{i\in N}[0, x_i] \to \mathbb{R}_+$ be the function minimized in (1). Namely

$$\mathcal{F}_x(\delta) = f_0(x-\delta) + \sum_{i\in N} f_i(\delta_i), \tag{2}$$

and $F(x) = \min_\delta \mathcal{F}_x(\delta)$. Since $\mathcal{F}_x$ is not necessarily continuous on the boundary of its domain it is necessary to establish that $\min_\delta \mathcal{F}_x(\delta)$ is well defined.

Let $M(x)$ be the set of minimizers of $\mathcal{F}_x(\delta)$.

Lemma 1 *For each $x \in \mathbb{R}^n_+$, $M(x)$ is nonempty and compact.*

Proof of Lemma 1 Clearly $\mathcal{F}_x(\delta)$ is bounded. Let $\alpha = \inf_\delta \mathcal{F}_x(\delta)$ where $0 \leq \delta_i \leq x_i$, $i \in N$. Let $(\delta^m)^\infty_{m=1}$ be a sequence in the do-

main of $\mathcal{F}_x(\cdot)$ s.t. $\lim_{m\to\infty} f_x(\delta^m) = \alpha$. Since $(\delta^m)^\infty_{m=1}$ is a bounded sequence ($0 \leq \delta^m \leq x$) it has a subsequence which converges, say to $\bar\delta$. W.l.o.g. assume that $\delta^m \to \bar\delta$ as $m \to \infty$. Then

$$\begin{aligned} \alpha = \lim_{m\to\infty} \mathcal{F}_x(\delta^m) &= \lim_{m\to\infty} [f_0(x-\delta^m) + \sum_{i\in N} f_i(\delta^m_i)] \\ &\geq \underline{\lim}_{m\to\infty} f_0(x-\delta^m) + \sum_{i\in N} \underline{\lim}_{m\to\infty} f_i(\delta^m_i). \end{aligned}$$

Since $f_0(x-\bar\delta) \leq \underline{\lim}\, f_0(x-\delta^m)$

$$\alpha \geq f_0(x-\bar\delta) + \sum_{i\in N} f_i(\bar\delta_i) = \mathcal{F}_x(\bar\delta).$$

Consequently, $\mathcal{F}_x(\bar\delta) \leq \alpha$. But $\alpha = \inf_\delta \mathcal{F}_x(\delta)$, therefore $\alpha = \mathcal{F}_x(\bar\delta)$ and thus $\bar\delta \in M(x)$.

Let us next prove that $M(x)$ is compact. Clearly $M(x)$ is bounded. Let $(\delta^m)^\infty_{m=1}$ be a sequence in $M(x)$. Suppose that $\delta^m \to \bar\delta$ as $m \to \infty$. It is sufficient to prove that $\bar\delta \in M(x)$. Let $0 \leq \delta \leq x$. Since $\mathcal{F}_x(\delta) \geq \mathcal{F}_x(\delta^m)$ for any m,

$$\mathcal{F}_x(\delta) \geq \underline{\lim}_{m\to\infty} \mathcal{F}_x(\delta^m) \geq \mathcal{F}_x(\bar\delta).$$

This implies that $\bar{\delta} \in M(x)$.

□

For each $x \in I\!R^n_+$ let $\delta(x) \in M(x)$. For instance $\delta(x)$ can be chosen to be the minimal element in $M(x)$ w.r.t. the lexicographic order. Since $M(x)$ is nonempty and compact $\delta(x)$ is well defined. A joint production plan (or for short an alternative) is an element $x \in I\!R^n_+$. An efficient production of x involves the production of $\sum_{i \in N}(x_i - \delta_i(x))$ units at A_0 and the production of $\delta_i(x)$ units at A_i, $i \in N$.

Suppose that the local demand in A_i is fixed at the level of x_i^0 units (e.g., due to a local price regulation). It is assumed that the marginal benefit from the production of an extra unit in A_i exceeds its marginal cost, as long as $x_i < x_i^0$. This assumption ensures that every $i \in N$ will supply the total demand x_i^0 in A_i.

The information structure of the model is as follows:

(i) Every agent $i \in N$ knows the demand x_i^0 in A_i, his cost function f_i, and the cost function f_0 of the center.

(ii) The *center* knows $f_0, f_1, \ldots, f_n$. The demands x_i^0, $i \in N$, are private information but it is common knowledge that every agent will supply the full demand of his customers.

A (balanced) *mechanism* is a family of n sets $S_1, S_2, \ldots, S_n$ (the strategy sets of the agents in N) and a function $g : S_1 \times S_2 \times \cdots \times S_n \to I\!R^n_+ \times I\!R^n$ which associates with any n-tuple of strategies $s = (s_1, \ldots, s_n) \in S = \prod_{i \in N} S_i$, a pair (y, t) where $y \in I\!R^n_+$ is the joint production in the *center* and $t = (t_1, \ldots, t_n)$

is a vector of individual charges satisfying the balance condition $\sum_{i \in N} t_i = f_0(y)$. Given a mechanism (S, g) every agent i chooses a strategy $s_i \in S_i$ and reports his choice to a coordinator. The corresponding outcome is $g(s) = (y, t)$. Every $i \in N$ is informed of his outcome (y_i, t_i). Then he self-produces in A_i the difference between y_i and x_i^0 provided that $y_i \leq x_i^0$. Denote

$$z_i = \max(0, x_i^0 - y_i). \tag{3}$$

That is, z_i is the number of units which will be produced in A_i. If i is of type x_i^0 his payoff is defined to be his total cost with negative sign

$$h^i(y_i, t_i | x_i^0) = -[f_i(z_i) + t_i], \tag{4}$$

where z_i is defined in (3) and $\sum_{i \in N} t_i = f_0(y)$.

