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An Algorithm for the Construction of Homogeneous Games

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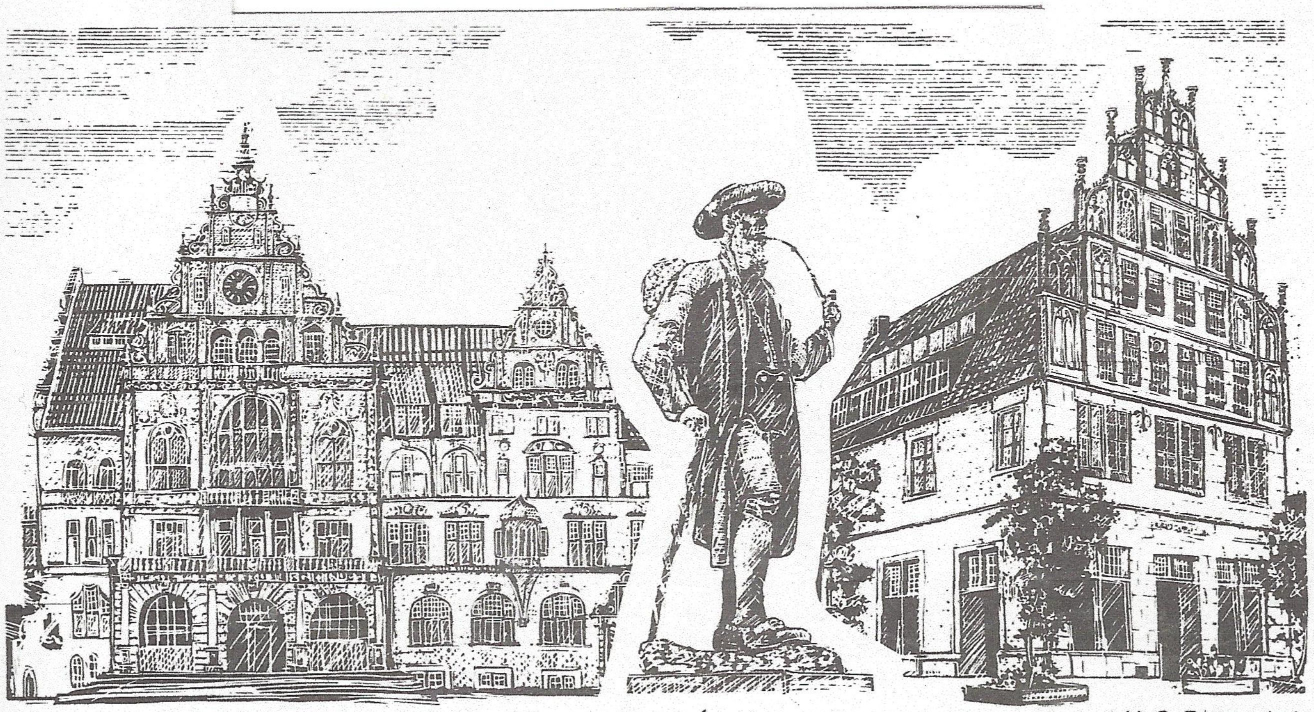

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Abstract

Suppose, a weighted majority simple n-person game is to be specified by allotting weight $g_o = 0$ to k_o players, weight g_1 to the next k_1 player,..., weight g_r to the last k_r players; here $g_i \in \mathbb{N}$ is increasing and $k_i \in \mathbb{N}$. $(i = 0,\ldots,r)$. A coalition is winning if the total weight of its members is at least $\lambda \in \mathbb{N}$. An algorithm is provided such that, given $(g_o, g_1,\ldots,g_r)$ and $(k_o, k_1,\ldots,k_r)$, every λ is produced which renders the resulting simple game to be homogeneous.

