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Construction of Homogeneous Zero-Sum Games

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Introduction

This paper presents two procedures to construct all equivalence-classes of homogeneous n-person zero-sum games without dummies, in form of certain canonical representatives, recursively on the number of persons. The concept of homogeneous games was introduced by VON NEUMANN und MORGENSTERN in [7]. In both methods only the weights of these canonical representatives of the equivalence-classes are of importance.

The first method, as described in the second paragraph, produces some unwanted objects, which only can be eliminated by testing whether they are homogeneous. This method, however, shows that the number of equivalence-classes of the n-person games mentioned above, grows more slowly than 3^{n-k}, for each natural number k. Up to now no smaller upper bound than (n-1)! was known ([2]).

The second method first divides the canonical representatives with lenght at least three - with the length being the number of members of the lex-max coalition (see [5,6]) - into disjoint sets, namely into the set of decomposable and of reducible games. For the definition of these sets I refer to § 3. From the first method the construction of the representatives with length two is kept, being bijective. The resulting recursive construction generates all equivalence-classes of homogeneous zero-sum games without dummies in an injective way, so that no further tests are required.

In the last paragraph a set of representatives of n-person games with length two is produced, whose cardinality grows faster than each 2^{n+k}, with k being a natural number. ISBELL ([1]) showed that 2^{n-4} is a lower bound, but no larger lower bound was known up to now.

§ 1 Some Notations and Definitions

A simple n-person game is a pair (Ω,v) with $\Omega = \{1,\dots,n\}$, which is called the set of players, and $v:\mathcal{P}(\Omega) \to \{0,1\}$, $v(\emptyset) = 0$. An element of $\mathcal{P}(\Omega)$, i. e. a subset of Ω, is a coalition. A coalition S is winning if $v(S) = 1$ and losing if $v(S) = 0$. If each proper sub-coalition of a winning coalition is losing, this winning coalition is called minimal.
The expression "n-person" is often deleted.

A nonnegative vector $m = (m_1,\dots,m_n) \in \mathbb{N}_0^n$ together with a natural number $\lambda, 0 < \lambda < m(\Omega)$ - where $m(S) = \sum_{i \in S} m_i$ is the weight of coalition S - defines a simple game v by

$$v(S) = \begin{cases} 1, & \text{if } m(S) \geq \lambda \\ 0, & \text{if } m(S) < \lambda, \end{cases}$$

or shortly by $v = 1_{[\lambda,m(\Omega)]} \circ m$, 1_T is the indicator function of T.
The pair (m,λ) is a representation of (Ω,v).
A weighted majority game is a simple game which has a representation.

If a weighted majority game has a representation (m,λ) such that all minimal winning coalitions have exactly weight λ, then both the simple game and the representation are called homogeneous.

Definition 1.1: A vector $m = (m_1,\dots,m_r) \in \mathbb{N}^r$ is said to be homogeneous with respect to $\lambda \in \mathbb{N}$ - written m hom λ - iff

1: $1 \leq \lambda \leq \sum_{i=1}^{r} m_i$ and

2: For $S \subseteq \{1,\dots,r\}$, $m(S) > \lambda$, there is $T \subseteq S$ such that $m(T) = \lambda$.

For this definition see also [4].

Thus a representation (m,λ) of a simple game is homogeneous iff m hom λ.
A simple game has zero-sum, iff $v(S) + v(\Omega\setminus S) = 1$ for all $S \in \mathcal{P}(\Omega)$.
A simple game (Ω,v) is equivalent to a simple game (Ω,v'), if $v = v'\circ\pi$ for some permutation π of Ω.
The representations (m,λ) of a weighted majority game are ordered by the total mass $m(\Omega)$. The minimal elements with respect to the total mass are called minimal representations. A member i of Ω is called dummy of (Ω,v), if for each subset S of Ω containing i with $v(S) = 1$ the equation $v(S\setminus\{i\}) = 1$ is valid.
OSTMANN ([3]) has shown that there is a unique minimal representation (m,λ) of a homogeneous zero-sum game (Ω,v) without dummies, which is automatically homogeneous and satisfies $m(\Omega) = 2\lambda-1$. The term "homogeneous zero-sum game without dummies" is abbreviated by "h. o-s. game w.d.".
Thus the representation (m,λ) can be identified with m, because $\lambda = \frac{m(\Omega)+1}{2}$. In this case we write $\lambda = \lambda(m)$ and call it the level of m.
The equivalence-class - for short class - of (Ω,v) is identified with the ordered representative, i. e. with the representation

$$m' = (m_{\pi(1)},\ldots,m_{\pi(n)})$$

such that

$$m_{\pi(1)} \geq m_{\pi(2)} \geq \cdots \geq m_{\pi(n)}.$$

The vector m' is called ordered minimal representation of (Ω,v).
If $m'(\{1,\ldots,k\}) = \lambda(m)$, then k is called the length of m - written $l(m)$. In the sense of [5,6] the set $\{1,\ldots,l(m)\}$ is the lex-max coalition of m'. In the following the class of a h. o-s. game w.d. is identified with its ordered minimal representation.

Definition 1.2: A (n-person) court m' is a vector of $\mathbb{N}^n$ such that there is an ordered minimal representation and a j, $1 \leq j \leq l(m)$, with $m' = (m_j,m_1,\ldots,m_{j-1},m_{j+1},\ldots,m_n)$, m' is a court of m.

Without the restriction $j \leq l(m)$ this definition of a court is equal to that of [2].
The ordered minimal representation m is called associated with m'.
Each ordered minimal representation has at least one court, namely itself. The number of courts of m is the number of types in the lex-max coalition. Two players are of the same type, if their weights are equal ([5,6]).
Let S_n resp. T_n be the set of all classes of homogeneous n-person zero-sum games without dummies resp. n-person courts and S_n^k resp. T_n^k the subset of classes resp. courts with length k. The length of an ordered minimal representation and of a court, which particularly are representations of simple games, was defined above.

The number of elements of sets S_n^k, T_n^k, S_n and T_n are defined as

s_n^k, t_n^k, s_n and t_n.

A member of S_n^k resp. S_n sometimes is called game in S_n^k resp. S_n.

From OSTMANN ([3]) and the "Basic Lemma" of ROSENMÜLLER ([4,5]) and with the help of the definition of S_n^k we know the following

Theorem 1.3: $m \in S_n^k$ iff

(i) $m_1 \geq m_2 \geq \ldots \geq m_n$, $m_i \in \mathbb{N}$,

(ii) $m_n = 1$,

(iii) $(m_{k+1}, \ldots, m_n)$ hom m_j, $j \leq k$,

(iv) $m(\{1,\ldots k\}) = m(\{k+1,\ldots,n\}) + 1$

(v) If $n \geq j > k$ then $m_j \leq \sum_{i=j+1}^{n} m_i$.

This important theorem will be used several times in the sequel and has the following consequences:

For $x \in \mathbb{R}$ define $[x] = \max\{z \in \mathbb{Z} | z \leq x\}$.

If $k \geq [\frac{n+2}{2}]$, then $S_n^k = \emptyset$ because of (i) and (iv).

$S_{2k-1}^k = \{(\underbrace{1,\ldots,1}_{(2k-1)\text{times}})\}$ because of (i), (ii), (iv).

§ 2 A Construction of the Classes of the Homogeneous n-Person Zero-Sum Games without Dummies from Certain Courts with Less Persons Providing an Upper Bound for the Number of these Classes.

In the sequel we presume $n \geq 3$. Indeed there is only one 1-person zero-sum game and none with two persons.

Lemma 2.1: Let m be in S_n^k and S be a proper nonempty subset of $K = \{1,\ldots,k\}$. Then there is a r, $k < r \leq n$, such that

$$\sum_{i \in S} m_i = \sum_{j=k+1}^{r} m_j .$$

Proof: For each i, $n \geq i \geq k$, define

$$S_i = S \cup \{k+1,\ldots,i\}.$$

Thus we have the following increasing chain:

$$S_k \subsetneqq S_{k+1} \subsetneqq \cdots \subsetneqq S_n .$$

S_n is a winning coalition because $\sum_{j=k+1}^{n} m_j = \lambda(m)-1$ and S is nonempty. S_k is losing because $m(S) < m(K) = \lambda(m)$. If S_j is winning but not minimal winning, then S_{j-1} is winning because m_j is the smallest weight in S_j and the representation is homogeneous.
We conclude that there is a j with $m(S_j) = \lambda(m)$.

Thus
$$\sum_{j \in S} m_i = \sum_{i=k+1}^{j} m_i .$$

q.e.d.

If $n \geq 4$ define for each $m \in T_n$

$$H(m) = \begin{cases} (m_1-m_2 , m_3,\ldots,m_n), & \text{if } m_1 \neq m_2. \\ (m_3,\ldots,m_n), & \text{if } m_1 = m_2. \end{cases}$$

For example $H(T_4) = \{(1,1,1)\}$ because $T_4 = \{(2,1,1,1),(1,2,1,1)\}$ (see [2]).

Theorem 2.2: Let m be in T_n^k, $n \geq 5$. Then there is a natural number r, $k < r \leq n$, such that $\min\{m_1, m_2\} = \sum_{j=k+1}^{r} m_j$ and $H(m)$ is either a court in T_{n-1}^{r-1} or a h. o-s. game w.d. in S_{n-2}^{r-2}.

Further the two mappings

$$H_{|S_n^k} \quad \text{and} \quad H_{|T_n^k \setminus S_n^k}$$

are injective.

Proof: Lemma 2.1 guarantees the existence of r. Let m_1 be different from m_2. (The other case can be treated analogously.)

In order to finish the first part of the proof we show that $H(m)$ has the properties (iii) and (iv) of Theorem 1.3 and that the inequality

$$|m_1 - m_2| \geq m_{r+1}$$

is valid.

The fact

$$|m_1 - m_2| = \max\{m_1, m_2\} - \min\{m_1, m_2\}$$

implies

$$|m_1 - m_2| + m_3 + \ldots + m_r = \max\{m_1, m_2\} + \lambda(m)\} - m_1 - m_2$$

$$= \lambda(m) - \min\{m_1, m_2\} = m(\{r+1, \ldots, n\}) + 1$$

and

$$|m_1 - m_2| \geq m_{r+1}$$

Define $R = (m_{r+1}, \ldots, m_n)$. In this first part it remains to show

(a) R hom $|m_1 - m_2|$ and (b) R hom m_j, $k+1 \leq j \leq r$,

because the assertion

$$R \text{ hom } m_j, \quad 3 \leq j \leq k$$

is obvious.

ad (b): Let S be a subset of $\{r+1,\dots,n\}$ with $m(S) > m_j$ for some j, $k+1 \le j \le r$. Then there is a subset T of S, such that $m_j \le m(T) < m_j+m_r$ because the weights m_t, $t \in S$, are at most m_r.