Let

$$h(\cdot | x^0) = (h^1(\cdot | x_1^0), \ldots, h^n(\cdot | x_n^0)).$$

Any mechanism (S, g) together with the n-tuple of payoff functions $h(\cdot | x^0)$ induces a "game" between the agents in N. (To make it a proper game of incomplete information (a la Harsanyi) a common prior over the set of possible types x^0 should be associated). We restrict the set of mechanisms and consider only mechanisms that have a dominant strategy for every agent in N and every vector of true demands.

Let x^0 be the true type vector of the agents. We say that $(\bar{y}, \bar{t})$ is an *efficient outcome with respect to* x^0 if it is a cost minimizer, namely

$$(\bar{y}, \bar{t}) \in \operatorname{argmax} \sum_{i \in N} h^i(y_i, t_i | x_i^0) = \operatorname{argmin} \left[f_0(y) + \sum_{i \in N} f_i(\max(0, x_i^0 - y_i)) \right] \tag{5}$$

where the max and min range over all outcomes $(y,t) \in \mathbb{R}^n_+ \times \mathbb{R}^n$, s.t. $\sum_{i\in N} t_i = f_0(y)$.

An outcome (y,t) is *individually rational (I.R.) with respect to* x^0 iff for all $i \in N$

$$t_i + f_i(z_i) \leq f_i(x_i^0), \tag{6}$$

where z_i is defined in (3). The left-hand side of (6) is the cost to supply x_i^0 units to the customers in A_i under the joint project, while the right-hand side of (6) is the production cost of this quantity in A_i.

Our goal is to find whether there are mechanisms (S,g) which would appear acceptable to all the agents before the game (induced by (S,g)) is actually played.

A mechanism is *satisfactory* if for any true type x^0 there exists $s^0 \in S$ and $t^0 \in \mathbb{R}^n$ s.t.

(i) The strategy $s_i^0 \in S_i$ is a dominant strategy for agent i of type x_i^0, $i \in N$.

(ii) $g(s^0) = (x^0 - \delta(x^0), t^0)$.

(iii) $\sum_{i\in N} t_i^0 = f_0(x^0 - \delta(x^0), t^0)$.

(iv) The outcome $(x^0 - \delta(x^0), t^0)$ is individually rational.

Thus a satisfactory mechanism removes the game theoretic aspect of the decision making procedure, at least as far as individual behavior is concerned. No one can conceivably gain by playing any other strategy than s^0 and no threats by individual

firms to deviate from their dominant strategy s^0 can be viable. In addition, the dominant strategy s^0 dictates the cost minimization outcome $(x^0 - \delta(x^0), t^0)$ and the total revenues of the center, $\sum_{i\in N} t_i^0$, from this outcome will just cover the total joint cost $f_0(x^0 - \delta(x^0))$. Finally, no individual firm will regret her participation in the game since the outcome is individually rational.

The possibility of another dominant strategy $s^* \neq s^0$ in (S,g) is not ruled out. Obviously, the payoff to every agent under s^* must coincide with that under s^0. Moreover, since we deal with a balanced mechanism

$$F(x^*) = \sum_{i\in n} [t_i^* + f_i(\delta_i(x^*))] = \sum_{i\in N} t_i^0 + f_i(\delta_i(x^0)) = F(x^0) \ ,$$

where $(x^* - \delta(x^*), t^*)$ is the outcome generated by s^*. Therefore, this outcome is efficient as well, and it is equivalent to $(x^0 - \delta(x^0), t^0)$.

The revelation principle allows us to restrict our attention to (balanced) revelation mechanisms where the truth is a dominant strategy for each player. A *revelation mechanism* is a pair $(\mathbb{R}^n_+, g)$. Every agent i chooses a nonnegative number x_i and reports his choice to the coordinator. In a revelation mechanism, truth revealing is a meaningful term. A revelation mechanism is *incentive compatible (I.C.)* if for any type vextor x^0 it is a dominant strategy for every agent $i \in N$ to submit his true type x_i^0. To guarantee that g generates efficient outcomes when the true type vector is reported one is led to consider revelation mechanisms of the form

$$g_t(x) = (x - \delta(x), t(x)),$$

where $\sum_{i\in N} t_i(x) = f_0(x - \delta(x))$.

A function $t : I\!R^n_+ \to I\!R$ is a *cost- allocation scheme* if $\sum_{i\in N} t_i(x) = f_0(x - \delta(x))$ for all $x \in I\!R^n_+$. It is incentive compatible (I.C.) if the revelation mechanism $(I\!R^n_+, g_t)$ is incentive compatible. That is, for each $x^0 \in I\!R^n_+$ and every $i \in N$

$$x_i^0 \in \operatorname*{argmax}_{x_i \geq 0} h^i(x_i - \delta_i(x), t_i(x)|x_i^0).$$

Equivalently, $t(\cdot)$ is I.C. iff for each $x^0 \in I\!R^n_+$ and every $i \in N$

$$x_i^0 \in \operatorname*{argmin}_{x_i \geq 0} [t_i(x) + f_i(z_i)]. \tag{7}$$

where $z_i = \max(0, x_i^0 - x_i + \delta_i(x))$.

