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**THEOREMS ON CLOSED COVERINGS
OF A SIMPLEX AND THEIR APPLICATIONS
TO COOPERATIVE GAME THEORY**

by

Tatsuro Ichiishi and Adam Idzik

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H. G. Bergenthal

**Institut für Mathematische Wirtschaftsforschung
an der**

Universität Bielefeld

Adresse / Address:

Universitätsstraße

4800 Bielefeld 1

Bundesrepublik Deutschland

Federal Republic of Germany

1. INTRODUCTION

Let N be a nonempty finite set, and let $\{e^j\}_{j \in N}$ be the unit vectors of the $(\#N)$ -dimensional Euclidean space \mathbb{R}^N ; $e_j^j = 1$ and $e_i^j = 0$ for every $i \neq j$. Denote by \mathcal{N} the family of nonempty subsets of N (i.e., $\mathcal{N} := 2^N \setminus \{\emptyset\}$). Given a subset X of \mathbb{R}^N , denote the convex hull of X by $\text{co } X$, the interior of X by $\overset{\circ}{X}$, the relative interior of X by $\text{ri } X$, and the affine hull of X by $\text{aff } X$. The faces of the unit simplex are then given by $\Delta^S := \text{co } \{e^i \mid i \in S\}$ for every $S \in \mathcal{N}$. The simplex Δ^N is endowed with the relativized Euclidean topology. For each $S \in \mathcal{N}$, its characteristic vector is given by $\chi_S := \sum_{i \in S} e^i$. Given two vectors x and y in \mathbb{R}^N , $x \cdot y$ denotes the Euclidean inner product, and the closed line segment joining the two (i.e., $\text{co } \{x, y\}$) is denoted by $[x, y]$.

It was sixty years ago when Sperner [25] published the following:

Theorem 1.1 (Sperner [25]). Let $\{C^i\}_{i \in N}$ be a closed covering of Δ^N such that $\Delta^{N \setminus \{i\}} \cap C^i = \emptyset$ for every $i \in N$. Then $\bigcap_{i \in N} C^i \neq \emptyset$.

A year later, Knaster, Kuratowski and Mazurkiewicz [17] published the following generalization of Theorem 1.1:

Theorem 1.2 (Knaster, Kuratowski and Mazurkiewicz [17]). Let $\{C^i\}_{i \in N}$ be a family of closed subsets of Δ^N such that $\Delta^S \subset \bigcup_{i \in S} C^i$ for every $S \in \mathcal{N}$.

Then $\bigcap_{i \in N} C^i \neq \emptyset$.

Actually, each of Theorems 1.1 and 1.2 is easily shown to be equivalent to Brouwer's fixed-point theorem, by using Brouwer's [3] technique which involves a partition of unity (see the independent work of Border [3] and Dugundji and Granas [5] for the equivalence of Theorem 1.2 and the Brouwer theorem). Fan [6] pointed out that Theorem 1.1 can be re-formulated as:

Theorem 1.3 (Sperner [25]). Let $\{C^i\}_{i \in N}$ be a closed covering of Δ^N such that $\Delta^{N \setminus \{i\}} \subset C^i$ for every $i \in N$. Then $\bigcap_{i \in N} C^i \neq \emptyset$.

(To show the equivalence of Theorems 1.1 and 1.3, use the Lebesgue number.)

Let K be a finite set such that $K \supset N$, and let $A := ((a_{ij}))_{i \in N, j \in K}$ and $c := (c_i)_{i \in N}$ be a $(\#N) \times (\#K)$ real matrix and a $(\#N) \times 1$ real matrix respectively such that

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \in N; \\ 0 & \text{if } i, j \in N \text{ but } i \neq j; \end{cases}$$

$$c_i \geq 0 \quad \text{for every } i \in N;$$

$$c_i > 0 \quad \text{for some } i \in N.$$

Notice that $\{x \in \mathbb{R}_+^K \mid Ax = c\} \neq \emptyset$. Theorem 1.3 is a special case of the following Scarf's theorem [20]:

Theorem 1.4 (Scarf [20]). Let $\{C^j\}_{j \in K}$ be a closed covering of Δ^N such that $\Delta^{N \setminus \{j\}} \subset C^j$ for every $j \in N$. Assume that the set $\{x \in \mathbb{R}_+^K \mid Ax = c\}$ is bounded. Then there exists $x \in \mathbb{R}_+^K$ such that $Ax = c$ and $\cap \{C^j \mid j \in K, x_j > 0\} \neq \emptyset$.