Assume $m(T) > m_j$, then

$$m((\{k+1,\dots r\} \setminus \{j\}) \cup T) > \min\{m_1,m_2\}.$$

By the homogeneity of m this superset of T has a subset S' with $m(S') = \min\{m_1,m_2\}$. Because of the fact $m(T) < m_j+m_r$ the form of S' must be

$$S' = (\{k+1,\dots,r\} \setminus \{j\}) \cup T'$$

for some $T' \subseteq S$.

This implies $m(T') = m_j$, thus R hom m_j.

The case (a) can be treated analogously.

The injectivity remains to be shown.

Let m be in S_n^k, $H(m)$ in T_{n-1}^j. Define

$$(m_1',\dots,m_{n-1}') := H(m).$$

Then we conclude

$$m = (m_1' + m_{k+1}' + \dots + m_j',\ m_{k+1}' + \dots + m_j',\ m_2',\ \dots,\ m_{n-1}').$$

If $H(m)$ is in S_{n-2}^j, define

$$(m_1',\dots,m_{n-2}') = H(m)$$

and conclude

$$m = (\sum_{i=1}^{j} m_i',\ \sum_{i=1}^{j} m_i',\ m_1',\dots,\ m_{n-2}')$$

The other function can be treated analogously, thus both mappings are injective.

q.e.d.

With the help of the function H two other mappings will be defined which allow us to construct inductively all S_n^k, T_n^k and which deliver upper bounds for the cardinalities of S_n^k and T_n^k.

Formally, let $k \ge 3$, $n \ge 2k$. Then define for all $m \in S_n^k$:

$$\sigma(m) = \begin{cases} H(m), \text{ if } m_{k+1} \neq m_2 \\ \text{the ordered minimal representation} \\ \text{associated with } H(m), \text{ if } m_{k+1} = m_2 \neq m_1 \\ H^{[\frac{k+l+1}{2}]}(m), \text{ if } m_1 = m_{k+l} > m_{k+k+1}, \\ \qquad 1 \leq l \leq k-2 \end{cases}$$

As for the expression "ordered minimal representation associated with H(m)" the passage after Definition 1.2. is referred to.

Under the prior assumptions $k \geq 3$ and $n \geq 2k$ define for all $m \in T_n^k$:

$$\tau(m) = \begin{cases} (o, \sigma(m)), & \text{if } m \in S_n^k \\ (1, H(m)), & \text{if } m \in T_n^k \setminus S_n^k \end{cases}$$

First it can be noticed that both mappings are well defined because for each m with $m_1 = m_{k+l} > m_{k+l+1}$ for some $l, 1 \leq l \leq k-2$, the term "$H^{[\frac{k+l+1}{2}]}(m)$" is defined. In the definition of σ no $m \in S_n^k$ was "forgotten" because in view of Theorem 1.3 the case $m_1 = m_{2k-1}$ can only occur if $m_{2k-1} = 1$ and therefore $n = 2k-1$, which contradicts the assumptions.

The "artificial" definition of τ guarantees that certain restrictions of τ are injective, which is very important for the enumeration.

<u>Lemma 2.3:</u> Let $n \geq 2k$, $k \geq 3$. Then

$$\sigma(S_n^k) \subseteq S_{n-1}^k \cup \bigcup_{j=k+1}^{[\frac{n}{2}]} T_{n-1}^j \cup \bigcup_{j=k}^{[\frac{n-1}{2}]} S_{n-2}^j \cup \bigcup_{l=1}^{k-2} \bigcup_{j=k-l}^{[\frac{n-k-l+1}{2}]} S_{n-k-l}^j$$

$$\tau(T_n^k) \subseteq (\{0\} \times \sigma(S_n^k)) \cup (\{1\} \times \bigcup_{j=k}^{[\frac{n}{2}]} T_{n-1}^j)$$

and the mappings

$$\sigma_{|S_n^k} , \quad \tau_{|T_n^k}$$

both are injective.

Proof: Let m be an ordered minimal representation in S_n^k. Distinguish the following four cases.

1. $m_1 \neq m_2 = m_{k+1}$

   The ordered minimal representation associated with H(m) is of the form

   $$(m_1', \underbrace{m_2, \ldots, m_2}_{(k-1) \text{ times}}, m_{k+2}, \ldots, m_n)$$

   or

   $$(\overbrace{m_2, \ldots, m_2}, m_1', m_{k+2}, \ldots, m_n)$$

   for some weight m_1'.

   Notice that m_1' equals $|m_1 - m_2|$.

   As the total weight of H(m) is $2\lambda(m)-1-2m_2$, the length of H(m) is k, thus S_{n-1}^k contains $\sigma(m)$. The injectivity of σ restricted to such elements is an obvious consequence of Theorem 2.2.

2. $m_1 = m_2 \neq m_{k+1}$

   Pick $t \in \mathbb{N}$ such that $m_{k+1} + \ldots + m_t = m_1$ (see Lemma 2.1). Then it follows from Theorem 2.2 that

   $$\sigma(m) \in S_{n-2}^{t-2}$$

   Moreover it is seen that the restriction of σ to such minimal representations is injective.

3. $m_{k+1} \neq m_2 \neq m_1$.

From Theorem 2.2 it is clear that $\sigma(m) \in T^j_{n-1}$ for some j and the restriction of σ to such representation is injective. It remains to consider the case

4. $m_1 = m_{k+l} > m_{k+l+1}$ for some $1 \leq l \leq k-2$.

Assume that k-l is even (the case "k-l is odd" can be treated analogously).

By definition of H it is clear that

$$H(m) = (\underbrace{m_1, \ldots, m_1}_{(k+l-2)\text{times}}, m_{k+l+1}, \ldots, m_n).$$

Successively applying H we yield the following equation:

$$H^{l+\frac{k-l}{2}}(m) = (m_{l+k+1}, \ldots, m_n) =: m'.$$

This new representation m' equals $\sigma(m)$ because $\left[\frac{k+l+1}{2}\right] = l + \frac{k-l}{2}$ by the assumption.

The fact that m' is an element of S_{n-k-l} is obvious.

The level of m' is $(\frac{k-l}{2}) \cdot m_1$, thus $l(m') \geq k-l$ because $m_1 \geq m_{k+l+1} + m_{k+l+2}$, $m_1 \geq m_{k+l+2} + m_{k+l+4}$ ...

The easy proof that σ restricted to these representations is injective is left to the reader.

As the images of σ restricted to these four cases are disjoint, the injectivity of $\sigma_{|S_n^k}$ is shown.

It remains to consider the mapping τ.

If m is a court in $T_n^k \setminus S_n^k$, then H(m) is in T_{n-1} because m_1 and m_2 do not coincide.

The assertions concerning τ are now direct consequences of the definition of τ, of Theorem 2.2, and the preceding proof.

q.e.d.

We proceed by defining the "inverse" of $\sigma_{|S_n^k}$, $k \geq 3$, in order to construct S_n^k from $S_{n'}$, $T_{n'}$, $n' < n$.

Formally define

$$\tilde{\sigma}(m) = \begin{cases} (m_1+m_2,\underbrace{m_2,\dots,m_2}_{k\text{-times}},m_{k+1},\dots,m_{n-1}) & \text{if } m = (m_1,\underbrace{m_2,\dots,m_2}_{(k-1)\text{times}},m_{k+1},\dots,m_{n-1}) \\ & \text{or } m = (\underbrace{m_2,\dots,m_2}_{(k-1)\text{times}},m_1,m_{k+1},\dots,m_{n-1}) \text{ and } m \in S^k_{n-1} \\ (m_1+m_k+\dots+m_j,m_k+\dots+m_j,m_2\dots,m_{n-1}) & \text{if } m \in T^j_{n-1} \text{ for some } j \geq k-1 \\ (m_{k-1}+\dots+m_j,m_{k-1}+\dots+m_j,m_1,\dots,m_{n-1}) & \text{if } m \in S^j_{n-2} \text{ for some } j \geq k \\ (\underbrace{m_l,\dots,m_l}_{(k+l)\text{times}},m_1,\dots,m_{n-k-l}), & \text{if } m \in S^j_{n-k-l} \text{ for some } 1 \leq l \leq k-2, \\ & j \geq k-l, \text{ where } m_l = \begin{cases} \frac{2(\lambda(m)-m_1)}{k-l-1}, & \text{if } k-l \text{ odd} \\ \frac{2\lambda(m)}{k-l}, & \text{if } k-l \text{ even.} \end{cases} \end{cases}$$

Define the following subset $\tilde{S}^k_{n-1}$ of S^k_{n-1} by

$$\tilde{S}^k_{n-1} = \{m \in S^k_{n-1} \mid m \text{ is of the form } (m_1,m_2,\dots,m_2,m_{k+1},\dots,m_{n-1}) \text{ or } (m_1,\dots,m_1,m_2,m_{k+1},\dots,m_{n-1})\}$$

Consequently the image of $\sigma_{|S^k_n}$ is a subset of the domain of $\tilde{\sigma}$. A simple computation shows that $\tilde{\sigma}$ is the left-inverse of σ, i. e.

$$\tilde{\sigma}(\sigma(m)) = m$$

for all $m \in S^k_n$.

Thus $\tilde{\sigma}$ allows to construct the elements of S^k_n from those of

$$\tilde{S}^k_{n-1} \cup \bigcup_{j \geq k+1} T^j_{n-1} \cup \bigcup_{j \geq k} S^j_{n-2} \cup \bigcup_{l=1}^{k-2} \bigcup_{j \geq k-l} S^j_{n-k-l}.$$

The fact that not all elements of the image of $\tilde{\sigma}$ are actually ordered minimal representations in S^k_n is the main disadvantage of this mapping $\tilde{\sigma}$. In order to construct all minimal ordered representations inductively by applying $\tilde{\sigma}$, those elements which are not representations have to be eliminated, a very difficult task.

The advantage of considering $\tilde{\sigma}$ is that this mapping together with the mapping $H_{|S_n^2}$ yields an upper bound of $|S_n^k|$. The next lemma shows that at least S_n^2 can be constructed in a satisfying way.

