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**The Nucleolus of a Game without
Side Payments**

by

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I. INTRODUCTION

The nucleolus is a very important notion in classical cooperative game theory. I know from private conversations that some scientists are thinking about an extension of this notion to a game without side payments. The following is a way for performing this extension. The notion of the nucleolus is based on the excess of a coalition. It is easy to test that the excess $v(S) - x(S)$ of the coalition S with respect to the vector x is proportional to the distance from x to the hyperplane $x(S) = v(S)$. The coefficient of proportionality is equal to $\sqrt{|S|}$ if $x(S) \leq v(S)$ and to $-\sqrt{|S|}$ if $x(S) > v(S)$. This circumstance allows to extend the definition of the nucleolus for games without side payments. It is proved that the nucleolus exists for each game and is unique for games with concave boundaries of sets $V(S)$ (complement of $V(S)$ is convex). The last condition is not so unfamiliar as it may seem because in this case the core is always convex.

2. Preliminaries

Let $\Gamma = \langle I, H, V \rangle$ be a game without side payments where $I = \{1, \dots, n\}$, H is a compact subset of R^n and $V : 2^n \rightarrow 2^{R^n}$ is a set-valued function with the properties:

- 1) $V(\emptyset) = \emptyset$,
- 2) $V(S)$ is closed,
- 3) $V(S)$ is comprehensive: if $x \in V(S)$ and $y_i \leq x_i, i \in S$,

then $y \in V(S)$.

Denote $b(V(S))$ the boundary of $V(S)$ and $\text{int}(V(S))$ the interior of $V(S)$. The set of such games is denoted G .

$$\text{Put } v_i = \max_{x \in V(\{i\})} x_i$$

Consider the subset of points undominated by coalition S :
 $H(S) = \{x \in R^n : x_i \geq v_i, i \in I\} - \text{int}(V(S))$.

From conditions 2 and 3 $H(S)$ is closed.

$$\text{The core of } \Gamma \text{ is } C(\Gamma) = H \cap \bigcap_{S \subset I} H(S)$$

Define the nucleolus $\mathcal{N}_E(Y)$ for $Y \subset R^n$ as usually but with respect to an arbitrary excess function $E(x, S), x \in R^n, S \subset I$.

Let $\theta_E(x) \in R^{2^n}$ is a vector of ordered excesses: $\theta_E^1(x) = E(x, S_1) \geq \theta_E^2(x) = E(x, S_2) \geq \dots \geq \theta_E^{2^n}(x) = E(x, S_{2^n})$. Put $x \succ y$ iff

$\theta_E(x) \prec \theta_E(y)$ i.e. $\theta_E(x)$ is lexicographical smaller than $\theta_E(y)$: there exists $k \leq 2^n$ such that $\theta_E^i(x) = \theta_E^i(y), i < k, \theta_E^k(x) < \theta_E^k(y)$.

The nucleolus $\mathcal{N}_E(Y)$ is the set of maximal elements in Y (the core in Y for relation \succ).

$$\mathcal{N}_E(Y) = \{x \in Y : \theta_E(x) \prec \theta_E(y), \forall y \in Y\}.$$

The initial notion of the nucleolus for classical cooperative games (Schmeidler (1969)) based on the excess-function $E(x, S) = v(S) - x(S)$. Later the extended nucleolus was defined, for example with $E(x, S) = f(S)(v(S) - x(S))$ (see Menshikova (1983)).

Consider the game $\Gamma = \langle I, v \rangle$ with side payments as a game without side payments $\Gamma = \langle I, V, H \rangle$ where

$$V(S) = \{x \in R^n : x(S) \leq v(S)\},$$

$$H = A = \{x \in R^n : x(I) = v(I), x_i \geq v(\{i\}), i \in I\}.$$

Denote G_0 the class of all such games.

Let the distance $\rho(x, y)$ is defined in R^n , for instance, the Euclidean distance $\rho_1(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.

Put $\delta_p(x, X) = \min_{y \in X} \rho(x, y)$ the distance from a point x to a set X and $\delta_p(A, B) = \max(\min_{x \in A} \delta_p(x, B), \max_{x \in B} \delta_p(x, A))$ the Hausdorff distance

between the sets A and B .

Proposition I. For game $\Gamma \in G_0$ the classical $E(x, S) = v(S) - x(S) = \delta_p(x, b(V(S)))f_x(S)$ where $f_x(S) = \sqrt{|S|}$ if $x \in V(S)$ and $f_x(S) = -\sqrt{|S|}$ otherwise.

