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Incentives in Market Games with Asymmetric Information: Approximate (NTU) Cores in Large Economies

by

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0. Abstract

Although incentive compatibility constraints can give rise to nonconvexities so that the cores of such exchange economies may be empty, large replica economies with incentives always have nonempty approximate cores which contain allocations satisfying the equal treatment property.

1. Introduction

Incentive compatibility considerations are certainly important for economies with asymmetric information whenever the information available to coalition members is exogenously determined. One wishes to permit strategic behavior with respect to information revelation, yet the potential withholding of information or the revelation of false information are serious concerns.

The idea followed in this paper is to pursue the interface between cooperative and noncooperative game theory--or, in other terms, the concepts of games with partial commitments or institutions with incomplete contracts--by focusing on informational motivation. Players may behave strategically within coalitions and the information of coalitions need not be exogenously given. This equilibrium determination of the extent of information sharing as the (noncooperative) strategic choices of economic agents then leads to the definition of the underlying cooperative game with asymmetric information in terms of the payoffs that can be achieved via incentive-compatible net trades. In this way, the underlying cooperative game is derived from an economy with asymmetric information and this derivation takes account of incentive considerations.

Yet the presence of incentive compatibility constraints can lead to nonconvexities and to games that are not balanced. For the (exact) NTU core of finite economies, these phenomena can give rise to games with empty cores. See Allen (1992) for an explicit--and complicated--example. Randomization can restore the existence of core allocations, although at the expense of resource feasibility on average rather than almost surely. The insufficiency of convexification alone (with almost sure feasibility) to restore the nonemptiness of the core suggests (but, of course, does not prove

conclusively) that the effects of large numbers will not suffice to ensure nonemptiness of the core in exact replicas.

However, rather than pursuing the "exact core of a large economy with mean resource feasibility," the strategy taken here is to explore the effects of weakening the solution concept to the approximate core. Thus, I can exploit the machinery developed in Shubik and Wooders (1983) and Wooders (1983, 1991) to analyze the ε-core of large replica economies with nontransferable utility.[2] Randomization is still used to guarantee convexity of coalitions' incentive compatible feasible payoff sets. However, this randomization features resource feasibility with probability one.

The remainder of this paper is organized as follows: Section 2 discusses examples of nonconvex feasible payoff sets for coalitions under incentive compatibility constraints and of resulting empty (NTU) cores. Section 3 presents the model while Section 4 is devoted to a discussion of incentives. Preliminaries are gathered in Section 5. Section 6 contains the initial analysis of approximate cores. Section 7 discusses randomizations. Main results appear in Section 8; their discussion follows in Section 9.

2. An Example

The purpose of this section is to display a simple example of an exchange economy with incentive compatibility constraints which induces a game having nonconvex sets of attainable utility vectors for the grand coalition. The NTU game also fails to be balanced, although it possesses core allocations. In the example, replication restores approximate convexity but not balancedness. Thus, the example helps to illustrate and explain my model but, strictly speaking, rigorously only demonstrates the need to convexify the sets of attainable payoffs in the games I consider.

The example features two goods (called x and y), three players (indicated by subscripts 1, 2, and 3), and two states of the world, heads (H) and tails (T), which each occur with probability one half. Initial endowments are assigned as follows: $e_1(H) = e_1(T) = (1,1)$, $e_2(H) = (0,4)$, $e_2(T) = (0,0)$, $e_3(H) = (4,0)$, $e_3(T) = (0,0)$. Players 2 and 3 know the state of the world (perhaps by observing their own initial endowment vectors) but player 1 knows the state only if another coalition member tells him and player 1 is unable to detect lies. State-dependent utilities are given by the following expressions:

$$u_1(x,y;H) = \min\left\{\frac{x + y - 1 + \sqrt{(x + y - 1)^2 + 8x}}{4}, \frac{x + y - 1 + \sqrt{(x + y - 1)2 + 8y}}{4}\right\}$$

$$u_1(x,y;T) = \begin{cases} (x + y)/2 & \text{if } x + y \le 2 \\ 1 + \min(x, y, (x + y) - 2) & \text{if } x + y \ge 2 \end{cases}$$

$$u_2(x,y;H) = x$$

$$u_3(x,y;H) = y$$

$$u_2(x,y;T) = u_3(x,y;T) = 0.$$

Although initial endowments are state dependent, they are incentive compatible. Note also that, to simplify calculations, I take total utility (rather than expected utility) as the payoff of each player.

