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Life-length of a process with
elements of decreasing importance

Axel Ostmann

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H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

Life-length of a process with elements of decreasing importance

Let (X_t), $t=1,2,\ldots$ be a Bernoulli-process with $p(X_t=1)=p$ and $q=1-p$. Then for $t<n+1$ and a simple game (N,v), $N=\{1,2,\ldots,n\}$ a process (V_t), $V_t=v(X_1,\ldots,X_t)$ is induced.
The random variable $T=\min\{t,n+1;V_t=1\}$ is a stopping time called life-length of the corresponding game. The following paper deals with the distribution of the life-length of ordered games. For another interpretation consider a system with independent subsystems. Subsystems can be ordered according to their importance. Subsystem t is tested at time t. Then $X_t=1$ means an error at time t (or subsystem t faulty or etc.), and $V_t=0$ means that the system is still alife (works well) at time t.
The distribution of life-length can be generated out of the family of Pascal-distributions.

Contents

1. Simple games, duality and constant sum

(1.1) The simple game

Definition: 1. A pair (N,v) with N={1,...,n} or N being the set of the natural numbers, and a function v on the power set 2^N of N with values 0 and 1, fulfilling $v(\emptyset)=0$ is called **simple game**.

The game is called **finite** if N is finite.

The elements i of N are called **players**, and the subsets S of N are called **coalitions**. We often identify the subsets S with the vectors s (or indicatorfunctions 1_S):

$s_i=1$ if $i\in S$ and $s_i=0$ if $i\in N\setminus S$.

Let us call $W=W(N,v)=\{S\in 2^N; v(S)=1\}$ the set of **winning coalitions** and $L=L(N,v)=2^N\setminus W$ the set of **losing coalitions**.

Let $|S|$ resp. $|s|$ denote the (cardinal) number of elements of S.

2. A simple game (N,v) is called **monotone** if for all coalitions S, T the set-inclusion $S\subset T$ induces $v(S)\leq v(T)$.

We write i instead of {i}, if this do not lead us to confusions; the corresponding vector is denoted by e(i).

We denote the union of two coalitions S and T with empty intersection by S+T.

3. A simple game (N,v) is **superadditive** if for all S, T (with an empty intersection) $v(S+T)\geq v(S)+v(T)$.

4. A simple game (N,v) has **constant sum** if $v(S)=1-v(N\setminus S)$ for all S.

(1.2) The dual game

Definition: Let $.^*:V\to V$, $(N,v)^*:=(N,v^*)$ and $v^*(S)=1-v(N\setminus S)$. $(N,v)^*$ is called the **dual game** with respect to (N,v).

A simple game (N,v) is called **dual superadditive** if its dual game $(N,v)^*$ is superadditive.

Remarks. 1. Constant sum games are selfdual, superadditive and dual superadditive.

2. $.^*$ is idempotent ($v^{**}=v$).

3. There are games neither superadditive nor dual superadditive. An example is $N=\{1,2,3,4\}$, $v(S)=1$ if and only if $|S|\geq 3$ or $S=\{1,2\}$ or $S=\{3,4\}$.

(1.3) **Dummies**

Definition: Let (N,v) be a simple game. Player $i\in N$ is called a **dummy** if $v(i+S)=v(S)$ for all S. $D=D(N,v)$ denotes the set (coalition) of all dummies of the game (N,v).

Remark. The set of finite games on $\{1,\dots,n\}$ can be embedded in the set of games on $\{1,\dots,n+1\}$ by adding a dummy $n+1$. By adding infinitely many dummies we embed the set of finite games in the set of games on the natural numbers. Conversely we can **identify** an infinite game with finitely many non-dummies with **the finite game** defined by dropping dummies.

Up from now N denotes the set of natural numbers. In case it does not lead to confusions, we write v instead of (N,v) and call the function v a game. Up from now a game is called **finite** if the set of non-dummies is finite.

(1.4) **Basic sets and finite approximations**

Definition:

An (open) basic set of order j is set $\{s\in 2^N;\ s_i=s'_i \text{ for } i\leq j\}$ denoted by the ternary representation $(s'_1,\dots,s'_j,*,*,\dots)$ or $[s'_1,\dots,s'_j]$.

Let $*$: $s \to (s_1,\dots,s_{o(s)},*,*,\dots)$ with $\mathbf{o(s)}:=\max\{i;s_i=1\}$.

Characteristic algebra and topology on the set of coalitions

The intersection of basic sets are basic sets. Any sum of basis sets is called an **open** set (if you want, a topology is defined on the set 2^N of coalitions - this topology is the product topology on 2^N; the topological space is isomorphic to the Cantor discontinuum (Cantor ternary set). The complement of any open set is a **closed** set. Let **cA** be the algebra of **clopen** (= closed and open) sets. This algebra is called characteristic algebra (of the topological space of coalitions). **cA** is the unique (up to an isomorphism) countable atomless algebra (cf. Bell/Slomson 1969, ch.1).

Remark. Assume that $W(N,v)\varepsilon$**cA**.

Then for every coalition s exists a "time" i(s) such that we can decide whether s is winning or losing.

Definition: A coalition s is said to be **finite** iff o(s) is finite.
A coalition s is said to have a **full tail** (or to be cofinite) iff $\mathbf{o'(s)}=\max\{i;s_i=0\}$ is finite.

Lemma: If there is an infinite element of W_*, then W is not clopen.

Proof: Let $s\varepsilon W_*$, s infinite. Assume W clopen. In that case there is a full neighborhood of s contained in W. This implies that s is not minimal.

Definition: For a given game (N,v) let
$v^n(s):=\max v(s_1,\dots,s_n)$ and
${}^n v(s):=\min v(s_1,\dots,s_n)$

By these definitions the game can be transformed into finite games of length n. We get $v^n \geq v \geq {}^n v$.

Definition: Let W^n and ${}^n W$ denote the corresponding sets of winning coalitions.

Correspondingly we get $^nW \subset W \subset W^n$.
If $s \varepsilon {}^nW$, then s wins up to time n.
^nW is open, W^n is closed. The interior int(W) of W is equal to the union of the finite inner approximations, the closure cl(W) of W is equal to the intersection of the finite outer approximations.
The corresponding games are denoted by

$$\max {}^n v \quad \text{resp.} \quad \min v^n$$

in fact: $s \varepsilon \mathrm{int}(W)$ iff $\max {}^n v(s) = 1$, etc.

(1.5) **Dual-equivalent and fair games**

Definition: Let (N,v) be a simple game, N finite. A game is called **fair** if $|W|=|L|$.

Remark. Constant-sum games are fair. Example (1.2) 3. proves that "fair" does not imply constant-sum property.

A game is called **dual-equivalent** if there is a permutation h of the players' set such that $v(h(s))=v^*(s)$ for all s.