Scarf [19, 20] used the "path-following technique" of Lemke and Howson [18] to establish a theorem on primitive sets (Theorem 4.6 of this paper), and then used the latter theorem to prove Theorem 1.4. An alternative proof of Theorem 1.4 was made by Kannai [15]; he used the Brouwer fixed-point theorem only.

A generalization of Theorem 1.2 was made by Shapley [21]. To formulate Shapley's result we need the following:

Definition 1.5. A subfamily \mathcal{B} of \mathcal{N} is called balanced, if there exists $\{\lambda_S\}_{S \in \mathcal{B}} \subset \mathbb{R}_+$ such that $\sum_{S \in \mathcal{B} : S \ni i} \lambda_S = 1$ for every $i \in N$.

Theorem 1.6 (Shapley [21]). Let $\{C^S\}_{S \in \mathcal{N}}$ be a family of closed
subsets of Δ^N such that $\Delta^T \subset \bigcup_{S \subset T} C^S$ for every $T \in \mathcal{N}$. Then there
exists a balanced family \mathcal{B} such that $\bigcap_{S \in \mathcal{B}} C^S \neq \emptyset$.

To see the relationship between the conclusions of Theorem 1.4 and of
 Theorem 1.6, let \tilde{A} be the $(\#N) \times (\#\mathcal{N})$ matrix whose rows (columns,
 resp.) are indexed by $i \in N$ (by $S \in \mathcal{N}$, resp.) such that column S is
 precisely χ_S . The set $\{x \in \mathbb{R}_+^{\mathcal{N}} \mid \tilde{A}x = \chi_N\}$ is nonempty and bounded.
 Then the conclusion of Theorem 1.6 is re-formulated as: There exists $x \in$
 $\mathbb{R}_+^{\mathcal{N}}$ such that

$$\tilde{A}x = \chi_N, \text{ and}$$

$$\bigcap \{C^S \mid S \in \mathcal{N}, x_S > 0\} \neq \emptyset.$$

Actually, motivated by Billera's generalization [1, 2] of Scarf's theorem
 [19] for nonemptiness of the core (Theorem 4.4 in this paper), Shapley [21]
 established a more general theorem (Theorem 1.6' below). Define $\Pi := \times_{S \in \mathcal{N}}$
 Δ^S .

Definition 1.5'. Choose any $\pi := (\pi_S)_{S \in \mathcal{N}} \in \Pi$. A subfamily \mathcal{B} of
 \mathcal{N} is called π -balanced, if $\pi_N \in \text{co} \{\pi_S \mid S \in \mathcal{B}\}$.

Theorem 1.6' (Shapley [21]). Let $\{C^S\}_{S \in \mathcal{N}}$ be a family of closed
subsets of Δ^N such that $\Delta^T \subset \bigcup_{S \supset T} C^S$ for every $T \in \mathcal{N}$. Choose any $\pi \in$

Π . Then there exists a π -balanced family \mathcal{B} such that $\bigcap_{S \in \mathcal{B}} C^S \neq \emptyset$.

The additional assumption in Shapley [21] that $\pi \in \text{ri } \Pi$ is non-essential:
 For an arbitrary $\pi \in \Pi$, choose a sequence in $\text{ri } \Pi$ which converges to π .
 Theorem 1.6 is a special case of Theorem 1.6' in which $\pi_S = \chi_S / (\#S)$.