Consider the state-dependent incentive compatible allocation ((5,0),(1,1)) to player 1, ((0,0),(0,0)) to player 2 and ((0,5),(0,0)) to player 3, which yields total utilities of $2 + 1 = 3$, $0 + 0 = 0$, and $5 + 0 = 5$ respectively. Similarly, the allocation (((0,5),(1,1)),((5,0),(0,0)),((0,0),(0,0))) gives the grand coalition the total utility vector (3, 5, 0). However, the convex combination $\frac{1}{2}(3, 0, 5) + \frac{1}{2}(3, 5, 0) = (3, 2\frac{1}{2}, 2\frac{1}{2})$ cannot be attained via a feasible and incentive compatible allocation, as the only allocations which give 3 to player 1 either equal one of the two allocations above or fail to satisfy incentive compatibility. Hence, in the NTU game generated by this economy, $V(I)$ is not a convex set.

However, replication restores approximate convexity. Indeed, in the 2-fold replica, one can give the clones of player 1 the allocations ((0,5),(1,1)) and ((5,0),(1,1)) respectively. This permits each copy of player 2 to receive $((2\frac{1}{2},0),(0,0))$ and each player 3 to obtain $((0,2\frac{1}{2}),(0,0))$, yielding total payoff vector of $(3, 3, 2\frac{1}{2}, 2\frac{1}{2}, 2\frac{1}{2}, 2\frac{1}{2})$, as desired.

To check that the derived NTU game is not balanced, examine the three two-player coalitions with balancing weights one half each. The allocations (((0,5),(1,1)),((1,0),(0,0))) and (((5,0),(1,1)),((0,1),(0,0))) show that $(3, 1) \in V(\{1,2\})$ and $(3, 1) \in V(\{1,3\})$. Moreover $(4, 4) \in V(\{2,3\})$ [using the allocation (((4,0),(0,0)),((0,4),(0,0)))]. However $(3, 2\frac{1}{2}, 2\frac{1}{2}) \notin V(I)$ as shown above. This game is not balanced, but its two-fold replica is balanced.

One can check that (3, 0, 5) and (3, 5, 0) belong to the core because no coalition can strictly improve the payoff of each of its members. A much more complex example (with five commodities, five states, and initial endowment vectors that are not state dependent) provided in Allen (1992) illustrates that incentive compatibility constraints can lead to emptiness of the (NTU) core.

3. The Model

Let Ω be a finite set of states of the world. Assume that all agents have the same (subjective) probability μ on (Ω, F) where $F = 2^\Omega$ with

$\mu(\omega) > 0$ for all $\omega \in \Omega$. [Otherwise Ω can be reduced by a null set.] Interpret Ω as a description of all of the payoff-relevant systematic risk or common uncertainty in the economy. Note that, if Ω is infinite but each agent's information is representable by a finite partition, one could redefine a finite set of states of the world by events in the pooled information partition.

Finitely many agents are present in the pure exchange economy. Let I be the set of traders or players in the induced game. Write $\#I$ for the cardinality of the player set and use subscripts i $(i \in I)$ to indicate individual agents. Each consumer has consumption set $\mathbb{R}^{\ell}_{+}$ in each state of the world, so that there are ℓ goods potentially available in each state.

Initial endowments are assumed to be independent of the state of the world in order to avoid the problem that endowments may convey information or may violate incentive compatibility (in which case the worth of coalitions may not be well defined). Write $e_i \in \mathbb{R}^{\ell}_{+}$ or $e_i : \Omega \to \mathbb{R}^{\ell}_{+}$ for i's initial endowment.

Preferences are state-dependent and are specified by cardinal utility functions, where expected utilities define payoffs. Write $u_i : \mathbb{R}^{\ell}_{+} \times \Omega \to \mathbb{R}$ and assume that for all $i \in I$ and all $\omega \in \Omega$, $u_i(\cdot;\omega)$ is continuous and concave on $\mathbb{R}^{\ell}_{+}$. For the version of the model with Bayesian incentive compatibility (see Section 4), assume also that $u_i(0;\omega) \le u_i(x_i;\omega)$ for all $i \in I$, all $\omega \in \Omega$, and all $x_i \in \mathbb{R}^{\ell}_{+}$. For the results in Section 6, assume that $\sum_{\omega \in \Omega} u_i(e_i(\omega);\omega)\mu(\omega) > 0$ for all $i \in I$.