Remark. Finite dual-equivalent games are fair. Constant-sum property is dual-equivalency for h=id. In (10.3) we shall give an example for a fair game that is not dual-equivalent.

For infinite N the game is called fair if $(|{}^nW|-|{}^nL|)/2^n$ converges to zero.

This is equivalent to $(|{}^nW(v)|-|{}^nW(v^*)|/2^n$ converges to zero $(|{}^nL(v)|=|{}^nW(v^*)|)$.

For a set A of coalitions let C(A) be the set of complements $N\backslash S$, $S \varepsilon A$.

Lemma: A open iff C(A) open, A closed iff C(A) closed.
Proof: Consider neighbourhoods of $s \varepsilon A$ resp.
$1-s \varepsilon C(A)$: $s \varepsilon A$ iff $s \varepsilon [a_1, \ldots, a_n]$
iff $1-s \varepsilon [1-a_1, \ldots, 1-a_n]$.
Corrollary: W(N,v) closed iff $W(N,v^*)$ open.
Proof: set $A=\{s; v(s)=0\}$
Corrollary: If v is constant-sum or finite dual-equivalent, and W is closed or open, then $W \varepsilon$ **cA**.

2. Directed games and the desirability relation

(2.1) The desirability relation

Definition: Let $(N,v)\varepsilon V$. The relation $\succeq=\succeq\subset N^2$ is called **desirability relation**, with $i\succeq j$ (i is more desirable than j) if $v(i+S)\geq v(j+S)$ for all S.

The players i and j are **symmetric** (or of the same type) if $i\succeq j$ and $j\succeq i$; we write $i\sim j$ and $\tilde{i}:=\{j\varepsilon N; i\sim j\}$.

$\tilde{N}:=\{\tilde{i}, i\varepsilon N\}$ is called the set of **types**.

Additionally let $\succ:=\succeq\setminus\sim$ be the **strict desirability relation** and $\|:=N^2\setminus(\succeq\cup\preceq)$ be the relation of incomparability; players i and j with $i\|j$ are called **incomparable.**

Remarks.

1. Analogue to the finite case the desirability relation is reflexive and transitive (Maschler/Peleg, Theorem 9.2).
2. There are games with an incomplete desirability relation (i.e. $\|$ is not empty).

(2.2) Ordered games

Definition: Let $(N,v)\varepsilon V$. (N,v) is an **ordered** game if its desirability relation is complete.

(2.3) Directed games

Definition: Let (N,v) be an ordered game on the natural numbers. It is called **directed** if $i\succeq j$ for all $j>i$.

Remarks.

Directed games are ordered games with "decreasing strength" of the players, that can be indexed by the natural numbers. Ordering players by non-increasing strength defines an ordinal number. Those and only those ordered games with the specific ordinal number 'natural numbers' can be identified with directed games. **If the game is directed and there is an infinite type, then this type is the smallest one and there are finitely many types.**

Finite approximations of ordered/directed games are ordered/directed. The desirability relation is a refinement of the finite desirabiliy relations (or equal).

Up from now we are dealing with directed games only.

Every finite ordered game can be identified with a directed game. Sometime we call a player i "time i". In this sense "time has non-increasing importance".

(2.4) **Dual-equivalent directed games**

Theorem: In case game is directed dual-equivalency implies constant-sum property.

Proof: Let h be the permutation named in definition (1.5). h(i) is more desirable than h(j) iff i is more desirable than j (see definition (2.1)). If the game is directed then there is a greatest type w.r.t. desirability. Thus, h(v)=v.

3. Representing games

(3.1) **Comparing coalitions**

Definition: Let $(N,v)\in V_{\succ}$, and fix coalitions S and T.
S is **more weighty** than T, denoted $S \succ T$,
if $|S\cap\{1,\ldots,t\}| \succ |T\cap\{1,\ldots,t\}|$ for all t.
We write $S \| T$ if S and T are incomparable with respect to $\succ$.

Example. Let $N=\{1,2,3,4\}$ and $i \succeq i+1$ $(i=1,2,3)$.
(1,0,0,1) and (0,1,1,0) are incomparable.

Lemma: $v(S) \succ v(T)$ for all $S \succ T$.
The proof is evident.

(3.2) **(Shift-)Minimal winning coalitions**

$W_* = W_*(N,v)$ be the set of minimal elements of W w.r.t. set-inclusion. Elements of W_* are called **minimal winning coalitions.**

Remark. If W is open, then $W=g(W_*)$. If s is minimal winning for $\max^n v$, the it is minimal winning for v. Both sets may differ only in infinite coalitions.

We say "s wins at time t" if $(s_1,\ldots,s_t)$ is minimal winning.
(3.1) ensures the existence of an economical representation of the game:
W^* be the set of all minimal elements of W with respect to $\succ$; W^* is called the set of **shift-minimal winning coalitions.** Write $\langle W^* \rangle$ for (N,v).
W^* is a subset of W_*.

Remark. Finite games are uniquely determined be W_* as well as by W^*.

(3.3) Aumanns ordered game

Let (N,v):=<{(1,0,0,1,1,0,0,1),(0,1,1,0,0,1,1,0)}.
The property of the game, we are interested in for the moment, is: this game is neither superadditive nor dual superadditive. On one side the two shift-minimal coalitions are winning and partioning N (the game is not superadditive), and on the other side coalitions (1,1,0,0,0,0,1,1) and (0,0,1,1,1,1,0,0) are winning and partioning N in the dual game (the game is not dual superadditive).

(3.4) Incidence matrix and interval matrix

X=X(N,v) denotes the **incidence matrix of the game**, i.e. the rows are the minimal winning coalitions given in the lexicographic order. Thus the column index is a natural number, but the row index is an ordinal number, and the row index-set depends on the game. For finite games let X be the finite matrix given by deleting dummies.

The *-transform of X(N,v) is denoted by Y=Y(N,v). This matrix is called the **interval matrix of the game**.

(3.5) A further example

Let N={1,2,3,4}, v(S)=1 if and only if ($1 \in S$ and $S \cap \{2,3\}$ not empty) or $\{2,3,4\} \subset S$. Then $1 \succ 2 \sim 3 \succ 4$.
Now consider $(N,v)^*$: $v^*(S)=1$ if and only if ($1 \in S$ and $S \cap \{2,3,4\}$ not empty) or $\{2,3\} \subset S$. Again we find $1 \succ 2 \sim 3 \succ 4$.
According to (3.2) (N,v)=<{(1,0,1,0),(0,1,1,1)}> and $(N,v)^*$=<{(1,0,0,1),(0,1,1,0)}>.
According to (3.4) we get:

$$X(N,v)=\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&1&1\end{pmatrix} \qquad X(N,v^*)=\begin{pmatrix}1&1&0&0\\1&0&1&0\\1&0&0&1\\0&1&1&0\end{pmatrix}$$

$$Y(N,v)=\begin{pmatrix} 1 & 1 & * & * \\ 1 & 0 & 1 & * \\ 0 & 1 & 1 & 1 \end{pmatrix} \qquad Y(N,v^*)=\begin{pmatrix} 1 & 1 & * & * \\ 1 & 0 & 1 & * \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & * \end{pmatrix}$$

(3.6) **Desirability for the dual game**

Lemma: The dual game exhibits the same desirability relation.