Lemma 2.4: The function

$$H_{|S_n^2} : S_n^2 \to \bigcup_{j\geq 2} T_{n-1}^j \cup \bigcup_{j\geq 2} S_{n-2}^j$$

is bijective, if $n \geq 5$.

Proof: By Theorem 2.2 this function is injective.
An inverse of this function remains to be defined.
Let $(m_1,\ldots,m_{n-1}) \in T_{n-1}^j$, then $\overline{m}:=(m_1+\ldots+m_j, m_2+\ldots+m_j, m_2,\ldots,m_{n-1})$ is an element of S_n^2 and $H(\overline{m}) = m$.
Analogously we have for $(m_1,\ldots,m_{n-1}) \in S_{n-2}^j$:

$$\overline{m} = \left(\sum_{r=1}^{j} m_r, \sum_{r=1}^{j} m_r, m_1,\ldots,m_{n-2}\right) \in S_n^2 \quad \text{and} \quad H(\overline{m}) = m.$$

q.e.d.

Up to the end of this chapter the maps σ, τ, H are used to give an upper bound of $|T_n^k|$ resp. $|S_n^k|$.

Corollary 2.5: (i) $s_{2k-1}^k = t_{2k-1}^k = 1$, if $k \geq 2$.

(ii)
$$s_{n+2}^{k+1} \leq s_{n+1}^{k+1} + \sum_{j=k+2}^{[\frac{n+2}{2}]} t_{n+1}^j + \sum_{j=k+1}^{[\frac{n+1}{2}]} s_n^j + \sum_{l=1}^{k-1} \sum_{j=k-l+1}^{[\frac{n-k-l+2}{2}]} s_{n-k-l+l}^j$$

and

$$t_{n+2}^{k+1} \leq s_{n+1}^{k+1} + 2\sum_{j\geq k+2} t_{n+1}^j + \sum_{j\geq k+1} s_n^j + \sum_{l=1}^{k-1} \sum_{j\geq k-l+1} s_{n-k-l+1}^j$$

if $k \geq 2$, $n \geq 3$.

(iii)
$$s_{n+2}^2 = \sum_{j\geq 2} t_{n+1}^j + \sum_{j\geq 2} s_n^j,$$

$$t_{n+2}^2 = 2\sum_{j\geq 2} t_{n+1}^j + \sum_{j\geq 2} s_n^j,$$

if $n \geq 3$.

Proof: (i) is obvious from the remark following Theorem 1.3.

(ii) is a direct consequence of Lemma 2.3.

(iii) follows from Lemma 2.4 because

$$(H_{|S^2_{n+1}})^{-1}(m) \text{ has exactly } \begin{cases} \text{two courts, if } m \in \bigcup_{j\geq 2} T^j_{n+1} \\ \text{one court, if } m \in \bigcup_{j\geq 2} S^j_{n+1}. \end{cases}$$

q.e.d.

In order to define an upper bound of s^k_n resp. t^k_n we need the following

Lemma 2.6: If $n \geq 5$ and $n \leq 3k-3$, then

$$s^k_n \leq s^{k-1}_{n-2} \quad \text{and} \quad t^k_n \leq t^{k-1}_{n-2}.$$

Later it will be shown that these inequalities in fact are equations (at the end of § 3).

Proof: Let m be an element of S^k_n and define

$$\tilde{m} = (m_1,\dots,m_{k-1},m_{k+2},\dots,m_n).$$

If $m_{k-1} = m_k = m_{k+1}$, then $\tilde{m}$ is an element of S^{k-1}_{n-2} and the number of courts of $\tilde{m}$ is still the number of courts of m. If all elements of S^k_n have this property the proof is finished because the map $m \mapsto \tilde{m}$ is injective and preserves the number of courts.

It remains to be shown that $m_{k-1} = m_{k+1}$ for all $m \in S^k_n$.
Assume $m_{k-1} \neq m_{k+1}$, then we have:

$$m_{k-1} \geq m_{k+1}+m_{k+2},\ m_{k-2} \geq m_{k+3}+m_{k+4},\dots,\ m_1 \geq m_{k+2(k-2)+1}+m_{3k-2}.$$

This is a consequence of the expression for the level of m, i. e.

$$\lambda(m) = \sum_{j=1}^{k} m_j = \sum_{j=k+1}^{n} m_j+1.$$

The last inequality shows that $n \geq 3k-2$, a contradiction.

q.e.d.

Now new sequences which majorize s^k_n, t^k_n can be defined in the following inductive way:

$$\tilde{s}_3^2 = \tilde{s}_4^2 = \tilde{t}_3^2 = 1,\ \tilde{t}_4^2 = 2.$$

$$\tilde{s}_n^k = \tilde{s}_{n-1}^k + \sum_{j=k+1}^{[\frac{n}{2}]} \tilde{t}_{n-1}^j + \sum_{j=k}^{[\frac{n-1}{2}]} \tilde{s}_{n-2}^j + \sum_{l=1}^{k-2} \sum_{j=k-l}^{[\frac{n-k-l+1}{2}]} \tilde{s}_{n-k-l}^j \quad \text{and}$$

$$\tilde{t}_n^k = \tilde{s}_n^k + \sum_{j=k}^{[\frac{n}{2}]} \tilde{t}_{n-1}^k, \text{ if } n \geq 3k-2,\ k \geq 3.$$

$$\tilde{s}_n^k = \tilde{s}_{n-2}^{k-1},\ \tilde{t}_n^k = \tilde{t}_{n-2}^{k-1}, \text{ if } k \geq 3,\ n \leq 3k-3,\ n \geq 2k-1.$$

$$\tilde{s}_n^2 = \sum_{j=2}^{[\frac{n}{2}]} \tilde{t}_{n-1}^j + \sum_{j=2}^{[\frac{n-1}{2}]} \tilde{s}_{n-2}^j,\ \tilde{t}_n^2 = \sum_{j=2}^{[\frac{n}{2}]} \tilde{t}_{n-1}^j + \tilde{s}_n^2,\ n \geq 5.$$

A direct consequence of Lemma 2.6 and Corollary 2.5 is

Corollary 2.7:

The sequence $\tilde{s}_n^k$ resp. $\tilde{t}_n^k$ majorizes s_n^k resp. t_n^k, i. e.

$$s_n^k \leq \tilde{s}_n^k,\ t_n^k \leq \tilde{t}_n^k,$$

if $n \geq 3$ and $2 \leq k \leq [\frac{n+1}{2}]$.

Some of the $\tilde{s}_n^k$, $\tilde{t}_n^k$ are tabulated in the supplement.

Lemma 2.8: $\tilde{s}_n^3 < \tilde{s}_{n+2}^4 < \ldots < \tilde{s}_{n+2k-6}^k < \ldots < \tilde{s}_{3n-14}^{n-4}$,

$\tilde{t}_n^3 < \tilde{t}_{n+2}^4 < \ldots < \tilde{t}_{n+2k-6}^k < \ldots < \tilde{t}_{3n-14}^{n-4}$,

if $n \geq 7$.

Proof (per induction on n):

If $n = 7$, nothing has to be shown because in this case $3n - 14 = n$.

Assume the assertion is valid for some $n \geq 7$, we are going to verify it for $n + 1$. Observe the following inequalities which follow from the inductive hypothesis and the definition:

$$\tilde{s}^{k}_{n+2k-5} = \tilde{s}^{k}_{n+2k-6} + \sum_{j\geq k+1} \tilde{t}^{j}_{n+2k-6} + \sum_{j\geq k} \tilde{s}^{j}_{n+2k-7} + \sum_{l=1}^{k-2} \sum_{j\geq k-l} \tilde{s}^{j}_{n+k-l-5}$$

$$\geq \tilde{s}^{k-1}_{n+2k-8} + \sum_{j\geq k} \tilde{t}^{j}_{n+2k-8} + \sum_{j\geq k-1} \tilde{s}^{j}_{n+2k-9} + \sum_{l=1}^{k-3} \sum_{j\geq k-1-l} \tilde{s}^{j}_{n+k-l-6} + \sum_{j\geq k-1} \tilde{s}^{j}_{n+k-6}$$

$$= \tilde{s}^{k-1}_{n+2k-7} + \sum_{j\geq k-1} \tilde{s}^{j}_{n+k-6}$$

$$> \tilde{s}^{k-1}_{n+2k-7} \quad \text{for each} \quad 4 \leq k \leq n-3.$$

The last inequality is strict because

$$\sum_{j\geq k-1} \tilde{s}^{j}_{n+2k-6} \geq \tilde{s}^{k-1}_{n+k-6} > 0.$$

The inequality $\tilde{t}^{k}_{n+2k-5} > \tilde{t}^{k-1}_{n+2k-7}$ can be shown analogously.

q.e.d.

Define

$$a^2_3 = a^2_4 = b^2_3 = 1, \quad b^2_4 = 2,$$

$$a_n := a^2_n = \sum_{j=2}^{[\frac{n}{2}]} b^j_{n-1} + \sum_{j=2}^{[\frac{n-1}{2}]} a^j_{n-2}, \text{ if } n \geq 5;$$

$$b_n := b^2_n = 2a^2_n - \sum_{j>2}^{[\frac{n-1}{2}]} a^j_{n-2}, \quad n > 5;$$

$$a^k_n = a^{k-1}_{n-2},\ b^k_n = b^{k-1}_{n-2}, \text{ if } k \geq 3,\ n \geq 2k-1.$$

Theorem 2.9:

The sequence a^k_n resp. b^k_n majorizes $\tilde{s}^k_n$ resp. $\tilde{t}^k_n$ - i. e. $\tilde{s}^k_n \leq a^k_n$ resp. $\tilde{t}^k_n \leq b^k_n$ - if $3 \leq n$, $2 \leq k \leq [\frac{n+1}{2}]$.

In the sequel this theorem allows us to give simple upper bounds of s^k_n, t^k_n, s_n, and t_n.

If $\tilde{s}^j_{n-1} \leq a^j_{n-1}$, $\tilde{t}^j_n \leq b^j_n$ then $\tilde{s}^2_{n+1} \leq a^2_{n+1}$, $\tilde{t}^2_{n+1} \leq b^2_{n+1}$, which is a consequence of the definitions of these sequences.