4. Information and Incentives

To begin, specify traders' information by $s_i : \Omega \to S_i$ where S_i is a finite set for all $i \in I$. Then s_i generates a finite partition P_i of Ω

and a finite sub-σ-field F_i of F. Interpret S_i as the set of signals that i can receive about the state of the world and P_i or F_i (equivalently) as i's initial information. Think of the sets S_i and the mappings s_i as common knowledge for all agents (and the planner or mechanism designer).[3] Take S_i also to be the set of messages that agents can communicate, in the sense that an agent can convey a (true or false) subset of his actual information partition. Note that the realizations $s_i(\omega)$ of i's signal are not observable to agents $j \neq i$ or to the planner or mechanism designer. Note also that random signals are allowed in this model in that otherwise identical copies of $\omega \in \Omega$ could be mapped to different elements of S_i, which is equivalent to expanding Ω to a larger (but still finite) set. Set $S = S_1 \times \ldots \times S_{\#I}$ with typical element $s = (s_1,\ldots,s_{\#I})$; write $(s_i', s_{-i}) = (s_1,\ldots,s_{i-1},s_i',s_{i+1},\ldots,s_{\#I}) \in S$ and $s_{-i} \in S_{-i} = \prod_{j \neq i} S_j$. Define $s : \Omega \to S$ by $s(\omega) = (s_1(\omega),\ldots,s_{\#I}(\omega))$ and define s_{-i} in the obvious way. Write $\mu_i(\cdot|s_i)$ and $\mu(\cdot|s)$ for the conditional probabilities on Ω given $s_i \in S_i$ or $s \in S$ respectively.

A state-dependent allocation $x_i : \Omega \to \mathbb{R}^{\ell}_{+}$ for trader i is <u>strongly incentive compatible</u> if $x_i(\cdot)$ is $\sigma(\bigcup_{i \in I} F_i)$-measurable [i.e., $x_i(\omega) = x_i(\omega')$ whenever $s(\omega) = s(\omega')$] and if $u_i(x_i(\omega);\omega) \ge u_i(x_i(\omega');\omega)$ for all $\omega, \omega' \in \Omega$. It is <u>Bayesian incentive compatible</u> if $x_i(\cdot)$ is $\sigma(\bigcup_{i \in I} F_i)$-measurable and if, for all $s_i \in S_i$ and all F_i-measurable $s_i' : \Omega \to S_i$, $\sum_{\omega \in \Omega} u_i(x_i(s(\omega);\omega)\mu_i(\omega|s_i) \ge \sum_{\omega \in \Omega} u_i(x_i(s_i',s_{-i}(\omega));\omega)\mu_i(\omega|s_i)$ where $x_i(s(\omega)) = x_i(\omega)$, $x_i(s_i',s_{-i}(\omega)) = x_i(\omega')$ if $s(\omega') = (s_i',s_{-i}(\omega))$ and $x_i(s_i',s_{-i}(\omega)) = 0$ if there is no $\omega' \in \Omega$ for which $s(\omega') = (s_i',s_{-i}(\omega))$. In both definitions, agents' state-dependent allocations must be measurable with respect to the joint information received--or reported--by all agents, so that allocations depend only on signals. In addition, both formulations have the property that no trader $i \in I$ wishes to misreport his signal (i.e., to say s_i' rather than s_i when the true signal is $s_i \in S_i$). Obviously strong incentive compatibility is stronger than Bayes incentive compatibility, where the difference involves whether incentive constraints apply separately to each possible realization of $\omega \in \Omega$ or whether incentive compatibility is given in terms of expected utility conditional on the player's received signal. Strong incentive compatibility is more appropriate if players do not know the probabilities affecting their opponents.[4] Either definition can be applied (consistently) in the remainder of this paper.

5. NTU Market Games with Incentive Compatibility Constraints

This section examines the structure of NTU games in characteristic function form that arise from exchange economies with incentive compatibility constraints. All of the previous assumptions of the model are maintained. The properties demonstrated here show that these games are well behaved and permit the application in the next section of Wooders (1983).

The data of my pure exchange economy under uncertainty generate a cooperative game which depends, of course, on agents' initial information and on incentive compatibility constraints. By definition, the correspondence $V : 2^I \to \mathbb{R}^{\#I}$ constitutes a cooperative game with nontransferable utility if $V(\emptyset) = \{0\}$ and for all $S \subseteq I$, $S \neq \emptyset$, $V(S)$ is a nonempty closed comprehensive cylinder set.

The cooperative games with nontransferable utility derived from my economic model with strong or Bayesian incentive compatibility are given by $V^S : 2^I \to \mathbb{R}^{\#I}$ and $V^B : 2^I \to \mathbb{R}^{\#I}$ with $V^S(\emptyset) = V^B(\emptyset) = \{0\}$ and, for $S \subseteq I$, $S \neq \emptyset$,

$$V^S(S) = \{(w_1,\ldots,w_{\#I}) \in \mathbb{R}^{\#I} \mid \text{for } i \in S, \text{ there are } \sigma(\cup_{i \in I} F_i)\text{-measurable } x_i : \Omega \to \mathbb{R}^\ell_+ \text{ such that } u_i(x_i(\omega);\omega) \geq u_i(x_i(\omega');\omega) \text{ for all } \omega, \omega' \in \Omega, \ \sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega) \text{ for all } \omega \in \Omega, \text{ and } w_i \leq \int u_i(x_i(\omega);\omega)d\mu(\omega)\}$$