Proof: analogue to the finite case, see Ostmann 1985, 3.8

(3.7) **f-vector and Banzhaf-value**

The following two vectors can be obtained by means of the matrices X and Y.

Definition: Let $f=\Sigma\{e(o(s));s\varepsilon W_*\}$.

This is a counter for exact winning time; f_t is the number of minimal winning coalitions at time t. $(f_1,\ldots,f_n)$ is determined by ${}^n v$ - that is why f is fully determined by $\max^n v$.

Definition: Let $l=\Sigma\{2^{-o(s)-1}(s+\max(*(s)));s\varepsilon W_*\}$.

Let us call $P=P(v)=\Sigma 2^{-i} f_i$ the performance of the game v.

Remark. This corresponds to Lapidot's counting-vector and to the Chow-numbers.

Definition: Now let $b=-1+2l/\Sigma 2^{-i} f_i=-1+2l/P$.

This vector is called **Banzhaf-value** of the corresponding game.

Remarks. This is a straight-forward generalization of the Banzhaf-value for directed games (the easy formula holds only for directed games).

$P(v)=1-P(v^*)$. A game v is fair iff $P(v)=P(v^*)$.

(3.8) **Examples**

1. Let W=(1)+(011)+(00111)+...+(0...011...1)+...
   This game is not directed: time 3 is strictly more desirable than time 2, since (01011) are not winning.
2. Let W=(111)+(11011)+(1101011...)+(10111)+(1011011...) +(01111). This game is not decidable: (1101011...) cannot be decided upon (in finite time).
   f=0010304, b=(11,10,10,7,5,1,1)/15.
3. Let (N,v)=<(0101...01...)>.
   This game is directed: (2i-1)~2i>(2i+1). There is no lexicographically first minimal winning coalition.
   f is the zero-vector.
4. Let (N,v)=<(01),(0011),(00001111),...>
   This game is not decidable. The empty coalition cannot be decided upon (in finite time).
   f=11010027.
5. Let W be the set of cofinite coalitions. This game is ordered. There is only one type and consequently the game is directed. But: W_* and W^* are empty.
   f is the zero-vector.
6. Let W=(1)+(01)+(001)+...+(0...01)+...
   This game is directed. W^* is empty.
   f=1..., b is the zero-vector.
7. W=(11)+(101...)+(1001...)+(011)+(0101...)

(3.9) **Constructivity of W**

Definition: For a set B of coalitions let g(B) resp. sh-g(B) be the set of all coalitions greater than some coalition of B w.r.t. set inclusion resp. shift inclusion ($\succ$).

What are the conditions for $W=g(W_*)$ and $W=sh\text{-}g(W^*)$?
It is clear that $W=sh\text{-}g(W^*)$ implies $W=g(W_*)$.

Theorem: g({s}) and sh-g({s}) are closed and
- open iff s is finite,
- finite iff s is cofinite,

Proof: Up to time n there are only finitely many coalitions greater resp. shift-greater than s. Let us call this set A_n. Then (sh-)g({s}) is the intersection of $\cup\{[a_1,\ldots a_n];a\varepsilon A_n\}$. The union is finite, thus (sh-)g({s}) is closed. Moreover if s is finite (sh-)g({s}) is equal to the finite intersection $n \leq o(s)$. If s is cofinite, then $a\varepsilon$(sh-)g({s}) iff $a_i=1$ for $i \geq o'(s)$. Thus $|(sh\text{-})g(\{s\})| \leq 2^{o'(s)}$.

Lemma: 1. A minimal winning coalition is either finite or an element of the boundary ∂W of W.
2. The set of minimal elements w.r.t. set-inclusion M of an open set O generates g(O) (i.e. g(O)=g(M)).
3. The set of minimal elements w.r.t. set-inclusion M′ of a closed set A generates g(A) (i.e. g(A)=g(M′)).

Proof: 1. Suppose s is minimal winning and not finite. Then every neighbourhood of s contains a losing coalition.
2. An open set is the sum of basic sets. A basic set B has the shape $[b_1,\ldots,b_n]$ and $(b_1,\ldots,b_n,0\ldots)$ is the only minimal element in B. Moreover, the union of minimal elements of all these basic sets is the set M of minimal elements of the open set (all these coalitions are finite).

3. It is enough to prove that A is a subset of g(M'). Let $a \varepsilon A$. Define a^n by $a^0=a$, $a^n=a^{n-1}-e(n)$ iff $a^{n-1}-e(n) \varepsilon A$, else $a^n=a^{n-1}$.
A is closed and lim a^n is minimal (i.e element of M').

Theorem: Suppose W is the sum of a closed set A and an open set O (W=A+O). Then $W=g(W_*)$.
Proof: W=g(W)=g(A+O)=g(M+M'). It is enough to prove $W_*=M+M'$. Let $s \varepsilon M$, $s' \varepsilon M'$. If there is a smaller one at all, then the smaller one has to be the finite one; but a finite one is element of int(W). Thus s||s' and all elements of M+M' are minimal winning.

Similar arguments proof the following
Theorem: Suppose W is closed. Then $W=sh\text{-}g(W^*)$.

(3.10) Problem
In what directed games there are at most finitely many minimal winning coalitions with a full tail?
$W=g(W_*)$ iff ... ?

(3.11) Convergence
Proposition: If we use the discrete topology on {0,1} and pointwise convergence (i.e. $(\lim v_n)(s)=\lim(v_n(s))$) on the space of games the following equivalences hold:

1. $\min v_n = \lim v^n$
2. $\max {}_n v = \lim {}^n v$
3. $\lim v^n = v$ iff W is closed
4. $\lim {}^n v = v$ iff W is open
5. $\lim v^n = \lim {}^n v$ iff $W \varepsilon cA$
   iff v continuous iff v finite

Proof is simple. Consider the intersection of the decreasing sequence (W^n) and the union of the increasing sequence $({}^n W)$.
(These sets are the closure resp. the inner of W, cf.1.4).

(3.12) Examples

$W(N, \lim v^n) = 2^N$ for the examples (3.8) 1.,4.-6.

$W(N, \lim {}^n v) = \emptyset$ for example (3.8) 5.

