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A Mathematical Note on the Structure of SYMLOG–Directions

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In social psychology SYMLOG–methods as conceptualized by Bales and his coworkers are valuable tools to measure the social field in a small group. The usual SYMLOG–model is derived from a factor–analysis of scores that are sums of ratings in specific questionnaires (Bales 1970, p.6f). These score–vectors are represented in a three–dimensional real space. In Berresheim/Ostmann/Schmitt 1991 it is argued that for use in multilateral experimental bargaining it is necessary to redefine some SYMLOG–concepts. A main argument was that the usual SYMLOG–tools are adequate for group tasks but not for conflict situations that allow for a strategic variation of socio–emotional aspects of communication. In consequence in the bargaining situations a spheric SYMLOG–model was used (computer programs for handling experimental data and explanations can be found in Schmitt 1991). In Ostmann 1991 and in a short note (Ostmann, unpublished) the structure of the modified SYMLOG–model is explained and a sketch is given how this model can be derived from a simple set of axioms on two relations called "being opposite to" and "being neutral to". It was conjectured that there is only one model that fits to those axioms: the model on the real sphere. But there was no proof that all finite models can be excluded. The present note closes this last gap.

In order to represent a single component of a judgement on the socio–emotional aspects of behaviour the SYMLOG–method uses 27 vectors of the three–dimensional real space. Call these vectors, defined by having components from the set $\{-1,0,1\}$, standard units. Corresponding to the locations of the 27 standard units Bales defines a set of behavioural types or group–roles, that can be idealized from experience with observations collected over more than twenty years. It can be discussed, if it would be better to conceptualize those types as predictions on behaviour (and value–orientations, and personality factors: cf. Bales 1970, p.4) that are indicated by observed values in the corresponding regions around the standard units; moreover there is a discussion, if the type corresponding to (0,0,0) lacks a rational interpretation. In this paper we give another approach to model the social field: this approach can be called an axiomatic one, whereas the standard approach is score–oriented. The main consequence of this change is that we ask first, what standard properties we can postulate in the theory of social field and social directions, and have to solve the problem of measurement in a second step.

1. Types and opposite directions

Let us first sketch some formal properties of the types. Bales 1970 uses the following set of types: AVE (average type), U, UP, UPF, UF, UNF, UN, UNB, UB, UPB, P, PF, F, NF, N, NB, B, PB, D, DP, DPF, DF, DNF, DN, DNB, DB, DPB. This scheme and the description of the corresponding behavior and group–role suggests:

1. that there are six basic types: U, P, F, D, N, B, and that the other ones can be seen as combinations or as lying in–between;
2. that there is a fundamental relation *, read as "is opposite to", between types that pairs all types except AVE; the relation is symmetric (that means X=*Y iff Y=*X) and is generated by U=*D, P=*N, F=*B by combining, for example we have UNB=*(DPF);
3. that there is a fundamental relation ⊥, read as "is neutral to", "is independent on" or "is indifferent to" that is an extension of the independence between the paired basic types U–D, P–N, F–B. But within the usual cube–model it is not clear how this extension can be formally defined. From the interpretation of the relation ⊥ it is clear that no type is neutral to itself (except possibly AVE), and that ⊥ is symmetric (X⊥Y iff Y⊥X); moreover the relatedness is a property of the pair: (*X)⊥Y iff X⊥Y.

For coding the answers in the SYMLOG–rating–questionnaires the numeric structure of a three–dimensional space is used with the following identifications: a count–vector near (n,0,0) is used as indicator of type U, near (0,n,0) for type P, near (–n,n,0) for DP, and so on (n is the maximal possible count). By these identifications the relations * and ⊥ can be translated into relations on the numerical space. Counts in opposite directions are cancelled.