Shapley [21] proved Theorem 1.6' by using the "path-following technique" of
 Lemke and Howson [18]. Todd [26, 27] has a proof of Theorem 1.6 which makes
 use of the Brouwer fixed-point theorem and a sequence of simplicial
 partitions. Shapley [22] has a shorter proof of Theorem 1.6 using
 Kakutani's fixed-point theorem. Ichiishi [12] has a yet shorter proof of
 Theorem 1.6 using Fan's [7] coincidence theorem; see also Ichiishi [13].

Recently Ichiishi [14] established the following theorem, which is dual
 to Theorem 1.6 just as Theorem 1.3 is dual to Theorem 1.2, and which is also
 a generalization of Theorem 1.3:

Theorem 1.7 (Ichiishi [14]). Let $\{C^S\}_{S \in \mathcal{N}}$ be a family of closed
subsets of Δ^N such that $\Delta^T \subset \bigcup_{S \supset N \setminus T} C^S$ for every $T \in \mathcal{N}$. Then there
exists a balanced family \mathcal{B} such that $\bigcap_{S \in \mathcal{B}} C^S \neq \emptyset$.

It was pointed out by David Schmeidler that Theorems 1.6 and 1.7 are equivalent; Schmeidler's argument is reproduced in Ichiishi [14]. Neither of Theorems 1.4 and 1.7 includes the other.

The first purpose of the present paper is to establish general theorems on closed coverings of a simplex in order to give a unified treatment of the above theorems. We prove these general theorems by using a certain geometric lemma and the following special case of Fan's [7, 9] coincidence theorem:

Theorem 1.8 (Fan [9]). Let X be a nonempty, compact and convex subset of \mathbb{R}^N , and let F and G be upper semicontinuous correspondences from X to the subsets of \mathbb{R}^N , such that both $F(x)$ and $G(x)$ are nonempty, compact, and convex for each $x \in X$, and such that

$$(\forall x \in X): (\forall p \in \mathbb{R}^N: p \cdot x = \min p \cdot X):$$

$$\exists u \in F(x): \exists v \in G(x): p \cdot u \geq p \cdot v.$$

Then there exists $x^* \in X$ such that $F(x^*) \cap G(x^*) \neq \emptyset$.

Other covering properties of convex sets were given, e.g., by Fan [6, 8, 10, 11] and Shih and Tan [23, 24].

The second purpose of the present paper is to clarify relationships between the above theorems on closed coverings of a simplex and certain theorems related to the core of a cooperative game without side-payments.

2. MAIN RESULTS

Let K , A , c be given as in the paragraph that precedes the statement of Theorem 1.4. Denote column j of the matrix A by a^j .

Theorem 2.1. Assume that $c \in \Delta^N$ and $a^j \in \text{aff } \Delta^N$ for every $j \in K$.

Let $\{C^j\}_{j \in K}$ be a closed covering of Δ^N such that

$$\forall T \in \mathcal{N} \setminus \{N\}: \Delta^T \subset \cup \{C^j \mid j \in K, a^j \in \Delta^T\}.$$

Then there exists a subset I of K such that $c \in \text{co} \{a^j \mid j \in I\}$ and

$$\bigcap_{j \in I} C^j \neq \emptyset.$$

Proof of Theorem 2.1. For each $x \in \Delta^N$ define $I(x) := \{j \in K \mid C^j \ni x\}$, $F(x) := \{c\}$, and $G(x) := \text{co} \{a^j \mid j \in I(x)\}$. Then the correspondences F and G from Δ^N to the subsets of $\text{aff } \Delta^N$ are upper semicontinuous with nonempty compact and convex values. Choose $x \in \Delta^N$ and $p \in \mathbb{R}^N$ such that $p \cdot x = \min p \cdot \Delta^N$. There exists a unique $S \subset N$ such that $x \in \text{ri } \Delta^S$. Thus we have $p \cdot y = \min p \cdot \Delta^N$ for all $y \in \Delta^S$. If $S = N$, then for all $u \in F(x)$ and all $v \in G(x)$, $p \cdot u = p \cdot v$. If $S \neq N$, then by the assumption of the present theorem there exists $j \in K$ such that $a^j \in \Delta^S$ and $x \in C^j$. For this j , $a^j \in G(x)$ and $p \cdot a^j = \min p \cdot \Delta^N \leq p \cdot c$. All the assumptions of Theorem 1.8 are now satisfied, so there exists $x^* \in \Delta^N$ such that $F(x^*) \cap G(x^*) \neq \emptyset$. The set $I(x^*)$ is the required set I . Q.E.D.