Lemma 2.8 suggests the following procedure: By the definitions of $\tilde{s}_n^j$, $\tilde{t}_n^j$, a_n^j, b_n^j in the case $n > 3j-3$, $j \geq 3$, it is realized that Theorem 2.9 is proved once the inequalities

$$\tilde{s}_{3k-2}^k \leq a_{3k-2}^k \quad \text{and} \quad \tilde{t}_{3k-2}^k \leq b_{3k-2}^k$$

are established. In view of the definitions of $\tilde{s}_n^k$, $\tilde{t}_n^k$ it is sufficient to show the following

Lemma 2.10:

$$a_{3k-2}^k \geq a_{3k-3}^k + \sum_{j \geq k+1} b_{3k-3}^j + \sum_{j \geq k} a_{3k-4}^j + \sum_{l=1}^{k-2} \sum_{j \geq k-l} a_{2k-l-2}^j ,$$

$$b_{3k-2}^k \geq b_{3k-3}^k + a_{3k-3}^k + 2 \cdot \sum_{j \geq k+1} b_{3k-3}^j + \sum_{j \geq k} a_{3k-4}^j + \sum_{l=1}^{k-2} \sum_{j \geq k-l} a_{2k-l-2}^j ,$$

if $k \geq 3$.

Preliminary Remarks:

From the definition it can be seen directly that

(1) $a_n^k = a_{n-2(k-2)}$, (2) $b_n^k = b_{n-2(k-2)}$ and

(3) $a_{n+1} = 2a_{n-1} + b_n$, (4) $b_{n+1} = b_{n-1} + 2b_n + a_{n-1}$, if $n \geq 5$.

From (3) and (4) it is easy to see the following equation:

(I) $a_{n+1} = 2b_{n+1} - 3b_n - 2b_{n-1}$,

thus

(II) $b_n = 2b_{n-1} + 3b_{n-2} - 3b_{n-3} - 2b_{n-4}$, if $n \geq 8$.

Proof of Lemma 2.10:

Put $x_k = \sum_{l=1}^{k-2} \sum_{j=1}^{[\frac{l+1}{2}]} a_{l-2+4}$.

According to the preliminary remarks (1),(2) it must be shown that

$$a_{k+2} \geq a_{k+1} + \sum_{j=1}^{[\frac{k-2}{2}]} b_{k-2j+1} + \sum_{j=1}^{[\frac{k-1}{2}]} a_{k-2j+2} + x_k$$

$$\underset{\text{def.}}{=} \quad a_{k+2} - b_{k+1} + a_{k+1} + x_k$$

resp. $\quad b_{k+2} \geq b_{k+2} - b_{k+1} + a_{k+1} + x_k \qquad (k \geq 3).$

Consequently it is sufficient to show that

$$b_{k+1} - a_{k+1} - x_k \geq 0.$$

From the preliminary remarks (1),(3) and the definition of b_n we get for each $n \geq 6$:

$$a_n = 2a_{n-2} + 2a_{n-1} - \sum_{j=1}^{[\frac{n-4}{2}]} a_{n-2j-1},$$

thus

$$\text{(III)}\quad x_n = \sum_{l=1}^{n-2} (2(a_{l+4} + a_{l+3}) - a_{l+5})$$

$$= 2a_4 + 4a_5 + 3\sum_{l=6}^{n+1} a_l + a_{n+2} - a_{n+3}$$

The case $k \leq 12$ is checked easily with the help of the list of the a_n^j, b_n^j ($n \leq 16$) in the supplement.

Let $k \geq 12$. Then we have the following equation:

$$z_k: = b_{k+1} - a_{k+1} - x_k \underset{(III)}{=} b_{k+1} - a_{k+1} + a_{k+3} - a_{k+2} - 3\sum_{l=6}^{k+1} a_l - 4a_5 - 2a_4$$

$$= b_{k+1} - 4a_{k+1} + a_{k+3} - a_{k+2} - 3\sum_{l=6}^{k} a_l - 4a_5 - 2a_4$$

$$\underset{(I)}{=} b_{k+1} - 8b_{k+1} + 12b_k + 8b_{k-1} + 2b_{k+3} - 3b_{k+2} - 2b_{k+1} - 2b_{k+2}$$

$$+ 3b_{k+1} + 2b_k - 3\sum_{l=6}^{k} (2b_l - 3b_{l-1} - 2b_{l-1}) - 14$$

$$= 2b_{k+3} - 5b_{k+2} - 6b_{k+1} + 14b_k + 8b_{k-1} - 6b_k + 3b_{k-1} - \sum_{l=6}^{k-2} 9b_l$$

$$+ 15b_5 + 6b_4 - 14$$

$$= 2b_{k+3} - 5b_{k+2} - 6b_{k+1} + 8b_k + 11b_{k-1} - \sum_{l=6}^{k-2} 9b_l + 15b_5 + 6b_4 - 14$$

$$= \begin{cases} 10b_{k-6} + 53b_{k-7} + 50b_{k-8} + 19b_{k-9} + \sum_{l=6}^{k-10} 9b_l + 15b_4 + 6b_4 - 14, & \text{if } k \geq 15 \\ 10b_8 + 53b_7 + 50b_6 + 25b_5 + 6b_4 - 14, & \text{if } k = 14 \\ 10b_7 + 56b_6 + 56b_5 + 16b_5 - 14, & \text{if } k = 13, \end{cases}$$

thus $z_k \geq 0$.

In this last equality the equation (II) is successively applied nine times.

The general case $k \geq 16$ is illustrated in the following diagram.

number of b_j \ j	k+3	k+2	k+1	k	k-1	k-2	k-3	k-4	k-5	k-6	k-7	k-8	k-9	k-10	...	6	5	4
former expression for z_k - 14	2	-5	-6	8	11	9	9	9	9	9	9	9	9	9	...	9	15	6
Application of (II) to the b_{k+3} yields		-1		2	7	9	9	9	9	9	9	9	9	9	...	9	15	6
Application of (II) to the b_{k+2} yields			-2	-1	10	11	9	9	9	9	9	9	9	9	...	9	15	6
Application of (II) to the b_{k+1} yields				-5	4	17	13	9	9	9	9	9	9	9	...	9	15	6
Application of (II) to the b_k yields					-6	2	28	19	9	9	9	9	9	9	...	9	15	6
Application of (II) to the b_{k-1} yields						-10	10	37	21	9	9	9	9	9	...	9	15	6
Application of (II) to the b_{k-2} yields							-10	7	51	29	9	9	9	9	...	9	15	6
Application of (II) to the b_{k-3} yields								-13	21	59	29	9	9	9	...	9	15	6
Application of (II) to the b_{k-4} yields									-5	20	68	35	9	9	...	9	15	6
Application of (II) to the b_{k-5} yields										10	53	50	19	9	...	9	15	6

q.e.d.

Lemma 2.12:

(i) $2b_n < b_{n+1}$, $n \geq 4$

(ii) $2a_{n+1} > b_{n+1}$, $n \geq 4$

(iii) $2a_n < a_{n+1}$, $n \geq 4$

(iv) $b_{n+1} < 3b_n$, $n \geq 3$

(v) $a_{n+1} < 3a_n$, $n \geq 5$.

Proof:

(i) $b_{n+1} \underset{(4)}{=} b_{n-1} + 2b_n + a_{n-1} > 2$, if $n \geq 5$

(the case n = 4 is trivial).

(ii) $b_{n+1} \underset{\text{def.}}{=} 2a_n - \sum_{j \geq 2} a^j_{n-2} < 2a_n$

(iii) For $n \leq 5$ the assertion is valid.

If $2a_{n-1}<a_n$, then by (i) we have the following inequality:

$$2a_{n+1} \underset{(3)}{=} 2(2a_{n-1} + b_n) < 2a_n + b_{n+1} = a_{n+2}$$

Thus the proof is finished by induction.

(iv) The list in the supplement shows the assertion in case $n \leq 7$. If the inequality is valid for $j \leq n \geq 7$ then

$$\begin{aligned} b_{n+1} &\underset{(II)}{=} 2b_n + 3b_{n-1} - 3b_{n-2} - 2b_{n-3} \\ &\underset{(i)}{\leq} 3b_n + b_{n-1} - 3b_{n-2} - 2b_{n-3} \\ &< 3b_n - 2b_{n-3} \\ &< 3b_n \end{aligned}$$

and the proof is finished by induction.

(v) $$\begin{aligned} a_{n+1} \underset{(3)}{=} 2a_{n-1} + b_n &\underset{(ii)}{<} 2a_{n-1} + 2a_n \\ &\underset{(iii)}{<} 3a_n . \end{aligned}$$

q.e.d.

Lemma 2.13.

(i) $b_{2n} \leq 2 \cdot 7^{n-2}$ and

$b_{2n+1} \leq 5 \cdot 7^{n-2}$, if $n \geq 2$.

(ii) $a_{2n} \leq 7^{n-2}$ and

$a_{2n+1} \leq 3 \cdot 7^{n-2}$, if $n \geq 2$.

Proof:

(i) For $n \leq 4$ the assertion follows from the list in the supplement. For $n \geq 5$ the inequalities are implied by induction on n with the help of Lemma 2.12. (iv):

$$\begin{aligned} b_l &= 2b_{l-1} + 3b_{l-2} - 3b_{l-3} - 2b_{l-4} \\ &= 7b_{l-2} + 3b_{l-3} - 8b_{l-4} - 4b_{l-5} \\ &\leq 7b_{l-2} \end{aligned}$$

because (iv) guarantees

$$8b_{l-4} + 4b_{l-5} > 9b_{l-4} > 3b_{l-3},$$

where $l = 2n$ resp. $l = 2n+1$.

(ii) Again for $n \leq 4$ the assertion follows from the list. For $n \geq 5$ the inequalities are implied by induction with the help of (i) and the preliminary remark (3):

$$a_{2n+2} = 2a_{2n} + b_{2n+1} \leq 2 \cdot 7^{n-2} + 5 \cdot 7^{n-2} = 7^{n-1} .$$

resp.

$$a_{2n+3} = 2a_{2n+1} + b_{2n+2} \leq 3 \cdot 7^{n-2} + 2 \cdot 7^{n-1} = 17 \cdot 7^{n-2} \leq 3 \cdot 7^{n-1}.$$

q.e.d.

A consequence of the last lemma is

Lemma 2.14: Let $k \geq 4$, $n \geq 3$.

(i) If $n \geq 9k-25$, then $b_n \leq 3^{n-k}$.

(ii) If $n \geq 9k-29$, then $a_n \leq 3^{n-k}$.

Proof of (i) if n is even (the other cases can be treated analogously):

Put $n = 2l \geq 9k-25$, then $l \geq \frac{k \cdot \ln 3 - \ln 49 + \ln 2}{\ln 9 - \ln 7}$, which is equivalent to $2 \cdot 7^{l-2} \leq 3^{2l-k}$, thus $b_{2l} \leq 3^{2l-k}$ by the last lemma.

q.e.d.

