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Statistically Varying K-Person Games with Incomplete Information

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Abstract

The paper provides a functional relationship between the amount of information available to the players and an equilibrium payoff in a game with statistically varying states of nature. Moreover, it is shown that for a supergame consisting of independent repetitions of a common matrix game equilibrium strategies are available which provide only "few" sequences of actions with positive probability.

Generalizing an earlier paper of the author on two-person zero-sum games [10] this paper is intended to present a functional relationship between the amount of information available on the statistically varying states of nature and an equilibrium payoff resulting thereof in a non-cooperative K-person game. Supposing the states of nature to vary according to some probability distribution may be viewed at as a more thorough introspection of explicit normal-form games. These are defined to be one-shot games in which a random move selecting a state of nature is performed at the beginning of the play. They are usually denoted by $\Gamma = ((\Omega, \mathfrak{A}, \mu), \mathfrak{A}_k, A_k, u_k)$ where $(\Omega, \mathfrak{A}, \mu)$ is a probability space of states of nature, $\mathfrak{A}_k$ denotes the information σ-algebra of player k, A_k his set of actions and $u_k : \Omega \times \prod_l A_l \longrightarrow \mathbb{R}$ defines his payoff depending on the state of nature and the actions selected. In those well-known games the information structure $\mathfrak{A}_k$ is given in advance. Consequently, player k's strategies are bound to be the $\mathfrak{A}_k$-measurable functions $\sigma_k : \Omega \longrightarrow \Delta\,(A_k)$, defining a probability on the set of available actions for each state of nature. (Of course, some assumptions are required to ensure the set of measurable function to behave well.) By means of the functional relationship of the states of nature and the probability-distributions on the set of actions the players make use of information. The information of player consists of the knowledge of the least $\mathfrak{A}_k$-measurable set containing a selected state of nature ω.

From a more long-run oriented point of view the information structure must be considered to be subject to the strategical decisions of the players. Thus, the players have to be endowed with some parameter characterizing their ability of analyzing their environment in contrast to the assumption of a specific information structure given in advance. We have to replace the fixed information σ-algebra and shall admit the players to choose their information structure. Quite trivially, on the selection of information structures there are some constraints to be faced which are related to the coarseness of the information σ-algebras describing the amount of information provided by the information structure. Consequently, we have to define the shape of of the set of available information structures. Not deleting the short-run aspects of information as to be used to select an action depending on the information available at the moment, we allow for a separate choice for the information structure and the mechanism making use of the particular information, the latter being the strategies in common models. This second part of the strategy may be viewed at as to translate the information on the unknown states of nature into actions to be performed. Since no new information is put in between the receipt of information and the choice of an action, we may just for convenience delete the translation process. By now, the information provided by the information structure takes the form of an action, proposed to be carried out by the player. It is an easy thing, of course, to put in these things separately into the formulae to be derived below. The crucial point allowing for this is known as "Data Processing Theorem" in information theory and related to the notion of a "sufficient statistic" in the field of mathematical statistics.

Our proceeding will be based upon the use of information-theoretical methods. We shall assume that the players, among other parameters, are characterized by a numerical upper bound to their ability of deriving information on their environment. In order to be justified to use the information-theoretical measure on information, known as the entropy, we have to take care of the implicit assumptions related to this approach.

A discussion of this point seems to be most important, since on the one hand the field of applications of information theory is overestimated by far and frequently springs off an excessive interpretation of the notion entropy. As a consequence, rather philosophical systems are erected than mathematical models are analyzed. On the other hand, applicability of information-theoretical methods is denied for economics and for decision-making in general. Both opinions in their generality have to be rejected. To oppose the overestimation, we quote R. Gallager [4], p. 13 (any other textbook on information theory contains a similar statement): "... the concept of information is far too broad and pervading to expect any quantitative measure of information to apply universally...". The measure of information, as provided by Shannon's entropy, is restricted to the analysis of the amount and the use of information emerging from a statistical structure. It is appropriate in probabilistic models and its usefulness is intimately related to the usefulness of the probabilistic model as a whole. The semantic aspects of information are completely left apart and it is only the underlying probability measure, which defines the amount of information via the entropy of the distribution. Thus, we cannot be perplexed by observing that information has no value for its own, but only in connection with its use on allowing its users to show a specific variety of behavior caused by variable information. A general criticism on the application of information theory, as e. g. provided by R. A. HOWARD ([5]), being based on the semantic aspect of information, therefore has to be rejected.

In contrast to Marshak's approach, which regards information as a particular kind of commodity having a value for its own (depending on the user), we shall investigate the impact on the payoff, which comes out of a specific amount of information. It is the alteration of the payoff, which attaches a value to information. In our model information shall only serve for defining the bound on the set of available strategies. Thereby, we shall be much more in the neighborhood of the common philosophy behind the use of Shannon's measure of information. Thus, only the justification of the description of the abilities of the players, in terms of entropy, - within an appropriate surrounding - is left to us.

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* R.A. HOWARD has put it this way: '...If losing all your assets in the stock market and having whale steak for supper, then the information associated with the occurence of either event is the same. Attempts to apply Shannon's information theory to problems beyond communications have, in large, come to grief. The failure of these attempts could have been predicted because no theory that involves just the probabilities of outcomes without considering their consequences could possibly be adequate in describing the importance of uncertainty to a decision maker.'

In our model the players use their partial information on the value of parameters of the game, randomly fixed at the beginning of a play, to choose well-adapted actions simultaneously. The value of the randomly chosen state of nature and the selected actions of the player define the payoff to the players. It is this assumption on the type of knowledge of the players, which divides the common semantic or qualitative aspect from the less familiar quantitative aspect of information. This, in turn, enables us making an appropriate use of information-theoretical methods and results.

Since we are interested in the economic aspects of the usefulness of information instead of the maximum throughput of a noisy communication system, it is not sufficient merely to assume equal weight to all reproduction errors on the states of nature. Thus, we are obliged to make use of a generalization of Shannon's result on source-coding. The so-called rate-distortion theory was originally concerned with the investigation of the exactness of reproductions of the information emitted by a source. Here, it will enable us to put attention to the variable utility related to different states of nature. We will thus be allowed to evaluate the expected quality of adapting the actions to the unknown states of nature by using the information. The quality of adaption or reproduction will be expressed in terms of payoff.

By an example, we shall now substantiate the usefulness of the information-theoretical approach towards the measurement of information via entropy within a game theoretical context.

An Example

To a large extent public contracts are conferred to the cheapest supplier of some performance. To ensure competition, the bids are made public simultaneously. Under those basic assumptions game theory is well-established as an instrument of the investigation of the costs bound up to some project.

Suppose performances 1,...,T being advertised to be carried out within some period. We may assume T to be relatively large by either considering long periods or assuming small sized orders. As the type of performances we think of: installing lamps of the street-lighting, carrying out a few square-meters of paving, shifting a fire-plug, transplanting some bushes and so forth. They originate from some limited field of demand, are - without too much idealizing - of a similar amount and may be viewed at as to reappear in each period.

The performances asked for, however, are not identical, since they vary with respect to the specific situations. In particular, carrying through a work, the supplier has to face, more or less expected, some intricacies, which have to be coped with. Since we assumed a fixed price (bid) for the work, the intricacies, augmenting the costs, reduce the net-gain of the supplier. The potentially arising obstacles may be classified as elements of some finite set $\mathfrak{X}$, comparably small to the number T of projects. We shall assume that it is only the degree of intricacy, which specifies the costs of a work. It does not seem to be overly restrictive, to suppose the obstacles to crop up independently according to some probability distribution, which is common knowledge of all competitors. In fact, due to observations in the past the competitors have some experience on the frequency of the degrees of intricacy arising and, since we consider distinct orders, we may assume independence on the occurence of intricacies. Thus, the suppliers of the sequence of performances, marked by the numbers 1 to T, are confronted with a sequence of x^T of degrees of intricacy subsequent to making their bids. The example as specified till now, may be summarized by the following glossary:

$\{1,\ldots,K\} = \mathcal{K}$

is the set of suppliers of

$\{1,\ldots,T\}$,

representing the (similar) performances. Each of the competitors is characterized by his cost function

$R_k : \mathcal{X} \longrightarrow \mathbb{R}$,

defining the cost related to providing work t subject to the degree x of intricacy the rules of conferring the contracts the payoff-function of player $k \in \mathcal{K}$.