A generalization of Theorem 2.1 is given by:

Theorem 2.2. Assume that $c \in \Delta^N$ and that the set $\{x \in \mathbb{R}_+^K \mid Ax = c\}$ is bounded. Let $\{C^j\}_{j \in K}$ be a closed covering of Δ^N such that

$$\forall T \in \mathcal{N} \setminus \{N\}: \Delta^T \subset \cup \{C^j \mid j \in K, a^j \in \Delta^T\}.$$

Then there exists $x \in \mathbb{R}_+^K$ such that $Ax = c$ and $\cap \{C^j \mid x_j > 0\} \neq \emptyset$.

We shall provide two proofs of this theorem. Both proofs make use of the following claim:

Claim 2.3. Let $n \leq k$, let A be an $n \times k$ matrix whose first n columns constitute the unit matrix, and let c be an $n \times 1$ nonnegative matrix. Then the following two conditions (i) and (ii) are equivalent.

(i) Set $\{x \in \mathbb{R}_+^k \mid Ax = c\}$ is bounded; and

(ii) $\neg \exists x \in \mathbb{R}_+^k \setminus \{0\}: Ax = 0$.

Moreover, for any $n \times 1$ nonnegative, nonzero matrix d , any of the conditions (i) and (ii) implies the following condition (iii).

(iii) $\neg \exists x \in \mathbb{R}_+^k: Ax = -d$.

Proof of Theorem 2.2, Using Theorem 2.1. Define $D := \{Ax \mid x \in \mathbb{R}_+^K,$

$\sum_{j \in K} x_j = 1\}$; it is a convex compact subset of \mathbb{R}^N . By Claim 1.3 (ii) and

(iii), $D \cap (-\mathbb{R}_+^N) = \emptyset$. There exists, therefore, a hyperplane H which

strictly separates D and $-\mathbf{R}_+^N$, in particular $0 \notin H$. Then for each $y \in D$ there exists a unique vector $\hat{y} \in [0, y] \cap H$. Notice that $c \in D$, and $a^j \in D$, for every j (in particular, $\Delta^N \subset D$). Define $\hat{\Delta}^S := \{\hat{y} \mid y \in \Delta^S\}$, and $\hat{C}^j := \{\hat{y} \mid y \in C^j\}$. Under the assumption of Theorem 2.2,

$$\forall T \in \mathcal{N} \setminus \{N\}: \hat{\Delta}^T \subset \cup \{\hat{C}^j \mid j \in K, \hat{a}^j \in \hat{\Delta}^T\}.$$

By Theorem 2.1 applied to $(\hat{\Delta}^N, \{\hat{a}^j\}_{j \in K}, \hat{c}, \{\hat{C}^j\}_{j \in K})$, there exists $I \subset K$ such that $\hat{c} \in \text{co}\{\hat{a}^j \mid j \in I\}$ and $\cap \{\hat{C}^j \mid j \in I\} \neq \emptyset$. We can now choose a suitable $x \in \mathbf{R}_+^K$ such that $x_j = 0$ for $j \in K \setminus I$, $Ax = c$ and $\cap \{C^j \mid x_j > 0\} \neq \emptyset$. Q.E.D.

Proof of Theorem 2.2, Using Theorem 1.8. Define

$$\forall j \in K: \hat{a}^j := a^j + (1 - \sum_{i \in N} a_{ij}) c.$$

Then, $\hat{a}^j \in \text{aff } \Delta^N$ for every $j \in K$, and $\hat{a}^j = a^j$ if $a^j \in \text{aff } \Delta^N$.