Now the aim to construct simple upper bounds is reached and can be expressed in form of the following

Proposition 2.15:

Assume $k \geq 4$, $n \geq 6$.

If $n \geq 9k-27$, then $s_{n-3} \leq 3^{n-k}$,
$t_{n-3} \leq 3^{n-k+1}$.

Proof: With the help of the preliminary remarks (1),(3) and the last lemma the following inequalities are implied:

$$s_{n-3} \leq \sum_{j=1}^{[\frac{n-4}{2}]} a_{n-2j-1} = 2a_{n-1} + 2a_{n-2} - a_n$$

$$\leq 2a_{n-1} + 2a_{n-2} \leq 2 \cdot 3^{n-k-1} + 2 \cdot 3^{n-k-2}$$

$$= 8 \cdot 3^{n-k-2} < 3^{n-k}.$$

The other inequality follows from the fact that

$$b_n \leq 2a_n \leq 3a_n.$$

q.e.d.

We have shown that the number of equivalence-classes of the homogeneous n-person zero-sum games without dummies is less than 3^{n-k}, for n sufficiently large.

§ 3 A Procedure to Construct all Classes of Homogeneous n-Person Zero-Sum Games Without Dummies From Those With Less Persons.

Unless otherwise specified, assume $n \geq 5$.
The games in S_n^2 can easily be constructed from those of S_{n-1} and S_{n-2} by means of

$$H_{|S_n^2}^{-1} : T_{n-1} \cup S_{n-2} \to S_n^2.$$

Indeed, Lemma 2.4 guarantees the bijectivity of this function.

Definition 3.1: For $m \in S_n^k$ define

$$R(m) := (m_{k+1}, \ldots, m_n).$$

A representation m' of S_j is called subgame of m, if the following conditions are fulfilled:

(i) $m_{k+r+i} = m'_i$, $i \in \{1,\ldots,j\}$, for some not negative integer r (especially $n \geq k + r + j$).

(ii) $m_i = 0 \bmod \lambda(m')$, if $1 \leq i \leq k + r$.

If $i \leq k$ and $k \geq 3$, we call m_i small in m, if

$$n \geq m_i+k, \quad m_{n-m_i+1} = m_{n-m_i+2} = \ldots = m_n = 1$$

and $$(m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_{n-m_i}) \in S_{n-m_i}^{k-1}.$$

For example the representation $m = (3,3,3,3,2,1,1,1)$ has the subgame $(2,1,1,1)$ but m_i $(i = 1,2,3)$ is not small in m although $m_{n-m_i+1} = 1$ and $n \geq m_i+k$.

If $m' = (m_i, \ldots, m_s)$ is a subgame of the game $m \in S_n^k$, then

$$n - s = c \cdot \lambda(m') \text{ for some } c \in \mathbb{N}.$$

In the following certain representations are "decomposed". This decomposition is based on the next important

Theorem 3.2: Let m be a game in S_n^k for some $k \geq 3$. If m has no subgame, then m_2 is small in m.

<u>Proof:</u> The cases n = 3,4 are trivial because then $S_3^k = \emptyset = S_4^k$. The cases n = 5,6 can be seen easily because all classes of homogeneous 5-resp.6-person zero-sum games without dummies are well-known (compare with the supplement).

The proof proceeds by induction on n.
Assume the assertion for $j \leq n_0 \geq 6$ and put $n = n_0 + 1$.
The following six cases are distinguished:

1. There is a natural number i, $3 \leq i \leq k$, with $m_{i-1} = m_i \geq m_{i+1}$:
A game $m' \in S_{n-1}^k$ is constructed in order to apply the inductive hypothesis:
If $i < k$, then there is a number j with $i < j \leq k$ such that

$$m' := (m_1,\ldots,m_{i-1},m_{i+1},\ldots,m_j,m_i-m_{k+1},m_{j+1},\ldots,m_k,m_{k+2},\ldots m_n)$$

is a member of S_{n-1}^k. If i = k, consider

$$m' := (m_1,\ldots,m_{k-1},m_k-m_{k+1},m_{k+2},\ldots,m_n),$$

which is an element of S_{n-1}^k, too.

If this newly formed game m' had a subgame m'', so would the old game m because with

$$m_{i-1} \equiv 0 \bmod \lambda(m''), \quad m_i - m_{k+1} \equiv 0 \bmod \lambda(m'')$$

it is also valid that

$$m_i \equiv 0 \bmod \lambda(m''), \quad m_{k+1} \equiv 0 \bmod \lambda(m'') \quad \text{as } m_{i-1} = m_i.$$

By hypothesis m_2 is small in m' and therefore in m as well.

2. $k \geq 4$ and $m_1 = m_2$:
Then

$$m' = (m_1+m_k,\ m_1,m_3,\ldots,m_{k-1},m_{k+1},\ldots,m_n)$$

is a game in S_{n-1}^{k-1} and has no subgame either, thus m_1 is small in this game as it is in m.

3. $m_{k-1} = m_k = m_{k+1}$:
If $k \geq 4$, then

$$m' = (m_1,\ldots,m_{k-1},m_{k+1},\ldots,m_n)$$

is in S_{n-2}^{k-1}. This game m' cannot have a subgame, otherwise m would have it as well. With the help of the inductive hypothesis it follows that m_2 is small even in m.

Now consider the case $k = 3$ and $m_k > 1$, the case $m_k = 1$ being obvious. It is $m_1 \neq m_2$, otherwise

$$(m_5, \ldots m_n)$$

would be a subgame of m.
In view of § 2 it is clear that $\sigma(m)$ is a game in S^3_{n-1} and $\sigma(m)$ has no subgame. As $\sigma(m)_2 = m_2$, we have shown that m_2 is small in $\sigma(m)$ by the hypothesis, thus m_2 is small in m.

4. $m_3 > m_4$:
Take $j \geq k+1$ such that $m_3 = m_{k+1} + \ldots + m_j$ and put $\widetilde{m}_3 := m_3 - m_j$.
If the game

$$m' = (m_1, m_2, \widetilde{m}_3, m_4, \ldots, m_{j-1}, m_{j+1}, \ldots, m_n)$$

which is in S^k_{n-1}, has no subgame, nothing further has to be shown.
Assume m' has a subgame

$$m'' = (m_i, \ldots, m_s) \in S^t_{s-i+1} \text{ resp. } S^t_{s-i}.$$

If $i \leq j-1$, then $t+s \leq j$ because of the fact

$$m_3 \equiv 0 \mod \lambda(m'').$$

This implies

$$m_{s+1} = 1 \text{ and } \sum_{l=s+1}^{n} m_l \geq m_1 + m_2 + \sum_{l=4}^{k} m_l - \lambda(m'')$$

$$\geq m_1 + m_2 - \lambda(m''),$$

thus m_2 is small in m.
If $i > j-1$, then because of $m_j \not\equiv 0 \mod \lambda(m'')$ we have $m_j = m_{j+1} + \ldots + m_x$ for some $x \geq i$. This and the fact that

$$m_1 \equiv 0 \mod \lambda(m'') \text{ has the consequence}$$

$$m_1 = m_{k+1} + \ldots + m_{j-1} + m_{j+1} + \ldots + m_{i+t-1} + c \cdot \lambda(m'')$$

for some not negative integer c, implying the following equation:

$$\sum_{j=s+1}^{n} m_j - m_2 + \sum_{j=i}^{i+t-1} m_j$$

$$= \widetilde{m}_3 + \sum_{j=4}^{k} m_j + (c-1) \cdot \lambda(m'') + \lambda(m'')$$

$$= c \cdot \lambda(m'') + \widetilde{m}_3 + \sum_{j=4}^{k} m_j.$$

By the inductive hypothesis the proof is finished in this case.

5. $m_1 > m_2 > m_3 = m_4$, $k = 3$.

Put $\tilde{m}_1 = m_1 - m_r$, where $m_1 = m_4 + \ldots + m_r$, and

$m' := (\tilde{m}_1, m_2, \ldots, m_{r-1}, m_{r+1}, \ldots, m_n)$, which is a game of S^k_{n-1}.

The case m' having no subgame is trivial and the case m' having a subgame $m'' = (m_i, \ldots, m_s)$ with $i \leq r-1$ can be treated analogously to 4. and is skipped here.

Assume $i \geq r+1$, which yields $m_3 = 2(m_{r+1} + \ldots + m_{i-1}) + k \cdot \lambda(m'')$ for some $k \geq 1$, thus $m(\{s+1, \ldots, n\}) + 2\lambda(m'') - 1 - m_2 = m_{r+1} + \ldots + m_{i-1} + k\lambda(m'') - 1 \geq m_{r+1}$ and $m_{r+1} + \ldots + m_n - m_2 = m_3 - 1$. It remains to be shown that $m_{n-m_2+1} = 1$. If $i > r+1$ or $i = r+1$ and $m_3 \geq 2\lambda(m'')$ the proof is finished directly. Otherwise $m' = (\tilde{m}_1, m_2, \underbrace{m_3, \ldots, m_3}_{3 \leq y \text{ times}}, m_i, \ldots, m_n)$. Observe that $m_2 - m_3$ is small in the game $(m_1 - m_3, m_2 - m_3, \underbrace{m_3, \ldots, m_3}_{y-1 \text{ times}}, m_r, \ldots, m_{n-m_3})$, as it has no subgame, thus m_2 is small in m.

6. $m_1 = m_2 > m_3 = m_4$, $k = 3$.

Put $m' = (2m_1 - m_3, m_3, m_3, m_4, \ldots, m_n)$ and consider that m' is a game in S^3_n. The following two cases may occur:

(A) m' has a subgame.