Proof. It follows from that $(v(S) - x(S)) \frac{1}{\sqrt{|S|}}$ is the distance from x to hiperplane $x(S) = v(S)$ if $x(S) \leq v(S)$ and the distance multiplied by -1 if $x(S) > v(S)$.

Proposition I allows us to extend the notion of the nucleolus to an arbitrary game $\Gamma \in G$.

For any distance ρ define the excess-function E :

$$E_{\rho f}(x, S) = \delta_p(x, b(V(S)))f(S) \quad \text{if } x \in V(S),$$

$$E_{\rho f}(x, S) = -\delta_p(x, b(V(S)))f(S) \quad \text{if } x \notin V(S),$$

where $f(S)$ is arbitrary positive function independent from x .

The nucleolus of Γ with respect to the excess-function $E_{\rho f}$ is denoted by $\mathcal{N}_{\rho f}(\Gamma)$.

3 An existence and uniqueness of the nucleolus

Theorem I. Every game $\Gamma \in G$ has a nonempty nucleolus $\mathcal{N}_{\rho f}(\Gamma)$ for any distance ρ and any bounded positive function f .

Proof. The proof of Schmeidler (1969) is easily extended on this case. Let give the another proof based on the result of Kulakovskaja (1976) about the von Neumann-Morgenstern solution for partial order on a compact set. Prove first that the relation \succ preserve in the limit i.e. if $x_n \rightarrow x_0$ and $x_{n+1} \succ x_n$ then $x_0 \succ x_n$ for some n . We'll prove for this that $\theta_{\rho f}^i(x)$ is continuous. Let j and $k, j \leq i \leq k$, be the number such that $\theta_{\rho f}^{j-1}(x) > \theta_{\rho f}^j(x) = \dots = \theta_{\rho f}^i(x) = \dots = \theta_{\rho f}^k(x) > \theta_{\rho f}^{k+1}(x)$. Put $\varepsilon = \min(\theta_{\rho f}^{j-1}(x) - \theta_{\rho f}^j(x), \theta_{\rho f}^k(x) - \theta_{\rho f}^{k+1}(x))$

and $\delta = \min_{j \leq l \leq k} \delta_l$ where δ_l is defined from conditions: if $|x - y| < \delta_l$ then $|E_{p_f}(x, S_l) - E_{p_f}(y, S_l)| < \frac{\epsilon}{2}$.

Denote $S_l(z): \theta_{p_f}^l(z) = E(z, S_l(z))$ then $|\theta_{p_f}^l(x) - \theta_{p_f}^l(y)| = |E(x, S_l(x)) - E(y, S_l(y))| \leq 2 \frac{\epsilon}{2} = \epsilon$ if $|x - y| < \delta$.

Consider now $\theta_{p_f}^1(x_n)$. From $x_{n+m} \succ \dots \succ x_n$ it follows that $\theta_{p_f}^1(x_{n+m}) \leq \theta_{p_f}^1(x_n)$. Since $\theta_{p_f}^1(x)$ is continuous then $\theta_{p_f}^1(x_0) \leq \theta_{p_f}^1(x_n)$. If $\theta_{p_f}^1(x_0) < \theta_{p_f}^1(x_n)$ then $x_0 \succ x_n$. If $\theta_{p_f}^1(x_0) = \theta_{p_f}^1(x_n)$ then each $\theta_{p_f}^1(x_{n+m}) = \theta_{p_f}^1(x_n)$, $m = 1, 2, \dots$. Consider $\theta_{p_f}^2(x_n)$, and so on. Because $x_{n+m} \succ x_n$ then there exists $k: \theta_{p_f}^k(x_0) < \theta_{p_f}^k(x_n)$ and $x_0 \succ x_n$. Therefore from theorem 2 Kulakovskaja (1976) it is follows that $\delta \mathcal{L}_{p_f}(\Gamma) \neq \emptyset$ q.e.d.

Schmeidler (1969) proved that the nucleolus of a classical game consists of a unique imputation $v(\Gamma)$. This result is extended in the following theorem.

Denote G^* the class of games $\Gamma \in G$ with all convex sets $H(S)$, $S \subset I$, and H . Note that $V(S)$, $S \subset I$, are not convex. This conditions are not unnatural because any game $\Gamma \in G^*$ has the convex core $C(\Gamma) = H \cap \bigcap_{S \subset I} H(S)$ if it is non-empty.