Define for each $x \in \Delta^N$,

$$F(x) := \{c\},$$

$$G(x) := \text{co}\{\hat{a}^j \mid j \in K, C^j \ni x\}.$$

As in the proof of Theorem 2.1, one can show that all the assumptions of

Theorem 1.8 are satisfied, so there exists $x^* \in \Delta^N$ such that $F(x^*) \cap$

$G(x^*) \neq \emptyset$. Define $I := \{j \in K \mid C^j \ni x^*\}$. Then there exists $\{z_j\}_{j \in I} \subset \mathbf{R}_+$

such that $c = \sum_{j \in I} z_j \hat{a}^j$. By substituting the definition of \hat{a}^j 's and by

setting $t_j := 1 - \sum_{i \in N} a_{ij}$, one obtains

$$c = \sum_{j \in I} z_j (a^j + t_j c).$$

To sum up, there exist $z_j \in \mathbb{R}_+$, $j \in I$, not all zero, such that

$$(1 - \sum_{j \in I} z_j t_j) c = \sum_{j \in I} z_j a^j.$$

By Claim 2.3,

$$1 - \sum_{j \in I} z_j t_j > 0,$$

thus there exists $z^* \in \mathbb{R}_+^K$ such that

$$A z^* = c, \text{ and}$$

$$\cap \{C^j \mid z_j^* > 0\} \supset \cap_{j \in I} C^j \neq \emptyset.$$

Q.E.D.

Now we generalize Theorem 1.7. We need the following geometric lemma:

Lemma 2.4. Let C be a compact, convex subset of \mathbb{R}^N , and let F be a finite subset of ∂C , the relative boundary of C . Choose any $c \in \text{ri co } F$. For each $x \in F$ choose $x' \in \partial C$ so that $c \in [x, x']$, and define $F' := \{x' \mid x \in F\}$. Then $c \in \text{co } F'$.

Proof of Lemma 2.4. There exists $\{\alpha_x\}_{x \in F} \subset \mathbb{R}_+$, $\sum_{x \in F} \alpha_x = 1$, such that $c = \sum_{x \in F} \alpha_x x$. For each $x \in F$ there exists β_x , $0 < \beta_x < 1$, such that $c = \beta_x x + (1 - \beta_x) x'$. Then $c = \sum_{x \in F} \alpha_x (c - (1 - \beta_x) x') / \beta_x$, so $((\sum_{x \in F} \alpha_x / \beta_x) - 1) c = \sum_{x \in F} (\alpha_x / \beta_x - \alpha_x) x'$, therefore $c \in \text{co } F'$. Q.E.D.

Theorem 2.5. Assume that $c \in \text{ri } \Delta^N$ and $a^j \in \text{aff } \Delta^N$ for every $j \in K$.
Assume also that for every $j \in K$ for which $a^j \in \partial \Delta^N$, there exists $j' \in K$
such that $a^{j'} \in \partial \Delta^N$ and $c \in [a^j, a^{j'}]$. Let $\{C^j\}_{j \in K}$ be a closed covering
of Δ^N such that

$$\forall T \in \mathcal{N} \setminus \{N\}: \Delta^T \subset \cup \{C^{j'} \mid j \in K, a^j \in \Delta^T\}.$$

Then there exists a subset I of K such that $c \in \text{co } \{a^j \mid j \in I\}$ and

$$\bigcap_{j \in I} C^j \neq \emptyset.$$

Proof of Theorem 2.5. Define $D^j := C^{j'}$ for every j for which $a^j \in \partial \Delta^N$, and $D^j := C^j$ for all other j . All the assumptions of Theorem 2.1 are satisfied for $(\Delta^N, \{a^j\}_{j \in K}, c, \{D^j\}_{j \in K})$, so there exists a subset I of K such that $c \in \text{co } \{a^j \mid j \in I\}$ and $\bigcap_{j \in I} D^j \neq \emptyset$. Define $I' := \{j' \mid j \in I\}$.