SEC. 1 The matrix of homogeneity

Let $k = (k_o,\ldots,k_r) \in \mathbb{N}_o^{r+1}$ satisfy

(1) $k_o \geq 0,\ k_1,\ldots,k_r \geq 1$;

a vector $s = (s_o,\ldots,s_r) \in \mathbb{N}_o^{r+1}$ is a feasible profile (for k) if $s \leq k$. Next, let $g = (g_o,\ g_1,\ldots,g_r) \in \mathbb{N}_o^{r+1}$ satisfy

(2) $0 = g_o \leq g_1 \leq g_2 \cdots \leq g_r \neq 0$.

g induces the function

(3)
$$g : \{s \leq k\} \to \mathbb{N}_o$$
$$g(s) = \sum_{i=0}^{r} s_i\, g_i .$$

The function g as well as the pair $M = (g,k)$ is called a measure. A measure and a constant $\lambda \in \mathbb{N}$ such that $g(k) \geq \lambda$ generate a characteristic function $v = v_\lambda^M : \{s \leq k\} \to \{0,1\}$ on the profiles of k via

(4)
$$v(s) = \begin{cases} 1 & g(s) \geq \lambda \\ 0 & g(s) < \lambda \end{cases} \qquad (s \leq k)$$

The familiar framework of n-person cooperative game theory is easily obtained; put $n = \sum_{i=0}^{r} k_i$ and $\Omega = 1,\ldots,n$. Decompose $\Omega = K_o + K_1 + \ldots + K_r$ (+ = "disjoint union") such that $|K_i| = k_i$. Ω is the "set of players" and any coalition $S \subseteq \Omega$ has a profile $s = (|S \cap K_o|,\ldots,|S \cap K_r|)$. Then $(M,\lambda) = (g,k;\lambda)$ induce a cf. (in the familiar sense) say, by

$$w(S) = v_\lambda^M (|S \cap K_o|,\ldots,|S \cap K_r|) = v(s).$$

Thus, players in K_i have the same weight g_i and there are k_i players with this property. Therefore, $i \in \{0,\ldots,r\}$ (or K_i) is called a fellowship. λ is the "majority level".

A profile $s \leq k$ is winning if $v(s) = 1$ (losing otherwise) and minimal winning if any winning profile $t \leq s$ satisfies $t = s$. Profiles will be ordered lexicographically (from right to left, i.e., s preceeds s' if $s_\rho > s'_\rho$ and $s_i = s'_i$ $(i > \rho)$). The lex-max profile is the lexicographically first min-win profile (containing the largest fellows).

A pair $M = (g,k)$ is said to be homogeneous w.r.t. $\lambda \in \mathbb{N}$ if $g(k) \geq \lambda$ and

(5) For any $s \leq k$, $g(s) > \lambda$ there is $t \leq s$ such that $g(t) = \lambda$

We use the notation M hom λ in order to indicate homogeneity; $M \text{ hom}_0 \lambda$ means that either M hom λ or $g(k) < \lambda$.

Assume that M hom λ and let us construct the lex-max profile, say s_M^λ, by collecting first the weights of the largest fellowship, then those of the second largest fellowship etc. until we have the total mass λ is combined. By homogeneity, the majority level is indeed exactly hit by this procedure, i.e., there is $i_0 \in \{1,\ldots,r\}$ and $c \in \mathbb{N}$, $1 \leq c \leq k_{i_0}$ such that

(6) $$\lambda = cg_{i_0} + \sum_{i=i_0+1}^{r} k_i g_i$$

and

(7) $$s_M^\lambda = (0,\ldots,0, c, k_{i_0+1},\ldots,k_r)$$

The remaining players (fellowships) constitute a smaller measure which is a projection of $M = (g,k)$; this measure is (for $c < k_{i_0}$)

(8) $$M_{i_0}^c := (g_0, g_1, \dots, g_{i_0}; k_0, k_1, \dots, k_{i_0} - c)$$

or (if $c = k_{i_0}$)

(9) $$M_{i_0}^{k_{i_0}} = M_{i_0 - 1} := (g_0, g_1, \dots, g_{i_0 - 1}; k_0, k_1, \dots, k_{i_0 - 1})$$