$u_k : \mathcal{X} \times \prod_{\ell} A_l \longrightarrow \mathbb{R}$

may be specified by

$$u_k (x, a^K) = \begin{cases} R_k (x) - a_k & \text{iff } a_k < \min \{a_l | k \neq l\} \\ 0 & \text{otherwise.} \end{cases}$$

The investigations as provided by this paper, however, are not restricted to the above form of payoff-functions, instead, we shall admit all functions $u_k : \mathcal{X} \times \prod A_l \longrightarrow \mathbb{R}$, $k \in \mathcal{K}$ for given finite sets $\mathcal{X}$ and A_l, $l \in \mathcal{K}$. In suppressing the index t within the definition of the payoff-functions, we express the costs of filling the contract to be independent of the specific task, but to depend solely on the degree of intricacy. It should also be noted that the assumption of all competitors being able to provide any performance, implicitly contained in our model, is not a crucial one: a service to contract which physically cannot be supplied by competitor n induces him to require a sufficiently high price.

The total payoff to competitor k obtained in the period is defined by the sequences a_k^T of bids, $k \in \mathcal{K}$ and the sequence x^T of degrees of intricacy. We assume additivity, thus,

$$u_k^T (x^T, (a_l^T)) = \Sigma\, u_k (x_t, (a_{l_t})).$$

In order to make the payoff resulting from finer or coarser descriptions of tasks or depending on different lengths of periods comparable, we normalize and consider the average payoff

$$T^{-1} u_k^T(x^T, (a_1^T)).$$

The difficulties set in the course of filling the contracts are predictable to a certain extent by a thorough analysis. However, physical, mental, and financial limitations on the analysis invoke that the uncertainty on the sequence x^T may only be reduced, but not completely removed. This implies that the information on the sequence x^T, which the competitors provide to themselves, may be assumed to be bound. Since the information may be viewed at to be condensed in the bids, it is the average information on the sequence of degrees of intricacy as contained in the bid, which is assumed to be bound. The bound on the average information is intended to describe the abilities of reasoning and analyzing of the competitors and is assumed to be common knowledge.

Remembering the assumptions of our example, in particular those assumptions concerning the unknown parameter, we find ourselves within a probabilistic framework, such that information theory becomes applicable to provide a measure on uncertainty, known as the entropy of a probability distribution. Being given a measure for uncertainty it sounds plausible to define the amount of information received in the course of an experiment or by some analysis by the reduction of uncertainty. Thus, since uncertainty in the context of information theory is measurable by a number, we define the information contained in the bids of player k on the degrees of uncertainty to be the difference of the a priori uncertainty on the sequence of degrees of intricacy minus the a posteriori uncertainty on x^T, given a_k^T. The main difference is denoted as average mutual information, the constraint on the capacity of analysis of the competitors will consequently have the form of an upper bound to the average mutual information.

As far as the available strategies of the competitors are concerned, it remains part of their considerations, how to divide up their capacity of analysis on the particular tasks. This means that the derivation of information on the degree of intricacy for specific projects is subject to their will. As an extreme case some competitor could try to determine the first T_k degrees of intricacy exactly to enable himself to raise optimum bids for them. Due to the restrictions on his capacity of reasoning, this necessarily carries along that for the tasks $T_k + 1,\ldots,T$ he has to base his decisions on his knowledge of the a priori distribution on $\mathfrak{X}$ solely.

The subsequent investigation will provide an answer to the question of optimum behavior of the competitors and to the payoff resulting from those strategies.

The model

The description of a game consists of two parts, first a list of the parameters of the game and secondly the rules according to which the game is played. In this paper, we shall investigate K-person non-cooperative games where the parameters are given as follows. $\mathfrak{X}$ is a finite set of states of nature, which is endowed with some probability distribution μ. We are given also finite sets of actions. A_k for each player $k \in \mathcal{K}$ and payoff-functions $u_k : \mathfrak{X} \times \prod A_l \longrightarrow \mathbb{R}$ relating the states of nature, the actions selected, and the utility obtained by the players. We shall be interested in games being based on T-fold replications of these quantities, namely $\mathfrak{X}^T = \prod_t \mathfrak{X}$, $\mu^T(x^T) = \prod_t \mu(x_t)$ for $x^T \in \mathfrak{X}^T$ and - with the canonical interpretation - $u_k^T(x^T, (a_l^T)) = \sum_t u_k(x_t, (a_{l_t}))$.

The above parameters are also contained in the description of explicit normal-form games. The difference will lie in the not yet specified information structure, since, following the lines of the previous discussion, we shall assume the information structure to be subject to the strategical considerations of the players.

We shall denote by (B_k^T) the actions as selected by the players, depending on the observed accent of the state of nature in the periods 1,...,T. More formally, (B_k^T) is a vector of random-variables with values in (A_k^T) where its conditional probability given X^T - the latter being distributed according to μ^T - expresses the strategical behavior of the players. The restriction on the set of available strategies will be formulated as an upper bound to the "average mutual information":

$$I\ (X^T \wedge B_k^T) \leq T \cdot C_k,\ k \in \mathcal{K}\ ;$$

the definition of the average mutual information may be found below. We emphasize that by the vector $(C_k)_k$ - its components are assumed to be positive - the missing parameters of the games to be considered are given; in analogy to explicit normal-form games the game will be denoted as

$$\Gamma^T = ((\mathfrak{X}^T, \mu^T), TC_k, A_k^T, u_k^T),\quad T \in \mathbb{N}.$$

The yet unspecified rules of the game define the relationship between the parameters of the game. They are condensed in the set of available strategies and the assumptive purport. The latter is expressed by using the common notion of normal-form games: we look for equilibria.

In order to obtain the common normal-form representation for the T-fold replicated parameters, we define the set of (mixed) strategies by

$$\Sigma_k^T = \{B_k^T / B_k^T \mid \mathfrak{X}^T \Longrightarrow A_k^{T^{*)}},\ T^{-1}\ I\ (X^T \wedge B_k^T) \leq C_k\}$$

and the average payoff by

$$U_k^T\ ((B_l^T)) = E\ [T^{-1}\ (u_k^T\ (X^T,\ (B_l^T))].$$

(The average payoff is considered in order to make the equilibrium payoffs obtained from distinct lengths of periods comparable.)

Whereas the existence of equilibria for games Γ^T may be ensured along the usual lines provided by game theory, the dynamics of the equilibrium payoffs may not be analyzed this way. In fact, due to the absence of constructivity in the proof of existence, we are not allowed to see what the equilibrium strategies are alike. Moreover, unless we find a bound on the speed of convergence (?) of equilibrium payoffs related to periods of length T, we may not compute the payoffs even for a modest size of T.

The way we shall proceed will provide the existence of an ε-equilibrium on the one hand, more insight to the shape of the equilibrium on the other - the existence of code-type ε-equilibria will be ensured for sufficiently large T - and it will provide a computable formular for code-type ε-equilibrium payoffs.

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*) The symbol $V|\mathfrak{X} \Longrightarrow A$ is meant to express V being a conditional probability on A given any element $x \in \mathfrak{X}$.

As an interesting feature, we shall also observe that a comparably small number of actions will be endowed with a positive probability of being used by the players. This reminds a little of the results of R. RADNER and R. ROSENTHAL [8], R. AUMANN et al [1] and P. MILGROM and R. WEBER [7] on purification and ε-purification on strategies in explicit normal-form games. Those results are derived under independency and non-atomicity conditions on the set of events observable by the players.

A strategy-vector $(B_k^T*)_k$, $B_k^T \mid \mathcal{X}^T \Longrightarrow A_k^T$, is called ε-equilibrium, if for all $k \in \mathcal{K}$

$$\left|\max_{B_k^T} \{E[T^{-1} u_k^T (X^T, B_k^T, (B_l^T*)_{l \neq k})]\} - E[T^{-1} u_k^T (X^T, (B_l^T*))]\right| < \varepsilon$$

By a series of intermediate results, among others, a source-coding theorem and its converse, the existence of a code-type ε-equilibrium is ensured.