Let $t(\cdot)$ be a cost-allocation scheme. By (6) the outcome $(x^0 - \delta(x^0), t(x^0))$ is individually rational with respect to x^0 iff

$$t_i(x^0) + f_i(\delta_i(x^0)) \leq f_i(x^0), \quad i \in N. \tag{8}$$

Thus a mechanism $(I\!R^n_+, g_t)$, where $t(\cdot)$ is a cost-allocation scheme, is satisfactory if for any $x^0 \in I\!R^n_+$ (7) and (8) hold.

Lemma 2 *A necessary and sufficient condition for $t(\cdot)$ to be incentive compatible is that for all $x^0 \in I\!R^n_+$ and every $i \in N$*

$$x_i^0 \in \operatorname*{argmin}_{x_i \geq 0} [t_i(\bar{x}) + f_i(\bar{z}_i)],$$

where $\bar{x} = (x^0_{-i}, x_i)$ and $\bar{z}_i = \max(0, x_i^0 - x_i + \delta_i(\bar{x}))$.

The lemma asserts that if for all $x^0 \in I\!R^n_+$, to report the true types x^0 is a Nash equilibrium strategy, then for all x^0 and every $i \in N$, x_i^0 is a dominant strategy for i.

Proof One direction is obvious. As for the other direction, suppose that

$$x_i^0 \in \operatorname*{argmin}_{x_i \geq 0} [t_i(\bar{x}) + f_i(\bar{z}_i)],$$

for all $x^0 \in I\!R^n_+$. In particular, this relation holds when x^0_{-i} is replaced by x_{-i} for any $x_{-i} \in I\!R^{n-1}_+$. In this case, condition (7) is satisfied, as required.

□

Corollary 1 *Let $t(\cdot)$ be a cost-allocation scheme. The revelation mechanism $(I\!R^n_+, g_t)$ is satisfactory if for all $x^0 \in I\!R^n_+$ and every $i \in N$*

(i) $t_i(x^0) + f_i(\delta_i(x^0)) \leq t_i(\bar{x}) + f_i(\bar{z}_i)$ and

(ii) $t_i(x^0) + f_i(\delta_i(x^0)) \leq f_i(x_i^0)$,

where $\bar{x}$ and $\bar{z}$ are defined as in Lemma 2.

Proof Follows from Lemma 2 and (8) and from $z_i = \delta_i(x^0)$ at the point where $x = x^0$.

□

The question under consideration is whether or not for any demand $x^0 = (x_1^0, \ldots, x_n^0)$ and any cost configuration $f_0, f_1, \ldots, f_n$ there exists an incentive compatible cost-allocation scheme $t(\cdot)$ for which $(I\!R^n_+, g_t)$ is a satisfactory mechanism. The answer to this question is negative, as the next proposition states.

Proposition 1 *Suppose that*

(i) f_i is strictly convex for all $i \in N$,

(ii) there exists $\mathrm{x}^0 \in \mathbb{R}^n_{++}$ *s.t.* $\delta(\mathrm{x}^0) = \mathrm{x}^0$ *and*

(iii) there exists $x^0 \in \mathbb{R}^n_+$ *and* $i \in N$ *s.t.* $x^0 \ll \mathrm{x}^0$ *and* $\delta_i(x_i^0) < x_i^0$.

Then there exists no cost-allocation scheme $t(\cdot)$ *for which the mechanism* $(\mathbb{R}^n_+, g_t)$ *is satisfactory.*

By strict convexity we mean that for any $x_i \geq 0$ and $y_i \geq 0$ and for any $0 < \lambda < 1$

$$f(\lambda x + (1-\lambda)y) \;<\; \lambda f(x) + (1-\lambda) f(y).$$

Condition (i) asserts that the local technologies exhibit decreasing returns to scale. Conditions (ii) and (iii) assert that the joint technology is inefficient for large demands while it is efficient for some smaller demands. This is for instance the case if the joint production cost has only a small fixed cost component and as production increases in A_0 the technology turns from increasing to relatively sharp decreasing returns to scale.

Proof Suppose that $t(\cdot)$ is a cost-allocation scheme and $(\mathbb{R}^n_+, g_t)$ is satisfactory. Since $\sum_{i \in N} t_i(\mathrm{x}^0) = f_0(\mathrm{x}^0 - \delta(\mathrm{x}^0))$ and $\delta(\mathrm{x}^0) = \mathrm{x}^0$ we have $\sum_{i \in N} t_i(\mathrm{x}^0) = 0$ and hence by individual rationality (applied to the agents of type x^0)

$$t_i(\mathrm{x}^0) = 0, \quad i \in N. \tag{9}$$

Since $t(\cdot)$ is incentive compatible then by Corrollary 1, for any $x^0 \in \mathbb{R}^n_+$ s.t. $x_i^0 < \mathrm{x}_i^0$ and for all $i \in N$,

$$t_i(\mathrm{x}^0) + f_i(\mathrm{x}_i^0) \leq t_i(x^0) + f_i(\mathrm{x}_i^0 - x_i^0 + \delta_i(x^0)) - f_i(\delta_i(x^0)) \ .$$

This, together with (9) implies that

$$f_i(\mathrm{x}_i^0) \leq t_i(x^0) + f_i(\mathrm{x}_i^0 - x_i^0 + \delta_i(x^0)) \ . \tag{10}$$

By individual rationality (applied to agents of type x^0)

$$t_i(x^0) + f_i(\delta_i(x^0)) \leq f_i(x_i^0) \ .$$

Hence by (10)

$$f_i(\mathrm{x}_i^0) - f_i(\mathrm{x}_i^0 - x_i^0 + \delta_i(x^0)) \leq f_i(x_i^0) - f(\delta_i(x^0)) \ . \tag{11}$$

Since $\delta_i(x^0) < x_i^0$ inequality (11) contradicts the strict convexity of f_i.

