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The rôle of nondegeneracy and homogeneity in n – person game theory: an equivalence theorem

Joachim Rosenmüller

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H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

Abstract

Nondegeneracy and homogeneity in the theory of finite side payment games serve as concepts which can be seen as surrogates for nonatomicity. Thus, nondegenerate convex games are extreme, and homogeneous superadditive games allow for nice v.N.–M.–solutions. When studying these phenomena one encounters the fact that the presence of many small players is required. Within this paper it is shown that (similar to the case of the L.P.–game) a suitable version of nondegeneracy can be employed for T.U.–market games in order to prove an equivalence theorem for the core and the Walrasian equilibrium.

1. Introduction

Let $\Omega = \{1,\ldots,n\}$ and $\mathbf{P} = \{S | S \subseteq \Omega\}$. Within the framework of Game Theory, the elements $i \in \Omega$ are called "the players" and $S \in \mathbf{P}$ is "a coalition". Any function $v : \mathbf{P} \to \mathbb{R}$, $v(0) = 0$, is a "characteristic function" (cf.) and the triple $(\Omega,\mathbf{P},v)$ is a (cooperative n–person) game. v (or the game) is simple if the range is $\{0,1\}$.

A system of coalitions (subsets) $\mathbf{Q} \subseteq \mathbf{P}$ is said to be nondegenerate (n.d.) if the linear system of equalations in variables $x_1,\ldots,x_n$ given by

$$\sum_{i \in S} x_i = 0 \qquad (S \in \mathbf{Q})$$

admits of the trivial solution $x^0 = 0 \in \mathbb{R}^n$ only, that is, if the "incidence matrix"

$$Q = (1_{S^{(i)}})_{\substack{S \in Q \\ i \in \Omega}}$$

has rank n. Here, 1_S is the indicator function of $S \in \mathbf{P}$, to be identified with the vector $1_S = (1_{S^{(1)}},\ldots,1_{S^{(n)}})$.

Vectors $m = (m_1,\ldots,m_n) \geq 0$ and "measures" $m : \mathbf{P} \to \mathbb{R}$ are identified via

$$m(S) = \sum_{i \in S} m_i \qquad (S \in \mathbf{P})$$

If $\alpha \in \mathbb{R}_+$, then m is nondegenerate w.r.t. α (m n.d. α)

$$\mathbf{Q} = \mathbf{Q}_\alpha = \mathbf{Q}_\alpha(m) = \{S | m(S) = \alpha\}$$

is n.d., i.e., if m is uniquely defined by its sets of measure α (its "α contour").

Moreover, m is called homogeneous w.r.t. α (m hom α) if $m \geq \alpha$ and, for any $S \in \mathbf{P}$, s.t. $m(S) > \alpha$, there is $T \subseteq S$ with $m(T) = \alpha$.

Our main thesis to be supported by the results of this paper is the following: non–degeneracy and homogeneity are related concepts which, in a finite framework (finitely many players), serve as a surrogate for nonatomicity (which is a concept for continuously many players).

This statement must be regarded in the light of the following evidence

1. Superficially homogeneity is satisfied in the continuous case: e.g. a non atomic measure in the Borel (or Lebesgue) sets of the unit interval is homogeneous with respect to every α within its range. The obvious objection that the quantifications are different is slightly weakened by the fact that (in the finite case) the homogeneity of m with respect to α implies the homogeneity with respect to various other values within the range; see [14] [15].

2. Analogously it is easily seen that, considering a non atomic measure on the unit intervall this measure is nondegenerate with respect to every α within it's range; that is every non atomic measure is uniquely determined by any "contour" $\mathbf{Q}_\alpha$.

3. It is possible to analyse sufficient conditions which ensure nondegeneracy and homogeneity in the finite case by introducing "types" or "fellowships", that is, to collect those players which, with respect to a given measure, have equal weight. Then sufficient conditions in order to ensure "n.d." or "hom" may be interpreted in a way such that, roughly speaking, it is the presence of many small players compared to the large ones which implies the desired properties. [7] [9].

4. Both concepts are closely related: Essentially it can be said that homogeneity is something like nondegeneracy with several degrees of freedom. In other words the incidence matrix of the system $\mathbf{Q}_\alpha$ in case of homogeneity is non singular if, in addition, we prescribe the value of certain variables x_i (see the definition of nondegeneracy above) in advance, i corresponding to a certain type (fellowship) of players, the so called steps, see [4] [12] [13].

5. There seems to be a wide range of applications in the framework of games and equilibrium theory where conditions n.d. and hom serve a similar purpose and frequently replace the requirement of non atomicity. E.g. the unique minimal homogeneous representation of a homogeneous simple weighted majority game (see [4] [12] [13]) coincides with the unique representation given already in [3] for the case that the game is zero sum; and if so, it follows that homogeneity implies nondegeneracy (that is, a zero sum game nessesarily has no degrees of freedom), see [4] [12] [13]. Many results have been obtained in the finite case which can be imitated in the continuous case by non atomic games (or economics). For instance the extreme point problem in the cone of non negative convex games may be attacked by composing extreme point functions with nondegenerate measures, see [6] [14] [15]. A similar result is obtained if nondegeneracy is replaced by nonatomicity in the continuous case. Similarly, extreme super additive games may be obtained by composing non degenerate and homogeneous measures (in the finite case; non atomic measures in the continuous case) with super additive point–to–point functions.

6. The construction of certain elements of the core and the construction of von Neumann–Morgenstern solutions has been particularly successfull when extreme games of the above type are being considered. ([7])

7. Finally, there is evidence that our concepts may be useful also in the context of equivalence theorems in equilibrium theory. For instance the L.P.–Game (in which case the convergence of the Core to the C.E. is finite) has been studied in [5], the continuous case was studied in [1], and finally the nondegenerate case turned out to be related to both of the previous ones as have been treated in [9].

In this paper we give further examples for the applicability of nondegeneracy concepts to the main equivalence theorem of equilibrium theory. It is shown that non–degeneracy can be meaningful defined in the framework of a T.U.–market game. In marked contrast to the commonly used version, we employ piecewise linear utility functions for the consumers or players. Using a version of nondegeneracy we shall then be able to represent regions in the space $\mathbb{N}^r$ of distributions of players over the r types in the market such that the core and the Walrasian equilibrium coincide.