Theorem

Given any equilibrium payoff vector $(D^{d_k}(C_k))_{k \in \mathcal{K}}$ of the one-shot game $\Gamma = ((\mathcal{X}, \mu), C_k, A_k, u_k)$ with positive entities C_k, there exists for any $\varepsilon > 0$ and T sufficiently large a code-type ε-equilibrium $(B_k^T*)_k$ of Γ^T such that

$$\left|E[T^{-1} u_k^T (X^T, (B_l^T*))] - D^{d_k}(C_k)\right| < \varepsilon \text{ for all } k \in \mathcal{K} .$$

The existence of an equilibrium for any one-shot game is readily ensured thereby showing the claim of the theorem to be non-void. The existence of an equilibrium for games Γ^T may also be performed along the following lines:

1.1 Lemma There exists an equilibrium for the game Γ.

Proof:

As it is well-known, $I(X \wedge B)$ is a convex function with respect to the conditional probability defined by B for any X. Thus, we conclude the convexity of Σ_k for any k, moreover, $I(X \wedge \cdot)$ being continuous implies Σ_k to be compact as a closed subset of $(\Delta\ (A_k))^{|\mathfrak{X}|}$. We remark, additionally, that U_k is a multi-affin-linear function, thereby we find the sufficient conditions for the existence of an equilibrium for $\Gamma = (\Sigma_k,\ U_k)$ to be satisfied.

We shall first derive a technical result on the probability of a large deviation from the expected value of the average of a sum of independent random-variables to be used in a random construction of codes. The proof uses Bernstein's version of the Markov-equality and insofar the lemma is folklore.

<u>1.2 Lemma</u> Let X_n, $n = 1,\dots,N$ denote independent, identically distributed random variables with values in $\{0,1\}$. Assume $\alpha \geq E[X_n]$ for all n. Then for any $x \in [0,1]$

$$\Pr\{N^{-1}\cdot\sum_n X_n < x\} \leq \exp\{\,N(x - \alpha\cdot\frac{\log e}{2})\}.$$

<u>Proof:</u> We have

$$\Pr\ \{N^{-1}\sum_n X_n < x\}$$
$$= \Pr\ \{-\sum_n X_n > -Nx\}$$
$$= \Pr\ \{\exp\ \{-\sum_n X_n\} > \exp\ \{-Nx\}\}$$
$$\leq \exp\{\,Nx\}\ E\ [\exp\ \{-\sum_n X_n\}].$$

We proceed by upperbounding the expectation:

$$E\ [\exp\ \{-\sum_n X_n\}]$$
$$= E\ [\exp\ \{-\ X_1\}]^N$$
$$= (\Pr\ \{X_1 = 0\}\ +\ \Pr\ \{X_1 = 1\}\cdot\frac{1}{2})^N$$
$$= (1 - \frac{1}{2}\cdot\Pr\ \{X_1 = 1\})^N$$
$$= (1 - \frac{1}{2}\,E\ [X_1])^N$$
$$\leq e^{-\frac{1}{2}\cdot\alpha\,.\,N}$$

Summarizing we infer

$$\Pr\{N^{-1} \sum_n X_n < x\}$$

$$\leq \exp\{N \cdot x\} \cdot e^{-\frac{1}{2}\alpha N}$$

$$= \exp\{N \cdot (x - \alpha \cdot \frac{\log e}{2})\}.$$

Prior to the application of this lemma within the information-theoretical context we have to provide some notation:

$N(x|x^T)$ denotes the number of occurences of x within the sequence x^T, i. e. $N(x|x^T) = |\{t/x_t = x\}|$. The vector $(T^{-1} N(x|x^T))_{x \in \mathfrak{X}}$ defines a probability distribution on $\mathfrak{X}$, denoted as P_{x^T}.

The set of all probability distribution constructed this way is denoted as $\Delta^T(\mathfrak{X})$. We define $\mathcal{T}_\nu$ to be the set of all sequences x^T giving rise to the same distribution $\nu \in \Delta^T(\mathfrak{X})$. As a natural extension we observe that $(N(x|x^T)^{-1} \cdot N(x,z|x^T,z^T))_{z \in \mathfrak{Z}}$ defines a conditional probability on $\mathfrak{Z}$ given $x \in \mathfrak{X}$, which is denoted as $P_{z^T|x^T}$. We define $\mathcal{T}_V(x^T) = z^T P_{z^T|x^T} = V$ for

$V|\mathfrak{X} \Longrightarrow \mathfrak{Z}$, $x^T \in \mathfrak{X}^T$.

For convenience we recall the definitions of <u>the entropy</u>:

$$H(\mu) = -\sum_x \mu(x) \log \mu(x) \quad \text{for } \mu \in \Delta(\mathfrak{X}),$$

of the <u>conditional entropy</u>:

$$H(V|\mu) = \sum_x \mu(x) \cdot H(V(\cdot|x)), \quad \text{where } V|\mathfrak{X} \Longrightarrow \mathfrak{Z},$$

and of the <u>average mutual information</u> $I(\mu,V) = H(V\circ\mu) - H(V|\mu)$,

where $V \circ \mu$ is defined to be a probability distribution on given by $V \circ \mu\,(z) = \sum_x \mu(x)\, V\,(z|x)$.

We shall make freely use of the well-known inequalities on the terms given above, as provided by information-theory. For more details any textbook may be consulted, see, for example the book of I. CSISZAR and J. KÖRNER [3]. Just for convenience we cite: For any $\varepsilon > 0$ and sufficiently large T

$$|\Delta^T(\mathcal{X})| \leq \exp\{T \cdot \varepsilon\},$$

$$\exp\{T \cdot (H(\mu) - \varepsilon)\} \leq |\mathcal{T}_\mu| \leq \exp\{T \cdot H(\mu)\}, \text{ for } \mu \in \Delta(\mathcal{X}),$$

$$\exp\{T \cdot (H(V|\mu) - \varepsilon)\} \leq |\mathcal{T}_V(x^T)| \leq \exp\{T \cdot (H(V|\mu))\},$$

$$\text{for } V|\Longrightarrow \mathcal{Z},\ x^T \in \mathcal{T}\mu.$$

Since the lemma to be derived now will investigate the abilities of some fixed, but arbitrary player, we shall delete the index and denote $A = A_k$ for his set of available actions. As an application of the foregoing lemma we prove that for an appropriate random mechanism the probability to find only a low number of codewords suited to encode some sequence of states of nature is super-exponentially small.

__1.3 Lemma__ Let $0 < \varepsilon < \log e$, $\nu \in \Delta(\mathcal{X})$ and $V|\mathcal{X} \Longrightarrow A$ be given such that $V \circ \nu \in \Delta^T(A)$, $I(\nu,V) \leq C - 3\varepsilon$. Assume $N \geq \exp\{T \cdot (C-3\varepsilon)\}$. Then there exists $T_0(|\mathcal{X}|,|A|,\varepsilon)$ such that for $T \geq T_0$ and independent and uniformly distributed random-variables Y_n, $n = 1,\dots,N$ with values in $\mathcal{T}_{V \circ \nu}$ the inequality

$$\Pr\{\sum_n 1_{\mathcal{T}_V(x^T)}(Y_n) < \exp\{T \cdot \varepsilon\}\} < \exp\{-\tfrac{\varepsilon}{2}\exp\{T \cdot \varepsilon\}\}$$

holds for any $x^T \in \mathcal{T}_\nu$.

__Proof:__ Observe $T^{-1} \log N \geq I(\nu,V)$ and define α by $\alpha = \exp\{-T \cdot (I(\nu,V) - 2\varepsilon)\}$. Due to a folklore inequality

$$\alpha \geq E[1_{\mathcal{T}_V(x^T)}(Y_n)] \text{ for all } n = 1,\dots,N \text{ and } x^T \in \mathcal{T}_\nu.$$

We infer by the preceding lemma

$$\Pr\{\sum_n 1_{\mathcal{J}_V(x^T)}(Y_n) < \exp\{T\cdot\varepsilon\}\}$$

$$\leq \exp\{N(N^{-1}\exp\{T\cdot\varepsilon\} - \alpha\cdot\frac{\log e}{2})\}$$

$$\leq \exp\{\exp\{T\cdot\varepsilon\} - \frac{\log e}{2}\exp\{T\cdot 2\varepsilon\}$$

whence the claim follows.

We continue to investigate the payoff from player k's point of view and thus regard his opponents as a whole. Therefore it is appropriate to define $\hat{A} = \prod_{l\neq k} A_l$.