(B) m' has **no** subgame.

ad (A): Let $m'' = (m_i, \ldots, m_s)$, $m'' \in S^t_{s-i+1}$, be a subgame of m. Take $r \in \mathbb{N}$ such that $m_1 = m_4 + \ldots + m_r$. The fact that $i+t-1 \leq r$ means that m_2 is small in m.
Now, assume $i+t-1 > r$. It is easy to see that $i \leq r$, otherwise m'' would be a subgame of m.
The fact $2m_1 \equiv 0 \bmod \lambda(m'')$ implies

$$m_i + \ldots + m_r = \frac{\lambda(m'')}{2}.$$

Put $m_3 = c \cdot \lambda(m'')$ and observe the following equality:

$$m_{s+1} + \ldots + m_n = m_1 + (c - \frac{3}{2})\lambda(m'').$$

Thus m_2 is small in m if $c \geq 2$.
If $c = 1$, then $m_3 = \ldots = m_{i-1} = \lambda(m'')$. Concerning the case $i-1 > 3$ look at

$$(m_1 - m_3, m_1 - m_3, \underbrace{m_3, \ldots, m_3}_{(i-4) \text{times}}, m_i, \ldots, m_{n-\lambda(m'')}),$$

which is a game in $S^3_{n-m_3-1}$ and has no subgame (otherwise m would have the same).
The inductive hypothesis guarantees that $m_1 - m_3$ is small in this newly formed game, thus m_1 is small in m. The case i-1 equals 3 can be treated analogously by looking at the game

$$(m_3, m_1-m_3, m_1-m_3, m_5, \ldots, m_{n-m_3}),$$

which is in $S^3_{n-m_3-1}$ again.

ad (B): In this case (3) guarantees that m_3 is small in m', thus

$$m'' = (m_1, m_1, m_3, m_5, \ldots, m_{n-m_3}) \text{ is a game in } S^2_{n-m_3-1}.$$

Put

$$\tilde{m} = \begin{cases} (m_1-m_3, m_1-m_3, m_3, m_5, \ldots, m_{n-m_3}), & \text{if } m_1-m_3 \geq m_3. \\ (m_3, m_1-m_3, m_1-m_3, m_5, \ldots, m_{n-m_3}), & \text{otherwise.} \end{cases}$$

Consequently m is the ordered minimal representation of a $(n-m_3-1)$-person game of length 3 and there is no subgame of $\tilde{m}$, otherwise m would have the same subgame. We conclude that (m_1-m_3) must be small in $\tilde{m}$, and therefore m_1 is small in m.

Each game in S^k_n, $k \geq 3$, has at least one of the previous properties (1)-(6), so that the proof is finished.
In this proof Theorem 1.3 has been used several times without being mentioned explicitly.

q.e.d.

<u>Definition 3.3:</u>

Let $m \in S^k_n$, $k \geq 3$.
Then m is called <u>reducible</u>, if

(i) $n \geq \sum_{j=2}^{k-1} m_j + k + 1$ and $m_{n-\sum_{j=2}^{k-1} m_j + 1} = 1$.

(ii) $(m_1, m_k, m_{k+1}, \ldots, m_{n-\sum_{j=2}^{k-1} m_j}) \in S^2_{n-\sum_{j=2}^{k-1} m_j - k+2}$

m is called <u>irreducible</u>, if it is not reducible.

<u>Lemma 3.4:</u> Let $m \in S^k_n$, which has a subgame.
If i resp. s is the minimal resp. maximal number such that there is a subgame of the form

$(m_i, \ldots, m_{s'})$ resp. $(m_{i'}, \ldots, m_s)$

then

$(m_i, \ldots, m_s)$

is a subgame, too.

<u>Definition 3.5:</u> With the notations of Lemma 3.4 the subgame $(m_i, \ldots, m_s)$ is called <u>maximal subgame</u> of m.

<u>Proof of Lemma 3.4:</u>

Put $m^1 = (m_i, \ldots, m_{s'})$, $m^2 = (m_{i'}, \ldots, m_s)$.
The fact $m_{i'-1} \equiv 0 \bmod \lambda(m^2)$ and the existence of at least two personswith weight one in m^1 imply $s' > i'$.
If i' is not a member of the lex-max coalition of m^1, i. e. $i + l(m^1) \leq i'$, then $m_{i'} + \ldots + m_{s'} = c\,\lambda(m^2)-1$ for some natural number c. If $c > 2$, then $s' > s$, a contradiction.
If $c = 1$, then $m_{i'-1} > m_{i'} + \ldots + m_{s'}$, which is a contradiction, too (see Theorem 1.3(v)).
Thus, we have $c = 2$ and $s' = s$.
If $i + l(m^1) > i'$, we conclude $i + l(m^1) < i' + l(m^2)$, as $s' \leq s$, thus $m_j \equiv 0 \bmod \lambda$, $1 \leq j < i$, where λ is the least common multiple of $\lambda(m^1)$ and $\lambda(m^2)$. The facts $m_{i-1} \leq m_i + \ldots + m_n$ and $\lambda \geq 2\lambda(m^1)$ imply the existence of some $p > s'$ such that

$m_i + \ldots + m_p = \lambda$, thus $(m_i, \ldots, m_{p+\lambda-1})$

is a subgame of m, which is a contradiction to the maximality of s.

q. e. d.

Lemma 3.6 Let $m \in S_n^k$, $k \geq 3$.

Then m is reducible, if and only if m has no subgame or the maximal subgame

$m' = (m_i, \ldots, m_s)$

of m has the property

$m_1 + m_k > m_{k+1} + \ldots + m_s$.

Proof:

1st STEP: Suppose that m is reducible.

Assume the contrary, i. e. m has a subgame and the maximal subgame has the property $m_1 + m_k \leq m_{k+1} + \ldots + m_s$. This last inequality is actually proper because of the congruence

$$m_j \equiv 0 \bmod \lambda(m'), \quad 1 \leq j \leq i-1.$$

As the total mass of m is $2\lambda(m)-1$, we have the following inequality:

$$m(\{s+1,\ldots,n\}) \leq m_2 + \ldots + m_{k-1} - \lambda(m'),$$

thus

$$m(\{i,\ldots,n\}) - m_2 - \ldots - m_{k-1} \leq \lambda(m')-1.$$

The reducibility together with Theorem 1.3 shows that

$$m_{i-1} \leq m(\{i,\ldots,n\}) - m_2 - \ldots - m_{k-1},$$

but

$$m_{i-1} \equiv 0 \bmod \lambda(m'),$$

which contradicts the assumption.

2nd STEP: Suppose that m has no subgame.

The assertion follows by induction on k.

For k = 3 the assertion is valid by the last theorem.

Assume the assertion for $k_0 \leq k-1 \geq 3$. From Theorem 3.2 we know that

$$\tilde{m} := (m_1, m_3, \ldots, m_{n-m_2})$$

is a game in $S^{k-1}_{n-m_2-1}$.

The case m having no subgame is clear by the inductive hypothesis.

Now consider the case m having a subgame $m' = (m_j,\ldots,m_s)$. Pick a nonnegative integer r such that $m_2 = m_{k+1}+\ldots+m_r$, then $j \leq r$, otherwise m' would be a subgame of m.
We conclude that there is a natural number c such that

$$m_1 = m_{k+1} + \ldots + m_{j-1} + \lambda(m') \cdot c,$$

thus

$$m_{s+1}+\ldots+m_n = m_2+\ldots+m_k+(c-2)\cdot\lambda(m'),$$

which implies the inequality

$$m_i+\ldots+m_n-m_2-\ldots-m_{k-1} \geq m_k,$$

thus

$$n - \sum_{j=2}^{k-1} m_j \geq s+1$$

and the proof is finished.

3rd STEP: Suppose that m has a subgame and the maximal subgame m' of m has the property

$$m_1 + m_k > m_{k+1}+\ldots+m_s.$$

We proceed by induction on n.
If n is less than six the assertion is certainly valid, since all such games are explicitly known.
Assume the assertion for $r \leq n-1 \geq 5$.
Let m be a game in S_n^k.
If $m_1 \geq m_{k+1}+\ldots+m_i$ the proof can be finished directly.
Otherwise look at

$$\tilde{m} := \Big(\frac{m_1}{\lambda(m')},\ldots,\frac{m_{i-1}}{\lambda(m')}, \underbrace{1,\ldots,1}_{(\frac{n-s}{\lambda(m')}+1)\text{times}}\Big),$$

which is in $S^k_{i+\frac{n-s}{\lambda(m')}}$.

If $\tilde{m}$ has no subgame resp. only subgames $(m_i,\ldots,m_s)$ with $s' \leq i-1$, then m is reducible by the 2nd step resp. the inductive hypothesis, consequently the assertion is true.
If $\tilde{m}$ had a subgame $\tilde{m}' = (m_{i'},\ldots,m_{s'})$ with $s' \geq i$, then

$(m_{i'},\dots,m_{s'},\underbrace{1,\dots,1}_{(\lambda(m'))\cdot(s'-1)\text{ times}})$

would be a subgame of m with level $\lambda(m')\lambda(\tilde{m}')$.
This would be a contradiction to the fact that m' is maximal because $i' < i$ or $s' > s$.

q.e.d.

Now "the decomposition" of irreducible games in S_n^k can be defined. Let m be irreducible and $m' = (m_i,\dots,m_s)$ be the maximal subgame (m' exists because of Lemma 3.6).
Then

$$\tilde{m} := \Big(\frac{m_1}{\lambda(m')},\dots,\frac{m_{i-1}}{\lambda(m')},\underbrace{1,\dots,1}_{(c+1)\text{times}}\Big), \quad \text{where}$$

$n - s = c\cdot\lambda(m')$, is in S^k_{i+c}.

<u>Definition 3.7:</u> With the previous notations m is called <u>decomposable</u> (with respect to i), written

$$m = \tilde{m} \otimes_i m'.$$

In order to "compose" two representations, first define for each $\tilde{m} \in S_n^k$, $k \geq 3$, the following:

$$p(\tilde{m}) = \max \{j \in \{k+1,\dots,n+1\} \,|\, \tilde{m}_1+\tilde{m}_k = \sum_{i=k+1}^{j} \tilde{m}_i \text{ or}$$

there is a subgame $(\tilde{m}_1,\dots,\tilde{m}_{j-1})$ of $\tilde{m}$ for some i}.

In view of the last lemma and Theorem 1.3 we know that

$$\tilde{m}_{p(m)-1} = \tilde{m}_{p(m)-2} = 1.$$

If $p(\tilde{m}) \leq n$, a natural number i, $p(\tilde{m}) \leq i \leq n$, and an arbitrary game $m' \in S_{n'}$ define the <u>composition</u> of $\tilde{m}$ and m' resp. i by

$$m = (\lambda(m')\cdot\tilde{m}_1,\dots,\lambda(m')\cdot\tilde{m}_{i-1},m'_1,\dots,m'_{n'},\underbrace{1,\dots,1}_{(n-i).\lambda(m')\text{times}}).$$

<u>Lemma 3.8.:</u> With the previous notations m' is the maximal subgame of m and m is decomposable with respect to i,

in addition
$m = \tilde{m} \otimes_i m'$.