2. The twofold use of summation: intensity and mean direction

One of the consequences of the use of the three–dimensional numeric count–oriented structure is that the count–vector allows for no clear statement on the intensity of behavior in a specific direction and on the mean–behavior. The count–vector is a mixture of both. Moreover the identification of the combinations of the letters U, N, P, etc. with sums, included the cancelations like 13U+11D=2U, are not based on an operation that is defined on types or directions of behavior; the procedure suggests, that the combinations are not entities like the basic types. In this sense it is suggested that the space of types is not homogeneous in contrast to the use of the homogeneous structure of the real vector-space. One possibility to avoid this strange inhomogeneity is to identify all type–corresponding combinations (except AVE) with vectors of the same length. This can happen by rescaling: the basic letter are represented by unit–vectors or their negatives, a twofold combination of non–opposite letters by the sum of such basic vectors divided by $\sqrt{2}$, and the complete combinations by the sum divided by $\sqrt{3}$. But if we can find no non–data oriented argument, why to use this specific identification and summation, the data aggregation procedure based on this new and homogeneous identification is as well an arbitrary act. In the following we shall show, that it is possible to find simple axioms to describe *our knowledge on properties of the types such that we can justify a model on behavioral directions represented on the unit sphere. This model will be the unique solution for the* axioms stated.

3. Neutrality and homogeneity

Taking the pattern of our empirical knowledge on types as found in Bales 1970, we shall construct a set of abstract social behavioral directions and relations based on axioms that exhibit properties analogues to those stated in the verbal descriptions found in the above book. For example statements like "Somebody of type A shows (a), but he never shows (b) nor (c)" are taken as indicating the following:
– a type can be represented by an (incomplete) list of behavioural descriptions
– the description (a) is element of the list A
– both descriptions (b) and (c) are "opposite" to (a)
– description (b) is "neutral" to (c)
From this structure on descriptions we derive a structure on lists or types, respectively. *As noted before these types are seen as an idealized subset of all possible behavioural directions. Not knowing what set of all possible directions we should assume, we assume that we can generalize the structure derived before: the relations "opposite" and "neutral" have to fulfill the same axioms as derived before.* Formally we state:

Let V be the space of directions, i.e. the set of directions V endowed with the structure given by the relations * and ⊥. As stated above we postulate the following three axioms:

(A1) * is a one–to–one mapping from V onto V; ** is the identity and there is no element X of V that fulfills X=*X.
(A2) ⊥ is a binary relation on V, that is symmetric (X⊥Y iff Y⊥X); and there is no element X of V that fulfills X⊥X.
(A3) * and ⊥ are compatible, this means: (*X)⊥Y iff X⊥Y
In order to guarantee a non–degenerate structure we further assume
(A4) there is a triple of elements that are pairwise related by ⊥, and there is one more element in V that is unrelated with those three elements according to ⊥.

These first axioms are direct generalizations of the properties used in the usual typology. An example for a triple of types corresponding to the triple in A4 is U, P, F; a type ⊥–unrelated to all of them is UPF. Before we state our next axiom let us examine a specific property of the type U in the usual model. In that model there exist one and only one pair of types that are neutral to U and X, for all types X not equal to U or D=*U. We obtain the following list (it is enough to look at types that are combinations with U):

type X	UPF	UPB	UNF	UNB	UP	UN	UF	UB
neutral to both U and X	PB NF	PF NB	PF NB	PB NF	F B	F B	P N	P N

In generalizing this property we assume:

(A5) there exists some direction Z, such that for all types X not equal to Z or *Z there exist two and only two types Y and Y' that are neutral to Z and X.

Remark. Consider Y and Y' as defined in (A5). If (A3) is fulfilled we get Y⊥X and (*Y)⊥X. Hence Y'=*Y.

A valuable tool to evalute the structure of V and to generate something like coordinates is the group Aut V of its automorphisms. A one–to–one mapping from V onto V is called automorphism if it preserves the relations of the structure: formally:

Definition 1: The bijection f is called an **automorphism** (denoted as f∈Aut V) if f(*X)=*(f(X)) and X⊥Y iff f(X)⊥f(Y) hold for all X,Y∈V.

Remark. * itself is an automorphism.

The next definition states the homogeneity condition in terms of the automorphism group.

Definition 2: *A structure S on a set S is called* **homogeneous** if the *group Aut S of* its automophisms acts transitively on S: this means for all pairs s,t∈S there is an element f∈Aut S that "translates" s into t, formally: f(s)=t.

The interpretation of this homogeneity is that no single element in S plays a special role in the structure. We propose to fix this property as axiom for the structure V.

(A6) V is homogeneous.

*In the following we use the set W of all paired directions x={X,*X}, X∈V, and* compress the structure of V into the simpler structure W defined by that set and a relation extending ⊥. Because of axiom A3 the following definition gives rise to a direct translation of the above stated axioms A2, A4 and A5 on V into the corresponding ones on W.