Now the remaining players may try to replace a member of the larger fellowships (those already engaged in the lex-max profile) in order to enter a min-win coalition (profile); more precisely, the measure $M_{i_0}^c$ and the weight g_j $(j > i_0)$ constitute a weighted majority game which, as it turns out, is a homogeneous one. Indeed we have

Lemma 1.1. (The BASIC LEMMA, see [8])

Let $M = (g,k)$ satisfy (1) and (2) and let $\lambda \in \mathbb{N}$, $g(k) \geq \lambda$. Then M hom λ if and only if there is $i_0 \in \{1, \dots, r\}$ and $c \in \mathbb{N}$, $1 \leq c \leq k_{i_0}$ such that (6) is satisfied and

(10) $$M_{i_0}^c \text{ hom}_0\ g_j \quad (i_0 \leq j \leq r)$$

holds true.

Note that $M_{i_0}^c$ hom$_0$ g_{i_0} is equivalent to $M_{i_0 - 1}$ hom$_0$ g_{i_0}.

Let us introduce the following notations

$$\mathfrak{M}^r := \{(g,k) \in \mathbb{N}_0^{2(r+1)} \mid k \text{ satisfies (1) and } g \text{ satisfies (2)}\}$$

(11) $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (r \geq 1)$

$$\mathfrak{M}^0 := \{(0,k_0) \mid k_0 \in \mathbb{N}_0\}$$

$$\mathfrak{M} = \bigcup_{r=0}^{\infty} \mathfrak{M}^r$$

For any $M \in \mathfrak{M}^r$, m denotes "total mass", i.e.

$$m = \sum_{i=1}^{r} k_i g_i$$

and indices are carried accordingly, i.e.

$$m_{i_0}^c = \sum_{i=1}^{i_0-1} k_i g_i + (k_{i_0} - c) g_{i_0}$$

etc.

Definition 1.2. Let $M = (g,k) \in \mathfrak{M}^s$ and for $1 \leq r \leq s$ consider $M_r = (g_0, g_1, \ldots, g_r; k_0\ k_1, \ldots, k_r)$.

For $1 \leq i_0 \leq r$ and $1 \leq c \leq k_{i_0}$ consider

$$\lambda_{i_0}^c = \lambda_{i_0,r}^c (g,k) = c\, g_{i_0} + \sum_{i=i_0+1}^{r} k_i g_i$$

and

$$c_{i_0}^r = c_{i_0}^r (g,k) = \min \{c \in \mathbb{N} \mid 1 \leq c \leq k_{i_0},\ M_r \text{ hom } \lambda_{i_0}^c\}$$

(where $\min \emptyset = \infty$). Then $C = (c_i^r)_{\substack{1 \leq r \leq s \\ 1 \leq i \leq s}}$

($c_i^r := 0$ for $r < i$) is called the matrix of homogeneity of $M = (g,k)$.

Lemma 1.3. Let $M \in \mathfrak{M}^s$. For $1 \leq i_0 \leq r \leq s$ and $1 \leq c \leq k_{i_0}$

$$M_r \text{ hom } \lambda_{i_0}^c$$

if and only if $c \geq c_{i_0}^r$.

Proof: This follows from the BASIC LEMMA since $M^c_{i_o}$ $\hom_o g_j$ for $j \geq i_o$ implies $M^{c+1}_{i_o}$ $\hom_o g_j$; see Lemma 2.1. of [8]

Thus, we have now slightly changed our view point: Fix $M \in \mathfrak{M}^s$ and consider any "projection" $M_r \in \mathfrak{M}^r$ $(r \leq s)$. The numbers $\lambda = \lambda^c_{i_o}$ are the only candidates such that $(M_r, \lambda^c_{i_o})$ generates a "homogeneous" cf. $v^{M_r}_{\lambda^c_{i_o}}$, i.e., sucht that M_r hom $\lambda^c_{i_o}$. Knowledge of the matrix C is sufficient in order to decide whether M hom $\lambda^c_{i_o}$ holds true. Now, Lemma 1.1. suggests that homogeneity is a property which is aquired or disturbed by a recursive procedure.

As has been shown in [8], C also allows for a recursive computation.

