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On the Harsanyi–Selten Value

by

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2. Mathematical notations

In the following let X, M, X_i, M_i, $i = 1, \ldots, k$, be sets. Let $\sum_{i=1}^{k} X_i$ denote the cartesian product of the sets X_i and X^M denote the set of all mappings from M into X.

If $\Psi_i \in X_i^{M_i}$, $i = 1, \ldots, k$, then let $\Psi_1 \times \ldots \times \Psi_k$ be that mapping in $(\sum_{i=1}^{k} X_i)^{\sum_{i=1}^{k} M_i}$ that is canonically defined by the mappings Ψ_i, $i = 1, \ldots, k$.

For a subset $V \subset \mathbb{R}^M$ we define

$$V_+ := \{ x \in V \mid x(m) \geq 0 \ \ \forall m \in M\} \text{ and}$$

$$V_{++} := \{ x \in V \mid x(m) > 0 \ \ \forall m \in M\}.$$

We write $\langle x,y \rangle := \sum_{m \in M} x(m)\, y(m)$ if $x,y \in \mathbb{R}^M$.

If M is finite then let $\Delta(M)$ denote the set of all completely mixed probability distributions on M:

$$\Delta(M) := \{ p \in (0,1]^M \mid \sum_{m \in M} p(m) = 1\}.$$

3. The model

We deal with a two person model which is discussed in detail by Myerson [3].

There is a decision set of possible decisions the players may agree on and a conflict outcome which results if the players don't come to an agreement. This conflict outcome is an element of the decision set.

Every player might be one type out of a given type set. A type is determined by two components:

- a von Neumann–Morgenstern utility function on the decision set which represents the utility of the decision
- a probability distribution on the type set of the other player which represents the beliefs about the frequency of the types of the opponent.

The cartesian product of the type sets is called the state space.

We will restrict ourself to the case that there exists a joint distribution on the state space which represents the beliefs of both players. This means that the belief of a player is the conditional distribution induced by the joint distribution.

The formal model:

A bargaining problem Γ is an object of the form

$$\Gamma=(D, d^*, T_1, T_2, u_1, u_2, p)$$

with the following components:

D, T_1,T_2 are finite sets with $T_1 \cap T_2 = \emptyset$,

d^* is an element of D,

u_1 and u_2 are functions from $D \times T_1 \times T_2$ into the real numbers $\mathbb{R}$,

p is a probability distribution on $T_1 \times T_2$.

D is the decision set, d^* is the conflict outcome, T_1 (resp. T_2) is the type set of player 1 (resp. 2), u_1 and u_2 are the utility functions of the players and p measures the probability that a pair of types occurs.

We suppose that $p(t_1,t_2) > 0$ for all $(t_1,t_2) \in T_1 \times T_2$.

A pair of types is called a state and the product $T_1 \times T_2$ is the state space. Speaking of two type sets we always claim them to be disjoint. If we have a function $x \in \mathbb{R}^{T_1 \times T_2}$ we may write $x(t_2,t_1)$ instead of $x(t_1,t_2)$ for $(t_1,t_2) \in T_1 \times T_2$ without causing any confusion.

To simplify the notation we write $-i$ instead of $3-i$ for $i=1,2$.

We use the following notations:

$$u_i(d,t_1,t_2) =: u_i(d\,|\,t_1,t_2) =: u_i(d\,|\,t_2,t_1)$$

$$\forall\, d \in D \;,\; t_1 \in T_1, t_2 \in T_2 \;,\; i=1,2.$$

For $i=1,2$ let q_i be the marginal distribution of p:

$$q_i(t_i) := \sum_{t_{-i} \in T_{-i}} p(t_1,t_2) \qquad \forall\, t_i \in T_i,$$

and p_i the conditional distribution:

$$p_i(t_{-i}\,|\,t_i) := \frac{p(t_1,t_2)}{q_i(t_i)} \qquad \forall\, (t_1,t_2) \in T_1 \times T_2.$$

Let the function

$$q : T_1 \cup T_2 \to \mathbb{R}$$

be defined by

$$q\,|_{T_i} = q_i \text{ for } i=1,2.$$

We say that the types are independently distributed if

$$p(t_1,t_2) = q_1(t_1) \cdot q_2(t_2) \qquad \forall\, (t_1,t_2) \in T_1 \times T_2,$$

and write $p = q_1 \otimes q_2$.