Definition 3: The set of all paired directions is denoted by W, and its elements are called points. *For all x,y∈W we set x⊥y if and only if there exist X∈x and Y∈y fulfilling X⊥Y. If x⊥y we say "x is neutral to y".*

Remark. Because of A3 we have x⊥y if and only if X⊥Y for all X∈x and Y∈y.

Proposition 1: The structure W=(W,⊥) has the following properties:

(B1) ⊥ is a binary and symmetric relation, and no element x∈P fulfills x⊥x

(B2) there is a triple of elements that are neutral to each other, and there is one more element in V that is not neutral to any of those three elements.

(B3) the structure is homogeneous

*Proof. If there would be an element x={X,*X} fulfilling x⊥x,* then either X⊥X or X⊥*X: in the latter case axiom A3 implies X⊥X. So property B1 is true. Property B2 follows directly from A3 and the remark. Now let y={Y,*Y} with x≠y. According to axiom A5 there exist an automorphism f∈Aut V such that f(X)=Y.

Define the mapping g by g(z)={f(Z),f(*Z)}; since f∈Aut V we got the equality f(*Z)=*f(Z), and g becomes an bijection on W. Moreover g(x)=y, and all will be proven if we have shown that g∈Aut W, i.e. z_1⊥z_2 iff g(z_1)⊥g(z_2) for all z_1,z_2∈W. By the definition of ⊥ on W we know that there are Z_i∈z_i, i=1,2 fulfilling Z_1⊥Z_2; by f∈Aut V we know f(Z_1)⊥f(Z_2) and the definition of g implies g(z_1)⊥g(z_2). The other direction of the proof works analoguously.

Remark. As a group Aut W is isomorphic to the factor group Aut V / {id,*}.

Proposition 2: The structure W=(W,⊥) has the following property:

(B4) for two different points there is one and only one that is neutral to both.

Proof. For x let $x^{||}$:={y; there is no z such that x⊥z⊥y} and x^{m}:={y; there are w≠z such that x⊥z⊥y and x⊥w⊥y}. According to axiom A5 we get $u^{||}$ and u^{m} are empty. Since the space is homogeneous, $x^{||}$ and x^{m} have to be empty too, for all x∈P. Now let x and y be two different points. The emptyness of the two sets $x^{||}$ and x^{m} implies that there is one and only one point z neutral to both x and y.

Definition 4: Define the operation ×:W^2–diag–W by x×y being that unique element neutral to both x and y.

4. First consequences

In this section we derive the following results: the space of direction–pairs can be structured as projective plane: furthermore there is no finite model that fulfills the neutrality conditions.

Definition 5: $x^{\perp}$={w∈W:w⊥x} defines the neutrality set of the point x (remember that according to definition 3 a point corresponds to a pair of directions). The set L of all neutrality sets is also called the set of lines.

Remark. x→$x^{\perp}$ defines a one–to–one mapping from W onto L.

Proposition 3: The structure G=(W,L,∈) is a projective plane, i.e. it fulfills the following axioms:

(P1) Any two different lines have one and only one point in common

(P2) any two different points are connected by one and only one line

(P3) there are 4 different points, such that no 3 of them are on a common line

Proof: For the proof of P1 and P2 we use property B4. Any two different points x and y are connected by $(x \times y)^{\perp}$, and the intersection of any two different lines, say $x^{\perp}$ and $y^{\perp}$, contains only the element $x \times y$.
For the proof of P3 we use property B2. We denote the pairwise neutral elements of the triple by u,p,f and the element *not neutral to all of them by w. There three points u,p,f cannot lie on one line: assume u is element of the line connecting p and f: but that line is* $u^{\perp}$ and this is a contradiction to property B1 (no point is neutral to itself). Now take two points of the triple u,p,f and connect them by a line. This line is the neutrality set of the remaining point of the triplet, but w is not neutral to all of them. This completes the proof of P3.

Definition 6: Let us consider a projective plane (W,L,∈). A one-to-one mapping α from W onto L and from L onto W is called a **polarity** *if α^2=id and if $x \in C$ implies $\alpha(C) \in \alpha(x)$ for all points x and all lines C. Any point x such that $x \in \alpha(x)$ holds is called an* **absolute point** (w.r.t. α).