2. Some applications

To make this paper self contained we shall consider a few of the examples mentioned in the first section.

Example 2.1. For $0 < \alpha < 1$, let $f^\alpha: [0,1] \to [0,1]$ be defined by

$$(1) \qquad f^\alpha(t) = \frac{1}{1-\alpha}(t-\alpha)^+$$

(where $\beta^+ = \max(0,\beta)$ $(\beta \in \mathbb{R})$). If m is a normalized measure $(m(\Omega) = 1)$, consider the cf. given by

$$(2) \qquad e = e^\alpha_m = f^\alpha \circ m.$$

Because f^α is convex, e is a convex cf. (see [6][14][15]). The (nonnegative) normalized convex cf.s form a compact convex polyhedron (in $\mathbb{R}^{2^n}$), say $\mathbb{C}^1$. It is well known ([7][14]) that e^α is an extreme point of $\mathbb{C}^1$ if m n.d. α. On the other hand, if e^α is an extreme point of $\mathbb{C}^1$ and $m_i \leq 1-\alpha$ $(i \in \Omega)$, then n.d. α.

The continuous analogue is found in ([6]): if m is a measure on the unit intervall, then again e is extreme if m is nonatomic.

Remark 2.2. The general theory of extreme convex games is established in [15]. It turns out that every convex (nonnegative, normalized) cf. v is represented by

$$v(\cdot) = \max(m^1(\cdot)-\alpha_1,\ldots,m^t(\cdot)-\alpha_t)$$

where $m = (m^1,\ldots,m^t)$ is a vector valued measure (i.e., m^τ is a measure, $\tau = 1,\ldots,t$) and $\alpha = (\alpha_1,\ldots,\alpha_t) \in \mathbb{R}^t_+$.

By suitably redefining "m n.d. α" the connections between extremality and n.d. are exactly equivalent to the ones stated in 2.1.

Example 2.3. A simple cf. is a "weighted majority" if there is a measure $m \geq 0$ and $\alpha \in \mathbb{R}_+$ s.t.

(3) $$v = 1_{[0,\alpha]} \circ m$$

i.e.

$$v(S) = \begin{cases} 1 & m(S) \geq \alpha \\ 0 & m(S) < \alpha \end{cases}$$

In this case we write $v = v^m_\alpha$ and (m,α) is called a representation of v. v is homogeneous if there is a representation (m,α) with m hom α. Given the condition that v is constant sum and dummies get zero weight (i.e. $m_i = 0$ whenever i is not in the carrier of v), the homogeneous representation is unique. In addition, in this case we have m n.d. α ([8]). (On the other hand, m n.d. α implies always that m and v have equal carrier.)

In order to emphasize the analogues between the convex case and the case of superadditive cf. s, again let $\alpha = (\alpha_{1,\dots,}\ \alpha_K) \in \mathbb{R}^K_+$, with increasing coordinates and $\alpha_K < 1$. Consider the function $f^c_\alpha : [0,1] \rightarrow \Theta$

(4) $$f^c_\alpha := \sum_{\varkappa=1}^{K-1} c_\varkappa 1_{[\alpha_\varkappa,\alpha_{\varkappa+1})} + 1_{[\alpha_\varkappa,1]}$$

where $c_\varkappa \in \{0, 1/N, 2/N, \dots, 1\} =: \Theta$ with a suitable large integer N. Pick a (normalized) measure m or Ω such that the range of m is included in Θ and put

(5) $$v^{c,m}_\alpha := f^c_\alpha \circ m$$

Then the following holds true ([7]): if f^α is extreme among the superadditive (normalized) functions $[0,1] \rightarrow \Theta$, then $v^{c,m}_\alpha$ is extreme among the superadditive (normalized) cf.s on Ω, provided that (essentially)

$$m \text{ hom } (\alpha_{\varkappa+1} - \alpha_\varkappa) \qquad \varkappa = 0,\dots,K-1;\ \alpha_0 := 0.$$

is satisfied (see [7]).

The continuous analogue (m a measure, say, on [0,1]) is not in the literature, but apparently straigthforward.

Remark 2.4. Note that the general superadditive (normalized) cf. v may be represented as

(6) $$v = \max_{\tau=1,\dots,T} f^{c^\tau}_{\alpha^\tau}$$

with suitable vectors c^τ, α^τ, $\tau=1,\dots,T$. However, the general characterization of extreme superadditive games (by means of hom or n.d. requirements) is not solved.

Example 2.5. The L.P.–game provides a first clue towards the connection of n.d. and the (first) equivalence theorem of general equilibrium theory (i.e., the "equivalence" of the core and the Walrasian equilibrium) : in this case nondegeneracy serves as a "surrogate" for either the replica version or the nonatomic measure space of agents (see e.g. [1] [5] [9]).

Let $A > 0$ be an $m \times l$ matrix and let $c \in \mathbb{R}^l_+$. Also, fix a vectorvalued measure $b = (b^1,\dots,b^m)$, i.e., b^μ ($\mu=1,\dots,m$) is a measure on Ω. Then a cf. $v = v^{A,b,c}$ is defined on Ω by

$$v(S) = \max \{cx \mid x \in \mathbb{R}^l_+,\ Ax \le b(S)\}$$

and $(\Omega,\mathcal{P}, v^{A,b,c})$ is called an L.P.–game. This game was treated by [5] (with a replica setup) while [1] considered the case of a nonatomic measure space of players.

Now, if $y^0 = (y^0_1,\dots,y^0_m)$ is an optimal solution of the "dual Ω–program", i.e.

$$v(\Omega) = y^0 b(\Omega) = \min \{yb(\Omega) \mid y \in \mathbb{R}^m_+,\ yA \ge c\}$$

then the measure $m^0 := y^0 b(\cdot)$ is an element of the core of v, say $m^0 \in \mathcal{C}(v)$.