In this case it holds:

$$p_i(t_{-i} \mid t_i) = q_{-i}(t_{-i}) \qquad \forall\, (t_1,t_2) \in T_1 \times T_2,\ i=1,2.$$

The set of all probability distributions on $T_1 \times T_2$ of the type $q_1 \otimes q_2$ is denoted by $\Delta^I(T_1 \times T_2)$.

4. The bargaining set

See Holmström & Myerson [2] and Rosenmüller [7].

In a given bargaining problem both players know both type sets, the two utility functions and the joint distribution. Every player knows his type and his beliefs about the probability of types of his opponent are described by the conditional distribution. Both players know how the opponent generates his beliefs.

The players are able to perform joint lotteries over the decision set including the conflict outcome. Both are risk neutral.

They have to agree on a mechanism which determines one lottery for every possible type combination. After the agreement both players announce a type out of their type set. The lottery will be performed according to the announcement and the resulting decision will be executed. After the agreement nothing can be changed. If they don't agree the conflict outcome is executed.

We want the players to reveal their real type in the announcement. Therefore we demand the mechanisms to be incentive compatible. That means: For every type of both players truthtelling gives the highest return of expected utility if the opponent always tells the truth. Truthtelling is a Nash–equilibrium in the game induced by this mechanism. Every incentive compatible mechanism defines an allocation of expected utility for the types of the players. An incentive compatible mechanism which gives a higher utility than the conflict outcome is called individual rational. The set of all those allocations is called the bargaining set.

A mechanism μ is a function $\mu : D \times T_1 \times T_2 \to [0,1]$, so that

$$\sum_{d \in D} \mu(d,t_1,t_2) = 1 \qquad\qquad \forall\, (t_1,t_2) \in T_1 \times T_2.$$

We write

$$\mu(d,t_1,t_2) =: \mu(d \mid t_1,t_2) =: \mu(d \mid t_2,t_1).$$

A decision $\hat{d} \in D$ is often identified with the mechanism $\mu_{\hat{d}}$ defined by:

$$\mu_{\hat{d}}(\hat{d} \mid \cdot\,,\cdot) = 1.$$

Let $\mathcal{M}\,(\Gamma)$ be the set of all mechanisms of the bargaining problem Γ. We often suppress the argument Γ.

The utility of a mechanism μ in a state $(t_1, t_2) \in T_1 \times T_2$ is the expected utility of the lottery $\mu(\cdot \mid t_1,t_2)$. We can denote this utility for player i by $u_i(\mu|t_1,t_2)$ and get an extension of the function $u_i : D \times T_1 \times T_2 \to \mathbb{R}$ to a function

$$u_i : \mathcal{M} \times T_1 \times T_2 \to \mathbb{R} \text{ defined by}$$

$$u_i(\mu,t_1,t_2) := u_i(\mu|t_1,t_2) := \sum_{d \in D} \mu(d|t_1,t_2)\, u_i(d|t_1,t_2)$$

$$\forall (t_1,t_2) \in T_1 \times T_2,\ \mu \in \mathcal{M}.$$

The expected utility of player i is the function

$$U_i : \mathcal{M} \times T_i \to \mathbb{R}$$

defined by

$$U_i(\mu|t_i) := U_i(\mu,t_i) := \sum_{t_{-i} \in T_{-i}} u_i(\mu|t_1,t_2)\, p_i(t_{-i}|t_i)$$