and

$$V^B(S) = \{(w_1,\ldots,w_{\#I}) \in \mathbb{R}^{\#I} \mid \text{for } i \in S, \text{ there are } \sigma(\cup_{i \in I} F_i)\text{-measurable } x_i : \Omega \to \mathbb{R}^\ell_+ \text{ such that } x_i(\cdot) \text{ is Bayesian incentive compatible, } \sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega) \text{ for all } \omega \in \Omega \text{ and } w_i \leq \int u_i(x_i(\omega);\omega)d\mu(\omega)\}.$$

For convenience, write $V : 2^I \to \mathbb{R}^{\#I}$ for either V^S or V^B.

By definition, the NTU game $V : 2^I \to \mathbb{R}^{\#I}$ is superadditive if $V(S) \cap V(T) \subset V(S \cup T)$ whenever $S \cap T = \emptyset$. This means that merging of disjoint coalitions does not necessarily decrease any agent's payoff; anything that disjoint coalitions can do separately can also be done by their union. Intuitively, incentive compatibility cannot destroy superadditivity because incentive compatibility constraints apply to an individual's allocation; they do not depend on the coalition or on the allocations of other players.

Proposition 5.1. Finite NTU games derived from exchange economies with incentive compatibility constraints as in the model are superadditive.

Proof. Let $S \subseteq I$, $T \subseteq I$ with $S \cap T = \emptyset$. Without loss of generality, reorder the players in I so that $S = \{1,\ldots,\#S\}$ and $T = \{\#S + 1,\ldots,\#S + \#T\}$. Pick $(v_1,\ldots,v_{\#I}) \in V(S)$ and $(v_1',\ldots,v_{\#I}') \in V(T)$ arbitrarily. I need to show that $(v_1,\ldots,v_{\#S},v'_{\#S+1},\ldots,v'_{\#S+T}, w_{\#S+\#T+1},\ldots,w_{\#I}) \in V(S \cup T)$ where the $w_{\#S+\#T+1},\ldots,w_{\#I}$ are arbitrary. By the definition of the games in coalition form derived from my economy, there are state-dependent allocations $x_i : \Omega \to \mathbb{R}^\ell_+$ for $i \in S$ and $x_i : \Omega \to \mathbb{R}^\ell_+$

for $i \in T$ such that $\int_\Omega u_i(x_i(\omega);\omega)d\mu(\omega) \geq v_i$ if $i \in S$, $\int_\Omega u_i(x_i(\omega);\omega)d\mu(\omega) \geq v'_i$ if $i \in T$, $\sum_{i\in S} x_i(\omega) = \sum_{i\in S} e_i(\omega)$, $\sum_{i\in T} x_i(\omega) = \sum_{i\in T} e_i(\omega)$, and for each $i \in S \cup T$, $x_i(\cdot)$ is incentive compatible. Taking the (same) allocations $x_i : \Omega \to \mathbb{R}^\ell_+$ for $i \in S \cup T$ gives $\int_\Omega u_i(x_i(\omega);\omega)d\mu(\omega) \geq v_i$ if $i \in S$, $\int_\Omega u_i(x_i(\omega);\omega)d\mu(\omega) \geq v'_i$ if $i \in T$, $\sum_{i\in S\cup T} x_i(\omega) = \sum_{i\in S\cup T} e_i(\omega)$, and, for all $i \in S \cup T$, $x_i(\cdot)$ is incentive compatible. This proves that $(v_1,\ldots,v_{\#S}, v'_{\#S+1},\ldots,v'_{\#S+\#T}, w_{\#S+\#T+1},\ldots,w_{\#I}) \in V(S \cap T)$, as desired. []

The game $V : 2^I \to \mathbb{R}^{\#I}$ is <u>comprehensive</u> if, for any $S \subseteq I$, $v \in V(S)$ and $w \leq v \Rightarrow w \in V(S)$, or, equivalently, if $V(S) \supseteq V(S) - \mathbb{R}^{\#I}_+$ for all $S \subseteq I$. Comprehensiveness of games derived from exchange economies is automatic from the definition of $V(S)$ for any coalition S; this is not affected by incentive compatibility.

<u>Proposition 5.2</u>. The NTU games generated by my model with incentives are comprehensive.