By Lemma 2.4, $c \in \text{co } \{a^j \mid j \in I'\}$. Moreover, $\bigcap_{j \in I'} C^j = \bigcap_{j \in I} D^j \neq \emptyset$.

Q.E.D.

Using the same method and Theorem 2.2, we can prove:

Theorem 2.6. Assume that $c \in \text{ri } \Delta^N$ and that the set $\{x \in \mathbb{R}_+^K \mid Ax = c\}$ is bounded. Assume also that for every $j \in K$ for which $a^j \in \partial \Delta^N$,

there exists $j' \in K$ such that $a^{j'} \in \partial\Delta^N$ and $c \in [a^j, a^{j'}]$. Let
 $\{C^j\}_{j \in K}$ be a closed covering of Δ^N such that

$$\forall T \in \mathcal{N} \setminus \{N\}: \Delta^T \subset \cup \{C^{j'} \mid j \in K, a^j \in \Delta^T\}.$$

Then there exists $x \in \mathbb{R}_+^K$ such that $Ax = c$ and $\cap \{C^j \mid x_j > 0\} \neq \emptyset$.

3. REMARKS

The K-K-M theorem (Theorem 1.2) follows from Theorem 2.1 if we take $K = N$ and $c \in \text{ri } \Delta^N$. Scarf's theorem (Theorem 1.4) for the case $c \in \text{ri } \Delta^N$ follows from Theorem 2.6 if we introduce new columns $\{a^S \mid S \in \mathcal{N} \setminus \{N\}, \#S \geq 2\}$ (so that if $\#N \geq 2$, the new matrix consists of $(\#K + \#\mathcal{N} - 1 - \#N)$ columns) defined by $a^S \in \Delta^S$ and $c \in [a^S, a^{N \setminus S}]$, and take $c^S = \prod_{i \in S} c^i$ for each $S \in \mathcal{N}$ (Theorem 1.4 would be trivial if $c \in \partial \Delta^N$). Shapley's theorem (Theorem 1.6') follows from Theorem 2.2 if we take $K = \mathcal{N}$, a^S (:= column S of the matrix A) $= \pi_S \in \Delta^S$, and $c = \pi_N$. Ichiishi's theorem (Theorem 1.7) follows from Theorem 2.6 if we take $K = \mathcal{N}$, $a^S = \chi_S$, and $c = \chi_N$.

All the results in Section 2 are valid for an arbitrary real matrix A of dimension $(\#N) \times (\#K)$, $N \subset K$, in which there are $\#N$ linearly independent columns, and $c (\neq 0)$ is a nonnegative linear combination of those columns.

Theorems similar to those of Section 2 can be proved for a compact polyhedron instead of a simplex.

4. CORE

The finite set N is now interpreted as the set of players, and \mathcal{N} as the family of nonempty coalitions.

Definition 4.1. A non-side-payment game is a function V from \mathcal{N} to the subsets of \mathbb{R}^N such that for every $S \in \mathcal{N}$, $V(S)$ is a cylinder, i.e., $[u, v \in \mathbb{R}^N, \forall i \in S: u_i = v_i]$ implies $[u \in V(S) \text{ iff } v \in V(S)]$.

The set $V(S)$, or rather its projection to \mathbb{R}^S , is interpreted as the set of utility allocations within S , each is made feasible by some coordination of strategies of the members of S .

Definition 4.2. The core of a non-side-payment game V is the set $C(V)$ of all $u \in \mathbb{R}^N$ such that (a) $u \in V(N)$ and (b) it is not true that there exist $S \in \mathcal{N}$ and $u' \in V(S)$ such that $u_i < u'_i$ for every $i \in S$.

The core is a typical solution concept; condition (a) says that the utility allocation u is feasible within the grand coalition N , and condition (b) says that no coalition can improve upon u .

Definition 4.3 A non-side-payment game V is called balanced if for every balanced subfamily \mathcal{B} of \mathcal{N} , $\bigcap_{S \in \mathcal{B}} V(S) \subset V(N)$.