Our present purpose is to exhibit regularity properties of the matrix C. This will provide faster algorithms for the actual computation of C. Besides, as has been elaborated in [5], [8], [9] the recursive structure allows for the definition of certain characters of players (as well as fellowships and types) in a homogeneous game. These types are the (familiar) dummy, the sum and the step. It will turn out that some properties of C also reflect the "strength" of certain players (fellowships) in the corresponding games in a way such that the character of a fellowship may be decided upon by inspection of C.

Thus, the matrix C at once yields all homogeneous games that may result from any projection M_r of $M \in \mathfrak{M}^s$ and shows something about the strength of the players in these games. Fast algorithms in order to compute C are therefore considered to be desirable.

SEC. 2 Properties of C

Lemma 1.3 suggests that, given $M = (g,k) \in \mathfrak{M}^s$, it suffices to know the matrix of homogeneity C in order to specify all "majority levels", $\lambda \in \mathbb{N}$ such that M hom λ. Let us, therefore, exhibit some properties of this matrix that will turn out to be useful for a recursive computation. Parts of these arguments we shall quote from [8], however, we want to use the fact that "many" entries of the matrix C are not finite, thus providing eventually a more effective algorithm.

Lemma 2.1. (cf. 2.3. of [8])

Let $M \in \mathfrak{M}^s$. Then, for $1 \leq i_0 \leq r \leq s$, we have

(1) $$c^r_{i_0} < \infty \text{ if and only if } M_{i_0-1} \text{ hom}_0\ g_i \ (i=i_0,\ldots,r).$$

Proof: $c^r_{i_0} < \infty$ is equivalent to M_r hom $\lambda^c_{i_0}$ for some c, $1 \leq c \leq k_{i_0}$ (Definition 1.2.) and by Lemma 1.3. this is equivalent

$$M_r \text{ hom } \lambda_{i_0}^{k_{i_0}} = \sum_{i=i_0}^{r} k_i\, g_i$$

Finally, by Lemma 1.1., this is equivalent to

$$M_{i_0-1} \text{ hom}_0\ g_i \qquad (i = i_i,\ldots,r)$$

Lemma 2.2. (cf. 2.3. of [8])

Let $M \in \mathfrak{M}^s$. Then

(2) $$c^1_1 = 1$$

and, for $r \geq 2$

(3) $$c^r_r = \begin{cases} 1 & \text{if } M_{r-1} \text{ hom}_0\ g_r \\ \infty & \text{otherwise.} \end{cases}$$

Proof: $c_1^1 = 1$ follows from the definition of C. In order to check (3), observe that Lemma 2.1. implies that $c_r^r < \infty$ if and only if M_{r-1} hom$_o$ g_r . But M_{r-1} hom$_o$ g_r implies M_r^c hom$_o$ g_r for all $c \in \mathbb{N}$, $1 \leq c \leq k_r$, q.e.d.

Next, let us define the quantity

(4) $$\gamma_{i_o}^r := \min \{c \mid 1 \leq c \leq k_{i_o}, M_{i_o}^c \text{ hom}_o\, g_r\},$$

then, using $\alpha \vee \beta$ in order to denote the maximum of reals α and β, we have

Lemma 2.3. (cf. [8])

Let $M \in \mathfrak{M}^s$ and $2 \leq r \leq s$. For $i_o < r$ it follows that

(5) $$c_{i_o}^r = c_{i_o}^{r-1} \vee \gamma_{i_o}^r$$

Proof: Obvious, in view of Lemma 1.1. we have

(6)
$$\begin{aligned} c_{i_o}^r &= \min \{c \in \mathbb{N} \mid 1 \leq c \leq k_{i_o}, M_{i_o}^c \text{ hom}_o\, g_i\ (i=i_o,\ldots,r)\} \\ &= \min \{c \in \mathbb{N} \mid 1 \leq c \leq k_{i_o}, M_{i_o}^c \text{ hom}_o\, g_i\ (i=i_o,\ldots,r-1)\} \\ &\quad \vee \min \{c \in \mathbb{N} \mid 1 \leq c \leq k_{i_o}, M_{i_o}^c \text{ hom}_o\, g_r\} \\ &= c_{i_o}^{r-1} \vee \gamma_{i_o}^r . \end{aligned}$$