□

3 The Main Result

We first show that if the local technologies exhibit constant returns to scale then, irrespective of the joint technology, a satisfactory mechanism does exist.

Let $\lambda \in \mathbb{R}^n_+$ s.t. $\sum_{k \in N} \lambda_k = 1$. Define an operator $\Psi^\lambda : \mathbb{R}^n_+ \to \mathbb{R}^n$ by

$$\Psi_i^\lambda(x) = f_i(x_i) - f_i(\delta_i(x)) - \lambda_i \left[\sum_{k \in N} f_k(x_k) - F(x) \right] , \tag{12}$$

for $i \in N$. Observe that

$$\sum_{k \in N} \Psi_i^\lambda(x) = F(x) - \sum_{k \in N} f_i(\delta_i(x)) = f_0(x - \delta(x)) \ .$$

Hence, Ψ^λ is a cost-allocation scheme.

Suppose that the true demand is $x \in \mathbb{R}^n_+$. Then the term $\sum_{k\in N} f_k(x_k) - F(x)$ measures the total saving due to the use of the joint technology in A_0. On the individual level, agent i saves before charges $f_i(x_i) - f_i(\delta_i(x))$ due to the cooperation. According to the scheme Ψ^λ every firm i pays the center her individual saving, $f_i(x_i) - f_i(\delta_i(x))$, and obtains from the center a fixed proportion (λ_i) of the total saving. Notice that when $\delta_i(x) = x_i$, namely if the entire demand is produced in A_i then the cooperation leaves firm i with a nonnegative amount, equal to her share in the total saving. This is the case where i's technology is relatively efficient and thus subsidises non efficient firms that need to use the center more.

Let us next relate Ψ^λ to the cooperative game theory literature. Consider the following cost game v_x defined by

$$v_x(S) = \begin{cases} \sum_{k\in S} f_k(x_k) & , S \neq N, \\ F(x) & , S = N . \end{cases}$$

This game describes the situation where only the grand coalition N can build the joint technology. The members of any other coalition will have to self produce their demands. It is easy to verify that $-v_x$ is a convex game for any cost configuration $f_0, f_1, \ldots, f_n$. Thus any extreme point of the inverse core of v_x corresponds to a certain order $\mathcal{R}$ of agents in N and it is defined by

$$C_i^{\mathcal{R}} = v_x(\mathcal{R}_i + i) - v_x(\mathcal{R}_i) ,$$

where $\mathcal{R}_i$ is the set of agents in N which preceed i in the order $\mathcal{R}$. That is, $C_i^{\mathcal{R}}$ is the marginal contribution of i to the cost of the agents which preceed i in the order $\mathcal{R}$. By the definition of v_x

for each $\mathcal{R}$ there exists $i \in N$, the last firm in the order $\mathcal{R}$, s. t.

$$C_j^{\mathcal{R}} = \begin{cases} f_j(x_j) & , j \neq i, \\ F(x) - \sum_{k\neq i} f_k(x_k) & , j = i . \end{cases}$$

Let $\lambda \in \mathbb{R}^n_+, \sum_{k\in N} \lambda_k = 1$. Then $\Psi^\lambda(x) = \sum_{i\in N} \lambda_i C^i$. If $\lambda = (\frac{1}{n}, \ldots, \frac{1}{n})$ then $\Psi^\lambda(x)$ is the Shapley value of v_x. Recently Monderer, Samet and Shapley (1990) proved that the core of a convex game coincides with its set of weighted Shapley values. Hence Ψ^λ can be interpreted also as a weighted Shapley value of v_x.

Theorem 1 *Suppose that $f_1, \ldots, f_n$ are concave on $\mathbb{R}_+$. Then the mechanism $(\mathbb{R}^n_+, g_{\Psi^\lambda})$ is satisfactory.*

Proof By Corollary 1 to lemma 2, it suffices to prove that for each $i \in N$, for each $x^0 \in \mathbb{R}^n_+$ and each $x_i \geq 0$

(i) $\Psi_i^\lambda(x^0) + f_i(\delta_i(x^0)) \leq \Psi_i^\lambda(\bar{x}) + f_i(\bar{z}_i)$, (I.C.)

(ii) $\Psi_i^\lambda(x^0) + f_i(\delta_i(x^0)) \leq f_i(x_i^0)$, (I.R.)

where $\bar{x} = (x^0_{-i}, x_i)$ and $\bar{z}_i = \max(0, x_i^0 - x_i + \delta_i(\bar{x}))$.

First we prove (ii):

$$\begin{aligned} \Psi_i^\lambda(x^0) + f_i(\delta_i(x^0)) &= -\lambda_i \left[\sum_{k\in N} f_k(x_k^0) - F(x^0)\right] + f_i(x_i^0) \\ &\leq f_i(x_i^0) . \end{aligned}$$

Next we prove (i). Observe that

$$\Psi_i^\lambda(x^0) - \Psi_i^\lambda(\bar{x}) = \lambda_i\left[F(x^0) - F(\bar{x})\right] + (1-\lambda_i)(f_i(x_i^0) - f_i(x_i)) - \left[f_i(\delta_i(x^0)) - f_i(\delta_i(\bar{x}))\right] ,$$

Hence the I.C. condition is

$$\lambda_i\left[F(x^0) - F(\bar{x})\right] + (1-\lambda_i)(f_i(x_i^0) - f_i(x_i)) \leq f_i(\bar{z}_i) - f_i(\delta_i(\bar{x})) \tag{$*$}$$

for all $x^0 \in \mathbb{R}_+^n$ and $x_i \in \mathbb{R}_+$. To prove $(*)$ we use the following observation.