Define

(7) $Q_0 := \{z \in R^m_+ \mid \min\{yz \mid y \in \mathbb{R}^m_+, Ay \geq c\} = y^0 z\}$

and a system **Q** of coalitions (subsets of Ω) by

(8) $\mathbf{Q} := \{S \in \mathbf{P} \mid b(S) \in Q_0, b(S^c) \in Q_0\}$.

Now we have ([9]): If **Q** is n.d., then $\mathcal{C}(v) = \{m^0\}$.

Thus, if there are sufficiently many coalitions having shadow prices equivalent to those of the grand coalition, then the core and the competitive equilibrium coincide. "Sufficiently many" means "in order to determine m^0 uniquely" – and will later be translated into "there are sufficiently many players".

3. Sufficient conditions

Introducing n.d. and hom, as has been argued, is a unifying procedure as well as a "finite surrogate" for the nonatomic measure space. However, the idea serves to little purpose unless (as a second step), we are capable of understanding our concepts structurally. By this we mean that operationable conditions should be specified e.g. in order to test whether a specified measure m is n.d. α or hom α. If such conditions or tests are sufficiently enlightening, then they should, simultaneously, provide additional hints concerning an interpretation of our concepts as versions of a player space with many small individuals.

These problems have been dealt with in the framework of several of the above mentioned papers, e.g. [7] [9] [14] [15]. Here we shall shortly review the n.d.–results related to Examples 2.1. and 2.5. Also, we want to hint at the (related) results concerning the minimal homogeneous representation of a homogeneous simple game. (Example 2.3.)

By well known reasons (see [14] [15]) we know that we may restrict ourselves to the domain of reationals when dealing e.g. with the question of a sufficient condition for m n.d. α. Of course this means that (by leaving the normalization of type $m(\Omega) = 1$) we may deal with integer–valued measures, say M (and integer constants, say λ, instead of rational α).

Next, it is reasonable to introduce groups of identical players, i.e., elements $i,j \in \Omega$ such that their measure or weight is equal. As types are generally considered to consist of players which are identical w.r. to the game, we use the term <u>fellowship</u> for a group of players with the same measure.

Up to permutations of players of the same fellowship a measure is then completely defined by specifying the distribution of the players over the fellowships and the (common) weight of (every player of) each fellowship.

This induces the following alternative formulation of our basic definitions.

Let $k = (k_1,\ldots,k_r) \in \mathbb{N}^r$; k_i represents the number of players in group or "fellowship" i where $i \in R = \{1,\ldots,r\}$. A vector $s \in \mathbb{N}^r$, is a profile feasible for k if $s \leq k$ (coordinate – wise).

Let $P = \{s \mid s \in \mathbb{N}^r, s \leq k\}$ denote the profiles; any profile enumerates a number of players within every fellowship – i.e., there are s_i players of fellowship i. A cf. (again called v) is a mapping taking profiles into th reals, i.e.,

$$v : P \to \mathbb{R}$$

or

$$v : P \to \{0,1\}$$

(the "simple" case). A game is given by the tupel (k,P,v). Frequently, however it is not necessary to mention P explicitely; so (k,v) is considered to constitute the game.

Next, suppose that $g = (g_1,\ldots,g_r) \in \mathbb{N}^r$ (frequently we assume $g_1 \leq g_2 \leq \ldots \leq g_r$). g is as well regarded as a function on profiles via

$$(1) \qquad g(s) = \sum_{i=0}^{r} s_i g_i \qquad (s \in P)$$

thus, g sometimes is referred to as a "measure" or a "weight vector". The pair $M = (k,g)$ is (by technical reasons, see [11]) somewhat sloppily, also called a "measure" – although it is rather a game.

Given $\lambda \in \mathbb{N}$ and $M = (k,g)$ such that $g(k) \leq \lambda$, the triple $(k;g;\lambda) = (M,\lambda)$ generates a cf. $v = v^M_\lambda$ by

$$(2) \qquad v(s) = \begin{cases} 1 & g(s) \geq \lambda \\ 0 & g(s) < \lambda \end{cases} \qquad (s \leq k)$$

and (M,λ) is a representation of v.

The familiar framework as established in sections 1 and 2 is closely related: put

$$n := \sum_{i=1}^{r} k_i$$

and define a decomposition of $\Omega = \{1,\ldots,n\}$ by specifying subsets of Ω appropriately, e.g.

(3) $$K_j := \{\omega\in\Omega \mid \sum_{i=0}^{j-1} k_i < \omega \le \sum_{i=0}^{j} k_i\}$$

(where $k_0 := 0$). Then, if v is defined on the profiles feasible for k,

$$w(S) := v((|S\cap K_1|,\ldots,|S\cap K_{r+1}|) \qquad (S \in \mathbf{P})$$

provides a cf. w in sense of section 1.

Let us now rephrase partially some of our previous formulation concerning surrogates for nonatomicity. E.g., we say that M n.d.λ if g is the unique solution of the linear system in variables $y_1,\ldots,y_r$ given by

$$\sum s_\varrho y_\varrho = \lambda$$

where $s = (s_1,\ldots,s_r)$ is taken from the set of feasible profiles such that $\sum s_\varrho g_\varrho = \lambda$. In other words, given $M = (k,g)$ and λ, n.d. is ensured if we can construct r profiles (integer vectors) s^σ $(\sigma=1,\ldots,r)$ satisfying

(4)

1. $0 \le s^\sigma_\varrho \le k_\varrho$ $\qquad (g=1,\ldots,r)$
2. $Q = (s^\sigma_\varrho)_{\varrho,\sigma=1,\ldots,r}$ $\qquad$ is nonsingular
3. $\sum_{\varrho=1}^{r} s^\sigma_\varrho g_\varrho = \lambda$

Thus, the problem "find integer vectors $s^\varrho \in \mathbb{N}^r_0$ such that (4) is satisfyed" amounts to testing n.d. (say in the framework of Example 2.1.).