$$\forall \mu \in \mathcal{M},\ t_i \in T_i.$$

$U_i(\mu|t_i)$ is the expected utility of the mechanism μ for player i if his type is t_i and both players are announcing their true types. If the true type of player i is $t_i \in T_i$ and he announces $s_i \in T_i$ while the other player announces truefully he will receive the expected utility $U_i^*(\mu,s_i|t_1)$, where

$$U_i^* : \mathcal{M} \times T_i \times T_i \to \mathbb{R}$$

is a function defined by

$$U_i^*(\mu,s_i|t_i) := \sum_{t_{-i} \in T_{-i}} \sum_{d \in D} \mu(d|s_i,t_{-i})\, u_i(d|t_i,t_{-i})\, p_i(t_{-i}|t_i)$$

$$\forall s_i,t_i \in T_i,\ \mu \in \mathcal{M}.$$

A mechanism $\mu \in \mathcal{M}$ is called incentive compatible iff

$$U_i(\mu|t_i) \geq U_i^*(\mu,s_i|t_i) \qquad \forall s_i,t_i \in T_i,\ i = 1,2.$$

To simplify the notation we define the function

$$U : \mathscr{M} \times (T_1 \cup T_2) \to \mathbb{R} \qquad \text{by}$$

$$U|_{\mathscr{M} \times T_1} := U_1 \qquad \text{and} \quad U|_{\mathscr{M} \times T_2} := U_2.$$

Let $B(\Gamma)$ be the set of all incentive compatible mechanisms of the bargaining problem Γ and define

$$\mathscr{U}(\Gamma) := \{x \in \mathbb{R}^{T_1 \cup T_2} \mid \exists\, \mu \in B(\Gamma) \text{ so that } x = U(\mu|\cdot)\}.$$

An incentive compatible mechanism $\mu \in B(\Gamma)$ is called individual rational iff

$$U_i(\mu|\cdot) \geq U_i(d^*|\cdot) \qquad \text{for } i = 1,2.$$

Define

$$B_r(\Gamma) = \{\, \mu \in B(\Gamma) \mid \mu \text{ is individual rational}\}$$

and

$$\mathscr{U}_r(\Gamma) = \{x \in \mathscr{U}(\Gamma) \mid x \geq U(d^*|\cdot)\}.$$

$\mathscr{U}_r(\Gamma)$ is called the bargaining set.

When we deal with a second bargaining problem $\tilde{\Gamma}$ all its components and the derived quantities are marked with ~.

A bargaining problem Γ is called regular iff there exists a mechanism $\mu \in B(\Gamma)$ so that

$$U(\mu|t) > U(d^*|t) \qquad \forall\ t \in T_1 \cup T_2.$$

Let $\mathscr{G}$ be the set of all regular bargaining problems.

For a decision set D a state space $T_1 \times T_2$ and a probability distribution $p \in \Delta(T_1 \times T_2)$ let

- $\mathscr{G}_{T_1 \times T_2}$ be the set of all regular bargaining problems with state space $T_1 \times T_2$

- $\mathscr{G}_{D \times T_1 \times T_2}$ be the set of all bargaining problems in $\mathscr{G}_{T_1 \times T_2}$ with decision set D
- $\mathscr{G}_p$ be the set of all bargaining problems in $\mathscr{G}_{T_1 \times T_2}$ with distribution p.

Let $\mathscr{G}^I$ denote the set of all regular bargaining problems with independently distributed types.

5. Some operations on bargaining problems

In this chapter we look at some mappings which transform a bargaining problem into another bargaining problem but don't really change the described situation.

Renaming

For a given bargaining problem we may generate another bargaining problem if we give other names to the decisions or the types of the players.