The subsequent lemma provides the basis for taking into account the opponents' behavior. Given any (K - 1) - vector of sequences of actions $\hat{a}^T \in \hat{A}^T$ we shall show that most of the sequences used for k's encoding of x^T induce jointly with $\hat{a}^T$ a conditional independent distribution on A x $\hat{A}$, given x. The independence is formalized by the following definition:

Given $\delta > 0$ we set

$$\mathcal{J}^\delta(x^T,\hat{a}^T) = \{a^T / |N(x,a,\hat{a}|x^T,a^T,\hat{a}^T) - V(a|x)N(x,\hat{a}|x^T,\hat{a}^T)| < T\cdot\delta \text{ for all } x,a,\hat{a}\}.$$

By $(\mathcal{J}^\delta(x^T,\hat{a}^T))^c$ we denote its complementary set within A^T.

<u>1.4 Lemma</u> For $\tau,\delta > 0$ there exists $T_0(|\mathcal{X}|,|A|,|\hat{A}|,\tau,\delta)$ such that for $T \geq T_0$, $\hat{a}^T \in \hat{A}^T$, $x^T \in \mathcal{X}^T$ and Y uniformly distributed on $\mathcal{J}_V(x^T)$

$$E[1_{\mathcal{J}^\delta(x^T,\hat{a}^T)}(Y)] \geq 1 - \tau.$$

Proof:

$$E\,[1_{\mathcal{T}^{\delta}(x^T,\hat{a}^T)}(Y)]$$

$$= Pr\{\bigwedge_{x,a,\hat{a}} |T^{-1}\, N(x,a,\hat{a}|x^T,Y,\hat{a}^{1T}) - T^{-1}V(a|x) \cdot N(x,\hat{a}|x^T,\hat{a}^T)| < \delta\}$$

$$= 1 - Pr\{\bigvee_{x,a,\hat{a}} |N(x,a,\hat{a}|x^T,Y,\hat{a}^{1T}) - V(a|x) \cdot N(x,\hat{a}|x^T,\hat{a}^T)| \geq T\delta\}$$

Due to the assumptions on Y, for any $x^T,\hat{a}^T$ $N(x,a,\hat{a}|x^T,Y,\hat{a}^{1T})$ is a hypergeometrical-distributed random-variable with expectation $N(x,\hat{a}|x^T,\hat{a}^T) \cdot V(a|x)$

and variance bounded from above by

$$N(x,\hat{a}|x^T,\hat{a}^T) \cdot V(a|x) \cdot (1 - V(a|x)) \leq \frac{1}{4} \cdot N(x,\hat{a}|x^T,\hat{a}^T).$$

Thus, by Chebychev's inequality

$$Pr\{\bigvee_{x,a,\hat{a}} N(x,a,\hat{a}|x^T,a^T,\hat{a}^T) - V(a|x) \cdot N(x,\hat{a}|x^T,\hat{a}^T)| \geq T \cdot \delta\}$$

$$\leq |\mathcal{X}| \cdot |A| \cdot |\hat{A}| \cdot \frac{N(x,\hat{a}|x^T,\hat{a}^T)}{4 \cdot T^2 \cdot \delta^2}$$

$\leq \tau$ for sufficently large T.

We learned that, given any $\hat{a}^T \in \hat{A}^T$, most sequences from $\mathcal{T}_V(x^T)$ induce a conditional independent distribution on $A \times \hat{A}$, given x; i. e. most sequences lie within the bounds of $\mathcal{T}^{\delta}(x^T,\hat{a}^T)$. Let us now choose a sufficiently large number of sequences according to the uniform distribution on $\mathcal{T}_V(x^T)$. Then the probability of their share with $\mathcal{T}^{\delta}(x^T,\hat{a}^T)$ being apart from one is exponentially bounded.

<u>1.5 Lemma</u> Let $\varepsilon,\delta > 0$ be given and assume $\log(1 + \tau) < \frac{\varepsilon}{2}$. Then there exists $T_0(|\mathcal{X}|,|A|,|\hat{A}|,\varepsilon,\delta,\tau)$ such that for all $T \geq T_0$ and independent, on $\mathcal{T}_V(x^T)$, uniformly distributed random-variables Y_l, $l = 1,\ldots,L$, $L \geq \exp\{T \cdot \varepsilon\}$, the inequality

$$\Pr \{L^{-1} \sum_{l} \mathbb{1}_{\mathcal{T}^{\delta}(x^T,\hat{a}^T)}(Y_l) < 1 - \varepsilon\} \leq \exp\{-L \cdot \frac{\varepsilon}{2}\}$$

holds for any $\hat{a}^T \in \hat{A}^T$.

Proof: Using the Markov-inequality, we obtain

$$\Pr \{L^{-1} \sum_{l} \mathbb{1}_{\mathcal{T}^{\delta}(x^T,\hat{a}^T)}(Y_l) < 1 - \varepsilon\}$$

$$= \Pr \{ \sum_{l} \mathbb{1}_{(\mathcal{T}^{\delta}(x^T,\hat{a}^T))^c}(Y_l) > L \varepsilon\}$$

$$\leq \exp \{-L\varepsilon\} E[\exp \{\sum_{l} \mathbb{1}_{(\mathcal{T}^{\delta}(x^T,\hat{a}^T))^c}(Y_l) \}].$$

Due to the independency of the random-variables Y_l, $l = 1,\ldots,L$ and using $\exp \{t\} \leq 1 + t$ for $t \in [0,1]$ we infer

$$E [\exp \{ \sum_{l} \mathbb{1}_{(\mathcal{T}^{\delta}(x^T,\hat{a}^T))^c}(Y_l) \}] \leq (1 + E[\mathbb{1}_{\mathcal{T}^{\delta}(x^T,\hat{a}^T))^c}(Y_l)])^L$$

As a consequence of the preceding lemma we may summarize

$$\Pr \{L^{-1} \sum_{l} \mathbb{1}_{\mathcal{T}^{\delta}(x^T,\hat{a}^T)}(Y_l) < 1 - \varepsilon\}$$

$$\leq \exp \{-L (\varepsilon - \log(1 + \tau))\}$$

whence the claim follows.

We have now provided all the material necessary to prove a source coding theorem which shall enable us to describe the abilities of the players as far as achieving a maximum payoff is concerned. The original version of this result, being occupied with deterministic encoding, dates back to 1959 when C. E. SHANNON [9] started the investigation on what information should be transmitted within a communication system. Our random encoding rule of the "source" emitting the states of nature will represent the type of analysis as being performed by a player. Given x^T, the outcome of the player's analysis is represented by one of the player's codewords for x^T.

As far as the formulation of the coding theorem is concerned, a comment seems to be necessary. Since the sequences x^T of states of nature are randomly selected according to a product distribution, it is a comparably small set of sequences which almost carries the total probability. It is given by

$$\mathcal{T}^\delta = \mathcal{T}^\delta_\mu = \{x^T / |N(x|x^T) - \mu(x)| < T \cdot \delta \text{ for all } x \in \mathfrak{X}\}$$

In fact, by Chebychev's inequality it is easily derived that for all $\varepsilon > 0$, $\delta > 0$ and sufficiently large $T(|\mathfrak{X}|, \varepsilon\ \delta)$: $\mu^T(\mathcal{T}^\delta_\mu) \geq 1 - \varepsilon$.

Observing $\mathcal{T}^\delta = \bigcup_{\substack{\nu: \\ \rho(\mu,\nu)<\delta}} \mathcal{T}_\nu$ for $\rho(\mu,\nu) = \max \{|\mu(x) - \nu(x)|\}$

we find restricting the attention on those sets $\mathcal{T}_\nu$ for which $\rho(\mu,\nu) < \delta$ to be sufficient. - The formal argument with respect to "sufficiency" will be given subsequent to the proof of the theorem.