(3.13) Remark

Dual-equivalent games with an open or closed set of winning coalitions are finite constant-sum games ((1.5),(2.4),(3.11)).

4. Life-length of the game

(4.1) Definition: Let (X_t), t=1,2,... be a Bernoulli-process with $p(X_t=1)=p$ and $q=1-p$. Then the random process (V_t) induced by a game (N,v)
- $V_t:=v(X_1,\dots,X_t)$ -
is called the game's **value process**. The random variable $T:=\min\{t;V_t=1\}$ is called **life-length** of the game. Let $p_t:=p(T=t)$ and $r:=\min\{t;p_t>0\}$.
The processes and variable are called **fair** if p=q=1/2.

Remarks. Product measure is also denoted by p, for example: $p(W)=E(V)=p(v(X)=1)=p(X\varepsilon W)$ - if W is a measurable set. $P\leqslant p(W)$. A set without inner point has measure zero (f. ex. the boundary of some set).
Sure win is defined by $P=\Sigma p_t=1$;

Remark: Sure win implies p(W)=1 and p(winning at infinity)=0.

Lemma: Fair processes fulfill $p_t=f_t2^{-t}$.

This follows from definition (3.7) and (4.1).

(4.2) Process-equivalency

Definition: v and v' are **process-equivalent** ($v\sim v'$) if $(p_t(v))=(p_t(v'))$ (for the the same p,q).

Lemma: 1. $v\sim v'$ iff the games have the same f-vector.
2. $v\sim\max^n v$.

Proof. 1. The f-vector and p fully determine the process.
2. ${}^n v$ contains all information up to time n.

(4.3) Lemma: If $W=g(W_*)$ or $W=sh\text{-}g(W^*)$ then:
1. $p(\bigcap W^n\setminus\bigcup^n W)=0$ (i.e. $\max^n v=\min v^n$ p-a.s.).
2. $\lim V^n=\lim {}^nV$ p-a.s., $p(W)=\Sigma p_t$,
and 3. - if T^n and ${}^nT$ are the corresponding life-lengths -
$\lim T^n=T=\lim {}^nT$ p-a.s.

Proof: The set $\bigcap W^n\setminus\bigcup^n W$ is a subset of the boundary.

(4.4) A simple class of sure games

Definition: Let $(N,v)^{(r)}$ the game "r players are minimal winning" ("r faults destroy the system"). $T^{(r)}$ denotes the corresponding life-length.

Remark: $P=p(W)=1$.

(4.5) Lemma: $T^{(r)}-r$ is Pascal-distributed with parameter r, $E(T^{(r)})=Var(T^{(r)})=2r$.

Proof: Let $f_t^{(r)}=\binom{t-1}{r-1}$

($f_t=0$ for $t<r$). f_t counts the minimal winning coalitions at time t distributing t-r "zeros". By lemma (5.1) we get $p_t^{(r)}$. The moment generating function is $P(s)=\Sigma f_t^{(r)} 2^{-t} s^t$ $=2^{-r}\Sigma p(\text{Pascal}(r))s^t$. $P(1)=1$, $P'(1)=2r$, $P''(1)=4r^2-r$. $E(.)=P'(1)$. $Var(.)=P'(1)+P''(1)-(P'(1))^2=2r$.

Remarks. 1. $\Delta f^{(r)}=f^{(r-1)}$.

2. $\lim (T^{(r)}/E(T^{(r)})) =$ Exponential (1) + 1.

The (binomial) coefficients f_t^r are given in the following table:

r\t	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	2	3	4	5	6	7	8	9	10	11
3	0	0	1	3	6	10	15	21	28	36	45	55
4	0	0	0	1	4	10	20	35	56	84	120	165
5	0	0	0	0	1	5	15	35	70	126	210	330
6	0	0	0	0	0	1	6	21	56	126	252	462
7	0	0	0	0	0	0	1	7	28	84	210	462
8	0	0	0	0	0	0	0	1	8	36	120	330

(4.6) Definition: $f^*_t := \max\{f_t^{(r)}; r \varepsilon N\}$.

Theorem: $\Sigma f^*_t 2^{-t} = \infty$

Proof: Raabe-criterium.

Remark: The set of f-vectors (numbers of wins at time ...) for all games F is not equal to $\{f; \Sigma f_t 2^{-t} \leq 1, 0 \leq f_t \leq f^*_t\}$ 10130... is not an element of F (see Appendix A). The increase of f is bounded by the history of "losses" related to the corresponding $f^{(r)}$.

(4.7) Theorem: The distribution of the stopping time T of the game (N,v) up to time t is the same as that one of $(N, \max\{{}^t v, v^{(t)}\})$.
The game $(N, \max\{{}^t v, v^{(t)}\})$ guarantees a sure win.

Proof: Up to time t both value processes are equal. The probability of a losing coalition s with r members of the game $(N, {}^t v)$ is equal to the probability of the set of minimal winning coalitions starting in $(s_1, \ldots, s_t)$. The hitting time of this set is a shifted Pascal(t)-variable.

(4.8) Corrollary: The maximal value of p_{t+1} is determined by the history ${}^t v$.
F can be generated this way.

Especially the family of all distributions of directed games is generated out of Pascal(r)+r - distributions by repeating the following two operations:

- deleting the r-coalitions smaller than some, and
- filling in that part of another Pascal(r')+r', r'>r losing in the r-game.(This is an apropriate sum of shifted Pascal(r")-distributions, 0≤r"≤r'.)

(4.9) The corresponding results can be obtained for non-fair processes - substituting the negative binomial distribution for Pascal's.

(4.10) An example

Consider the game max(<00111>,$v^{(7)}$).

f=(0,0,1,3,6,0,1,6,22,63,...).

Ones up to time 5	Number of events	Time 6	7	8	9	10
3	10	0	0	0	0	0
2	10	0	1	5	15	35
1	5	0	0	1	6	21
0	1	0	0	0	1	7
sum		0	1	6	22	63

(4.11) A formula for v=<s,s'>

Let f and f' be the f-vectors of <s> resp. <s'>, and r, r' the length of the corresponding lexicographically first coalition.
Then we get f_t up to time r'-1 and after that time $g_t - \Sigma\{f_j; j=1,...,r'-r\}$.

(4.12) Pascal-type

Definition: $\{i \in N; i=|s|, s \in W_*\}$ is called
Pascal-type of the corresponding game.

In the appendices the Pascal-type is denoted by a corresponding letter-combination, f.ex. AB, BDE or CG (example (4.10)).

(4.13)

An f-vector does not determine the game (even in the finite case). The smallest example is f=0111 (see Appendix A.).
It is easy to construct all (process-equivalent) games for a given f-vector by successive construction of ^nW (with $(f_1,...,f_n)$).