Similarly, Example 2.5 (the L.P.–game) is rewritten as follows: with a suitable "tableau matrix" Λ, the set Q_0 (cf. (7) of section 2) may be written

$$Q_0 = \{z \in \mathbb{R}^m_+ \mid \Lambda z \geq 0\}.$$

The vector valued measure b of initial resources in 2.5 amounts to a vector valued measure, say

$$G = (k; g^1, \ldots, g^m)$$

such that $g^j \in \mathbb{N}^r$ describes the distribution of factor j over the fellowships 1,...,r. The required nondegeneracy of **Q** (in (8), section 2) is reflected by the assumption that profiles

$$\{s \mid \Lambda \sum_{\varrho=1}^{r} s_\varrho G_\varrho \geq 0, \quad \Lambda \sum_{\varrho=1}^{r} (k_\varrho - s_\varrho)\, G_\varrho \geq 0)$$

$$= \quad \{s \mid 0 \leq \Lambda \sum_{\varrho=1}^{r} s_\varrho\, G_\varrho \leq \Lambda \sum_{\varrho=1}^{r} k_\varrho\, G_\varrho\}$$

constitute a matrix of full rank. (Of course, we use $G_\varrho = (g^1_\varrho, \ldots, g^m_\varrho)$, i.e., fellowship ϱ holds this vector of resources 1,...,m).

Thus, if we can solve the problem

"Find integer vectors $s^\sigma \in \mathbb{N}^r_0$, $\sigma = 1, \ldots, r$, such that

1. $0 \leq s^\sigma_\varrho \leq k_\varrho$ $\quad (\sigma,\ \varrho = 1, \ldots, r)$

(5) 2. $Q = (s^\sigma_\varrho)_{\varrho,\sigma=1,\ldots,r}$ is nonsingular

3. $0 \leq \Lambda \sum_{\varrho=1}^{r} s^\sigma_\varrho\, G_\varrho \leq \Lambda \sum_{\varrho=1}^{r} k_\varrho\, G_\varrho$ "

then we have "**Q** n.d." – and hence the core and the c.e. of the L.P.–game coincide. Note, that in (5), G specifies the distribution of factors over fellowships while Λ (the tableau matrix of the linear program

A, $\sum_{\varrho=1}^{r} k_\varrho G_\varrho$, c, corresponding to "the" optimal dual solution y^0

(cf. (7) in section (2))) represents the production process involved with the L.P.–game.

The combinatorial (number theoretical) problems suggested by (4) and (5) obviously are of similar type. Similar problems arise in the framework of Remark 2.2.

It remains to find reasonable sufficient conditions in order to establish that solutions to (4) and (5) exist.

With respect to (4), this completed already in [14]; the result is, roughly speaking, as follows:

Given $g = (g_1,\dots,g_r) \in \mathbb{N}^r$, there are "reasonable" bounds R_r, L_ϱ $(\varrho=1,\dots,r) \in \mathbb{N}$ (depending on g in a constructive way) such that, whenever

(6) $$k_\varrho \geq L_\varrho \qquad (\varrho=1,\dots,r)$$

and

(7) $$R_r \leq \lambda \leq \sum k_\varrho g_\varrho - R_r$$

as well as

(8) $$\lambda \equiv 0 \bmod \text{g.c.d. of } g_1,\dots,g_r$$

then (4) can be solved, i.e., there are profiles $s^1,\dots,s^r$ with the properties required for n.d. In other words, if there are sufficiently many players (see (6), k_ϱ is the number of players of fellowship ϱ), then there is a nonempty (reasonable) intervall $[R_r, g(k)-R_r]$ (see (7)) within the ideal spanned by $g_1,\dots,g_r$ (see (8)) such that all λ's within this intervall allow for M n.d.λ, i.e. e.g., such that e^{α}_{m} ((m,α) obtained by normalizing (M,λ)) is an extreme convex set function.

Similarly, the question of solving (5) leads to a problem in geometric number theory.

For, the set

$$E^k_{\Lambda,G} = \{x \in \mathbb{R}^r | 0 \leq x \leq k, 0 \leq x \Lambda G \leq k \Lambda G\}$$

defines a compact convex polyhedron in $\mathbb{R}^r$.

Obviously solving problem (5) is equivalent to finding r integer vectors within this polyhedron.

Given the date of the L.P. – game and the distribution of a factor with the fellowships (i.e. the tableau matrix Λ and the matrix G), we may therefore attempt to specify regions in $\mathbb{N}^r$ such that, if $k \in \mathbb{N}^r$ is an element of such a region, then the corresponding $E^k_{\Lambda,G}$ admits of r linearly independent integer vectors, hence (5) can be solved, hence the core and the Walrasian equilibrium coincide. The tools for this task are provided by geometric number theory; in fact MINKOWSKY's (2nd) theorem tells us that the number of linearly independent integer vectors in a compact convex polyhedron can be linked to the volume of this polyhedron. Now, obviously, if k increases then the volume of the polyhedron $E_{\Lambda,G}$ will increase, hence we conclude that for large k (that is, "many players") we solve problem (5) and establish the equivalence theorem for the core and the Walrasian equilibrium.

In the case of the L.P. – game the convergence of the core towards the equilibrium is finite; this fact of course depends strongly on the linearity of the model. The question is indeed, whether the method produced in order to describe this type of conversions can be extended to nonlinear models when the convergence is not finite.

This is the aim of the present exposition: We want to show that, for a side payment (transferable utility) market game, similar mechanisms can be established. We shall use a simple model (and assume piecewise linear and convex utility functions), so the existence of core and equilibrium as well as the shrinking of the first concept towards the second one are well established facts. However, we do neither hinge on the replica nor on the non atomic measure space agents. Instead, we shall argue that the appropriate version of nondegeneracy ensures that for large k (distribution of players over the various types of fellowships, in a suitable sense) any core payoff equals the payoff in equilibrium. Hence, our arguments are referring to utility space: as yet, it is not clear whether the procedure may be employed in the space of allocations.

Let us close this section by a few remarks concerning the connection between the concepts of homogeneity and nondegeneracy. Speaking in terms of profiles, "M hom λ" is of course specified by requiring that $g(k) > \lambda$ and, whenever $g(s) > \lambda$, then there exists t such that $g(t) = \lambda$.