1.6 Theorem

For any $\varepsilon > 0$, $\delta > 0$ $\mu \in \Delta(\mathfrak{X})$ and $V|\mathfrak{X} \Longrightarrow A$ such that $I(\mu; V) \leq C - 2\varepsilon$ there exists $T_0(|\mathfrak{X}|, |A|, |\hat{A}|, \varepsilon, \delta)$ such that for $T \geq T_0$ and for $\nu \in \Delta^T(\mathfrak{X})$ such that $\rho(\mu,\nu) \leq \delta$ there exists $\mathcal{C}_\nu \subset A^T$ such that $|\mathcal{C}_\nu| \geq \exp\{T \cdot (C - 2\varepsilon)$,

$$\Big|\bigcup_{\substack{\nu: \\ \rho(\mu,\nu)<\delta}} \mathcal{C}_\nu\Big| \leq \exp\{T \cdot C\}$$

and for all ν satisfying $\rho(\mu,\nu) < \delta$:

$$(*) \quad \frac{|\mathcal{C}_\nu \cap \mathcal{T}_V(x^T) \cap \mathcal{T}^\delta(x^T, \hat{a}^T)|}{|\mathcal{C}_\nu \cap \mathcal{T}_V(x^T)|} \geq 1 - \varepsilon$$

for all $x^T \in \mathcal{T}_\nu$ and $\hat{a}^T \in \hat{A}^T$.

Henceforth inequality (*) will be called joint type condition.

Proof: At first recall $I(\nu;V)$ to be a continuous function with respect to the probability $\nu \in \Delta(\mathcal{X})$ and the conditional probability $V|\mathcal{X} \Longrightarrow A$. The compactness of $\Delta(\mathcal{X})$ and $(\Delta(A))^{|\mathcal{X}|}$ thereby yield $I(\cdot;\cdot)$ to be uniformly continuous; as a consequence to every $\varepsilon > 0$ we may find a $\delta_\varepsilon > 0$ such that $\rho(\mu,\nu) < \delta_\varepsilon$ implies $|I(\nu;V) - I(\mu;V)| < \varepsilon$ for all $V|\mathcal{X} \Longrightarrow A$. Thus, $I(\mu;V) \le C - 2\varepsilon$ gives $I(\mu;V) \le C - \varepsilon$ for all μ such that $\rho(\mu,\nu) < \delta_\varepsilon$.

Now the Lemmas 1.3 and 1.5 will allow for proving the existence of sets $\mathcal{C}_\nu \subset A^T$ with $|\mathcal{C}_\nu| \le \exp\{T(C - \varepsilon)\}$ such that for all $x^T \in \mathcal{T}_\nu$ and $\hat{a}^T \in \hat{A}^T$ the inequality

$$\frac{|\mathcal{C}_\nu \cap \mathcal{T}_V(x^T) \cap \mathcal{T}^\delta(x^T,\hat{a}^T)|}{|\mathcal{C}_\nu \cap \mathcal{T}_V(x^T)|} \ge 1 - \varepsilon$$

holds. In fact, let $N_\nu \in \mathbb{N}$ be defined such that $\exp\{T(C - \varepsilon)\} \ge N_\nu \ge \exp\{T(C - 2\varepsilon)\}$. Suppose Y_n^ν, $n = 1,\dots,N_\nu$ to denote independent random variables, uniformly distributed on $\mathcal{T}_{V\circ\nu} \subset A^T$. Then

$$\Pr\{\sum_n \mathbb{1}_{\mathcal{T}_V(x^T)\cap\mathcal{T}^\delta(x^T,\hat{a}^T)}(Y_n^\nu) < (1-\varepsilon)\sum_n \mathbb{1}_{\mathcal{T}_V(x^T)}(Y_n^\nu)$$

$$\text{for some } x^T \in \mathcal{T}_\nu \text{ or } \hat{a}^T \in \hat{A}^T\}$$

$$\le \sum_{x^T\in\mathcal{T}_\nu}\sum_{\hat{a}^T\in\hat{A}^T} \Pr\{\sum_n \mathbb{1}_{\mathcal{T}_V(x^T)\cap\mathcal{T}^\delta(x^T,\hat{a}^T)}(Y_n^\nu) < (1-\varepsilon)\sum_n \mathbb{1}_{\mathcal{T}_V(x^T)}(Y_n^\nu)\}$$

The latter probability may be rephrased by introducing explicitly the sets of indices n such that $Y_n^\nu \in \mathcal{T}_V(x^T)$. We obtain the expression

$$\sum_{L=1}^{N_\nu} \sum_{\substack{A_L \subset \{1,\dots,N_\nu\} \\ |A_L|=L}} \Pr\{\mathbb{1}_{\tilde{\mathcal{J}}_V(x^T)}(Y_n^\nu) = 1_{A_L}(n),\ L^{-1} \sum_n \mathbb{1}_{\tilde{\mathcal{J}}_V(x^T)\cap\mathcal{J}^\delta(x^T,\hat{a}^T)}(Y_n^\nu) < 1-\varepsilon\}$$

$$= \sum_L \sum_{\substack{A_L \\ |A_L|=L}} \Pr\{\mathbb{1}_{\tilde{\mathcal{J}}_V(x^T)}(Y_n^\nu) = 1_{A_L}(n)\}$$

$$\cdot \Pr\{L^{-1} \sum_n \mathbb{1}_{\mathcal{J}_V(x^T)\cap\mathcal{J}^\delta(x^T,\hat{a}^T)}(Y_n^\nu) < 1-\varepsilon \mid \mathbb{1}_{\tilde{\mathcal{J}}_V(x^T)}(Y_n^\nu) = 1_{A_L}(n)\}$$

Referring to the size of L we may upperbound the latter expression by

$$\sum_{L=1}^{\exp\{T\cdot\varepsilon\}} \sum_{\substack{A_L \\ |A_L|=L}} \Pr\{\mathbb{1}_{\mathcal{J}_V(x^T)}(Y_n^\nu) = 1_{A_L}(n)\}$$

$$+ \sum_{L > \exp\{T\cdot\varepsilon\}} \sum_{\substack{A_L \\ |A_L|=L}} \Pr\{\mathbb{1}_{\tilde{\mathcal{J}}_V(x^T)}(Y_n^\nu) = 1_{A_L}(n)\}$$

$$\cdot \Pr\{L^{-1} \sum_n \mathbb{1}_{\mathcal{J}_V(x^T)\cap\mathcal{J}^\delta(x^T,\hat{a}^T)}(Y_n^\nu) < 1-\varepsilon \mid \mathbb{1}_{\tilde{\mathcal{J}}_V(x^T)}(Y_n^\nu) = 1_{A_L}(n)\}$$

In order to give a bound on the first item we use Lemma 1.3. The bounding expression is just

$$\exp\{-\frac{\varepsilon}{2}\cdot\exp\{T\cdot\varepsilon\}\}.$$

Bounding the second term is slightly more complicated. Given $Y_n^\nu \in \tilde{\mathcal{J}}_V(x^T)$ we find the random variables Y_n^ν to be independent and uniformly distributed on $\tilde{\mathcal{J}}_V(x^T)$! Thus, an upper bound to

$$\max_{\substack{L \\ L>\exp\{T\cdot\varepsilon\}}} \{\Pr\{L^{-1} \sum_1 \mathbb{1}_{\tilde{\mathcal{J}}^\delta(x^T,\hat{a}^T)}(Y_n^\nu) < 1-\varepsilon\}$$

for independent uniformly distributed random variables Y_1 with values in $\tilde{J}_V(x^T)$ is required. The appropriate bound

$$\max_{\substack{L \\ L > \exp\{T\cdot\varepsilon\}}} \exp\{-L\cdot\frac{\varepsilon}{2}\}$$

$$\leq \exp\{-\frac{\varepsilon}{2}\exp\{T\cdot\varepsilon\}\}$$

is provided by Lemma 1.5.

Summarizing, we obtain

$$\Pr\{\sum_n 1_{\tilde{J}_V(x^T)\cap\tilde{J}^\delta(x^T,\hat{a}^T)}(Y_n^\nu) < (1-\varepsilon)\sum_n 1_{\tilde{J}_V(x^T)}(Y_n^\nu)$$

$$\text{for some } x^T \in \tilde{J}_\nu \text{ or } \hat{a}^T \in \hat{A}^T\}$$

$$\sum_{x^T\in\tilde{J}_\nu}\sum_{\hat{a}^T\in\hat{A}^T}\exp\{-\frac{\varepsilon}{2}\exp\{T\cdot\varepsilon\}\} + \exp\{-\frac{\varepsilon}{2}\cdot\exp\{T\cdot\varepsilon\}\}$$

$$\leq |\mathcal{X}^T|\cdot|\hat{A}^T|\cdot\exp\{-\exp\{T\cdot\varepsilon\}\cdot\frac{\varepsilon}{4}\}$$

$$= \exp\{-\frac{\varepsilon}{4}\cdot\exp\{T\cdot\varepsilon\} + T(\log|\mathcal{X}| + \log|\hat{A}|)\}$$

< 1 for sufficiently large T.