5. Characters

(5.1) **Structure of X(N,v)**

Lemma: If there is a positive probability of stopping in finite time then there is a lexicographically first coalition s° in W_{*}.

Proof: If there is a finite coalition s of order o(s)=j in **W**, then the coalition 1+...+j is more weighty and winning too. By successively deleting the last element we find a minimal winning coalition. This coalition is the lexicographically first in W_{*}.

(5.2) **Characters for the players**

Definition: 1. i is called **dummy** ($i\varepsilon D$) iff i is not a member of any minimal winning coalition;

2. i is called **sum** ($i\varepsilon\Sigma$) iff there are minimal winning coalitions s and t coincident up to time i-1 fulfilling $s_i=t_{i+1}=1$, $s_{i+1}=t_i=0$ and $t_j \geq s_j$ for all j>i.

In this case t-s+(i+1) is called the **substitute** of i.

The sum is called **proper** iff there is a finite substitute (= that does not contain a full tail).

3. i is called **step** if it is neither a dummy nor a sum. A step is called **improper** if it is not a member of a finite minimal winning coalition.

(5.3)

Lemma: Any game is process-equivalent to a game without improper characters.

Proof: consider the 'inner game' $\lim^n v$.

6. Weighted majority games and homogeneous games

(6.1) Weighted majority games

Definition: $(N,v)\varepsilon V$ is a **weighted majority game** (w.m.g) if there exist a natural number μ and a nonnegative integral measure $m=(m_1,...,m_n)$ fulfilling $v=1_{[\mu,m(N)]}\circ m$.

Write $m(S)$ for $\Sigma\{m_i;i\varepsilon S\}=1_S m$.

Definition: $(N,v)\varepsilon V$ is a **rational weighted majority game** (r.w.m.g) if there exist a natural number μ and a nonnegative rational measure $m=(m_1,...,m_n)$ fulfilling $v=1_{[\mu,m(N)]}\circ m$, and $m(N)$ finite.

A coalition S is winning if and only if $m(S)\geq\mu$.

(μ,m) is called **representation** of (N,v).

Write $\langle\mu,m\rangle$ for (N,v). The set of all w.m.g. is denoted by V_m resp. by V_{rm}.

(6.2) Homogeneous games

Definition: A (r.)w.m.g. (N,v) is called **homogeneous** if there is a representation (μ,m) of this game fulfilling $m(S)=\mu$ for all minimal winning coalitions.
The set of all homogeneous games is denoted by V_h resp. by V_{rh}.

(6.3) Characters are not revealed

Rational (homogeneous) games (their characters and representations are introduced in Rosenmüller 1987.
(Integer) w.m.g. without dummies are sure games.
In many processes characters are not revealed by the life-length distribution. We give an example: Both of the games given by the representations (13;7,6,3,3,1,1) and (11;7,4,3,1,1,1) generate the f-vector (0,1,0,1,1,1) - they are process-equivalent; but in the first game time 1 is a sum while it is a step in the second.

(6.4) **Fellowship representation**

The pair (k,g) of an integer l-vector and a decreasing rational l-vector can be identified with the measure having k_i 'fellows' of weight g_i.

(6.5) **Cantor games**

Games generated by (k,g)=(1,2,3,...;1/2!,1/3!,1/4!,...) are called Cantor games. Every finite coalition has a rational weight. Coalitions can be identified with the unit interval doubleing up the rationals - m(N)=1. The game is homogeneous for μ=m(s), s=(1,...,1,0,...).

(6.6) Lemma: If (N,v) is a (r.)w.m.g., then W is closed and $W=g(W_*)$.

Proof: m is continuous.

Remark. See (3.9).

Conclusions: 1. P=p(W)

2. **If a game and its dual are r.w.m.g., then the game is finite.**

(6.7) Theorem: 1. Fair w.m.games are p-a.s. constant-sum games.

2. Constant-sum games are finite.

Proof: 1. The symmetric difference $W(v)\Delta W(v^*)$ is a subset of $\partial W(v)\cup\partial W(v^*)$, (4.1).

2. (3.13)+(6.6)

Remark. It is well known that a finite w.m.g. is either superadditive or dual superadditive or both (cf. Ostmann 1985).

7. Construction

(7.1) In Aumann/Peleg/Rabinowitz 1965 and Kopelowitz 1967 tables of w.m.g. (n<9) are found. Rosenmüller 1986 an recursive procedure is given to construct all homogeneous w.m.g.. The following table gives the number of w.m.g..

Number of...

players	1	2	3	4	5	6	7
games	1	3	8	25	117	927	
constant-sum games	1	1	2	3	7	21	135

(7.2) **Theorem**: Up to 5 players all ordered games are w.m.g.; n=6 is the first number at which both classes of games differ.
Proof: The first part is proven by explicit construction according to 'W and to the Pascal-type (see Appendix B.). The second part is proven by examples in (7.3).

(7.3) In Ostmann 1985 there are two crucial examples reported: Aumann's game <10011001,01100110> stands for an ordered non-w.m.g. and Ostmann's game is a 13-person ordered non-w.m.g. with constant sum.
Here we list some 6-person ordered non-w.m.g.:

1. The following game is a fair one:
<100101,011001>, f=001344.
There are four types. (101000) and (010111) are losing, (100101) and (011010) are winning coalitions. Assume that the game is w.m. then the two losing coalitions together have the same weight than the two winning ones - namely the total weight - this is impossible.

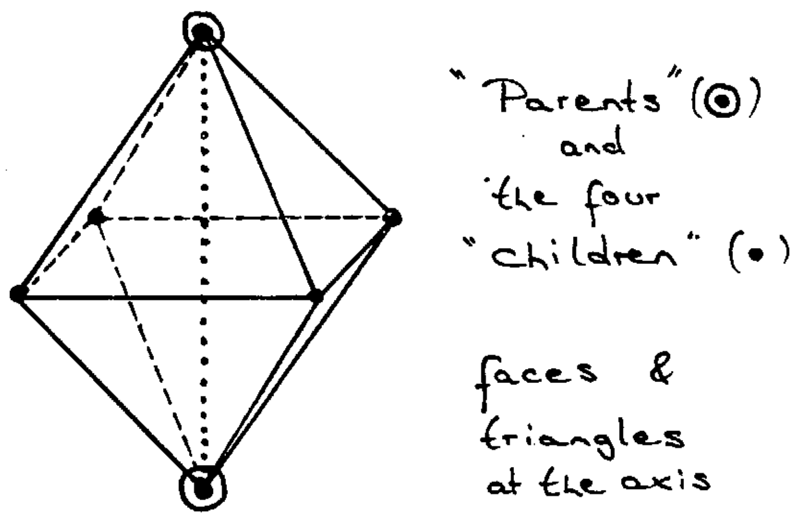
Arguments of the same type can be given for the following examples:

2. <100011,010101>, f=001356
3. <010011>, f=001357

This example is called "the parents and their four children" - for it can be seen as corresponding decision rule.