<u>Proof</u>. Fix $S \subseteq I$ arbitrarily. Then $V(S) = \{(v_1,\ldots,v_{\#I}) \in \mathbb{R}^{\#I} |$ for $i \in S$, there exist $x_i : \Omega \to \mathbb{R}^\ell_+$ satisfying incentive compatibility, $\sum_{i\in S} x_i(\omega) = \sum_{i\in S} e_i(\omega)$, and $\int_\Omega u_i(x_i(\omega);\omega)d\mu(\omega) \geq v_i$ if $i \in S\}$. By definition, if $v \in V(S)$ and $w \leq v$, then $w \in V(S)$. []

<u>Proposition 5.3</u>. Any economy satisfying my assumptions induces a game $V : 2^I \to \mathbb{R}^{\#I}$ in which, for all $S \subseteq I$, $V(S)$ is closed and $V(S) \cap \{v \in \mathbb{R}^{\#I} | v_i = 0$ if $i \notin S\}$ is compactly generated.

<u>Proof</u>. Fix $S \subseteq I$. Then $X(S) = \{x \in \mathbb{R}^{(\#S)(\#\Omega)\ell}_+$ such that, if one writes $x_i : \Omega \to \mathbb{R}^\ell_+$ for $i \in S$, x_i satisfies incentive compatibility and $\sum_{i\in S} x_i(\omega) = \sum_{i\in S} e_i(\omega)\}$ is a compact set because the set of feasible allocations is closed and bounded while the incentive compatibility constraints (for $i \in S$) define a closed subset. Hence $V(S) \cap \{v \in \mathbb{R}^{\#I} | v_i = 0$ if $i \notin S\}$ is closed as the sum of the closed set $-\mathbb{R}^{\#I}_+ \cap \{v \in \mathbb{R}^{\#I} | v_i = 0$ if $i \notin S\}$ and the compact image in $\mathbb{R}^{\#I} \cap \{v \in \mathbb{R}^{\#I} | v_i = 0$ if $i \notin S\}$ of $X(S)$ under the continuous functions $u_i, i \in S$. The proof of this assertion involves a well known argument with asymptotic cones. Taking the cylinder set $V(S)$ in $\mathbb{R}^{\#I}$ generated by this closed set does not destroy closedness. []

6. Approximate Cores

This section considers the possibilities for approximate core allocations in exchange economies with asymmetric information and strong or Bayesian incentive compatibility. In particular, sufficiently large replica economies with incentive constraints have weak ε-cores. The more economically appealing approximate core notion of the strong ε-core is nonempty for some subsequence of replica economies with incentive compatibility constraints.

<u>Definition 6.1</u>. The sequence $(V_r)_{r=1}^\infty$ of (NTU) <u>replica games</u> derived from the exchange economy with (strong or Bayesian) incentive compatibility constraints has $V_1(S) = V(S)$ for all $S \subseteq I$ and, for $r \geq 1$, is defined as in the preceding section except that in the underlying (r-th replica) economy, there are exactly r agents of each <u>type</u> $i \in I$. If $S \subseteq I$, write rS for the set of players in the r' replica, for $r' \geq r$, containing precisely r copies of each player type belonging to S. Write $rS = \{ir" | i \in S$ and $1 \leq r" \leq r\}$. Note that rI is the player set for the game V_r, $r = 1,2,\ldots$.

<u>Definition 6.2</u>. Given $\varepsilon > 0$, a payoff vector $w = (w_{11},\ldots,w_{1r}, w_{21},\ldots,w_{2r},\ldots,w_{\#I1},\ldots,w_{\#Ir})$ belongs to the <u>strong ε-core</u> of V_r $(r \geq 1)$ if (i) the payoffs are feasible--i.e., $w \in V_r(rI)$, and (ii) for all coalitions $S_r \subseteq rI$, the payoff w cannot be improved upon by ε or more for

each player in S_r--i.e., $w + (\epsilon,\ldots,\epsilon) \notin \text{int}\ V_r(S_r)$. [In (ii), the coalition S_r may have different numbers of players of each type.]

<u>Definition 6.3</u>. Given $\epsilon > 0$, a feasible payoff vector $w = (w_{11},\ldots,w_{1r}, w_{21},\ldots,w_{2r},\ldots,w_{\#I1},\ldots,w_{\#Ir}) \in V_r(rI)$ belongs to the <u>weak ϵ-core</u> of V_r. $(r \geq 1)$ if there is some payoff vector $\bar{w} = (\bar{w}_{11},\ldots,\bar{w}_{1r},\bar{w}_{21},\ldots,\bar{w}_{2r},\ldots,\bar{w}_{\#I1},\ldots\bar{w}_{\#Ir}) \in \mathbb{R}^{\#Ir}$ such that (i) $\#\{\bar{ir} \in rI \mid \bar{w}_{\bar{ir}} \neq w_{\bar{ir}}\}/r\#I < \epsilon$, and (ii) for all coalitions $S_r \subseteq rI$, the payoff $\bar{w}$ cannot be improved upon by ϵ or more for each player in S_r--i.e., $\bar{w} + (\epsilon,\ldots,\epsilon) \notin \text{int}\ V_r(S_r)$. [Again in (ii), S_r need not contain the same number of players of each type.]