The above calculations ensure the existence of a set $\mathcal{C}_\nu$ such that all sequences $x^T \in \tilde{J}_\nu$ may be encoded by elements from $\tilde{J}_V(x^T)$ from which most generate a conditional independent distribution with arbitrary elements of $\hat{A}^T$. Forming the union of all these ensembles of codewords $\mathcal{C}_\nu$ for those $\nu \in \Delta(\mathcal{X})$ for which $\tilde{J}_\nu \neq \emptyset$ and $\rho(\mu,\nu) < \delta_\varepsilon$, we find an ensemble $\mathcal{C}$ for which

$$|\mathcal{C}| = |\bigcup_\nu \mathcal{C}_\nu| \leq \exp\{T\cdot\varepsilon\}\exp\{T\cdot(C-\varepsilon\}$$

$$\leq \exp\{T\cdot C\}.$$

(The first factor comes from $|\Delta^T(\mathcal{X})| \leq \exp\{T\cdot\varepsilon\}$.)

On the basis of the foregoing source-coding theorem code-type strategies may be defined. They are given as random encoding rules

$$B_k^T \mid \mathcal{X}^T \Longrightarrow A^T$$

such that

$$\Pr\{B_k^T(x^T) = a^T\} = \frac{1}{|\mathcal{C}_\nu \cap \mathcal{T}_V(x^T)|}$$

for $x^T \in \mathcal{T}_\nu$ and $a^T \in \mathcal{C}_\nu \cap \mathcal{T}_V(x^T)$.

Since $T^{-1}\, I\cdot(X^T \wedge B_k^T)$

$$\leq T^{-1}\, H\,(B_k^T)$$

$$\leq T^{-1} \log \Big| \bigcup_{\nu:\ \rho(\mu,\nu)<\delta_\varepsilon} \mathcal{C}_\nu^{\ *)} \Big|$$

$$\leq C_k$$

the code-type strategy defined above falls within the bounds of the strategies available to player k.

We shall now provide a result on the consequences of using code-type strategies as far as the payoff is concerned. Define for $V|\mathcal{X} \Longrightarrow A$ a function $d_l : \mathcal{X} \times \prod_{l \neq k} A_l \longrightarrow \mathbb{R}$ by

$$d_l(x,\hat{a}) = \sum_a V_l\,(a|x) \cdot u\,(x,a,\hat{a})^{*)}$$

Assume $d_l^T : \mathcal{X}^T \times \hat{A}^T \longrightarrow \mathbb{R}$ to be additive, which means $d_l^T(x^T,\hat{a}^T) = \sum_t d_l(x_t,\hat{a}_t)$. On deriving a bound on the payoff derivable we shall make heavily use of the "joint type condition".

*) We omit the mark of the player in accordance with our proceeding on the source-coding theorem.

Let B_k^T denote a code-type strategy as given above. We find:

<u>2.1 Lemma</u> Suppose T to be sufficiently large, then

$$T^{-1} \mid E[u^T(x^T,B_l^T,\hat{a}^T)] - d_l^T(x^T,\hat{a}^T) \mid < \varepsilon$$

for all $l \in \mathcal{K}$, $x^T \in \mathcal{T}^\delta$ and $\hat{a}^T \in \prod_{m \neq l} A_m$.

<u>Proof:</u> Denote max $u(x,a,\hat{a}) = ||u||$ and define $\tilde{\varepsilon} > 0$ such that $2 \cdot \tilde{\varepsilon} \cdot ||\cdot u|| < \varepsilon$.

Assume $\delta > 0$ to satisfy

$|\mathcal{X}| \cdot \prod_m | A_m | \cdot \delta < \varepsilon$. Then, for $x^T \in \mathcal{T}_\nu$

$E\ [u^T(x^T,B_l^T,\hat{a}^T)]$

$$= \sum_{\substack{a^T \in \\ \mathcal{C}_\nu \cap \mathcal{T}_V(x^T)}} |\mathcal{C}_\nu \cap \mathcal{T}_V(x^T)|^{-1} \; u^T(x^T,a^T,\hat{a}^T)$$

$$= \sum_{\substack{a^T \in \\ \mathcal{C}_\nu \cap \mathcal{T}_V(x^T) \cap \mathcal{T}^\delta(x^T,\hat{a}^T)}} |\mathcal{C}_\nu \cap \mathcal{T}_V(x^T)|^{-1} \; u^T(x^T,a^T,\hat{a}^T)$$

$$+ \sum_{\substack{a^T \in \\ \mathcal{C}_\nu \cap \mathcal{T}_V(x^T) \cap (\mathcal{T}^\delta(x^T,\hat{a}^T))^c}} |\mathcal{C}_\nu \cap \mathcal{T}_V(x^T)| \; u^T(x^T,a^T,\hat{a}^T).$$

Since $T^{-1} \mid u^T(x^T,a^T,\hat{a}^T) - d_V^T(x^T.\hat{a}^T) \mid$

$$= T^{-1} \Big| \sum_{x,a,\hat{a}} (N(x,a,\hat{a}|x^T,a^T,\hat{a}^T) - N(x,\hat{a}|x^T,\hat{a}^T) \cdot V(a|x)) \cdot u(x,a,\hat{a}) \Big|$$

$$\leq |\mathcal{X}| \cdot \prod_m |A_m| \cdot \delta \cdot ||u||$$

$$< \tilde{\varepsilon} \cdot ||u||$$

- where the second to last inequality follows from $a^T \in \mathcal{J}^{\delta}(x^T, \bar{a}^T)$ - we may infer, using the "joint type condition",

$$T^{-1} \mid E\,[u^T(x^T, B_1^T, \bar{a}^T)] - d_1^T\,(x^T, \bar{a}^T) \mid$$

$$< (1 - \tilde{\varepsilon}) \cdot (\tilde{\varepsilon} + \tilde{\varepsilon}) \cdot \|u\|$$

$$< \varepsilon.$$

Assume now all players with potential exception of player k to use a code-type strategy. Those strategies are defined with respect to conditional probabilities $V_l \mid \mathfrak{X} \Longrightarrow A_l$, $l \neq k$. We shall show that player k is faced with the problem of encoding a "discrete, memoryless source" $\{\mathfrak{X}, \mu\}$ along a "single-letter fidelity criterion" d.

Define

$$d = d_{\tilde{k}} : \mathfrak{X} \times A_k \longrightarrow \mathbb{R} \quad \text{by}$$

$$d_{\tilde{k}}\,(x, a_k) = \sum_{(a_l)_{l \neq k}} \prod V_l(a_l \mid x) \cdot u_k(x, a_k, (a_l)),$$

and $d^T(x^T, a_k^T) = \sum_t d(x_t, a_{k_t})$.

The preceding lemma may be applied iteratively to give a lower bound to himself - provided the opponents stick to their code-type strategies.

<u>2.2 Lemma</u> Suppose T to be sufficiently large, then

$$T^{-1} \mid E[u_k^T(x^T,(B_l^T)_{l\neq k},a_k^T] - d^T(x^T,a_k^T) \mid < K \cdot \varepsilon$$

for $k \in \mathcal{K}$, $x^T \in \mathcal{J}^\delta$ and $a_k^T \in A_k^T$.

<u>Proof:</u> Assume without loss of generality $k = K$.
Using the identity

$$\sum_t \sum_{a_l} V_l(a_l|x_t)\, u_k\, (x_t,(a_{m_t})_{m<l},(a_l),(a_{m_t})_{m>l})$$

$$= \sum_{a_l^T} (\prod_t V_l(a_{l_t}|x_t)\, u_k^T\, (x^T,(a_m^T)_{m<l},a_l^T,(a_m^T)_{m>l}).$$

we may paraphrase the preceding lemma to read

$$T \cdot \varepsilon > \mid E[u_k^T(x^T,(a_m^T)_{m<l},B_l^T,(a_m^T)_{m>l})]$$

$$- \sum_{a_l^T} (\prod_t V_l(a_{l_t}|x_t))\, u_k^T\, (x^T,(a_m^T)_{m<l},a_l^T,(a_m^T)_{m>l}) \mid$$

for all $(a_m^T)_{m\neq l}$ and $k \in \mathcal{K}$.