4. <011001,010110>, f=001334
5. The games dual to 1.-4.

(7.2f)



Appendix A. Four-person games

The following list uses the reference letters of Straffin, pp.310ff

f-vector	game	W*	16 p(W)	Pascal-type
1111	f*	0001	15	A
1110	c*	0010	14	A
1101	g*	0100,0011	13	AB
1100	b*	0100	12	A
1012	h*	1000,0011	12	AB
1011	j*	1000,0101	11	AB
1010	d*	1000,0110	10	AB
1001	m*	1000,0111	9	AC
1000	a	1000	8	A
0123	i*	0011	11	B
0122	k*	0101	10	B
0121	l*	1001,0110	9	B
0120	e	0110	8	B

f-vector	game	W*	16 p(W)	Pascal-type
0112	n	1001,0111	8	BC
0111	l	1010,0111	7	BC
	m	1001	7	B
0110	d	1010	6	B
0102	k	1100,0111	6	BC
0101	j	1100,1011	5	BC
0100	b	1100	4	B
0013	i	0111	5	C
0012	h	1011	4	C
0011	g	1101	3	C
0010	c	1110	2	C
0001	f	1111	1	D

Remarks: The games o, o* and the fair game p (example (1.2) 3.) (cf. Straffin's list) are not ordered.

Appendix B. 5-person games

According to their Pascal-type for n=4
we got the following number of games (App.A.):
A:4
B:7 AB:4
C:4 BC:4 AC:1
D:1
For n=5 we can construct the following:
A: 5
B:15 AB:11
C:15 BC:35 AC:5 ABC:4
D: 5 CD:11 BD:5 BCD:4 AD:1
E: 1
The following list uses the reference numbers of Aumann/Peleg/ Rabinowitz.

Pascal-type	no.	W*	f	32P	dual	ref.
A	1	1	1	16	1	1-1
A	2	01	11	24	6	2-2
A	3	001	111	28	32	3-4
A	4	0001	1111	30	91	4-13
A	5	00001	11111	31	117	5-76
B	6	11	01	8	2	2-1
B	7	101	011	12	21	3-2
B	8	1001	0111	14	82	4-3
B	9	10001	01111	15	116	5-4
B	10	011	012	16	10	3-3
B	11	0101	0122	20	50	4-11
B	12	01001	01222	22	108	5-52
B	13	0011	0123	22	38	4-10
B	14	00101	01233	25	100	5-55

Pascal-type	no.	W*	f	32P	dual	ref.
B	15	00011	01234	26	95	5-49
B	16	01001,0011	01232	24	101	5-68
B	17	10001,0011	01231	23	103	5-65
B	18	10001,0101	01221	21	112	5-72
B	19	10001,011	01211	19	109	5-60
B	20	1001,011	0121	18	58	4-12
AB	21	1,011	101	20	7	3-5
AB	22	1,0101	1011	22	47	4-17
AB	23	1,01001	10111	23	107	5-84
AB	24	1,0011	1012	24	35	4-14
AB	25	1,00101	10122	26	97	5-86
AB	26	1,00011	10123	27	94	5-77
AB	27	1,01001,0011	10121	25	98	5-91
AB	28	01,0011	1101	26	33	4-16
AB	29	01,00101	11011	27	96	5-89
AB	30	01,00011	11012	28	93	5-80
AB	31	001,00011	11101	29	92	5-85
C	32	111	001	4	3	3-1
C	33	1101	0011	6	28	4-4
C	34	11001	00111	7	86	5-6
C	35	1011	0012	8	24	4-2
C	36	10101	00122	10	88	5-12
C	37	10011	00123	11	85	5-3
C	38	0111	0013	10	13	4-6
C	39	01101	00133	13	74	5-27
C	40	01011	00135	15	53	5-25
C	41	00111	00136	16	41	5-18
C	42	11001,1011	00121	9	89	5-17
C	43	11001,0111	00131	11	75	5-41
C	44	10101,0111	00132	12	78	5-46

Pascal-type	no.	W*	f	32P	dual	ref.
C	45	10011,0111	00133	13	67	5-38
C	46	10011,01101	00134	14	61	5-34
BC	47	11,1011	0101	10	22	4-5
BC	48	11,10101	01011	11	87	5-15
BC	49	11,10011	01012	12	85	5-8
BC	50	11,0111	0102	12	11	4-8
BC	51	11,01101	01022	14	73	5-35
BC	52	11,01011	01024	16	52	5-22
BC	53	11,00111	01025	17	40	5-53
BC	54	11,10101,0111	01021	13	77	5-48
BC	55	11,10011,0111	01022	14	66	5-43
BC	56	11,10011,01101	01023	15	60	5-37
BC	57	101,10011	01101	13	83	5-13
BC	58	101,0111	0111	14	20	4-9
BC	59	101,01101	01111	15	72	5-32
BC	60	101,01011	01113	17	56	5-64
BC	61	101,00111	01114	18	46	5-61
BC	62	101,10011,0111	01111	15	65	5-47
BC	63	101,10011,0110	01112	16	63	5-29
BC	64	1001,0111	0112	16	64	4-7
BC	65	1001,01101	01121	17	62	5-74
BC	66	1001,01011	01122	18	55	5-71
BC	67	1001,00111	01123	19	45	5-66
BC	68	10001,0111	01121	17	110	5-67
BC	69	10001,01101	01122	18	115	5-58
BC	70	10001,01011	01123	19	114	5-54
BC	71	10001,00111	01124	20	105	5-50

Pascal-type	no.	W*	f	32P	dual	ref.
BC	72	011,10011	01201	17	59	5-59
BC	73	011,01011	01202	18	51	5-62
BC	74	011,00111	01203	19	39	5-56
BC	75	0101,00111	01221	21	43	5-69
BC	76	01001,00111	01223	23	102	5-51
BC	77	1001,011,01011	01211	19	54	5-75
BC	78	1001,011,00111	01212	20	44	5-73
BC	79	10001,011,01011	01212	20	113	5-63
BC	80	10001,011,00111	01213	21	104	5-57
BC	81	10001,0101,00111	01222	22	106	5-70
AC	82	1,0111	1001	18	8	4-15
AC	83	1,01101	10011	19	57	5-88
AC	84	1,01011	10012	20	49	5-83
AC	85	1,00111	10013	21	37	5-78
AC	86	01,00111	11001	28	34	5-81
ABC	87	1,011,01011	10101	21	48	5-90
ABC	88	1,011,00111	10102	22	36	5-87
ABC	89	1,0101,00111	10111	23	42	5-92
ABC	90	1,01001,00111	10112	24	99	5-82
D	91	1111	0001	2	4	4-1
D	92	11101	00011	3	31	5-10
D	93	11011	00012	4	30	5-5
D	94	10111	00013	5	26	5-2
D	95	01111	00014	6	15	5-19