Proof: From the definition of $p(\tilde{m})$ and the last lemma it is easy to see that m is irreducible, so it suffices to show that m' is the maximal subgame of m.
The fact m' being a subgame of m is a direct consequence of Definition 3.1.

Let $m'' = (m_{i'},\dots,m_{s'})$ be the maximal subgame.
Assume $i' < i$, then

$$(\tilde{m}_{i'},\dots,\tilde{m}_{i-1},\underbrace{1\dots,1}_{(\frac{s'-s+1}{\lambda(m')})\text{times}})$$

is a subgame of $\tilde{m}$, thus $p(\tilde{m}) \geq i+1$, a contradiction.
Moreover $\tilde{m}_{p(\tilde{m})-1}$ equals one, that means

$$m_{p(\tilde{m})-1} = \lambda(m'),$$

thus $s' = s$.

q.e.d.

Notice that $p(m) = i+n'$.

Corollary 3.9. Let $m \in S_n^k$, $k \geq 3$, irreducible. Then m is decomposable and the decomposition is unique, i. e. if

$$m = \tilde{m}^1 \otimes_{i_1} m' = \tilde{m}^2 \otimes_{i_2} m'',$$

then $\tilde{m}^1 = \tilde{m}^2$, $i_1 = i_2$ and $m' = m''$.
Furthermore $i_1 \geq p(\tilde{m}^1)$.

The last results can be summarized in the following important

Theorem 3.10: Let $m \in S_n^k$, $k \geq 3$.

Then m is irreducible, iff there is exactly one "complete decomposition" of m, that means:

$$m = \underbrace{(\dots(}_{(r-1)\text{brackets}} m^1 \otimes_{i_1} m^2) \otimes_{i_2} m^3) \otimes_{i_3} \dots) \otimes_{i_r} m^{r+1}$$

with a reducible game m^1 in $S^k_{n_1}$ and games $m^t \in S_{n_t}$ and

$i_1 \geq p(m^1)$, $i_t \geq i_{t-1} + n_t$. In addition the parameters r, i_t, m^t

are uniquely determined by m and vice versa.

This theorem yields a procedure to construct all games in S^k_n, $k \geq 3$, which are irreducible from the sets S_m, $m \leq n-2$.

It remains to look at the reducible games of S^k_n, which may be constructed in another way, shown in the sequel.

Remark 3.11: If m is a reducible element of S^k_n for some $k \geq 3$, then as well

$f(m) := (m_1, m_3, \ldots, m_{n-m_1})$

is in $S^{k-1}_{n-m_2-1}$ as

$g(m) := (m_1, m_2, m_4, \ldots, m_{n-m_3})$

is in $S^{k-1}_{n-m_3-1}$ and in addition f(m) and g(m) are

reducible, if $k \geq 4$.

The other way round, if

m^1 is in $S^{k-1}_{n_1}$ and m^2 is in $S^{k-1}_{n_2}$, both are reducible,

if $k \geq 4$, such that

$m^2_1 = m^1_1$, $m^2_2 \geq m^1_2$, $m^2_3 = m^1_3, \ldots, m^2_{n_1} = m^1_{n_1}$ (especially

$n_2 \geq n_1$) then

$$m = (m^1_1, m^2_2, m^1_2, m^2_3, \ldots, m^2_{n_2}, \underbrace{1, \ldots, 1}_{m^1_2 \text{ times}})$$

is a reducible game in $S^k_{n_2+m^2_2+1}$ and in addition

$f(m) = m^1$, $g(m) = m^2$.

So we have proved the following

Theorem 3.12: Let $m \in S^k_n$, $k \geq 3$.

m is reducible, iff there are two -in case $k \geq 4$ reducible- games $m^1 \in S^{k-1}_{n_2}$ with $n_2 \geq n_1$ and $m^1_i = m^2_i$ for all $i \in \{1, 3, \ldots, n_1\}$ such that

$$m = (m_1^1, m_2^2, m_2^1, m_3^2, \ldots, m_{n_2}^2, \underbrace{1, \ldots, 1}_{m_2^1 \text{ times}}).$$

Moreover these games m^1 and m^2 are uniquely determined by m and vice versa. We say that m can be <u>reduced</u> to m^1 and m^2.
The last two theorems together with the bijections $(H_{|S_n^2})^{-1}$ yield a procedure to construct all classes of homogeneous zero-sum games without dummies successively.
The last section of this chapter refers to Lemma 2.6 in order to show that the inequalities in fact are equations, applying the previous results. More precisely it must be shown that s_n^k and s_{n-2}^{k-1} resp. t_n^k and t_{n-2}^{k-1} coincide, if $3k-3 \geq n \geq 5$. In view of the proof of Lemma 2.6 it suffices to show that m_i equals $m_{k-1} \cdot c_i$ for all $1 \leq i \leq k-2$ (for some natural numbers c_i) and all $m \in S_{n-2}^{k-1}$ - because then

$$(m_{k-1}, \ldots, m_n) \text{ hom } m_i,\ 1 \leq i \leq k-1,$$

thus

$$(m_1, \ldots, m_{k-1}, m_{k-1}, m_{k-1}, m_k, \ldots, m_n)$$

is a game in S_n^k, implying the other inequalities.
This can be done by induction on n:
Assume the assertion is true for all $l \leq n-1 \geq 5$ as it is for $l = 5$.
Let m be a game in S_{n-2}^{k-1}.

<u>1st STEP:</u> Propose m is decomposable, let us say that

$$m = m^1 \otimes_j m^2,$$

$m^1 \in S_l^{k-1}$ for some $l \leq n-3$, by induction hypothesis $m_i^1 = c_i \cdot m_{k-1}^1$, $i \leq k-1$ because $k \geq \frac{l+3}{3}$. The fact $m_i = \lambda(m^2) \cdot m_i^1$ for $i \leq k-1$ finishes the proof in this case.

<u>2nd STEP:</u> Propose m is reducible or k = 3. Assume there is $i \leq k-2$ such that $m_i \not\equiv 0 \mod m_{k-1}$, then $m_i \geq m_k + m_{k+1}$, thus $m_1 \geq m_k + m_{k+1}$ and $m_{k-1} \geq 1$.

Consequently we have

$$\sum_{i=2}^{k-2} m_i \geq 2(k-3),$$

thus

$$n-2 \geq 2(k-3) + 2 + k = 3k-4$$

because

$$(m_1, m_{k-1}, m_k, \ldots, m_{n-2-\sum_{i=2}^{k-2} m_i})$$

is an ordered minimal representation by the fact that m is reducible resp. k = 3. This shows that n must be greater than 3k-3, a contradiction.

As all cases were treated the proof is finished.

§ 4 A Lower Bound for the Number of Classes of Games with Length Two

At first the definition of a set of n-person games of length two, introduced by ISBELL ([1]), is recalled, from which he derived the well-known lower bound 2^{n-4}, i. e. the cardinality of this set.

Let

$$\tilde{m} = (m_1, \underbrace{m_2,\ldots,m_2}_{(k_2+1)\text{times}}, \underbrace{m_3,\ldots m_3}_{k_3\text{times}}, \ldots, \underbrace{m_j,\ldots,m_j}_{k_j\text{times}})$$

be a vector in $\mathbb{N}^n$, such that

(1) $k_2+\ldots+k_j = n-2$, $k_l \geq 1$, for $2 \leq l \leq j$;

(2) $m_1 = \sum_{l=2}^{j} k_l m_l - m_2 + 1$;

(3) $$m_l = \begin{cases} \sum_{r=1}^{\frac{j-l-2}{2}} k_{l+2r-1} \cdot m_{l+2r-1} + 1, & \text{if } j-l \equiv 0 \bmod 2. \\ \sum_{r=1}^{\frac{j-l-1}{2}} k_{l+2r-1} \cdot m_{l+2r-1}, & \text{if } j-l \equiv 1 \bmod 2. \end{cases}, \quad l \geq 2.$$

Then m is a game in S_n^2.

Exactly these games possess a "minimal" incidence-matrix, i. e. one with n rows. For this property and the definition of the incidence-matrix I refer to ISBELL ([1,2]) and OSTMANN ([3]).

Define

$$C_n = \{m \in S_n | m \text{ has a minimal incidence-matrix}\}.$$

It is known from the paper mentioned above [1], that $C_3 = \{(1,1,1)\}$ and $|C_n| = 2^{n-4}$ in case $n \geq 4$.

The rest of this chapter is used to enlarge the sets C_n.

Continue the composition $\otimes_k$ of the last paragraph formally on C_n, which yields a mapping

$$\varphi_n : \bigcup_{k+l=n+1} C_k \otimes S_l \rightarrow S_n^2$$

$$(m,m') \mapsto m \otimes_k m',$$

where

$$m \otimes_k m' = (m_1\lambda(m'),\ldots,m_{k-1}\lambda(m'),m_1',\ldots,m_l').$$

Lemma 4.1: A game in C_n has no subgame.

Proof: Assume the contrary and $n \geq 4$, the case $n = 3$ being trivial. Let $m' = (m_j, \dots, m_s)$ be a subgame. From the fact that $m_2 = m_3$ we know that j must be larger than three.

Put

$$c = \frac{m_{j-1}}{\lambda(m')}, \quad c' = \frac{m_{s+1}+\dots+m_n}{\lambda(m')}$$

It follows from Theorem 1.3.(v) that c is a positive and c' a nonnegative integer, such that

$$c' \geq c-1$$

The game m has the property (3) of the beginning of this paragraph. Applying (3) twice it follows that

$$m_{j-1} = \sum_{r=j+1}^{n} m_r + 1,$$

thus

$$m_{j-1} = 2 \cdot \lambda(m') + c'\lambda(m') - m_j.$$

On the other hand $m_{j-1} = c\lambda(m')$, which leads to

$$(c' + 2 - c)\lambda(m') = m_j,$$

thus

$$m_j \geq \lambda(m'),$$

which contradicts the fact that m' is a game in some S_t.

q.e.d.

Corollary 4.2: The mapping φ_n is injective and the image of φ_n does not contain any game with a minimal incidence-matrix, formally

$\text{im } \varphi_n \cap C_n = \emptyset$.

Proof: The second assertion is implied by the last lemma because of the fact that m' is a subgame of $\varphi_n(m,m')$. The first assertion follows directly by considering m' as the maximal subgame of $\varphi_n(m,m')$, otherwise m would have a subgame, too.

q.e.d.

Consequently the set S_n^2 contains two disjoint subsets

C_n and $\varphi_n\,(\bigcup_{k=3}^{n-2} C_k \times S_{n-j+1})$.