Consequently,

$$T \cdot \varepsilon > \mid E[u_k^T(x^T,(B_l^T)_{l\neq k},a_k^T)]$$

$$- \sum_{a_{k-1}^T} (\prod_t V_{k-1}(a_{k-1_t}|x_t))\, E[u_k(x^T,(B_l^T)_{l\leq K-2},a_{K-1}^T a_K^T)] \mid$$

Continuing we find

$$T^{-1} \mid E[u_K^T(x^T,(B_l^T)_{l\neq K},a_K^T) - d(x^T,a_K^T) \mid$$

$$\leq T^{-1} (\mid E[u_K^T(x^T,(B_m^T)_{m\neq K},a_K^T)]$$

$$- \sum_{a_{K-2}^T} (\prod_t V_{K-2_t}(a_{K-2_t} \mid x_t)) \; E[u_k^T(x^T,(B_m^T)_{m<K-2}(a_m^T)_{m>k-2})] \mid$$

$$+ \mid \sum_{a_{K-2}^T} (\prod_t V_{K-2} (a_{K-2_t} \mid x_t)) \; E[u_K^T(x^T,(B_m^T)_{m\leq K-2},(a_m^T)_{m>K-2})]$$

$$- \sum_{a_{K-3}^T} (\prod_t V_{K-3} (a_{K-3_t} \mid x_t)) \; E \; [u_K^T(x^T,(B_m^T)_{m\leq K-3},(a_m^T)_{m>K-3})] \mid$$

$$+$$

$$\vdots$$

$$+ \mid \sum_{(a_m^T)_{m\geq 2}} \prod_{m\geq 2} (\prod_t V_m(a_{m_t} \mid x_t)) \cdot E[u_K^T(x^T,(B_1^T),(a_m^T)_{m>1})]$$

$$- \sum_{(a_m^T)_{m\geq 1}} \prod_{m\geq 1} (\prod_t V_m(a_{m_t} \mid x_t)) \, u_K^T \; (x^T,(a_m^T)_{m\geq 1})] \mid$$

$$\leq K \cdot \varepsilon.$$

Having analyzed the situation player $k \in \mathcal{K}$ is faced with when his opponents use code-type strategies, we now have to investigate his payoff in order to provide a lower bound.

A dual of this problem was already looked at by C. E. SHANNON [9] in 1959. He was then interested in minimizing the distortion whence reproducing a discrete, memoryless source and was able to give a computable formula for the symptotic minimum distortion. Since we are interested in the maximum payoff achievable for the players instead of the minimum distortion, the formulae of SHANNON and the one derived here differ by the operator. Transformed to one type of problem SHANNON's result reads (informally):

for deterministic encoding

$$D(C) = \max_{\substack{V: \\ I(\mu;V) \leq C}} \{E[d(X,B)]\}$$

is attainable as the average payoff. Here the joint distribution of the random variable (X,B) is given by $\mu(x) \cdot V(a|x)$.

In our game theoretical context SHANNON's result is not completely satisfying, since a deterministic encoding allows for taking advantage of its structure destroying the equilibrium property. However, our approach as given by Theorem 1.6 takes this into account and will thereby be used twofold: Firstly, in order to show player k to be faced with encoding a discrete, memoryless source (on the basis of the opponents of player k sticking to presumable equilibrium strategies) and then, as usual, to provide a code for player k.

2.3 Theorem:

Let $d : \mathfrak{X} \times A \longrightarrow \mathbb{R}$ be any fidelity criterion. For any $\varepsilon_* > 0$ and sufficiently large T there exists a code-type strategy $B^T \mid \mathfrak{X}^T \Longrightarrow A^T$ for player k, satisfying $I(X^T \wedge B^T) \leq C$ such that

$$T^{-1} E[d^T(X^T,B^T)] \geq D(C) - \varepsilon_*.$$

<u>Proof:</u> Use the continuity of $D(\cdot)$ in order to find ε such that $D(C) - \frac{\varepsilon_0}{3} \leq D(C-2\varepsilon)^{*)}$.

Assume V to be chosen as to achieve

$$D(C-2\varepsilon) = \max_{I(\mu;V)\leq C-2\varepsilon} \{E[d(X,B)]\}$$

subject to the constraint $I(\mu;V) \leq C-2\varepsilon$.

Then let for $\nu \in \Delta(\mathfrak{X})$ such that $\rho(\mu;\nu) < \delta_\varepsilon$ (to ensure $| I(\mu;V) - I(\nu;V) | < \varepsilon$) sets of codeword $\mathcal{C}_\nu$ be given in accordance with the conditions of Theorem 1.6. The code-type strategy B^T now encodes $x^T \in \tilde{\mathfrak{J}}_\nu$ by elements of $\mathcal{C}_\nu \cap \tilde{\mathfrak{J}}_V(x^T)$. Consequently for those pairs (x^T,a^T) we have $N(x,a|x^T,a^T) = T \cdot \nu(x) \cdot V(a|x)$. Thereby we may infer

$$\begin{aligned} T^{-1}d^T(x^T,a^T) &= \sum_{x,a} \nu(x) \cdot V(a|x) \cdot d(x,a) \\ &\geq E_{\mu,V}[d(X,B)] - \frac{\varepsilon_*}{3} \\ &= D(C-2\varepsilon) - \frac{\varepsilon_*}{3} \\ &\geq D(C) - \frac{2}{3}\varepsilon_* , \end{aligned}$$

where the first inequality is provided by continuity; summarizing, we get

$$T^{-1}d^T(x^T,B^T) \geq D(C) - \frac{2}{3}\varepsilon_*$$

for all $x^T \in \bigcup_{\nu:\ \rho(\mu,\nu)<\delta_\varepsilon} \tilde{\mathfrak{J}}_\nu = \tilde{\mathfrak{J}}^\delta$, where $\delta = \delta_\varepsilon$.

Define ε_0 (sufficiently small) such that

$$- \varepsilon_0 \cdot ||u|| + (1 - \varepsilon_0)\ (D(C) - \frac{2}{3}\ \varepsilon_*)$$

$\geq$ D (C) - ε_*, then, since for sufficiently large T

T, $\mu^T(\mathcal{J}^\delta) > 1 - \varepsilon_0$, we find

$$T^{-1}\ E\ [d^T(X^T, B^T)]$$

$$= T^{-1} \sum_{x^T} \mu(x^T)\ d^T(x^T, B^T)$$

$$\geq - \varepsilon_0\ ||u|| + (1 - \varepsilon_0)\ (D(C) - \frac{2}{3}\ \varepsilon_*)$$

$\geq$ D (C) - ε^* , completing the proof.

As a corollary we obtain abound on the payoff achievable by player k, provided the opponents stick to their code-type strategies.

2.4 Corollary:

For any $\varepsilon > 0$, sufficiently large T and code-type strategies for the opponents of player k being based on $(V_l)_{l \neq k}$ there exists a code-type strategy $B_k^T \mid \mathfrak{X}^T \Longrightarrow A_k^T$ of player k, satisfying $I(X^T \wedge B^T) \leq C$ such that

$$T^{-1}\ E\ [u_k^T(X^T, B_k^T, (B_l^T)_{l \neq k})] \geq D(C_k) - \varepsilon.$$

Proof: According to Lemma 2.2 we may assume

$$T^{-1}\ |\ E[u_k^T(x^T, a_k^T, (B_l^T)_{l \neq k})] - d^T(x^T, a_k^T)\ | < \frac{\varepsilon}{2},$$

then applying the foregoing theorem with $\varepsilon^* = \frac{\varepsilon}{2}$ yields

$$T^{-1} E [u_k^T(X^T, B_k^T, (B_l^T)_{l \neq k})]$$
$$\geq T^{-1} E [d^T(X^T, B_k^T)] - \frac{\varepsilon}{2}$$
$$\geq D(C_k) - \varepsilon.$$

In order to prove the equilibrium property of code-type strategies we have to show that no player can gain by unilateral deviation. Such a result is provided by the companion piece to the source-coding Theorem 2.3 and is called its "converse". The impossibility result on achieving higher payoff (or lower distortion) was found by J. WOLFOWITZ [11] in 1966. In its information-theoretic context it defines a bound on the distortion which cannot be transgressed by any technical system. What makes the coding theorem and its counterpart, the converse, so important, is that the two divide up the payoffs either to be achievable or non-achievable, tertium non datur. We shall give a very short proof of WOLFOWITZ's result taking pattern from a proof given by T. BERGER [2], pp. 71 - 72.