Denote by $[\alpha]$ the largest integer less than or equal to α and put, for $1 \leq i \leq s$

(7) $$l_1^i := \begin{cases} 1 & \text{if } g_1 \mid g_i \text{ or } k_1 g_1 \leq g_i \\ k_1 - [\frac{g_i}{g_1}] & \text{otherwise} . \end{cases}$$

Lemma 2.4. (cf. [8])

Let $M \in \mathfrak{M}^s$ and $1 < r \leq s$. Then

(8) $$c_1^r = c_1^{r-1} \vee 1_1^r .$$

Proof: This follows from 2.3., since

$$\begin{aligned}\gamma_1^r &= \min\{c \mid 1 \leq c \leq k_1,\ M_1^c \text{ hom}_0\ g_r\} \\ &= \min\{c \mid 1 \leq c \leq k_1,\ g_1 \mid g_r \text{ or } k_1 g_1 < g_r\} \\ &= 1_1^r\end{aligned}$$

Corollary 2.5. Let $M = (g,k) \in \mathfrak{M}^s$ and let

$C = C(M) = (c_{i_o}^r)_{1 \leq i_o \leq r \leq s}$

denote the matrix of homogeneity of M.

Each column of C is monotone increasing. If there are finite entries in a column at all, then the first entry equals 1.

This is obvious since monotonicity follows from (8) and the shape of the first entry (the diagonal element of C) is specified by (3).

Corollary 2.6. Let $M \in \mathfrak{M}^s$ and $2 \leq r \leq s$. Suppose that $c_1^r > c_1^{r-1}$. Then

(9) $$c_2^r = \ldots = c_r^r = \infty$$

(and, by monotonicity, $c_i^j = \infty$ for $j \geq r$ and $2 \leq i \leq r$).

<u>Proof:</u> If $c_1^r > c_1^{r-1}$, then $1_1^r = \gamma_1^r > 1$; i.e., in view of (7)

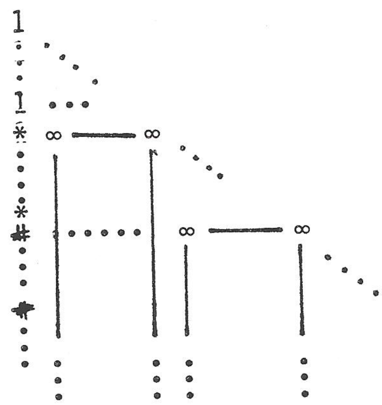
$$g_1 \nmid g_r \text{ and } k_1 g_1 > g_r$$

It follows that, for any i_o, $2 \leq i_o \leq r$,

$$M_{i_o-1} \; h\o m_o \; g_r$$

That is, $c_{i_o}^r = \infty$ by Lemma 2.1., q.e.d.

Consider the matrix $C = C(M)$. Given r, i_o we may say that the entries c_i^j , $j \geq r$, $i_o+1 \leq i \leq r$ constitute the "south-east-stripe" of $c_{i_o}^r$. Thus, whenever there occurs a jump in the first column, then the entries in the south-east-stripe are all ∞ ; this suggests vaguely the following form of C:



Now, it turns out, that the principle is a general one: whenever jumps occur in <u>any</u> column, then the south-east-stripe is rendered ∞.

<u>Theorem</u> 2.7. Let $M \in \mathfrak{M}^s$ and C the matrix of homogeneity. Whenever, for some $2 \leq i_o \leq r-1 \leq r \leq s$,

$$c_{i_o}^r > c_{i_o}^{r-1}$$

then

(10) $$c^r_{i_o+1} = \dots = c^r_r = \infty$$

(and, by monotonicity $c^j_i = \infty$ for $j \geq r$, $i_o + 1 \leq i \leq r$).