Lemma 3 *Let $f_0, f_1, \ldots, f_n$ be any cost configuration. Then for any $x^0 \in \mathbb{R}_+^n$, any $i \in N$ and any $x_i \in \mathbb{R}_+$,*

$$F(x^0) - F(\bar{x}) \leq f_i(\bar{z}_i) - f_i(\delta_i(\bar{x})) .$$

Proof By the definition of $F(x^0)$ as the minimum production cost of x^0

$$F(x^0) \leq \sum_{k \neq i} f_k(\delta_k(\bar{x}_k)) + f_0(\bar{x} - \delta(\bar{x})) + f_i(\max(0, x_i^0 - x_i + \delta_i(\bar{x}))) . \tag{13}$$

Observe that the righthandside of (13) is one feasible way to produce x^0. Agent i may obtain more than x_i^0 units if $x_i - \delta_i(\bar{x}) > x_i^0$. In this case the free disposal assumption is applied. By the definition of $F(\bar{x})$,

$$F(\bar{x}) = \sum_{k \in N} f_k(\delta_k(\bar{x})) + f_0(\bar{x} - \delta(\bar{x})) . \tag{14}$$

By (13) and (14),

$$F(x^0) - F(\bar{x}) \leq f_i(\bar{z}_i) - f_i(\delta_i(\bar{x}))$$

as claimed.

□

Let us complete now the proof of Theorem 1.

By Lemma 3 inequality $(*)$ holds if

$$f_i(x_i^0) - f_i(x_i) \leq f_i(\bar{z}_i) - f_i(\delta_i(\bar{x})) . \tag{$**$}$$

Since the f_i are linear and nondecreasing on $\mathbb{R}_+$ inequality $(**)$ is equivalent to $x_i^0 - x_i \leq \bar{z}_i - \delta_i(\bar{x})$ whenever $f_i \not\equiv 0$. But this certainly can be written as $x_i^0 - x_i + \delta_i(\bar{x}) \leq \max(0, x_i^0 - x_i + \delta_i(\bar{x}))$, which certainly holds.

□

Let us now show that the scheme which allocates cost via prices which are proportional to marginal costs is not incentive compatible, even if the individual cost functions are all linear.

Example Consider the cost-allocation scheme τ which allocates the total cost among the agents via prices which are proportional to the marginal cost prices (wherever they exist). That is, let $\alpha : \mathbb{R}_+^n \to \mathbb{R}$ be defined for each $x \in \mathbb{R}_+^n$ as the number α for which $\alpha x \nabla F(x) = F(x)$, where $\nabla F(x)$ is the gradient of F at x. Let

$$\tau_i(x) = \alpha(x) x_i \frac{\partial F}{\partial x_i}(x) - f_i(\delta_i(x)) , \quad i \in N ,$$

and let $\tau(x) = (\tau_1(x), \ldots, \tau_n(x))$. Obviously τ is a cost-allocation scheme. Let us show that τ may not be incentive compatible.

Consider a model with two agents ($n = 2$) of types $x^0 = (5, 100)$. Suppose that the joint project at A_0 results in production cost of \$1 per unit for the first 100 units and \$7 per unit from there on. The joint project involves a fixed cost, E, of \$710. The per unit transportation cost from A_0 to A_1 is \$1 and it is zero from A_0 to A_2. The agents can self-produce an unlimited amount for a fixed per unit cost of \$20 and \$8.1 respectively and with no fixed costs. The cost configuration is depicted in the following table.

	A_1	A_2		
A_0	2	1	100	
A_0	8	7	∞	
A_1	20	∞	∞	$E = 710$
A_2	∞	8.1	∞	
	5	100		

An optimal solution is to send the first 100 units to A_2 and the next 5 units to A_1. The total cost is then $F(5, 100) = 850$. If, however, the agents do not use the center and self-produce their entire needs they will pay a total of $20 \cdot 5 + 8.1 \cdot 100 = 910$. Hence, it pays them to jointly build the plant at A_0, while it does not pay anyone individually to do so. Let us examine now the cost-allocation scheme, τ. Observe that here $\delta_i(x) = 0$ for all x hence $F(x) = f_0(x)$ and $\tau_i(x) = \alpha(x) \cdot x_i \frac{\partial f_0}{\partial x_i}(x)$. Since $\nabla f_0(5, 100) = (8, 7)$, we have $\alpha = \frac{850}{5 \cdot 8 + 100 \cdot 7} = \frac{850}{740}$. Hence

$$\tau_2 = \alpha \cdot 7 \cdot 100 = 804.05 .$$

Suppose next that agent 1 truly reports his type $x_1 = 5$ while the second agent reports $x_2 = 94$ and self-produces the remaining

6 units. Now $F(5, 94) = 710{+}2{\cdot}5{+}1{\cdot}94 = 814$, $\nabla f_0(5, 94) = (2, 1)$ and $\alpha = \frac{814}{104}$. Thus $\tau_2 = \frac{814}{104} \cdot 1 \cdot 94 = 735.73$. In addition 2 spends 48.6 to self-produce the remaining 6 units. His total cost is therefore 784.33, which is below his cost under a true report. Consequently, τ is not incentive compatible.