We first have to mention that

$$D(\cdot) = \max_{I(X \wedge B) \leq \cdot} \{E[d(X,B)]\}$$

is an isotonic and concave ($\cap$) function.

<u>3.1 Theorem:</u>

Given any fidelity criterion $d: \mathcal{X} \times A \longrightarrow \mathbb{R}$, then for any $T \in \mathbb{N}$ and encoding rule $B^T \mid \mathcal{X} \overset{T}{\Longrightarrow} A^T$ satisfying $T^{-1} I(X^T \wedge B^T) \leq C$ the inequality

$$T^{-1} E [d^T(X^T, B^T)] \leq D(C)$$

holds.

Proof: Due to the additivity assumption we get

$$E\,[d^T(X^T,B^T)]$$

$$= \sum_t E\,[d(X_t,B_t)].$$

Now, using the definition of $D(\cdot)$, we may give an upperbound to the latter item by $\sum_t D\,(I(X_t \wedge B_t))$, which, by concavity may be bound by

$$T\,D(T^{-1} \sum_t I(X_t \wedge B_t)).$$

As a consequence of DOBRUSHIN's inequality

$$I(X^T \wedge B^T) \quad \geq \quad \sum_t I(X_t \wedge B_t)$$

- valid for independent X_t -, we infer, using isotonicity of $D(\cdot)$, the claim.

The converse may be directly applied to the fidelity criterion as induced by the code-type strategies as assumed to be used by the opponents of player k.

In fact, since the fidelity criterion $d_{\tilde{k}}$ as defined preceding Lemma 2.2 and the payoff to player k differ by an arbitrary small amount "almost everywhere", provided T is sufficiently large, we may use the converse to give an upper bound to the payoff attainable by player k.

3.2 Corollary:

For any $\varepsilon_o > 0$ and sufficiently large T and $B_k^T \mid \mathcal{X}^T \Longrightarrow A^T$ satisfying $T^{-1}\, I(X^T \wedge B_k^T) \leq C_k$ the inequality

$$T^{-1}\, E\,[u_k^T(X^T,(B_k^T),(B_l^T)_{l\neq k})]$$

$$\leq D\,(C_k) + \varepsilon_o$$

holds.

Proof: Define $\varepsilon = \frac{\varepsilon_0}{2N}$ and observe that by Lemma 2.2 for all $x^T \in \mathcal{T}^\delta$ and $a_k^T \in A_k^T$

$$| u^T(x^T, a_k^T, (B_l^T)_{l \neq k}) - d_k^T (x^T, a_k^T) | < \frac{\varepsilon}{2} .$$

Assume further ε to satisfy

$$\varepsilon \cdot ||u|| < \frac{\varepsilon_0}{4} .$$

Then

$$T^{-1} E [u_k^T(X^T, B_k^T, (B_l^T)_{l \neq k})]$$

$$< \sum_{x^T \in \mathcal{T}^\delta} T^{-1} E [u_k^T(x^T, B_k^T, (B_l^T))] \cdot \mu^T(x^T)$$

$$+ \sum_{x^T \notin \mathcal{T}^\delta} T^{-1} E [u_k^T(x^T, B_k^T, (B_l^T))] \cdot \mu^T(x^T)$$

$$\leq (1 - \varepsilon) \cdot (T^{-1} E [d_k^T(x^T, a_k^T) + \frac{\varepsilon_0}{2}) + \varepsilon \cdot ||u||$$

$$\leq T^{-1} E [d_k^T(x^T, a^T)] + \frac{\varepsilon_0}{2} + 2 \cdot \frac{\varepsilon_0}{4} ,$$

which proves our statement by application of Theorem 3.1.

By now, we collected all information-theoretical material to infer the ε-equilibrium property of code-type strategies as claimed by our main theorem. Its proof consequently is an easy thing to perform.

Proof (of the main theorem):

Following Lemma 1.1 there exists an equilibrium in the game Γ^1. Let it be denoted as $(V_1^* \ldots V_k^*)$. Let us analyze the presumable equilibrium property from player k's point of view. The equilibrium property yields for him the payoff

$$E_{(V_l^*)}[u_k] = \max_{\substack{V_k: \\ I(X \wedge V_k) \leq C_k}} \{E_{V_{\mu,k},(V_l^*)}[u_k]\} ,$$

where the latter expression is equal to $D(C_k)$. (Observe, since we analyze from player k's point of view, we suppress the dependence on the other players strategies.) Given $\varepsilon > 0$, Corollary 2.4 - the coding theorem - shows that the average payoff $D(C_k)$ is ε-attainable for player k, provided the opponents stick to their code-type strategies based on $(V_l^*)_{l \neq k}$ assuming T sufficiently large. On the other hand player k cannot achieve an ascertainably higher payoff than $D(C_k) + \varepsilon$ due to Corollary 3.2 - the converse. Thus, $(D(C_k))_{k \in \mathcal{K}}$ is an ε-achievable payoff-vector and no player can obtain an ascertainably higher payoff by unilateral deviation, proving the equilibrium property for code-type strategies being based on an equilibrium (V_k^*) in the one-shot game.

Two remarks seem to be useful. In the course of deriving our existence result on ε-equilibria we did not intend to suggest the impression that those are obtained constructively in contrast to the existence-proofs on equilibria via fixed-point theorems. Of course, we needed a fixed-point argument to ensure the existence of an equilibrium in the one-shot game by ourselves. This, however, could be argued to be not as bad as that, since an equilibrium may be found by numerical analysis by some effort depending on the size of the strategy sets and the number of states of nature. What is more counteracting the constructivity is that also the code-type strategies are obtained non-constructively via a probability argument. These are for "practical" purposes, though bad enough, considerably to be preferred to a proof of existence via fixed-point theorems. Also, as was already mentioned, we inferred some structural properties in contrast to the usual derivation of equilibria in supergames.

Our second point concerns the synchronization assumption related to the investigation of periods of fixed length T. It is easily observed - but takes pains within its formalization - that the fixed-length periods only facilitated the analysis, but are of no relevance as far as the validity of our results is concerned. Thus, since every player may use his own length of period, no game-theoretic problems arise from a selection of the length of periods which are to be considered.

Let us conclude with an easy consequence of the main theorem. Consider a K-person matrix game $\Gamma = (A_k, u_k)$. An equilibrium vector of this game is given by $(P_1^*, \ldots, P_K^*) \in (\Delta(A_k))_{k \in \mathcal{K}}$, say. This probability vector may be viewed at as consisting of conditional probabilities $(V_1^*, \ldots, V_K^*)$, for which $V_k \mid \mathfrak{X} \Longrightarrow A_k$ with a one-elementary set $\mathfrak{X}$. In this case $I(\varepsilon_x; V) = 0$ for all $V \mid \mathfrak{X} \Longrightarrow A$, where ε_x denotes the unique probability distribution on $\mathfrak{X}$. Thus, the inequality $I(\varepsilon_x; V) \leq C$ provides no restriction on V or P, respectively. Consequently the equilibrium payoff-vector $D(C_k) = D_k$ is independent of the vector of capacities (C_k).

Now let Γ^T denote the T-fold replication of the above game as usual. Our main theorem shows that the equilibrium vector (D_k) of the one-shot game may arbitrarily well approximated by code-type strategies in supergames Γ^T for sufficiently large T. Moreover, since the equilibrium payoff in the one-shot game did not depend on the capacities C_k, there exists an ε-equilibrium code-type strategy for which the carrier contains less than $\exp\{T \cdot \varepsilon\}$ elements. Thereby we derived

Corollary

Let us be given any K-person game $\Gamma = (A_k, u_k)$. Then, for any $\varepsilon > 0$ and sufficiently large T there exists an ε-(code-type) equilibrium in the T-fold replicated game $\Gamma^T = (A_k^T, T^{-1} u_k^T)$ such that for each player $k \in \mathcal{K}$ the cardinality of those set of sequences a_k^T used with positive probability is bound by $\exp \{T \cdot \varepsilon\}$.

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