Proof: Let $c = c^{r-1}_{i_o}$. Then we have

$$M^c_{i_o} \text{ hom}_o \, g_j \qquad (i_o \leq j \leq r - 1)$$

(see e.g. formula (6)). On the other hand, we have $\gamma^r_{i_o} > c$, thus

$$M^c_{i_o} \text{ h}\not{o}\text{m}_o \, g_r \, .$$

Now, $M^c_{i_o}$ is a projection of M_{i_o} and hence we have a fortiori

$$M_{i_o} \text{ h}\not{o}\text{m}_o \, g_r \, ,$$

which by Lemma 2.1. implies $c^r_{i_o+1} = \infty$. Similarly, for any i, $i_o \leq i \leq r$, $M^c_{i_o}$ is a projection of M_i and thus M_i h$\not{o}$m$_o$ g_r , implying $c^r_i = \infty$, q.e.d.

Remark 2.8. The computational procedure for obtaining the matrix C is greatly simplified by the fact that frequently $c^r_{i_o} = \infty$ is implied by the "south-east-stripe" rule indicated via Theorem 2.7.

However, in some cases we actually need to compute $c^r_{i_o}$ given that the entries $c^{r'}_i$ are already known for $r' = r$, $i < i_o$ and for $r' < r$, $i \leq r'$. To this end we proceed as follows.

2.8. A. If $i_o = r$, then we know that $c^r_r = 1$ if and only if M_{r-1} hom$_o$ g_r. The matrix of homogeneity w.r.t. M_{r-1} is (recursively) known; this is $(c^{r'}_i)_{1 \leq i \leq r' \leq r-1}$.

2.8.B. Consider the case that $1 < i_o < r$. We may assume $c_{i_o}^{i_o} = 1$ and $c_{i_o}^{r'} < \infty$ $(i_o \leq r' \leq r-1)$, for otherwise we have $c_{i_o}^{r} = \infty$ by monotonicity. Thus we have

(11) $$M_{i_o-1} \ \mathrm{hom}_o \ g_{i_o}$$

(by Theorem 2.1.). Given this hypothesis, we have to compute

(12) $$\gamma_{i_o}^{r} = \min \{c \in \mathbb{N} \mid 1 \leq c \leq k_{i_o} , M_{i_o}^{c} \ \mathrm{hom}_o \ g_r\} .$$

To this end we first apply a test in order to check whether

(13) $$M_{i_o-1} \ \mathrm{hom}_o \ g_r$$

holds true. (Again, the C-matrix for M_{i_o-1} is already known)

2.8.B.a. If the answer is no and M_{i_o-1} $\mathrm{h\o m}_o$ g_r then

$$\gamma_{i_o}^{r} = \infty$$

as $M_{i_o-1} = M_{i_o}^{k_{i_o}}$.

2.8.B.b. If the answer is yes and $M_{i_o-1} \ \mathrm{hom}_o \ g_r$, then, by the same reasoning we have $\gamma_{i_o}^{r} \leq k_{i_o}$ (i.e. $c = k_{i_o}$ is admitted in (12); thus, in particular the "min" operation in (12) is not taken w.r.t. the empty set). Our next test consists of a check whether

(14) $$g_{i_o} \mid g_r$$

holds true.

2.8.B.b.α. If "yes", and $g_{i_o} \mid g_r$, then (13) implies $M^1_{i_o}$ $\hom_o g_r$ and hence

$$\gamma^r_{i_o} = 1.$$

Indeed, if M_{i_o-1} hom g_{i_o} then $M^1_{i_o}$ $\hom_o g_r$ is trivial and if $m_{i_o-1} < g_{i_o}$ then $M^1_{i_o}$ $\hom_o g_r$ is a simple exercise.

2.8.B.b.β. If "no", and $g_{i_o} \nmid g_r$, then we call upon Lemma 3.4. in [8] which tells us that, given the present conditions

$$M^c_{i_o} \hom_o g_r$$

is equivalent to

$$M_{i_o-1} \hom_o (g_r - (k_{i_o} - c)\, g_{i_o})$$

if $(k_{i_o} - c)\, g_{i_o} < g_r$, that is, if $c\, g_{i_o} > k_{i_o} g_{i_o} - g_r$.