Consider next the allocation scheme, Ψ^λ for $\lambda = (\frac{1}{2}, \frac{1}{2})$. Since

$$f_0(5, 0) = 100 , \quad f_0(0, 100) = 810 \text{ and } f_0(5, 100) = 850 ,$$

we have

$$\Psi_1^\lambda = \frac{1}{2} \cdot 100 + \frac{1}{2}(850 - 810) = 70 ,$$

$$\Psi_2^\lambda = \frac{1}{2}(850 - 100) + \frac{1}{2} \cdot 810 = 780 .$$

If agent 2 misrepresents his true demand and rather reports $x_2 = 94$, then the corresponding costs are

$$f_0(5, 0) = 100 , \quad f_0(0, 94) = 761.4 \text{ and } f_0(5, 94) = 814 .$$

Thus

$$\Psi_1^\lambda = \frac{1}{2} \cdot 100 + \frac{1}{2}(814 - 761.4) = 76.3 ,$$

$$\Psi_2^\lambda = \frac{1}{2} \cdot 761.4 + \frac{1}{2}(814 - 100) = 737.7 .$$

In addition, 2 self-produces the remaining 6 units for the cost of 48.6. His total cost is therefore 786.3 and he is worse off relative to the case where he reports his true type. Observe that since $f_1(x_1) = 20x_1$ and $f_2(x_2) = 8.1x_2$ are linear, Theorem 1 can be applied to conclude that for the cost structure of this example, $\Psi^\lambda(x)$ is satisfactory for each $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ s.t. $\lambda_1 + \lambda_2 = 1$.

In practice cooperation between firms often happens when the firms posses decreasing returns to scale technology. We could extend Theorem 1 to deal with cost functions which are subadditive. Subadditivity and convexity of a function f are not inconsistent properties unless $f(0) = 0$ and f is continuous at zero. Subadditivity of a cost function can be the result of a significant fixed cost component even if the technology exhibits decreasing returns to scale. For f_i to be subadditive it is sufficient that $f_i(x_i) - f_i(0^+) \le f_i(0^+)$. Although we will state the assumption that the f_i are subadditive everywhere it will be enough to assume it on an interval which contains the true demand x_i^0. If it is common knowledge that x_i^0 is bounded above by a_i then it is sufficient to assume subadditivity on $[0, a_i]$ only (e.g. $f_i(x_i) = x_i^2 + 2$ is subadditive on $[0, 1]$). Public utilities often do not operate an additional turbine, even if they have one installed, largely because of a high fixed cost associated with its operation. In this case they prefer to buy their excess demand from other ressources that have excess capacity.

For any $x \in I\!R^n$ and any $S \subseteq N$ let $x^S \in I\!R^n$ be defined by $x_i^S = x_i$ for $i \in S$ and $x_i = 0$ for $i \notin S$.

Theorem 2 *Suppose that*

(i) f_i is subadditive for $i \in N$,

(ii) $M(x) \cap \{x^S \mid S \subseteq N\} \neq \emptyset$, for any $x \in I\!R^n_+$.

Let $\delta(x) \in M(x) \cap \{x^S \mid S \subseteq N\}$ for any $x \in I\!R^n_+$. Then $(I\!R^n_+, g_{\Psi^\lambda})$ is a satisfactory mechanism.

The second condition in Theorem 2 requires that for any $i \in N$, the optimal production in A_i is either $\delta_i(x) = x_i$ or else it is $\delta_i(x) = 0$. As an illustration consider the case discussed on page 5, where f_0 has the form

$$f_0(y_1, \ldots, y_n) = s\left(\sum_{i \in N} y_i\right) + \sum_{i \in N} r_i(y_i) ,$$

where $s : I\!R_+ \to I\!R_+$ is the joint production cost and $r_i : I\!R_+ \to I\!R_+$ is the transportation cost from A_0 to A_i. Both s and r_i, $i \in N$, are nondecreasing on $I\!R_+$ and continuous on $I\!R_{++}$ and may include fixed cost components. Suppose that for any $y_i > 0$ the marginal production and transportation cost to provide A_i with an extra unit from A_0 is smaller than the marginal production cost of this unit in A_i. Then either $\delta_i(x) = 0$ or $\delta_i(x_i) = x_i$, depending on the magnitude of the fixed costs of s and r_i. This scenario is described in Corollary 1 to this theorem. Another scenario is described in Corollary 2.

Proofof Theorem 2 As in the proof of Theorem 1 it is sufficient to prove inequality (**), which is

$$f(x_i^0) - f_i(x_i) \le f_i(\bar{z}) - f_i(\delta_i(\bar{x})) .$$

By our assumption for any $x \in I\!R^n_+$ $\exists S \subseteq N$ s.t. $\delta(x) = x^S$. Hence $\delta_i(\bar{x}) \in \{0, x_i\}$. If $\delta_i(\bar{x}) = x_i$ then (**) stands as equality. If $\delta_i(\bar{x}) = 0$ then (**) is equivalent to $f_i(x_i^0) - f_i(x_i) \le f_i(\max(0, x_i^0 - x_i))$ which hold by the monotonicity and subadditivity of f_i. □

Corollary 2 *Suppose that*

(i) f_i is subadditive for all $i \in N$,

(ii) For all $x \in \mathbb{R}^n_+$ $\mathcal{F}_x$ is concave, where $\mathcal{F}_x$ is defined in (2).

Then $(\mathbb{R}^n_+, g_{\Psi^\lambda})$ is a satisfactory mechanism.