If M hom λ, then consider v^M_λ (cf.(2)). The representation theory of homogeneous simple games as developed by OSTMANN [4] and ROSENMÜLLER [12] shows that it is possible to recursively label fellowships, the labels being called "dummy", "sum", and "step", and referred to as characters. This in other words leads to a decomposition of fellowships into three classes of characters. The role of the dummies is clear. The role of a sum is described as follows. In certain minimal winning coalitions a sum may be replaced by a specific group of smaller players (not the immidiate followers), the weight of which sums up to the weight of the "sum". As for steps, they are of quite a contrary nature: They dominate their successors in a way such that in every minimal coalition including a smaller player, all preceeding steps must as well appear completely. Steps in a sense cannot be replaced properly by groups of smaller players.

For the details compare the theory as presented in [4], [12], [13].

The process of introducing characters has consequences, the most important one being the existence of the unique minimal representation of a homogeneous game. This representation is as well computed recursively. As a byproduct of the theory, it turns out that the smallest player is always a step. If there are no further steps then M n.d. λ. A particular interesting case is presented by the zero–sum simple homogeneous game; here we have the theorem that there are no steps but the players of the smallest type. Hence, the minimal representation yields the unique representation of von NEUMANN and MORGENSTERN [3].

Concluding, it is seen that nondegeneracy, in a sense, is a special case of homogeneity, given additional requirements. In fact, in a homogeneous simple game the weight of the steps may be chosen more or less arbitrary within certain constrains imposed upon by the representation theory. The weight of the sums, however, is dictated uniquely by the structure of the game once the weight of the steps is fixed. Hence, a homogeneous game has as many "degrees of freedom" as steps are present. If there is just one degree of freedom (only the smallest type is a step) then the game is nondegenerate.

In other words homogeneous games with the minimal degree of freedom are nondegenerate.

As for sufficient conditions in order to render a given pair $M = (k,g)$ homogeneous w.r.t. a certain λ, compare the theory of the "matrix of homogeneity" as introduced in ROSENMÜLLER [11]. Here again it is seen that essentially there should be many small players in order to induce homogeneity.

4. T.U. – market games

A T.U. – market ("transferable utility") is a pair

$$u = (U,A)$$

where $U = (u^i)_{i\in R}$; $u^i : \mathbb{R}^m_+ \to \mathbb{R}_+$;

and $A = (a^i)_{i\in R}$; $a^i \in \mathbb{R}^m_{++}$. a^i represents the initial allocation of each player of fellowship i. u^i is the utility function of (each member of fellowship) i. u^i is assumed to be (weakly) monotone, continuous, and concave. However, we do not assume differentiability, rather, each u^i is required to be piecewise linear. More precisely, we assume the existence of finitely many affine functions

(1) $\qquad p^{il} : \mathbb{R}^m_+ \to \mathbb{R};\ p^{il}(x) = c^{il} x + d^{il}$

($x\in\mathbb{R}^m_+$, $c^{il}\in\mathbb{R}^m_+$, $d^{il}\in\mathbb{R}$), $i\in R$ $l\in L$ (where L is a finite index set which, w.l.o.g., does not depend on i) such that

(2) $\qquad u^i = \min_{l\in L} p^{il}$

We denote by

$$P^{il} = \{x\in\mathbb{R}^m_+ \mid u^i(x) = p^{il}(x)\}$$

and P^{0il} is the interior of the convex polyhedron P^{il}.

Given $u = (U,A)$, define the function

$$f = f^u : \mathbb{R}^r_+ \to \mathbb{R}$$

$$f(t) = \max \{ \sum_{i\in R} t_i u^i(x^i) \mid x^i\in\mathbb{R}^m_+ \ (i\in R);\ \sum_{i\in R} t_i x^i = \sum_{i\in R} t_i a^i \}$$

Whenever $k\in\mathbb{N}^r$, then $v^k = v^{k,u}$ denotes the restriction of f to the feasible profiles of k and (k, v^k) is the market game generated by u (and k) where k_i players of fellowship i are present. For any feasible profile $s \leq k$, $v^k(s)$ denotes the maximal joint utility players in a coalition with profile s may obtain by pooling their initial allocations and redistributing them. Therefore, for $t \in \mathbb{R}^r_+$

$$A_t = \{X = (x^i)_{i=1,\dots,r} | x^i \in R^m_+, (i \in \mathbb{R}), \sum_{i \in R} t_i x_i = \sum_{i \in R} t_i a^i\}$$

is the set of "feasible allocations for t".

Clearly, the function f^u is continuous, concave, and positively homogeneous (the proofs given in SHAPLEY–SHUBIK [16] apply at once). However, we have to spent some time for exhibiting the structural properties of f^u resulting from the piecewise linearity of the u^i.

A feasible allocation X^0 is <u>optimal</u> for t if

$$f(t) \quad = \sum_{i \in R}^{r} t_i u^i(x^{0i})$$

i.e., if X^0 is a maximizer in (1). Let

(5) $\qquad X_t = \{X \in A_t \mid X \text{ is optimal for } t\},$

this is a convex, closed set which is compact for $t > 0$. It is a well known consequence of a "Kuhn–Tucker" principle that we have

<u>Lemma 4.1.</u> $X^0 \in A_t$ is optimal for $t \in \mathbb{R}^r_+$ if and only if there is $p^0 \in \mathbb{R}^m_+$ such that, for any $y \in \mathbb{R}^m_+$

(6) $$u^i(x^{0i}) - u^i(y) \geq p^0(x^{0i} - y)$$

holds true for all i such that $t_i > 0$.

p^0 represents a joint hyperplane supporting the graph of each u^i in x^{0i}; also if $t = k \in \mathbb{N}^r$ then p^0 can be interpreted as an equilibrium price vector (normalized as to let the $m+1$–coordinate ("money") appear as the numeraire; see [16] [8]. Clearly, if $x^{0i} \in P^{0il}$ for some i and l, then, in view of the special form of u^i, we have necessarily $p^0 = c^{il}$ (see (1) and (2)).

Definition 4.2. $p^0 \in \mathbb{R}^m_+$ is a Λ–price for $t \in \mathbb{R}^r_+$ if there is $X^0 \in A_t$ such that (6) is satisfied for all i such that $t_i > 0$. If so, then $X^0 \in X_t$ automatically and we say that p^0 corresponds to X^0.