<u>Theorem 6.4</u>. For any fixed $\epsilon > 0$, there is an integer R such that for all $r \geq R$, the weak ϵ-core of V_r is nonempty. In particular, for every $\epsilon > 0$, sufficiently large replica economies with (strong or Bayesian) incentive compatibility have weak ϵ-core allocations.

<u>Proof</u>. The games V^S and V^B satisfy the basic conditions [for each coalition S, $V(S)$ is a nonempty proper closed cylinder set containing some strictly positive vector such that $V(S) \cap \{v \in \mathbb{R}^{\#I} \mid v_j = 0 \text{ if } j \notin S\}$ is bounded] of Shubik and Wooders (1983). Moreover $(V_r)_{r=1}^{\infty}$ is a sequence of superadditive and per-capita bounded [i.e., equal treatment payoffs in $V_r(rI)$ are uniformly bounded above for $r = 1,2,3,\ldots$] replica games. Hence Theorem 1 of Shubik and Wooders (1983) applies, and the weak ϵ-core of V_r is nonempty whenever r is sufficiently large. []

<u>Theorem 6.5</u>. For arbitrary $\epsilon > 0$ and any integer R, there exists $r \geq R$ for which the strong ϵ-core of V_r is nonempty. In particular, every sequence of replica exchange economies with (strong or Bayesian) incentive compatibility has a subsequence for which there are strong ϵ-core allocations.

<u>Proof</u>. As in the previous proof, the conditions in Shubik and Wooders (1983) are satisfied; their Theorem 2 states that, for some subsequence $V_{r'}$ of V_r, the strong ϵ-core of $V_{r'}$ is nonempty. []

<u>Remark 6.6</u>. Recall that, by Proposition 5.2, my games V_r (derived from exchange economies with incentive compatibility constraints) are comprehensive. Hence, as Shubik and Wooders (1983) observe, there exist strong ϵ-core payoffs satisfying the equal treatment property for the subsequence of replicas in Theorem 6.5. [By definition, this means that all players of the same type can be assigned identical utility allocations in the strong ϵ-core along the subsequence.]

7. Randomization

Consider the game induced by exchange economies with randomizations over allocations satisfying incentive compatibility constraints and feasibility with probability one.[5] I claim that permitting coalitions to use such "mixed strategies" generates a game $\hat{V} : 2^I \to \mathbb{R}^{\#I}$ for which $\hat{V}(S)$ is a convex, closed, and comprehensive cylinder set for all S. In fact $\hat{V}(S) = \text{conv}(V(S))$, the closed convex hull of $V(S)$. Unfortunately, $\hat{V}$ is not necessarily balanced, as demonstrated by the example in Allen (1992).

Somewhat more formally, the game generated by an exchange economy with almost surely feasible randomizations and incentive compatibility constraints is given by $\hat{V} : 2^I \to \mathbb{R}^{\#I}$ where, for $S \subseteq I$, $\hat{V}(S) = \{0\}$ if $S = \emptyset$ and otherwise (for $S \neq \emptyset$)

$$\hat{V}(S) = \{(v_1,\ldots,v_{\#I}) \in \mathbb{R}^{\#I} \mid \text{there is a probability measure } \nu \text{ on } \tilde{x} \in \mathbb{R}_+^{(\#I)(\#\Omega)\ell} \text{ such that } \sum_{i\in S} \tilde{x}_i(\omega) = \sum_{i\in S} e_i(\omega) \text{ for } \nu\text{-almost all } \tilde{x}(\cdot) \text{ and all } \omega \in \Omega, \ \nu\text{-almost all } \tilde{x} \text{ satisfy, for } i \in S, \text{ incentive compatibility of } \tilde{x}_i : \Omega \to \mathbb{R}_+^{\ell} \text{ and } v_i \leq \iint u_i(\tilde{x}_i(\omega);\omega)d\mu(\omega)d\nu(\tilde{x})\}.$$

<u>Proposition 7.1</u>. The game $\hat{V} : 2^I \to \mathbb{R}^{\#I}$ with randomization over almost surely feasible and incentive compatible allocations has sets $\hat{V}(S)$ of attainable payoff vectors which are, for all $S \subseteq I$, $S \neq \emptyset$, nonempty, closed, comprehensive, and compactly-generated (in $\mathbb{R}^{\#S}$) cylinder sets.