Proof A concave function on a box attains its minimum on one of the edges. The set of edges of the domain of $\mathcal{F}_x$ is $\{x^S \mid S \subseteq N\}$. Hence $M(x) \cap \{x^S \mid S \subseteq N\} \neq \emptyset$ and Theorem 2 can be applied.

□

Corollary 3 *Suppose that*

(i) f_i is subadditive for all $i \in N$,

(ii) for all $x \in \mathbb{R}^n_+$ $\mathcal{F}_x$ is nonincreasing.

Then $(\mathbb{R}^n_+, g_{\Psi^\lambda})$ is satisfactory.

Proof Follows from the fact that either x or 0 is a minimizer of $\mathcal{F}_x$, depending on the magnitude of the fixed cost of f_0 relative to that of the f_i's.

□

It can be easily verified that $\mathcal{F}_x$ in the example above is nonincreasing but convex.

We have shown that whenever Ψ^λ is incentive compatible it generates an outcome which is individually rational and efficient. The question is whether this outcome is also group rational, or else, is there a coalition S of agents that can improve their payoffs by establishing a center by their own (if they can). It turns out

that whenever Ψ^λ is incentive compatible it generates a group rational outcome. Furthermore, if Ψ^λ is applied to allocate the joint costs of S, then no agent in S will be made better off relative to his payoff under the grand coalition.

Proposition 2 *Suppose that Ψ^λ is incentive compatible. Let x^0 be the true demand and let $S \subseteq N$. Then*

$$\Psi_i^\lambda(x^0) + f_i(\delta_i(x^0)) \leq \Psi_i^\lambda(x^{0,S}) + f_i(\delta_i(x^{0,S}))$$

for all $i \in S$. In particular the outcome $(x^0 - \delta(x^0), \Psi^\lambda(x^0))$ is group rational.

Proof By the definition off Ψ^λ, for any $i \in S$

$$\Psi_i^\lambda(x^0) + f_i(\delta_i(x^0)) - [\Psi_i^\lambda(x^{0,S}) + f_i(\delta_i(x^{0,S}))]$$
$$= F(x^0) - F(x^{0,S}) - \sum_{k \notin S} f_k(x^0) \leq 0$$

This follows from the observation that $F(x^{0,S}) + \sum_{k \notin S} f_k(x^0)$ is the production cost of x^0 under one feasible way, while $F(x^0)$ is the minimal production cost of x^0.

□

Finally, consider a situation where each firm i has virtually limited capacity Q_i, and capacity expansion is expensive. The technology f_i is more efficient than f_0 up to Q_i (due, for example, to significant transportation costs) but beyond this level it is cheaper to produce and to transport the remaining units from the center. In this case $\delta_i(x) = \min(x_i, Q_i)$. The center in this scenario can be interpreted as a group of public utilities that wish to sell their excess capacity to a group of regional public utilities that are temporarily short of capacity. The buyers who cooperate to minimize total cost find for each $x \in I\!R^n_+$ the optimal transportation configuration which provides every $i \in N$ with $x_i - Q_i$ units in A_i (provided that $x_i \geq Q_i$). Then the total cost is allocated according to a prespecified scheme.

It will be shown below that if the cost configuration is such that $\delta_i(x)$ depends on x_i only (as it is when $\delta_i(x) = \min(x_i, Q_i)$) then there exists a satisfactory mechanism. This also applies to the case where $F(x) = f_0(x)$. That is, cost minimization is achieved when total production is taken place in the center only, and $\delta_i(x) \equiv 0$, $i \in N$. This is the case of the example above. The two agents may have incentive to misreport their true demand and to self-produce the incremental quantity locally.

Let $\mathcal{R}$ be an order of N. Recall that $\mathcal{R}_i$ is the set of agents that preceed i in the order of $\mathcal{R}$. Let λ be a probability distribution over the set of $n!$ permutations of N. Define $\varphi^\lambda : I\!R^n_+ \to I\!R$ by

$$\varphi_i^\lambda(x) = \sum_{\mathcal{R}} \lambda(\mathcal{R}) \left[F(x^{\mathcal{R}_i + i}) - F(x^{\mathcal{R}_i})\right] - f_i(\delta_i(x)) .$$

That is, the cost to i is $\varphi_i^\lambda(x) + f_i(\delta_i(x))$ which is an average of his marginal contribution to a random coalition. If $\lambda(\mathcal{R}) = \frac{1}{n!}$ for all $\mathcal{R}$ then $\varphi_i^\lambda(x) + f_i(\delta_i(x))$ is the Shapley value of agent i in the game $\mathcal{W}_x$. This game is defined by

$$\mathcal{W}_x(S) = F(x^S) ,$$

where $F(x^S)$ is the minimum cost to provide every $i \in S$ with x_i units in A_i when S owns the joint technology in A_0.

Theorem 3 *Let $f_0, f_1, \ldots, f_n$ be any cost configuration such that $\delta_i(x)$ depends on x_i only. Then the mechanism $(I\!R^n_+, g_{\varphi^\lambda})$ is satisfactory.*

The statement "$\delta_i(x)$ depends on x_i only" formally means that if x and x' are in $I\!R^n_+$ and if $x_i = x'_i$ then $\delta_i(x) = \delta_i(x')$. It is interesting to note that the mechanism $(I\!R^n_+, g_{\Psi^\lambda})$ where Ψ^λ is defined in (12) is in general not satisfactory under this condition. To see this consider the case where $f_0(x) \equiv 0$ and $f_i(x_i) > 0$ for all $x_i > 0$. Then $\delta_i(x) \equiv 0$ whenever $x_i > 0$ and $F(x) = 0$. The incentive compatibility condition (*) (see the proof of Theorem 1) is then

$$(1 - \lambda_i)(f_i(x_i^0) - f_i(x_i)) \leq f_i(x_i^0 - x_i)$$

for $x_i < x_i^0$. Suppose that $\lambda_i = \frac{1}{n}$. For this inequality to hold for all n it must be that $f_i(x_i^0) \leq f_i(x_i) + f_i(x_i^0 - x_i)$. Hence, f_i must be subadditive, but this is not assumed in Theorem 3.