Any game $\Gamma \in G^*$ has other good properties. Define operation $\max(\Gamma_1, \Gamma_2) = \Gamma$ for games with the same I as $v(S) = v_1(S) \vee v_2(S)$. The class $G^*(I, H)$ of games with the same I and H is closed with respect to the operation "max". If $\Gamma_1, \Gamma_2 \in G_0$ then $\Gamma = \max(\Gamma_1, \Gamma_2)$ has a function $v(S) = \max(v_1(S), v_2(S))$. Pachterski; and Sobolev (1983) prove that the characteristic properties of the classical nucleolus is related with this operation. Probably this holds in a general case.

Lemma. If $C \subset R^n$ is convex and $\alpha(x, C) = \gamma(x) \delta_p(x, b(C))$ where $\gamma(x) = 1$ if $x \notin C$ and $\gamma(x) = -1$ if $x \in C$ then $\alpha(\lambda x + (1-\lambda)y, C) \leq \max(\alpha(x, C), \alpha(y, C))$, $0 < \lambda < 1$, $x \neq y$, and if $\alpha(x, C) \neq \alpha(y, C)$ then inequalities are strong.

Proof. Consider first the case $\delta(x) = \delta(y) = I$. Construct a cylinder of rotation K with an axis $[x, y]$ and a radius $r = \max(\alpha(x, C), \alpha(y, C))$. Let \bar{x} be the nearest to x point of $b(C)$, and \bar{y} be the same to y . For construction K both x and y belong to K . Because of convexity of K and C any $z(\lambda) = \lambda\bar{x} + (1-\lambda)\bar{y}$ belongs to $K \cap C$ for any $\lambda: 0 < \lambda < 1$ and belongs to $\text{int}(K \cap C)$ if $\alpha(x, C) \neq \alpha(y, C)$. Therefore each sphere with the radius r and the center $z(\lambda)$ has a nonempty intersection with the set C . It means $\delta_\rho(z(\lambda), b(C)) \leq r$ or $\delta_\rho(z(\lambda), b(C)) < r$ if $\alpha(x, C) \neq \alpha(y, C)$ what is needed.

Next let $\delta(x) = \delta(y) = -I$ for $x, y \in C$. Put $r = \min(\delta_\rho(x, b(C)), \delta_\rho(y, b(C)))$ and construct a cylinder K with the radius r as above. Consider the set $C' = R^n - C$. Note that the distance from any point belonged to $[x, y]$ to $b(C)$ is equal to the distance to \bar{C}' . Because of $\delta_\rho(x, b(C)) \geq r$ any point of the set $K_x = \{t: \rho(x, t) \leq r\}$ belongs to C . The same is true for K_y . The set C is convex then $K \subset C$. Hence there are no point of K belonging to C' , i.e. $\delta_\rho(z, b(C)) \geq r$ or $\delta_\rho(z, b(C)) > r$ if $\alpha(x, C) \neq \alpha(y, C)$. Because of $\delta(x) = -I$ we have the needed.

If $\delta(x) = I, \delta(y) = -I$ then consider a point y' of an intersection of $[x, y]$ and $b(C)$. As ^{it} have been proved the conclusion of lemma holds for $[x, y']$ and $[y', y]$. Note that $\alpha(y', b(C)) = 0$. So lemma holds for all cases.

Theorem 2. The nucleolus $\delta_{\rho_f}(\Gamma)$ of any game $\Gamma \in G^*$ consists of a unique point for any distance ρ and any bounded positive function f .

Proof. $\delta_{\rho_f}(\Gamma) \neq \emptyset$ from theorem I. Let $x, y \in \delta_{\rho_f}(\Gamma)$ and $x \neq y$. Consider $z = \lambda x + (1-\lambda)y$ with the fixed $\lambda: 0 < \lambda < 1$. We'll prove that $z \succ x$ or $z \succ y$. Note that $\theta_{\rho_f}^i(x) = \theta_{\rho_f}^i(y), i = 1, 2, \dots$