Lemma 4.3. Let $p^0 \in \mathbb{R}^m_+$ then

$$T_{p^0} := \{t \in \mathbb{R}^r_+ \mid p^0 \text{ is a } \Lambda\text{–price for } t\}$$

is a convex cone.

Proof: T_{p^0} is a cone because of the positive homogeneity of f. Convexity follows by means of a standard procedure (employed in [16] [8]) as follows:

Suppose, t and t' are such that p^0 is a Λ–price for both of them, corresponding to the maximizers $X^0 \in X_t$ and $X^1 \in X_{t'}$. Define,

for $i \in R$ (and $t_i + t'_i > 0$)

$$x^{+i} := \frac{t_i x^{0i} + t'_i x^{1i}}{t_i + t'_i}$$

($x^{+i} = 0$ if $t_i + t'_i = 0$). Then clearly $X^+ \in A_{t+t'}$. For any $y \in \mathbb{R}^m_+$ we find ($i \in R$; $t_i + t'_i > 0$)

$$
\begin{aligned}
& u^i(x^{+i}) - u^i(y) \\
&= u^i\left(\frac{t_i x^{0i} + t'_i x^{1i}}{t_i + t'_i}\right) - u^i(y) \\
&\geq \frac{t_i}{t_i + t'_i}(u^i(x^{0i}) - u^i(y)) + \frac{t'_i}{t_i + t'_i}(u^i(x^{1i}) - u^i(y)) \\
&\geq \frac{t_i}{t_i + t'_i}(p^0(x^{0i} - y)) + \frac{t'_i}{t_i + t'_i}(p^0(x^{1i} - y)) \\
&= p^0(x^{+i} - y),
\end{aligned}
$$

meaning that p^0 is a Λ–price for $X^+ \in X_{t+t'}$.

Note that any boundary point $t\in\mathbb{R}^m_+$ of T_{p^0} with strictly positive coordinates belongs to T_{p^0}, i.e., for any $\epsilon > 0$ the set $T_{p^0} \cap \{t \geq (\epsilon,\ldots,\epsilon)\}$ is closed.

<u>Lemma 4.4.</u> Let $t \in \mathbb{R}^r_+$ and let p^0, p^1 be Λ–prices corresponding to $X^0\in\mathcal{X}_t$ and $X^1\in\mathcal{X}_t$. Then p^0 corresponds to x^1 as well (and vice versa).

<u>Proof:</u> Because of

$$u^i(x^{0i}) - u^i(y) \geq p^0(x^{0i}-y)$$
$$u^i(x^{1i}) - u^i(y) \geq p^1(x^{1i}-y)$$

we have

$$p^1(x^{0i}-x^{1i}) \geq u^i(x^{0i}) - u^i(x^{1i}) \geq p^0(x^{0i}-x^{1i}) \tag{7}$$

for all $i\in R$ with $t_i > 0$ and $y\in\mathbb{R}^m_+$. Clearly, any $>$ – sign for some i in (7), $t_i > 0$, implies

$$\sum_{i\in R} t_i p^1(x^{0i}-x^{1i}) > \sum_{i\in R} t_i p^0(x^{0i}-x^{1i}),$$

contradicting the fact that X^0 and X^1 are feasible for t.

Thus, (7) implies

$$p^1(x^{0i}-x^{1i}) = u^i(x^{0i}) - u^i(x^{1i}) = p^0(x^{0i}-x^{1i}) \tag{8}$$

for all $i\in R$ such that $t_i > 0$. Hence, for $y\in\mathbb{R}^m_+$:

$$u^i(x^{0i}) - p^1x^{0i} = u^i(x^{1i}) - p^1x^{1i} \geq u^i(y) - p^1y,$$

meaning that p^1 is a Λ–price for t corresponding to X^0.

<u>Remark 4.5.</u> Let $t\in\mathbb{R}^r_+$ and define

$$\Lambda_t := \{p^0\in\mathbb{R}^m_+ | p^0 \text{ is a } \Lambda\text{–price for } t\}. \tag{9}$$

The set Λ_t is a closed convex polyhedron, the extreme points of which are elements of $\{c^{il} | i\in R,\ l\in L\}$. If, for some $X^0\in\mathcal{X}_t$ we have $x^{0i} > 0$ $(i\in R)$, then Λ_t is indeed compact.

Given $p^0 \in \Lambda_t$, define

(10) $$\mu^t = \mu^{t,p^0} = (\mu^t_1,\dots,\mu^t_r)$$

by

$$\mu^t_i = \begin{cases} u^i(x^{0i}) - p^0(x^{0i}-a^i) & (i\in R,\ t_i > 0) \\ 0 & (i\in R,\ t_i = 0) \end{cases}$$

μ^t is interpreted as an "equilibrium" payoff (for t at price p^0). As indicated by Lemma 4.4. (and by (8)), μ^t does not depend on the particular choice of $X^0 \in X_t$. If Λ_t is a singleton, i.e. if some $X^0 \in X_t$ satisfies $x^{0i} \in P^{0il}$, then μ^t is uniquely defined (i.e., does not vary with p^0.

Next, introduce the sets

$$T^{il} = \{t \in \mathbb{R}^r_+ | c^{il} \in \Lambda_t\}$$

$$= \{t \in \mathbb{R}^r_t | c^{il} \text{ is a } \Lambda-\text{price for t}\}$$

It is seen at once that $\mathbb{R}^r_+ = \bigcup_{\substack{i\in R \\ l\in L}} T^{il}$.