Remark. The definition of a polarity is equivalent to the following:
α^2=id and $\alpha(B) \in C$ implies $\alpha(C) \in B$ for all lines B,C∈L.
Proof: For $\alpha(B) \in C$ put $x=\alpha(B)$ and use the original definition: this implies $\alpha(C) \in \alpha(x) = \alpha(\alpha(B)) = B$. Conversely, suppose $x \in C$ and put $x=\alpha(B)$. The second description *of a polarity then implies $\alpha(C) \in B = \alpha^2(B) = \alpha(\alpha(B)) = \alpha(x)$.*

Lemma: The relation ⊥ induces a polarity α on the projective plane of paired directions.
Proof: Let $\alpha(x)=x^{\perp}$ for $x \in W$ and $\alpha(B)=x$ if $B=x^{\perp}$ for B∈L. Note that the latter expression is well-defined (Remark following definition 5). For $B=x^{\perp}$ and $x \in C=y^{\perp}$ we get $\alpha(C)=\alpha(y^{\perp})=y$. By the symmetry in (B1) we get $y \in x^{\perp}$ and therefore $\alpha(C)=y \in x^{\perp}=B$.

Theorem A: *The projective plane G of the paired directions is not finite.*

Proof: Assume G to be finite. According to Lemma 12.3 of Hughes/Piper 1982 any polarity ϑ on a finite projective plane has at least one absolute point. But an absolute point x with respect to α fulfills $x \in x^{\perp}$. This is a contradiction to property (B1) (and (B2)) of proposition 1.

Remark. Theorem A has the consequence that the space of directions is infinite too. In *order to handle this space adequately we will introduce some topology and use methods of topological geometry. One possibility to do this is the following axiom:*

(A6) V is a connected topological (Hausdorff-)space, the correspondence $X \mapsto X^{\perp}$ is upper hemi-continuous and closed-valued; moreover * is a homeomorphism.

Remark. *It is clear that some continuity should be assumed. We find it natural to assume the closedness of the neutrality regions too.*
Remember: A correspondence $f: Z \to Z' : z \mapsto f(z)$ is a set-valued mapping with $f(z) \neq \emptyset$. A correspondence f is said to be upper hemi-continuous (uhc) if the strong inverse image $f^{-}(G)=\{z \in Z; f(z) \subset G\}$ is open for all open sets G in Z′. The property uhc ensures that the only possible discontinuities are implosions at certain points (see f.ex. Hildenbrand 1974, pp.21-6). A single-valued upper hemi-continuous correspondence that maps z to {z′} induces a continuous mapping $z \mapsto z'$.

Proposition 4: The topology on V induces topologies on the set W of points and on the set L of lines of G. With respect to those topologies the geometrical operations of intersection and joining are continuous (i.e. G is a topological geometry). Moreover the polarity α is a homeomorphism.

The proof uses the following lemma (Hildenbrand 1978, p.23f, proposition 2): if f and g are upper hemicontinuous and closed-valued then f∩g is upper hemi-continuous too. By this lemma the operation × of Definition 4 becomes continuous (where defined).
Proof. *The topology on the direction space V induces a topology on the point space.* T⊂W is open if {X∈V: there exist some x such that X∈x∈T} is an open set in V. According to (A6) $X \mapsto X^{\perp}$ is uhc. Axiom (A3) implies $X^{\perp}=(*X)^{\perp}$. Hence $x \mapsto x^{\perp}$ is uhc. By the Lemma stated above $(x,y) \mapsto x^{\perp} \cap y^{\perp}$ is uhc. Moreover by (B4) and the definition of $\times : W^2 - \text{diag} \to W$ for $x \neq y$ $x \times y$ is the only element of $x^{\perp} \cap y^{\perp}$. Hence $\times : W^2 - \text{diag} \to W$ is continuous. In proposition 3 we used that joining and intersection can be defined by using operation ×. Since ⊥ is closed-valued ⊥ induces an homoeomorphism between W and L. The continuity of (the geometrical operations) intersecting and joining and the continuity of the polarity follow. (Remember that according to the remark following definition 5 ⊥ induces a one-to-one mapping and a polarity as defined becomes a homeomorphism).

5. Introducing local coordinates

We find it natural to add the following axiom:

(A7) The region of neutrality $U^{\perp}$ is locally homeomorphic to the reals.