Now, for $c = k_{i_o}$ this is satisfied, indeed, we know already that $\gamma^r_{i_o} \leq k_{i_o}$. Hence, for $c = k_{i_o} - 1, k_{i_o} - 2, \ldots, 1$ let us check wether

(15) $$(k_{i_o} - c)\, g_{i_o} < g_r$$

and

(16) $$M_{i_o-1} \hom_o (g_r - (k_{i_o} - c)\, g_{i_o})$$

are simultaneously satisfied. Again the c-matrix for M_{i_o-1} is known. Once for some c the answer is "no" we put $\gamma^r_{i_o} = c + 1$. If the answer is "yes" for all c, we put $\gamma^r_{i_o} = 1$.

Presumably, it is preferable to check for $t = 1,2,\ldots,k_{i_o}-1$ (i.e. $t = k_{i_o} - c$) whether

(17) $$t\, g_{i_o} < g_r$$

(18) $$M_{i_o-1}\ \mathrm{hom}_o(g_r - t\, g_{i_o})$$

If, for some t the answer is "no" put

(19) $$\gamma^r_{i_o} = k_{i_o} - t + 1$$

and if the answer is "yes" always, put

$$\gamma^r_{i_o} = 1.$$

<u>Remark 2.9.</u> Any test for homogeneity, given that the matrix C is known, takes place according to Lemma 1.3. That is, given M_r and λ, check first whether there is i_o, $1 \leq i_o \leq r$ and c, $1 \leq c \leq k_{i_o}$ such that $\lambda = \lambda^c_{i_o}$. Then check whether $c \geq c^r_{i_o}$.

SEC. 3 The algorithm

Collecting the pieces we now want to describe an algorithm for the matrix of homogeneity of a given measure $M = (g,k)$.

Now, for the sake of a consistent representation it is useful to carry a fellowship with players having weight 0; i.e. to consider k and g as specified by (1) and (2) of SEC. 1, where $g_0 = 0$. However, for the present algorithm this is not necessary; thus we deal with $g = (g_1,\dots,g_r) \in \mathbb{N}^r$ and $k = (k_0,\dots,k_r) \in \mathbb{N}^r$.

The algorithm is described in terms of "functions" defined on vectors of integers. The essential one is the last function, called CE, which yields the matrix C. However, the functions defined by I, II, III are necessary because of the recursive nature of our procedure.

I. Function IOC $(g; k; \lambda)$.

Entries : $g = (g_1,\dots,g_r) \in \mathbb{N}^r$; $k = (k_1,\dots,k_r) \in \mathbb{N}^r$; $\lambda \in \mathbb{N}$.

Output: $(i_0, c) \in \mathbb{N}_0 \times \mathbb{N}_0$.

Task: Determines i_0, c such that $\lambda = \lambda_{i_0}^c$ or otherwise reports failure if no such quantities exist.

Procedure:

1. Choose $i_0 \in \mathbb{N}$ such that

$$\sum_{i=i_0+1}^{r} k_i g_i < \lambda \leq \sum_{i=i_0}^{r} k_i g_i$$

2. Put $\Delta := \lambda - \sum_{i=i_0+1}^{r} k_i g_i$

3. If $g_{i_0} \mid \Delta$, then ⇒ 4; otherwise ⇒ 5

4. Put $c := \Delta / g_{i_0}$; put IOC $(g; k; \lambda) = (i_0, c)$ ⇒ 6

5. Put IOC $(g; k; \lambda) = (0, 0)$ ⇒ 6

6. END.

II. Function HOMN $(g; k; \lambda; c_{\cdot})$

Entries: $g = (g_1, \ldots, g_r) \in \mathbb{N}^r$

$k = (k_1, \ldots, k_r) \in \mathbb{N}^r$

$\lambda \in \mathbb{N}$

$c_{\cdot} = (c_1, \ldots, c_r) \in \mathbb{N}^r$

Output: + or - (HOMN is a "Boolean function")

Task: Given (the last row of the matrix C) $c_{\cdot}$, HOMN decides whether $M = (g,k)$ $\text{hom}_0\, \lambda$ or not.