Thus, $\mathbb{R}^r_+$ is covered by finitely many convex (and hence polyhedral) cones. Moreover, we have

<u>Lemma 4.6.</u> f restricted to T^{il} is affine. Within the interior of each T^{il}, f is differentiable and, if $t \in T^{il}$, $X^0 \in X_t$ and $p^0 = c^{il}$ (a $\Lambda-$price for t) then

$$\frac{\partial f}{\partial t_i}(t) = u^i(x^{0i}) - p^0(x^{0i}-a^i) = \mu^t_i$$

<u>Proof:</u> Let t and X^0, p^0 as above and consider any $t' \in T^{i1}$, $X^1 \in \mathcal{X}_{t'}$ which also admits $p^0 = c^{i1}$ as a Λ–price. Because of

$$u^i(x^{0i}) - u^i(y) \geq p^0(x^{0i} - y)$$
$$u^i(x^{1i}) - u^i(y) \geq p^0(x^{1i} - y)$$

$(y \in \mathbb{R}^m_+)$ we have as usual

$$u^i(x^{0i}) - u^i(x^{1i}) = p^0(x^{0i} - x^{1i})$$

and hence

$$\begin{aligned} f(t') - f(t) &= \sum (t'_i u^i(x^{1i}) - t_i u^i(x^{0i})) \\ &= \sum t'_i (u^i(x^{1i}) - p^0(x^{1i} - a^i)) \\ &\quad - \sum t_i (u^i(x^{0i}) - p^0(x^{0i} - a^i)) \\ &= \sum (t'_i - t_i)(u^i(x^{0i}) - p^0(x^{0i} - a^i)) \\ &\quad + \sum t'_i \underbrace{(u^i(x^{1i}) - p^0(x^{1i} - a^i) - [u^i(x^{0i}) - p^0(x^{0i} - a^i)])}_{0} \\ &= \sum t'_i (u^i(x^{0i}) - p^0(x^{0i} - a^i)) + \text{const} \\ &= \sum (t'_i - t_i) \mu^t_i . \end{aligned}$$

After having delt with the structure of $f = f^{\mathbf{u}}$ as resulting from the piecewise linearity of the utility functions, we now turn to the properties of the market games induced by the T.U. market **u**. Denote by $v^k = v^{\mathbf{u},k}$ the restriction of f to the set of feasible profiles of k; thus

$$(k, v^k)$$

is the <u>market game</u> generated by **u** (and k) with k_i players of fellowship i $(i \in R)$. Now, $v^k(s)$ indeed denotes the maximum joint utility players in a coalition with profile s may obtain by pooling their initial allocations and redistributing them.

The <u>core</u> of a characteristic function v, defined on the profiles of k is

(11) $\mathcal{C}(v) = \{m \in \mathbb{R}^r_+ \mid m(k) = v(k),\ m(s) \geq v(s)\ (s \leq k)\}.$

Intuitively, we impose a restriction on our treatment by introducing definition (11): we shall only consider "equal treatment elements" of the core. However, by symmetry reasons it is advisable (see [9]) to require that the greatest common divisor of k is a multiple of 2 and thus (see e.g. GREEN [2]), equal treatment in the core will be a consequence.

Next, if some $\mu^k = \mu^{k,p^0}$ is considered as a function on $\{s \leq k\}$ via

$$\mu^k(s) = \sum_{i=1}^{r} s_i \, \mu^k_i$$

as usual, then, for any $X^0 \in X_k$ we have

(12) $$\mu^k(k) = \sum k_i \, \mu^k_i = \sum k_i u^i(x^{0i}) = v^k(k)$$

and, if p^0 is a Λ–price for k and $X^1 \in X_s$ for some profile $s \leq k$,

(13) $$\begin{aligned} \mu^k(s) &= \sum s_i \, \mu^k_i \\ &= \sum s_i(u^i(x^{0i}) - p^0(x^{0i} - a^i)) \\ &\geq \sum s_i(u^i(x^{1i}) - p^0(x^{1i} - a^i)) \\ &= \sum s_i u_i(x^{1i}) = v^k(s) \end{aligned}$$

is also satisfied. This indicates the well known fact that (see [5] [9])

$$\mu^k \in \mathcal{C}(v^k).$$

Define now, for $k \in \mathbb{N}^r$, $i \in R$ and $l \in L$

(14) $$E^{il}_k := \{t \in \mathbb{R}^r \,|\, t \leq k,\ t \in T^{il},\ k - t \in T^{il}\}$$

<u>Theorem 4.7.</u> Let $k \in \mathbb{N}^r$ and $k \in T^{il}$. If there are r linearly independent $\mathbb{N}^r$–vectors (i.e. profiles) in E^{il}_k, then

$$\mathcal{C}(v^k) = \{\mu^k\}.$$

<u>Proof:</u> Clearly, $p^0 = c^{il}$ is a Λ–price for all $s \in E^{il}_k$ as well as for k.

Fix some profile $s \in E^{il}_k$ and let $X^1 \in \mathcal{X}_k$, $X^2 \in \mathcal{X}_s$. Then

$$v^k(s) \;=\; f(s) \;=\; \sum s_i u^i(x^{2i})$$
$$=\; \sum s_i (u^i(x^{2i}) - p^0(x^{2i} - a^i))$$
(15) $$\geq\; \sum s_i (u^i(x^{0i}) - p^0(x^{0i} - a^i))$$
$$=\; \mu^k(s)$$

Therefore, any $m \in \mathcal{C}(v^k)$ satisfies

(16) $$m(s) \;\geq\; v^k(s) \;\geq\; \mu^k(s).$$

However, as $k - s \in E^{il}_k$ as well, we have also

$$m(k-s) \geq \mu^k(k-s)$$

i.e., $m(s) \leq \mu^k(s)$ as $m(k) = \mu^k(k) = v^k(k)$.

Hence, we come up with

(17) $$m(s) \;=\; \mu^k(s) \qquad s \in E^{il}_k$$

Now, if $s^\varrho(\varrho = 1,\ldots,r)$ are r linearly independent vectors (profiles) in E^{il}_k, then the system of equations

$$\sum_{i=1}^{r} s^\varrho{}_i m_i \;=\; \sum_{i=1}^{r} s^\varrho{}_i\, \mu^k{}_i \qquad (\varrho = 1,\ldots,r)$$

shows that $m = \mu^k$; the theorem follows now. q.e.d.

<u>Remark 4.8.</u> f, regarded as a function on $\mathbb{R}^r{}_+$ is affine in any T^{il}; thus, if $T^{0il} \neq 0$, then f (a concave function) admits of a unique gradient at some $t \in T^{0il}$, this (by Lemma 4.6.) is μ^t.