See, e.g., Ichiishi [13, Chapter 5] for further discussions of Definitions 4.1, 4.2 and 4.3. Scarf's [19] fundamental theorem for nonemptiness of the core:

Theorem 4.4 (Scarf [19]). Let $V : \mathcal{N} \rightarrow 2^{\mathbb{R}^N}$ be a non-side-payment game, and define $b \in \mathbb{R}^N$ by $b_i := \sup \{u_i \in \mathbb{R} \mid u \in V(\{i\})\}$ for each $i \in N$. The core of V is nonempty if (i) $V(S) - \mathbb{R}_+^N = V(S)$ for every $S \in \mathcal{N}$; (ii) there exists $M \in \mathbb{R}$ such that for every $S \in \mathcal{N}$, $[u \in V(S) \cap [(b) + \mathbb{R}_+^N]]$ implies $[u_i < M$ for every $i \in S]$; (iii) $V(S)$ is closed in \mathbb{R}^N for every $S \in \mathcal{N}$; and (iv) V is balanced.

Scarf [19] established the following Theorem 4.6, and then derived from it Theorem 4.4. Let K, A, c be given as in the paragraph that precedes the statement of Theorem 1.4. Choose vectors $P := \{\pi^j\}_{j \in K}$ in \mathbb{R}^N such that

$$\pi^i = (\overbrace{R_1, \dots, R_1}^i, 0, R_1, \dots, R_1) \quad \text{if } i \in N;$$

$$\pi^j \in (\Delta^N - \mathbb{R}_+^N) \cap \mathbb{R}_+^N \quad \text{if } j \in K \setminus N,$$

where $R_i > 1$ for each $i \in N$.

Definition 4.5. A subset of P , $\{\pi^j\}_{j \in I}$, is called a primitive set, if there does not exist $\pi \in P$ such that

$$\forall i \in N: \pi_i > \min \{\pi_i^j \mid j \in I\}.$$

Theorem 4.6 (Scarf [19, 20]). If the set $\{x \in \mathbb{R}_+^K \mid Ax = c\}$ is bounded, then there exists $x \in \mathbb{R}_+^K$ such that $Ax = c$ and $\{\pi^j \mid j \in K, x_j > 0\}$ is a primitive set.

Remark that the vectors π^j , $j \in K \setminus N$, can actually be chosen arbitrarily from \mathbb{R}_+^N , provided that the R_i , $i \in N$, are suitably re-defined. Due to arbitrariness of the finite set K (provided that it contains N), and hence the generality of matrix A compared with \tilde{A} (the matrix \tilde{A} was introduced in a paragraph between the statement of Theorem 1.6 and Definition 1.5'), Theorem 4.6 together with a certain nondegeneracy assumption summarizes an analytical feature of Scarf's algorithm to compute a member of the core.

It was pointed out earlier that Scarf [20] derived Theorem 1.4 from Theorem 4.6. Conversely, Theorem 4.6 can be derived from Theorem 1.4; the proof is based on the idea in Vohra [28]:

Derivation of Theorem 4.6 from Theorem 1.4. Define $C_1^j := \{\pi^j\} - \mathbb{R}_+^N$.

Denote by F the boundary of $\bigcup_{j \in K} C_1^j$, and define for each $j \in K$,

$$C^j := \{z \in \Delta^N \mid \exists y \in C_1^j \cap F: z = y / \sum_{i \in N} y_i\}.$$

If $\pi \notin \overset{\circ}{\mathbb{R}}_+^N$ for any $\pi \in P$, then the assertion of Theorem 4.6 is trivial.