<u>Proof</u>. Randomization over the sets $X(S)$ in the proof of Proposition 5.3 gives a compact and convex set of probabilities for the topology of weak convergence of probability measures. The image of $X(S)$ under the continuous and concave functions u_i $(i \in S)$ thus is compact and convex. Taking its comprehensive hull and then the cylinder set which this generates thus gives sets $\hat{V}(S)$ which are nonempty, closed, convex, comprehensive, and compactly-generated (in $\mathbb{R}^{\#S}$) cylinder sets. In fact, by definition $\hat{V}(S) = \text{conv}(V(S))$ for all $S \subseteq I$. []

Thus, the NTU game $\hat{V} : 2^I \to \mathbb{R}^{\#I}$ satisfies all of the properties needed to show nonemptiness of its core except balancedness. The next section sidesteps this difficulty by considering large replicas and demonstrating approximate balancedness, which then yields approximate cores.

To be pedantic, I could write $\hat{V}^S : 2^I \to \mathbb{R}^{\#I}$ and $\hat{V}^B : 2^I \to \mathbb{R}^{\#I}$ where, for all $S \subseteq I$, $\hat{V}^S(S) = \text{conv}(V^S(S))$ and $\hat{V}^B(S) = \text{conv}(V^B(S))$. The core of the economy with strong or Bayesian incentive compatibility constraints equals the (strong or Bayesian incentive compatible) state-dependent allocations giving rise to expected utility allocations in the core of the NTU game $V^S : 2^I \to \mathbb{R}^{\#I}$ or $V^B : 2^I \to \mathbb{R}^{\#I}$ respectively. The core of the economy with strong or Bayesian incentive compatibility constraints and almost surely resource feasible randomization equals the (strong or Bayesian incentive compatible) state-dependent allocations giving rise to expected utility allocations

in the (NTU) core of the (convexified) game $\hat{V}^S : 2^I \to \mathbb{R}^{\#I}$ or $\hat{V}^B : 2^I \to \mathbb{R}^{\#I}$ respectively.

8. Approximate Cores with Randomization

The main result of this paper states that all sufficiently large replica economies with incentive compatibility constraints have nonempty strong ε-cores which contain at least one equal treatment allocation. This requires convexity of the NTU game's sets of attainable payoff vectors, so that randomization is necessary.

<u>Definition 8.1</u>. The sequence $(\hat{V}_r)_{r=1}^{\infty}$ of (NTU) <u>replica games with randomization</u> derived from the exchange economy with incentive compatibility constraints has $\hat{V}_1(S) = \hat{V}(S)$ for all $S \subseteq I$ and, for $r \geq 1$, is defined as the NTU game induced by an economy with incentives and randomization as in the preceding section except that in the underlying (replica) economy, there are r agents of each type $i \in I$. If $S \subseteq I$, write rS for the set of players in the r' replica, for $r' \geq r$, consisting of exactly r copies of each player type belonging to S. Write $rS = \{ir'' \mid i \in S \text{ and } 1 \leq r'' \leq r\}$.

<u>Definition 8.2</u>. Given $\varepsilon > 0$, a payoff vector $w = (w_{11},\dots,w_{1r}, w_{21},\dots,w_{2r},\dots,w_{\#I1},\dots,w_{\#Ir})$ belongs to the <u>(NTU) strong ε-core with randomization</u> if it belongs to the <u>strong ε-core</u> of $\hat{V}_r$ $(r \geq 1)$. This is equivalent to following two conditions: (i) the payoffs are feasible--i.e., $w \in \hat{V}_r(rI)$, and (ii) for all coalitions $S_r \subset rI$, the payoff w cannot be improved upon by ε or more for each player in S_r--i.e., $w + (\varepsilon,\dots,\varepsilon) \notin \text{int}\,\hat{V}_r(S_r)$. [Note that in (ii), the coalition S_r may have different numbers of clones of each type.]

<u>Definition 8.3</u>. The (feasible) payoff vector $w \in \hat{V}_r(rI)$ for $r \geq 1$ has the <u>equal treatment property</u> (ETP) if all players of the same type receive the

same utility allocation or payoff in w. In symbols, $w \in \hat{V}_r(rI)$ satisfies ETP if $w_{ir'} = w_{ir''}$ for all $i \in I$ and all $r', r'' \leq r$. [If $r = 1$, this requirement is vacuous but true.]

Theorem 8.4. Fix $\epsilon > 0$ arbitrarily. Then if r is sufficiently large, the NTU strong ϵ-core with randomization in the r-th replica economy or the strong ϵ-core of $\hat{V}_r$ is nonempty and moreover contains a payoff satisfying the equal treatment property.