Clearly, if it so happens that $T^{0il} \cap T^{0i'l'} \neq 0$, then f has the same gradient within both sets. Thus, by omitting some of the T^{il} if necessary, $\mathbb{R}^r{}_+$ may be decomposed into a finite number of convex sets with nonempty interior in each of which f has the same gradient, clearly, those sets are polyhedral.

Now, if $k \in T^{0il}$, then μ^k represents the unique gradient – and the equilibrium payoff for v^k as well. If the vector μ^k is uniquely defined by the values $\mu^k(s)$, s vary through the profiles of E^{il}_k, then $C(v^k) = \{\mu^k\}$.

Now, T^{il} is a convex polyhedral cone with 0 as a boundary point. Suppose Λ^{il} is an r' x r–matrix such that

$$T^{il} = \{t \in \mathbb{R}^m_+ | \Lambda^{il} t \geq 0\}.$$

Then

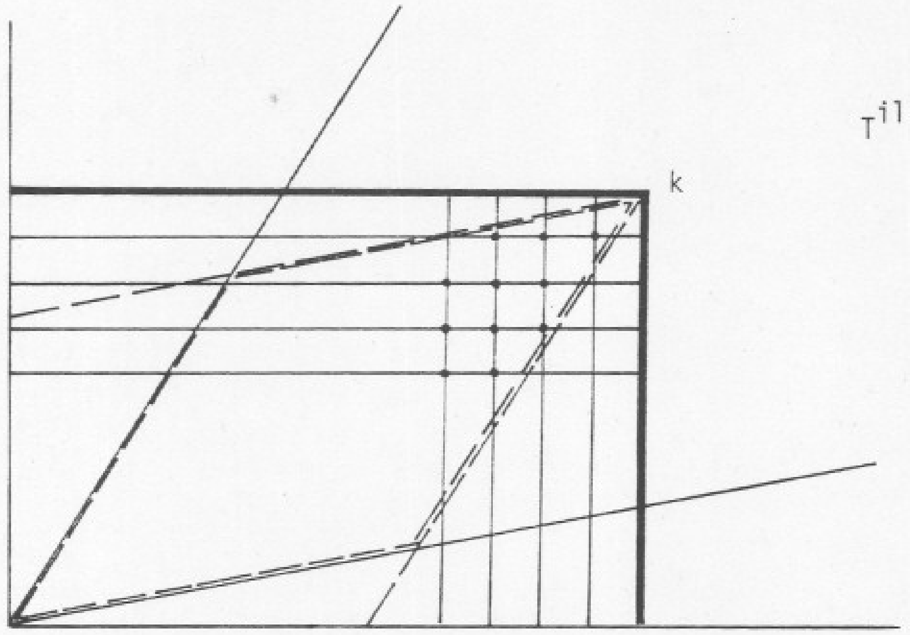
$$(18) \qquad E^{il}_k = \{t \in \mathbb{R}^m | \quad \begin{array}{l} 0 \leq t \leq k \\ 0 \leq \Lambda^{il} t \leq \Lambda^{il} k\} \end{array}$$

Thus, in order to check for $C(v^k) = \{\mu^k\}$ we have to solve the following problem:

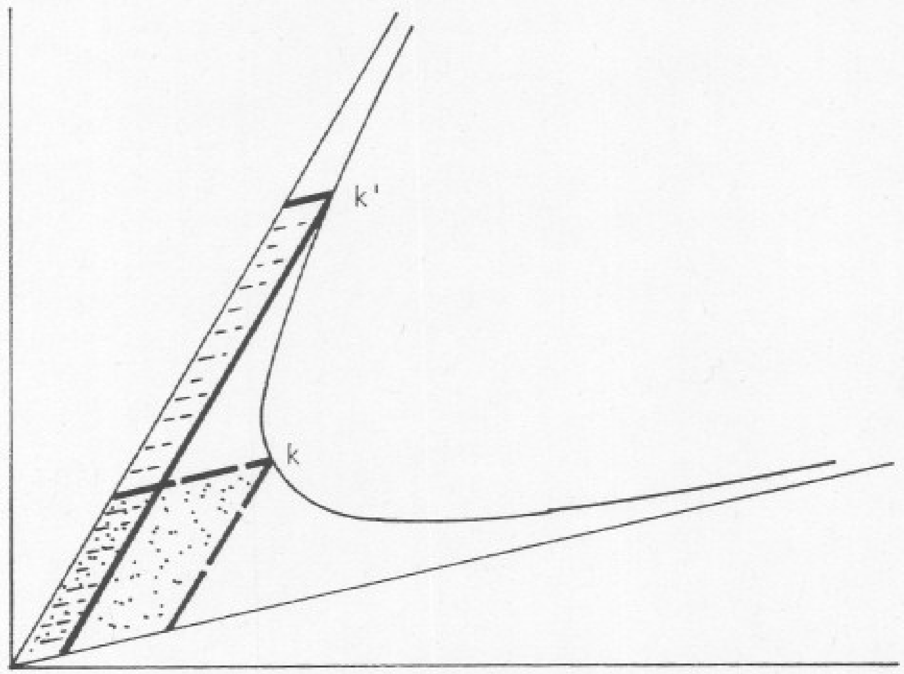
"Find r integer vectors $s^\sigma \in \mathbb{N}^r_0$, $\sigma = 1,\ldots,r$
such that

(19)
1. $0 \leq s^\sigma \leq k \qquad (\sigma = 1,\ldots,r)$
2. $Q = (s^\sigma_i)_{i,\sigma=1,\ldots,r}$ is nonsingular
3. $0 \leq \Lambda^{il} s^\sigma \leq \Lambda^{il} k$

This obviously corresponds to (5) in Section 3. Accordingly, we may classify any $k \in \mathbb{R}^r_+$ (any distribution of players over the types). In order to have the "equivalence property" (i.e. $C(v^k) = \{\mu^k\}$), k must be within the interior of some T^{il} and the volume of (18) must be sufficiently large in order to admit r linearly independent elements (MINKOWSKI's 2nd theorem) i.e., a solution of (19). Thus, we may characterize the set of distributions k with the equivalence property in a geometrical way which essentially involves the boundaries of the convex polyhedron T^{il} and the volume of the E^{il}_k.



T^{i1}
k



k'
k

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