Assume, therefore, that there exists $\pi \in P \cap \overset{\circ}{\mathbb{R}}_+^N$. Then 0 is in the interior of $\bigcup_{j \in K} C_1^j$, so $\{C^j\}_{j \in K}$ is a closed covering of Δ^N . Observe that $y \in F$, if $y \in \bigcup_{j \in K} C_1^j$ and $y_i \geq R_i$ for some $i \in N$. By this observation, it is easy to check $\Delta^{N \setminus \{j\}} \subset C^j$. Thus $\{C^j\}_{j \in K}$ satisfies the assumption of Theorem 1.4, so there exists $x^* \in \mathbb{R}_+^K$ such that $A x^* = c$ and $\bigcap \{C^j \mid j \in K, x_j^* > 0\} \neq \emptyset$. Set $I := \{j \in K \mid x_j^* > 0\}$, choose $z^* \in \bigcap_{j \in I} C^j$, and consider $y^* \in F$ defined by $z^* = y^* / \sum_{i \in N} y_i^*$. Then $\pi^j \geq y^*$ for all $j \in I$, so $\{\pi^j\}_{j \in I}$ is the required primitive set. Q.E.D.

Many alternative proofs of Theorem 4.4 have appeared in the literature: Shapley [21] derived Theorem 4.4 from Theorem 1.6. Keiding and Thorlund-Petersen [16] and Vohra [28] proved Theorem 4.4, using Theorem 1.2 and Kakutani's fixed-point theorem, respectively. Ichiishi [14] pointed out that the geometric insights of Keiding and Thorlund-Petersen and of Vohra can be re-formulated as Theorem 1.7. It will be shown here that Theorem 4.4 follows simply from a theorem which is weaker than Theorem 1.4 and weaker than Theorem 1.7; the proof is based on the idea in Vohra [28]:

Derivation of Theorem 4.4 either from Theorem 1.4 or from Theorem 1.7.

The special case of Theorem 1.4 and of Theorem 1.7, in which $K = \mathcal{N}$, $A = \tilde{A}$ and $c = \chi_{\mathcal{N}}$, will be used here. Assume without loss of generality that

$b = 0$. If $0 \notin \overset{\circ}{V}(S)$ for every $S \in \mathcal{N}$, then there is nothing to prove (indeed, in this case, $0 \in C(V)$). Assume, therefore, that $0 \in \overset{\circ}{V}(S)$ for some $S \in \mathcal{N}$. Choose two real numbers M_1 and M_2 such that $M_1 > M_2 > M$, and denote by F the boundary of the set,

$$\bigcup_{i \in N} \{u \in V(\{i\}) \mid \forall k \in N \setminus \{i\}: u_k \leq M_1\} \\ \bigcup_{S \in \mathcal{N}: \#S \geq 2} \{u \in V(S) \mid \forall k \in N \setminus S: u_k \leq M_2\}.$$

For each $\pi \in \Delta^N$, consider the unique point $f(\pi) \in F \cap \mathbb{R}_+^N$ defined by $\pi = f(\pi) / \sum_{i \in N} f_i(\pi)$. Define $C^S := \{\pi \in \Delta^N \mid f(\pi) \in V(S)\}$ for every $S \in \mathcal{N}$. The family $\{C^S\}_{S \in \mathcal{N}}$ is a closed covering of Δ^N , and it is easy to check $\Delta^{N \setminus \{j\}} \subset C^{\{j\}}$ for every $j \in N$. All the assumptions of Theorem 1.4 and of Theorem 1.7 are satisfied, so there exist $x^* \in \mathbb{R}_+^{\mathcal{N}}$ and $\pi^* \in \Delta^N$ such that $\tilde{A} x^* = \chi_N$ and $\pi^* \in \bigcap \{C^S \mid S \in \mathcal{N}, x_S^* > 0\}$. The point $f(\pi^*)$ will be shown to be a member of $C(V)$. The family $\mathcal{B} := \{S \in \mathcal{N} \mid x_S^* > 0\}$ is balanced and $f(\pi^*) \in \bigcap_{S \in \mathcal{B}} V(S)$. So by the balancedness assumption on V , $f(\pi^*) \in V(N)$. Consequently, $f(\pi^*) \in \{u \in F \mid \forall i \in N: u_i < M\}$, which implies that the utility allocation $f(\pi^*)$ cannot be improved upon by any coalition. Q.E.D.

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