Proof. Because the sequence $(\hat{V}_r)_r$ of replica games (generated by my model with incentive compatibility and randomization) is a sequence of replica games in the sense of Wooders (1983) satisfying superadditivity, per capita boundedness, and comprehensiveness, Theorems 1 and 2 of Wooders (1983) can be applied. []

While Theorem 8.4 is stated in terms of payoffs in the NTU core, they give rise to state-dependent core allocations for which the corresponding (expected) utility allocations satisfy the equal treatment property.

9. Remarks

This paper analyzes replica economies rather than more general sequences of large but finite economies (with incentives and almost surely feasible randomization) because the interpretation of the ϵ-core for the latter is problematic. The analogue of the first part of Theorem 8.4 for nonreplicas (see Wooders (1991)) would give existence of the ϵ-core at the expense of the possibility that the fraction ϵ of players receive payoffs w that may fail to satisfy $w + (\epsilon,\ldots,\epsilon) \notin \mathrm{int}(\hat{V}_r(S_r))$ or, in other words, a small group S_r (of size at most ϵ) receives allocations that could be blocked so as to gain more than an ϵ improvement in their payoffs. Such a use of the weak

ϵ-core is troublesome here because in this framework with asymmetric information, the information of such small groups may nevertheless be essential to the achievement of state-dependent allocations for other players. If the small group could receive large gains from its participation in blocking, these players may not be willing to reveal their information (truthfully) to others. Recall that the possibility for endogenous strategic information sharing is the primary economic question addressed here by incentive compatibility.

Similarly, I focus on games with nontransferable utility instead of the more tractable case of transferable utility because I believe that the transferable utility assumption contradicts the basic premise that incentive compatibility concerns. If utility were transferable, so that all coalition members maximize the coalition's total utility as their sole objective function, then they should naturally share all of their information fully and correctly. Yet, when this happens, incentive compatibility constraints are unnecessary.

Needless to say, this paper leaves unanswered the natural question of whether large economies necessarily have nonempty incentive compatible cores. In all of the examples I've analyzed with empty incentive compatible cores, replication restores the existence of state-dependent allocations in the incentive compatible (exact) core. Unfortunately, I have been unable to show that replication guarantees the existence of such a core as, in particular, I see no intuitive reason for all sufficiently large replicas to be (exactly) balanced. On the other hand, I conjecture that exchange economies with asymmetric information and an atomless continuum of agents must have nonempty incentive compatible cores.

Footnotes

*Most of this research was performed during my visit to Graduiertenkolleg/Institut fur Mathematische Wirtschaftsforschung at Universitat Bielefeld.

**Discussion with Myrna Wooders convinced me to consider the ϵ-core. Two anonymous referees provided helpful comments regarding final revisions.

[1]The literature on the core of an economy with asymmetric information begins with Wilson (1978) and includes quite recent work by Yannelis (1991) and Allen (1991a, 1991b, 1991c, 1992). Marimon (1990) has examined the core with adverse selection while Berliant (1992) and Boyd and Prescott (1986) consider the cores of economies with incentive compatibility constraints having special structures based on taxation and financial intermediation respectively. Myerson (1984) and Rosenmuller (1990) have studied NTU games with incentive compatibility constraints, but they focus on solution concepts other than the core. The very recent paper by Koutsourgeras and Yannelis (1991) considers a different definition of incentive compatibility and asks whether (NTU) core allocations--defined without incentive compatibility constraints--satisfy their definition.

[2]More specifically, Wooders (1983, 1991) shows that an approximate balancedness condition implies that the approximate core is nonempty and that her condition is always satisfied for replica exchange economies with sufficiently many agents providing that a hypothesis (called efficacy of small groups) is not violated.

[3]The planner or mechanism designer does not play a formal role in my model but is mentioned here as a referee who is able to verify that (for the classical case of strong incentive compatibility) the proposed state-dependent allocations for coalitions are indeed incentive compatible or that (for the case of Bayesian

incentive compatibility) the mechanism really is Bayesian incentive compatible. However the planner/mechanism designer is unable to observe or verify individuals' realized information or signals (as opposed to the nature of their information partition structures or signal mappings from the set of states of the world to the set of signal values, which are considered to be common knowledge).

[4]See d'Aspremont and Gerard-Varet (1979) for a further discussion and Holmstrom and Myerson (1983) for alternative notions of efficiency (or, implicitly, the allocations that can be achieved by the grand coalition) with incentives.

[5]This is reminiscent of the randomizations introduced in Prescott and Townsend (1984a, 1984b) to obtain the existence of competitive equilibrium and welfare theorems in general equilibrium exchange economies under uncertainty.

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