and put $\theta_{pf}^1(x) = \theta_{pf}^1(y) = a$. Consider $\theta_{pf}^1(z)$ and let $\theta_{pf}^1(z) = E_{pf}(z, S_1)$. Because of $E_{pf}(x, S_1) \leq a$ and $E_{pf}(y, S_1) \leq a$ then $E_{pf}(z, S_1) \leq a$ from lemma. If $E_{pf}(z, S_1) < a$ then $z \succ x$ and $z \succ y$. It remains $\theta_{pf}^1(z) = a$. If $\theta_{pf}^1(z) = \dots = \theta_{pf}^k(z) = a$ then from lemma all corresponding $E_{pf}(x, S_j) = E_{pf}(y, S_j) = a$ therefore also $\theta_{pf}^1(x) = \dots = \theta_{pf}^k(x) = a$ and $\theta_{pf}^1(y) = \dots = \theta_{pf}^k(y) = a$. Consider the reduced vectors $\bar{\theta}_{pf}(x) = (\theta_{pf}^{k+1}(x), \dots, \theta_{pf}^{2^n}(x))$ and similar $\bar{\theta}_{pf}(y)$ and $\bar{\theta}_{pf}(z)$. Repeat the above consideration. And so on. Because $x \neq y$ then S_0 exists that $E_{pf}(x, S_0) \neq E_{pf}(y, S_0)$ so we'll obtain $z \succ x$ and $z \succ y$ what is impossible. Therefore theorem is proved.

Corollary. The nucleolus of a side-payment game is non-empty and consists of a unique point for any ρ and any f specifically for $E(x, S) = v(S) - x(S)$ (see Schmeidler (1969)) and

$$E(x, S) = \frac{v(S) - x(S)}{d(S)} \quad (\text{see Menshikova (1983)}).$$

Proof. It follows from proposition I, theorems I and 2.

4 Properties of the extended nucleolus

In Pecherskij, Sobolev (1983) the characteristic properties of the classical nucleolus is investigated. The extended nucleolus satisfies the similar properties.

Proposition 2. $\delta \mathcal{N}_{pf}(\Gamma) \subset C(\Gamma)$ for any ρ and positive f .

Proof. If $f(S) > 0$ then from definition all $E_{pf}(x, S) \leq 0$ for $x \in C(\Gamma)$. If $y \notin C(\Gamma)$ then such S_0 exists that $E_{pf}(y, S_0) > 0$. Hence $x \succ y$, $x \in C(\Gamma)$ and $y \notin \delta \mathcal{N}_{pf}(\Gamma)$. Therefore $\delta \mathcal{N}_{pf}(\Gamma) \subset C(\Gamma)$ q.e.d.

Proposition 3. $\mathcal{N}_{pf}(\Gamma) \subset P(H)$ where $P(H)$ is the Pareto boundry of H .

Proof. We'll prove that if $x_i \geq y_i, i \in I$, and $x \neq y$ then $x \succ y$. Because $V(S)$ is closed there exists for each $S \subset I$ such $x^S \in V(S)$ that $\rho(x, x^S) = \delta_\rho(x, b(V(S)))$. Put $y^S = x^S - (x - y) \leq x^S$.

From the property 3 of $V(S)$ we have $y^S \in V(S)$. From definition y^S we have $\rho(y, y^S) = \rho(x, x^S)$. Therefore $\delta_\rho(y, b(V(S))) \leq \delta_\rho(x, b(V(S)))$ if $y \notin V(S)$ and $\delta_\rho(y, b(V(S))) \geq \delta_\rho(x, b(V(S)))$, $y \in V$. In any case $E_{\rho_f}(y, S) \geq E_{\rho_f}(x, S)$, $S \in I$, and $x \succ y$ because of $x \neq y$ and proof is over.

Proposition 4. If V' is received from V with the mapping of equivalence $x'_i = \alpha x_i + a_i, i=1, \dots, n$, then if $v \in \mathcal{N}_{\rho_f}(\Gamma)$ then $v' \in \mathcal{N}_{\rho_f}(\Gamma')$ where $v'_i = \alpha v_i + a_i, i=1, \dots, n$.

Proof follows from properties of a distance ρ .

References

- Shmeidler D (1969) The nucleolus of a characteristic function game. SIAM J Appl Math 17:1163-1169
- Menshikova O, Menshikov I (1983) The generalized nucleolus as a solution of a cost allocation problem. Collaborative paper, International Institute for Applied Systems Analysis A-2361 Laxenburg, Austria.
- Kulakovskaia T.E (1976) Classical principles of the optimality in infinite cooperative games. In: Recent Direction in Games Theory, Mosklas, Vilnius pp 94-108
- Pecherskiĭ S.L, Sobolev A.I (1983) A problem of an optimal distribution in social-economic problems and cooperative games. Nauka, Leningrad.