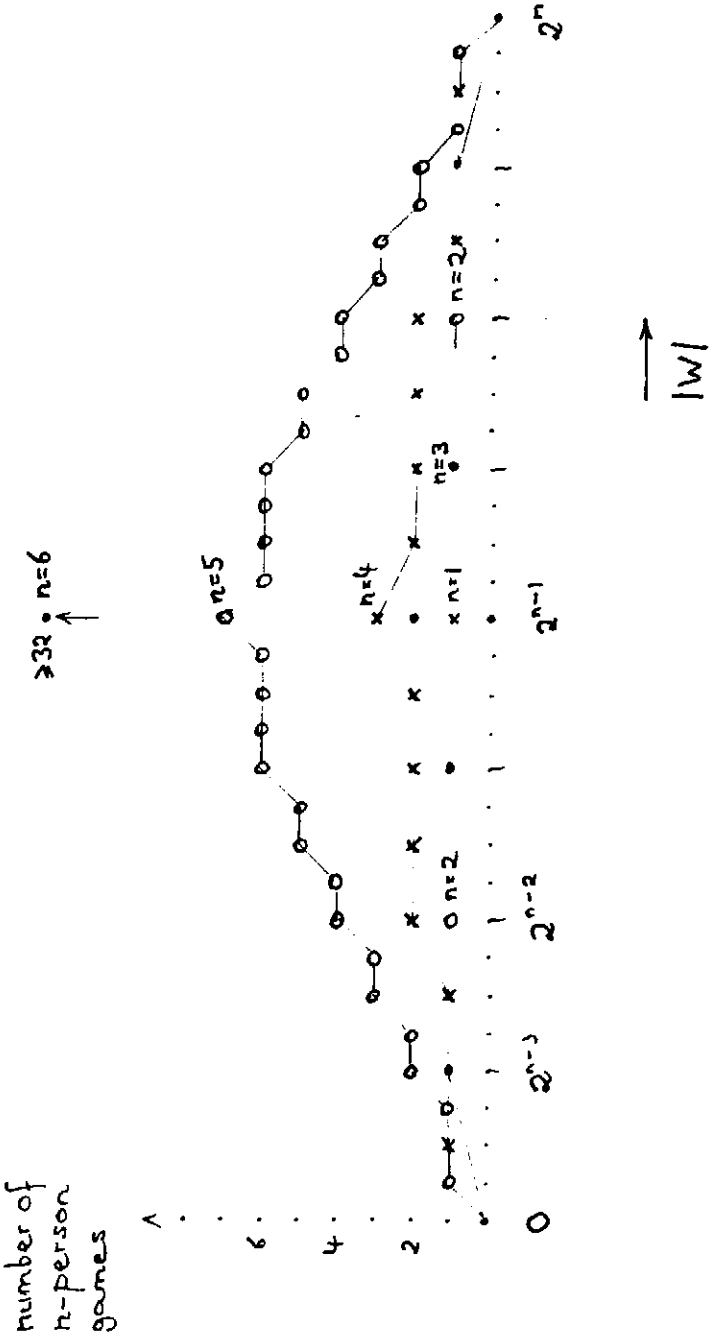
Pascal-type	no.	W*	f	32P	dual	ref.
CD	96	111,11011	00101	5	29	5-14
CD	97	111,10111	00102	6	25	5-11
CD	98	1101,10111	00111	7	27	5-16
CD	99	11001,10111	00112	8	90	5-7
CD	100	111,01111	00101	7	14	5-28
CD	101	1101,01111	00112	8	16	5-42
CD	102	11001,01111	00113	9	76	5-23
CD	103	1011,01111	00121	9	17	5-39
CD	104	10101,01111	00123	11	80	5-30
CD	105	10011,01111	00121	12	71	5-20
CD	106	11001,1011,01111	00122	10	81	5-44
BD	107	11,10111	01001	9	23	5-9
BD	108	11,01111	01002	10	12	5-24
BD	109	101,01111	01101	13	19	5-33
BD	110	1001,01111	01111	15	68	5-40
BD	111	10001,01111	01112	16	111	5-21
BCD	112	11,1011,01111	01011	11	18	5-45
BCD	113	11,10101,01111	01012	12	79	5-36
BCD	114	11,10011,01111	01013	13	70	5-26
BCD	115	101,10011,01111	01102	14	69	5-31
AD	116	1,01111	10001	17	9	5-79
E	117	11111	00001	1	5	5-1

The list according to ref.

Pascal	no.		f	|W|	dual	ref.
A	1	1	1	16	1	1-1
B	6	11	01	8	2	2-1
A	2	01	11	24	6	2-2
C	32	111	001	4	3	3-1
B	7	101	011	12	21	3-2
B	10	011	012	16	10	3-3
A	3	001	111	28	32	3-4
AB	21	1,011	101	20	7	3-5
D	91	1111	0001	2	4	4-1
C	35	1011	0012	8	24	4-2
B	8	1001	0111	14	82	4-3
C	33	1101	0011	6	28	4-4
BC	47	11,1011	0101	10	22	4-5
C	38	0111	0013	10	13	4-6
BC	64	1001,0111	0112	16	64	4-7
BC	50	11,0111	0102	12	11	4-8
BC	58	101,0111	0111	14	20	4-9
B	13	0011	0123	22	38	4-10
B	11	0101	0122	20	50	4-11
B	20	1001,011	0121	18	58	4-12
A	4	0001	1111	30	91	4-13
AB	24	1,0011	1012	24	35	4-14
AC	82	1,0111	1001	18	8	4-15
AB	28	01,0011	1101	26	33	4-16
AB	22	1,0101	1011	22	47	4-17
E	117	11111	00001	1	5	5-1
D	94	10111	00013	5	26	5-2
C	37	10011	00123	11	85	5-3
B	9	10001	01111	15	116	5-4
D	93	11011	00012	4	30	5-5
C	34	11001	00111	7	86	5-6
CD	99	11001,10111	00112	8	90	5-7
BC	49	11,10011	01012	12	85	5-8
BD	107	11,10111	01001	9	23	5-9
D	92	11101	00011	3	31	5-10
CD	97	111,10111	00102	6	25	5-11
C	36	10101	00122	10	88	5-12
BC	57	101,10011	01101	13	83	5-13
CD	96	111,11011	00101	5	29	5-14
BC	48	11,10101	01011	11	87	5-15
CD	98	1101,10111	00111	7	27	5-16
C	42	11001,1011	00121	9	89	5-17