Now we define subsets of S_n^2, which contain c_n, inductively by

$$\tilde{C}_3 := C_3,\quad \tilde{C}_4 := C_4,\quad \tilde{C}_{n+1} := C_{n+1} \cup \varphi_{n+1}\,(\bigcup_{k=3}^{n-1} C_k \times \tilde{C}_{n+2-k}).$$

Let c_n denote the cardinality of $\tilde{C}_n$.

<u>Remark 4.3:</u> In the case $n \geq 7$ the set $\tilde{C}_n$ does not contain the image of φ_n because the game

$(1,1,1,1,1)$

is in S_5 but not in $\tilde{C}_5$.

In order to get a lower bound of the cardinality of S_n^2, the cardinality of $\tilde{C}_n$ is determined in the next

<u>Lemma 4.4:</u>

(i) $c_3 = c_4 = 1$, $c_5 = 3$.

(ii) If $n \geq 5$, then $c_{n+1} = 2c_n + c_{n-1} - c_{n-2}$.

(iii) $c_{n+1} \geq 2c_n$, if $n \geq 4$.

<u>Proof:</u> ad (i) The equation $c_3 = c_4 = 1$ is obvious. The fact $c_5 = 3$ is a direct consequence of the last corollary.

ad (ii) The last corollary implies for all $n \geq 5$ that

$$c_n = 2^{n-4} + \sum_{k=4}^{n-2} 2^{k-4} c_{n-k+1} + c_{n-2}.$$

A simple induction on n shows the assertion, applying the last formula.

(iii) is a direct consequence of (ii), since the increase of c_n with respect to n is obvious.

q.e.d.

Furthermore in case $n \geq 5$ we have - by applying Lemma 4.4(ii) twice:

$$c_{n+2} = 5c_n + c_{n-1} - 2c_{n-2}.$$

This equality yields the following inequality by Lemma 4.4(iii):

$$c_{n+2} \geq 5c_n, \; n \geq 6.$$

From the list in the supplement it can be seen that $c_6 = 6$ and $c_7 = 14$, thus

$$c_{6+2_j} \geq 6 \cdot 5^j, \quad j \geq 0.$$

$$c_{7+2_j} \geq 14 \cdot 5^j, \; j \geq 0.$$

With the help of the preceding results we can give a real number n, such that 2^{l+k} is at most the cardinality of $\tilde{C}_l$ for all $l \geq n$. Indeed, the next theorem yields this natural number.

Theorem 4.5: If

$$n \geq 2 \frac{(k+4)\ln 2 - \ln 3}{\ln 5 - \ln 4} + 7 \;, \quad k \geq 1 \text{ fixed},$$

then

$$c_n \geq 2^{n+k}.$$

The proof can be treated analogously to the proof of Lemma 2.14 and is therefore omitted.

SUPPLEMENT

Some lists:

In List 1 the entries in the rectangles represent " $\tilde{s}_n^k \; \tilde{t}_n^k$ ".

In List 2 they represent " $a_n^k \; b_n^k$ ".

List 1:

n \ k	2	3	4	5	6	7	8	9	10	11	12
3	1 1										
4	1 2										
5	3 5	1 1									
6	7 13	1 2									
7	19 34	3 5	1 1								
8	48 88	6 12	1 2								
9	125 227	16 30	3 5	1 1							
10	318 581	37 73	7 13	1 2							
11	ooo	95 183	18 33	3 5	1 1						
12	ooo	234 456	44 83	7 13	1 2						
13	ooo	ooo	111 209	19 34	3 5	1 1					
14	ooo	ooo	278 527	46 86	7 13	1 2					
15	ooo	ooo	ooo	118 219	19 34	3 5	1 1				
16	ooo	ooo	ooo	295 554	47 87	7 13	1 2				
17	ooo	ooo	ooo	ooo	120 222	19 34	3 5	1 1			
18	ooo	ooo	ooo	ooo	302 564	47 87	7 13	1 2			
19	ooo	ooo	ooo	ooo	ooo	121 223	19 34	3 5	1 1		
20	ooo	ooo	ooo	ooo	ooo	304 567	47 87	7 13	1 2		
21	ooo	ooo	ooo	ooo	ooo	ooo	121 223	19 34	3 5	1 1	
22	ooo	ooo	ooo	ooo	ooo	ooo	305 568	47 87	7 13	1 2	
23	ooo	ooo	ooo	ooo	ooo	ooo	ooo	121 223	19 34	3 5	1 1
ooo	ooo	ooo	ooo	ooo	ooo	ooo	ooo	ooo	ooo	ooo	ooo

In the marked area of List 1 there are only entries with respect to k and n with $\tilde{s}_n^k = \tilde{s}_{n-2}^{k-1}$ and $\tilde{t}_n^k = \tilde{t}_{n-2}^{k-1}$ because there the inequality $n \leq 3k-3$ is valid.

<u>List 2:</u>

n \ k	2	3	4	5	6	7	8
3	1 1						
4	1 2						
5	3 5	1 1					
6	7 13	1 2					
7	19 34	3 5	1 1				
8	48 88	7 13	1 2				
9	126 229	19 34	3 5	1 1			
10	325 594	48 88	7 13	1 2			
11	846 1543	126 229	19 34	3 5	1 1		
12	2193 4005	325 594	48 88	7 13	1 2		
13	5697 10399	846 1543	126 229	19 34	3 5	1 1	
14	14785 26996	2193 4005	325 594	48 88	7 13	1 2	
15	38398 70088	5687 10399	846 1543	126 229	19 34	3 5	1 1
16	99658 181957	14785 24996	2193 4005	325 594	48 88	7 13	1 2

Next the first eleven c_n are tabulated (see §4):

n	3	4	5	6	7	8	9	10	11	12	13
c_n	1	1	3	6	14	31	70	157	353	793	1782

As an example all games in S_9 with length at least three are listed with the help of the results of §3, the sets $S_3,\dots,S_7$ being explicitly known from ISBELL ([2]).

The reducible games in S_9^k, $k \leq 3$:

(3,3,1,1,1,1,1,1,1) $\in S_9^3$	can be reduced to	(3,1,1,1,1) and (3,1,1,1,1).
(4,2,1,1,1,1,1,1,1) $\in$ "	" " " "	(4,1,1,1,1,1) " (4,2,1,1,1,1,1).
(4,2,2,2,1,1,1,1,1) $\in$ "	" " " "	(4,2,2,1,1,1) " (4,2,2,1,1,1).
(5,1,1,1,1,1,1,1,1) $\in$ "	" " " "	(5,1,1,1,1,1,1) " (5,1,1,1,1,1,1).
(3,3,2,2,1,1,1,1,1) $\in$ "	" " " "	(3,2,2,1,1) " (3,3,2,1,1,1).
(3,2,2,1,1,1,1,1,1) $\in$ "	" " " "	(3,2,1,1,1,1) " (3,2,1,1,1,1).
(5,2,2,2,2,1,1,1,1) $\in$ "	" " " "	(5,2,2,2,1,1) " (5,2,2,2,1,1).
(2,2,1,1,1,1,1,1,1) $\in S_9^4$	" " " "	(2,1,1,1,1,1) " (2,2,1,1,1,1,1).
(3,1,1,1,1,1,1,1,1) $\in$ "	" " " "	(3,1,1,1,1,1,1) " (3,1,1,1,1,1,1).
(1,1,1,1,1,1,1,1,1) $\in S_9^5$	" " " "	(1,1,1,1,1,1,1) " (1,1,1,1,1,1,1).

The decomposable games in S_9:

(1,1,1,1,1)	$\otimes_5$ (1,1,1,1,1)	=	(3,3,3,3,1,1,1,1,1)	$\in S_9^3$
(1,1,1,1,1)	$\otimes_5$ (3,1,1,1,1)	=	(4,4,4,4,3,1,1,1,1)	$\in S_9^3$
(1,1,1,1,1)	$\otimes_5$ (2,2,1,1,1)	=	(4,4,4,4,2,2,1,1,1)	$\in S_9^3$
(1,1,1,1,1)	$\otimes_5$ (3,2,2,1,1)	=	(5,5,5,5,3,2,2,1,1)	$\in S_9^3$
(2,1,1,1,1,1)	$\otimes_6$ (2,1,1,1)	=	(6,3,3,3,3,2,1,1,1)	$\in S_9^3$
(3,1,1,1,1,1,1)	$\otimes_7$ (1,1,1)	=	(6,2,2,2,2,2,1,1,1)	$\in S_9^3$
(2,2,1,1,1,1,1)	$\otimes_7$ (1,1,1)	=	(4,4,2,2,2,2,1,1,1)	$\in S_9^3$
(1,1,1,1,1,1,1)	$\otimes_7$ (1,1,1)	=	(2,2,2,2,2,2,1,1,1)	$\in S_9^4$

As last example the elements of C_9 are written down:

(7,1,1,1,1,1,1,1,1)	(7, 6, 6,1,1,1,1,1,1)
(11,5,5,5,1,1,1,1,1)	(11, 6, 6,5,1,1,1,1,1)
(13,4,4,4,4,1,1,1,1)	(13, 9, 9,4,4,1,1,1,1)
(14,5,5,5,4,1,1,1,1)	(14, 9, 9,5,4,1,1,1,1)
(13,3,3,3,3,3,1,1,1)	(13,10,10,3,3,3,1,1,1)
(14,7,7,7,3,3,1,1,1)	(17,10,10,7,3,3,1,1,1)
(15,4,4,4,4,3,1,1,1)	(15,11,11,4,4,3,1,1,1)
(18,7,7,7,4,3,1,1,1)	(18,11,11,7,4,3,1,1,1)
(11,2,2,2,2,2,2,1,1)	(11, 9, 9,2,2,2,2,1,1)
(16,7,7,7,2,2,2,1,1)	(16, 9, 9,7,2,2,2,1,1)
(17,5,5,5,5,2,2,1,1)	(17,12,12,5,5,2,2,1,1)
(19,7,7,7,5,2,2,1,1)	(19,12,12,7,5,2,2,1,1)
(14,3,3,3,3,3,2,1,1)	(14,11,11,3,3,3,2,1,2)
(19,8,8,8,3,3,2,1,1)	(19,11,11,8,3,3,2,1,1)
(18,5,5,5,5,3,2,1,1)	(18,13,13,5,5,3,2,1,1)
(21,8,8,8,5,3,2,1,1)	(21,13,13,8,5,3,2,1,1)

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