Remark. Another equivalent assumption would be an axiom that gives a local linear order on that region. Consider a behaviour $X \in U^{\perp}$ and some slightly different behaviour $Y \in U^{\perp}$, then we know the two directions A and B neutral to both X and U, and we can ask if Y deviates from X in direction A or in direction B. We assume that we can answer these questions. Locally this gives an order on equivalence classes of $U^{\perp}$. By practical reasons it makes no sense to distinguish directions that cannot be distinguished by the relations $\perp$ and *. *So let us assume these equivalence classes to be singletons. After having introduced a local linear order we can assume that for all two points on* $U^{\perp}$ *it is possible to find some* point between. Together with the technical assumption of locally completeness of the order it is possible to show that the induced topology on $U^{\perp}$ would fulfill axiom (A7).

Proposition 5: Any line in the geometry of the paired–directions is locally homeomorphic to the reals. The point space of the plane is a 2–manifold.
The first property follows directly. The second part follows by the continuity of the *geometrical operations: the space is locally the product of two distinct lines.*
By *Theorem 2.0 of Salzmann 1968 we know:*

Proposition 6: Each line is homeomorphic to the circle and the point space is homeomorphic to the point space of the ordinary real projective plane.

6. The main theorem

Proposition 7 *(Salzmann 1962, main theorem, p.418): A two–dimensional projective plane is isomorphic to the (Desarguesian) real projective plane if the group G of* automorphisms acts transitively on the set of points; moreover G contains a subgroup isomorphic to the group of spatial rotations (SO_3).

Proposition 8: There are no more automorphisms that preserve the neutrality relation, *i.e. the space of paired directions is the usual elliptic geometry.*
Since the set {u,p,d} *forms a polarity triangle it is possible to use the usual coordinates* and the usual scalar product to represent the neutrality–relation $\perp$ by $\langle x,y \rangle = 0$. This is equivalent to the usual definition of the elliptic geometry.

Theorem B: The geometry $(V,\{U^{\perp}:U \in V\})$ is isomorphic to the usual geometry of the sphere (having as point space the 2–sphere and as distinguished subsets all great circles).

Proof: Let $p:V \to W$ *be the quotient map which identifies every pair* U,*U *to one point.* By the assumptions on * the map is a covering map, the number of sheets being #{id,*}=2, see for instance Stöcker/Zieschang 1988, 6.1.7. We have assumed that V is connected and we know that W is the real projective plane with fundamental group $\Pi_1(W)=\mathbb{Z}_2$. By the classification theorem for covering spaces (Stöcker/Zieschang 1988, 6.6.3) the covering *spaces over W correspond to the conjugacy classes of subgroups of* $\Pi_1(W)$, *i.e. to id or* $\mathbb{Z}_2$. Since the covering is 2–sheeted we have the first case represented by $V=S^2$ (space of all directions) and $p:V \to W$ identifies each pair of opposite directions. The preimages of the lines of the projective plane under p are the great circles of S^2.

References

Baer, R. (1946): *Polarities in finite projective planes*, Bulletin Amer.Math.Soc. 52, 77–93

Bales, R.F. (1970): *Personality and Interpersonal Behavior*, New York: Holt, Rinehart and Whinston

Berresheim, A., Ostmann, A. and Schmitt, H. (1991): Verhandeln als sozialer Prozeß – eine Verhaltenstheorie für Verhandlungen, University of the Saar at Saarbrücken. Institute of Psychology, Working Paper 153

Betten, D. (1972): Topologische Geometrien, University of Nantes, Institute of Mathematics

Hildenbrand, W. (1974): Core and equilibria of a large economy, Princeton: Princeton University Press

Hughes, D.R. & Piper, F.C. (1973, ²1982): Projective Planes, New York/Heidelberg/Berlin: Springer

Lingenberg, R. (1969): Grundlagen der Geometrie I, Mannheim: BI

Orlik, P. (1987): Ein semantischer Atlas zur Kodierung alltagssprachlicher Verhaltensbeschreibungen nach dem SYMLOG–Raummodell, International Journal of Small Group Research 3, 88–111

Salzmann, H. (1962): Kompakte zweidimensionale projektive Ebenen, Math.Annalen 145, 401–28

Salzmann, H. (1968): Topological planes, Advances in Mathematics 2, 1–60

Seib, M (1970): Unitäre Polaritäten endlicher projektiver Ebenen, Ann.d.Math. 21, 103–12

Stöcker, R. & Zieschang, H. (1988): Algebraische Topologie, Stuttgart: Teubner