Pascal	no.		f	$\lvert W \rvert$	dual	ref.
C	41	00111	00136	16	41	5-18
D	95	01111	00014	6	15	5-19
CD	105	10011,01111	00121	12	71	5-20
BD	111	10001,01111	01112	16	111	5-21
BC	52	11,01011	01024	16	52	5-22
CD	102	11001,01111	00113	9	76	5-23
BD	108	11,01111	01002	10	12	5-24
C	40	01011	00135	15	53	5-25
BCD	114	11,10011,01111	01013	13	70	5-26
C	39	01101	00133	13	74	5-27
CD	100	111,01111	00101	7	14	5-28
BC	63	101,10011,0110	01112	16	63	5-29
CD	104	10101,01111	00123	11	80	5-30
BCD	115	101,10011,01111	01102	14	69	5-31
BC	59	101,01101	01111	15	72	5-32
BD	109	101,01111	01101	13	19	5-33
C	46	10011,01101	00134	14	61	5-34
BC	51	11,01101	01022	14	73	5-35
BCD	113	11,10101,01111	01012	12	79	5-36
BC	56	11,10011,01101	01023	15	60	5-37
C	45	10011,0111	00133	13	67	5-38
CD	103	1011,01111	00121	9	17	5-39
BD	110	1001,01111	01111	15	68	5-40
C	43	11001,0111	00131	11	75	5-41
CD	101	1101,01111	00112	8	16	5-42
BC	55	11,10011,0111	01022	14	66	5-43
CD	106	11001,1011,01111	00122	10	81	5-44
BCD	112	11,1011,01111	01011	11	18	5-45
C	44	10101,0111	00132	12	78	5-46
BC	62	101,10011,0111	01111	15	65	5-47
BC	54	11,10101,0111	01021	13	77	5-48
B	15	00011	01234	26	95	5-49
BC	71	10001,00111	01124	20	105	5-50
BC	76	01001,00111	01223	23	102	5-51
B	12	01001	01222	22	108	5-52
BC	53	11,00111	01025	17	40	5-53
BC	70	10001,01011	01123	19	114	5-54
B	14	00101	01233	25	100	5-55
BC	74	011,00111	01203	19	39	5-56
BC	80	10001,011,00111	01213	21	104	5-57
BC	69	10001,01101	01122	18	115	5-58
BC	72	011,10011	01201	17	59	5-59
B	19	10001,011	01211	19	109	5-60
BC	61	101,00111	01114	18	46	5-61

Pascal	no.		f	|W|	dual	ref.
BC	73	011,01011	01202	18	51	5-62
BC	79	10001,011,01011	01212	20	113	5-63
BC	60	101,01011	01113	17	56	5-64
B	17	10001,0011	01231	23	103	5-65
BC	67	1001,00111	01123	19	45	5-66
BC	68	10001,0111	01121	17	110	5-67
B	16	01001,0011	01232	24	101	5-68
BC	75	0101,00111	01221	21	43	5-69
BC	81	10001,0101,00111	01222	22	106	5-70
BC	66	1001,01011	01122	18	55	5-71
B	18	10001,0101	01221	21	112	5-72
BC	78	1001,011,00111	01212	20	44	5-73
BC	65	1001,01101	01121	17	62	5-74
BC	77	1001,011,01011	01211	19	54	5-75
A	5	00001	11111	31	117	5-76
AB	26	1,00011	10123	27	94	5-77
AC	85	1,00111	10013	21	37	5-78
AD	116	1,01111	10001	17	9	5-79
AB	30	01,00011	11012	28	93	5-80
AC	86	01,00111	11001	28	34	5-81
ABC	90	1,01001,00111	10112	24	99	5-82
AC	84	1,01011	10012	20	49	5-83
AB	23	1,01001	10111	23	107	5-84
AB	31	001,00011	11101	29	92	5-85
AB	25	1,00101	10122	26	97	5-86
ABC	88	1,011,00111	10102	22	36	5-87
AC	83	1,01101	10011	19	57	5-88
AB	29	01,00101	11011	27	96	5-89
ABC	87	1,011,01011	10101	21	48	5-90
AB	27	1,01001,0011	10121	25	98	5-91
ABC	89	1,0101,00111	10111	23	42	5-92



number of
n-person
games
0
2
4
6
2^{n-3}
2^{n-2}
2^{n-1}
2^n
|W|
n=1
n=2
n=3
n=4
n=5
≥32 n=6

Appendix C. Fair ordered games (n≤6)

The following list gives fair n-person-games without dummies according to their f-vector. Up to n=5 all games are constant-sum w.m.games. If a game is w.m. (and therefore constant-sum) we give its minimal representation, else W* is listed.

n<6

f-vector	no.	(μ,m)
1	1.	1;1
012	2.	2;1,1,1
0112	3.	3;2,1,1,1
01112	4.1	4;3,1,1,1,1
	4.2	5;3,2,2,1,1
01024	5.	4;2,2,1,1,1
00136	6.	3;1,1,1,1,1

The following list uses the reference symbols (ref.) of Kopelowitz.

n=6

f-vector	no.	(μ,m)	W* / ref.	
011112	7.1	5;4,1,1,1,1,1	6-WZ-2	
	7.2	7;4,3,3,1,1,1	6-WZ-6	
	7.3	7;5,2,2,2,1,1	6-WZ-8	
	7.4	8;5,3,3,2,1,1	6-WZ-11	
	7.5	not c.s.	101,10011,0111,010111	dual to 10.3
	7.6	not c.s.	101,01101,010111	dual to 10.2

f-vector	no.	(μ,m)	W*
011104	8.	not c.s.	101,0111,001111 dual to 15.3
011024	9.1	6;4,2,2,1,1,1	6-WZ-5
	9.2	not c.s.	101,10011,001111 dual to 15.2
010232	10.1	9;5,4,3,2,2,1	6-WZ-14
	10.2	not c.s.	11,10011,011001 dual to 7.6
	10.3	not c.s.	11,100101,01101 dual to 7.5
	10.4	not c.s.	11,10011,01101,001111 dual to 14.2
010224	11.1	6;3,3,2,1,1,1	6-WZ-7
	11.2	7;4,3,2,2,1,1	6-WZ-10
010136	12.	5;3,2,1,1,1,1	6-WZ-3
001436	13.	6;3,2,2,2,1,1	6-WZ-9
001352	14.1	7;3,3,2,2,2,1	6-WZ-12
	14.2	not c.s.	101001,01011 dual to 10.4
001344	15.1	8;4,3,3,2,2,1	6-WZ-13
	15.2	not c.s.	100011,01101 dual to 9.2
	15.3	not c.s.	100101,011001 dual to 8.
001336	16.	5;2,2,2,1,1,1	6-WZ-4
001248	17.	4;2,1,1,1,1,1	6-WZ-1

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