Procedure:

1. If $\sum_{i=1}^{r} k_i\, g_i \leq \lambda$, then put HOMN $(\cdot,\cdot,\cdot,\cdot)$

   = + and ⇒ 4. Otherwise ⇒ 2.

2. If IOC $(g; k; \lambda) = (0,0)$, then put HOMN $(\cdot,\cdot,\cdot,\cdot)$

   = - and ⇒ 4.

   Otherwise put $(i_0, c) :=$ IOC $(g; k; \lambda)$ and ⇒ 3.

3. If $c \geq c_{i_o}$, then put HOMN $(\cdot,\cdot,\cdot,\cdot)$ $= +$ and $\Rightarrow$ 4. Otherwise, put HOMN $(\cdot,\cdot,\cdot,\cdot)$ $= -$ and $\Rightarrow$ 4

4. END.

III. <u>Function GAM $(g; k; g_{i_o}; g_r; c_.)$</u>

Entries: $g = (g_1,\ldots,g_{i_o-1}) \in \mathbb{N}^{i_o-1}$

$k = (k_1,\ldots,k_{i_o-1}) \in \mathbb{N}^{i_o-1}$

$g_{i_o}, g_r \in \mathbb{N}$

$c_. = (c_1,\ldots,c_{i_o-1}) \in \mathbb{N}^{i_o-1}$

Output: $\gamma^r_{i_o} \in \mathbb{N} \cup \{\infty\}$

Task: Computes $\gamma^r_{i_o}$ given row i_o of C.

Procedure:

1. If HOMN $(g; k; g_r; c_.) = -$, then put GAM $(\cdot,\cdot,\cdot,\cdot,\cdot) = \infty$ and $\Rightarrow$ 9. Otherwise $\Rightarrow$ 2.

2. If $g_{i_o} \mid g_r$ then put GAM $(\cdot,\cdot,\cdot,\cdot,\cdot) = 1$ and $\Rightarrow$ 9 . Otherwise $\Rightarrow$ 3.

3. Let $t = 1$ and $\Rightarrow$ 4.

4. If $t\, g_{i_o} < g_r$ and HOMN $(g; k; g_r - t g_{i_o}; c_.) = +$, then $\Rightarrow$ 5. Otherwise $\Rightarrow$ 7.

5. $t \to t + 1$; $\Rightarrow$ 6 .

6. If $t \leq k_{i_o} - 1$ then ⇒ 4. Otherwise ⇒ 7

7. Put GAM (•,•,•,•,•) = 1 and ⇒ 9.

8. Put GAM (•,•,•,•,•) = $k_{i_o} - t + 1$ and ⇒ 9 .

9. END.

IV. Function CE (g, k)

Entries: $g = (g_1,\dots,g_s) \in \mathbb{N}^s$

$k = (k_1,\dots,k_s) \in \mathbb{N}^s$

Output: Matrix $C = (c_{ir})_{i,r=1,\dots,s} \in \mathbb{N}^{s\times s}$

$c_{ir} = 0$ $(i > r)$ (or C triangular and

c_{ir} not defined for $i > r$)

Procedure:

1. Put $c_i^r = 0$ for all i and r.

2. Put $c_1^1 = 1$

3. For $j = 1,\dots,s$ put $l_1^j = 1$ if $k_1 g_1 < g_j$ or $g_1 \mid g_j$.
   Otherwise, put $l_1^j = k_1 - [\frac{g_j}{g_1}]$.

4. If $k_1 g_1 < g_2$ or $g_1 \mid g_2$, then put $c_1^2 = 1$ and $c_2^2 = 1$.
   Otherwise put $c_1^2 = l_1^2$ and $c_2^2 = \infty$.

5. Put $r = 3$ and $i_o = 1$.

6. If $c_{i_o}^r = \infty$, then ⇒ 12. Otherwise ⇒ 7.

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