Proof of Theorem 3 (i) Let us first prove that, for any $x^0 \in I\!R^n_+$, $(x^0 - \delta(x^0), \varphi^\lambda(x^0))$ is an individually rational outcome. By the definition of F, for each $S \subseteq N$ and every $i \in S$

$$F(x^{0,S}) + f_i(x_i^0) \geq F(x^{0,S+i}),$$

where $x^{0,S}$ is given by $x_j^{0,S} = x_j^0$ if $j \in S$ and $x_j^{0,S} = 0$ if $j \notin S$. Thus

$$\begin{aligned} \varphi_i^\lambda(x^0) + f_i(\delta(x^0)) &= \sum_{\mathcal{R}} \lambda_{\mathcal{R}} \left[F(x^{0,\mathcal{R}_i + i}) - F(x^{0,\mathcal{R}_i})\right] \\ &\leq f_i(x_i^0) \sum_{\mathcal{R}} \lambda_{\mathcal{R}} = f_i(x_i^0) . \end{aligned} \qquad (15)$$

(ii) Let us prove now that for any type x^0, φ^λ is incentive compatible. By Lemma (2) it is sufficient to prove that for all $x^0 \in \mathbb{R}^n_+$ and $x_i \in \mathbb{R}_+$

$$\varphi_i^\lambda(x^0) - \varphi_i^\lambda(\bar{x}) \leq f_i(\bar{z}_i) - f_i(\delta_i(x^0)), \qquad (*)$$

where $\bar{x} = (x^0_{-i}, x_i)$ and $\bar{z}_i = \max(0, x_i^0 - x_i + \delta_i(\bar{x}))$.

For every $i \in N$

$$\begin{aligned} \varphi_i^\lambda(x^0) - \varphi_i^\lambda(\bar{x}) &= \sum_{\mathcal{R}} \lambda_{\mathcal{R}} \left[F(x^{0,\mathcal{R}_i+i}) - F(x^{0,\mathcal{R}_i} + x_i e_i)\right] \\ &\quad - \left[f_i(\delta_i(x^0)) - f_i(\delta_i(\bar{x}))\right] . \end{aligned} \qquad (16)$$

By (16) and (*) it is sufficient that for all $x^0 \in \mathbb{R}^n_+$, $x_i \geq 0$ and $S \subseteq N$

$$F(x^{0,S+i}) - F(x^{0,S} + x_i e_i) \leq f_i(\bar{z}_i) - f_i(\delta_i(\bar{x})). \qquad (**)$$

Let $x^0 \in \mathbb{R}^n_+$, $x_i \geq 0$ and $S \subseteq N$. Denote $\beta_k = \delta_k(x^{0,S+i})$, $\alpha_k = \delta_k(x^{0,S} + x_i e_i)$ and $\bar{\alpha}_k = \delta_k(\bar{x})$. Then

$$F(x^{0,S} + x_i e_i) = f_0(x^{0,S} + x_i e_i - \alpha^{S+i}) + \sum_{k \in S+i} f_k(\alpha_k), \qquad (17)$$

and

$$F(x^{0,S+i}) = f_0(x^{0,S+i} - \beta^{S+i}) + \sum_{k \in S+i} f_k(\beta_k), \qquad (18)$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$.

By (17), (18) and (**) it is sufficient to prove that

$$\begin{aligned} & f_0(x^{0,S+i} - \beta^{S+i}) - f_0(x^{0,S} + x_i e_i - \alpha^{S+i}) \\ & + \sum_{k \in S+i} [f_k(\beta_k) - f_k(\alpha_k)] \;\leq\; f_i(\bar{z}_i) - f_i(\delta(\alpha_k)) \,. \end{aligned} \qquad (***)$$

Next observe that

$$f_0(x^{0,S+i} - \beta^{S+i}) + \sum_{k \in S+i} f_k(\beta_k) \qquad (19)$$

$$\begin{aligned} &= F(x^{0,S+i}) \\ &\leq f_0(x^{0,S} + x_i e_i - \alpha^{S+i}) \\ &\quad + \sum_{k \in S} f_k(\alpha_k) + f_i(\max(0, x_i^0 - x_i + \alpha_i)) \,. \end{aligned} \qquad (20)$$

The right-hand side of (19) is a feasible way to produce $x^{0,S+i}$. By (19) and $(***)$ it is sufficient to prove that

$$f_i(\bar{\alpha}_i) - f_i(\alpha_i) \leq f_i(\bar{z}_i) - f_i(z_i),$$

where $z_i = \max(0, x_i^0 - x_i + \alpha_i)$. Now we use the assumption that $\delta_i(y)$ depends on y_i for any $y \in \mathbb{R}^n_+$ to obtain

$$\bar{\alpha}_i = \delta_i(\bar{x}) = \delta_i(x^{0,S} + x_i e_i) = \alpha_i$$

Hence also $z_i = \bar{z}_i$ and the last inequality holds as an equality.

□

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