Given two decision sets D, $\tilde{D}$ and two state spaces $T_1 \times T_2$, $\tilde{T}_1 \times \tilde{T}_2$ and three bijections

$$\phi_D : D \to \tilde{D},\ \phi_1 : T_1 \to \tilde{T}_1,\ \phi_2 : T_2 \to \tilde{T}_2$$

we define the following functions:

$$\phi := \phi_D \times \phi_1 \times \phi_2 : D \times T_1 \times T_2 \to \tilde{D} \times \tilde{T}_1 \times \tilde{T}_2$$

$$\phi_T := \phi_1 \times \phi_2 : T_1 \times T_2 \to \tilde{T}_1 \times \tilde{T}_2$$

and

$$\phi_0 : T_1 \cup T_2 \to \tilde{T}_1 \cup \tilde{T}_2 \text{ is defined by}$$

$$\phi_0|_{T_i} = \phi_i \quad \text{for } i = 1,2.$$

Let $\phi^* : \mathcal{G}_{D \times T_1 \times T_2} \to \mathcal{G}_{\tilde{D} \times \tilde{T}_1 \times \tilde{T}_2}$ be defined by

$$\phi^*(\Gamma) = (\tilde{D}, \phi_D(d^*), \tilde{T}_1, \tilde{T}_2, u_1 \circ \phi^{-1}, u_2 \circ \phi^{-1}, p \circ \phi_T^{-1})$$

$$\forall\, \Gamma = (D, d^*, T_1, T_2, u_1, u_2, p) \in \mathcal{G}_{D \times T_1 \times T_2}.$$

We may also define the bijection $\hat{\phi} : \mathbb{R}^{T_1 \cup T_2} \to \mathbb{R}^{\tilde{T}_1 \cup \tilde{T}_2}$ by

$$\hat{\phi}(x) = x \circ \phi_0^{-1} \qquad \forall\, x \in \mathbb{R}^{T_1 \cup T_2}.$$

Of course we have

$$\mathcal{U}_r(\phi^*(\Gamma)) = \hat{\phi}(\mathcal{U}_r(\Gamma))$$

This operation is related to the procedure 'changing of types' in [1]. Harsanyi & Selten only look at type sets which are subsets of the natural numbers and are ordered therefore. We look at type sets without ordering. Interchanging the names of two types has the same result as 'changing of types'.

Interchanging the players

We may change the names of the players.

Define the mapping

$$\psi: \mathcal{G} \rightarrow \mathcal{G} \quad \text{by}$$

$$\psi(\Gamma) = (D, d^*, T_2, T_1, u_2, u_1, \tilde{p}) \quad \forall\, \Gamma = (D, d^*, T_1, T_2, u_1, u_2, p) \in \mathcal{G}.$$

$$\text{with } \tilde{p}(t_2, t_1) = p(t_1, t_2) \qquad \forall\, (t_2, t_1) \in T_2 \times T_1$$

ψ is bijective and we have:

$$\psi = \psi^{-1} \quad \text{and}$$

$$\mathcal{U}(\psi(\Gamma)) = \mathcal{U}(\Gamma) \qquad \forall\, \Gamma \in \mathcal{G}.$$

Linear utility transformations

We claim both players to have a von Neumann–Morgenstern utility function. These are only determined up to linear transformations.

Let $T_1 \times T_2$ be a state space.

If $a \in \mathbb{R}^{T_1 \cup T_2}$, $\lambda \in \mathbb{R}_{++}^{T_1 \cup T_2}$ and $i \in \{1,2\}$ then let the mapping

$$\check{\Lambda}_{a,\lambda,i} : \mathbb{R}^{T_1 \times T_2} \rightarrow \mathbb{R}^{T_1 \times T_2} \quad \text{be defined by:}$$

$$(\check{\Lambda}_{a,\lambda,i}(x))\,(t_1,t_2) = \lambda(t_i)\, x(t_1,t_2) + a(t_i) \quad \forall\, (t_1,t_2) \in T_1 \times T_2,\ x \in \mathbb{R}^{T_1 \times T_2}.$$

(a,λ) defines a mapping $\Lambda^*_{a,\lambda} : \mathcal{G}_{T_1 \times T_2} \rightarrow \mathcal{G}_{T_1 \times T_2}$ by:

$$\Lambda^*_{a,\lambda}(\Gamma) = (D, d^*, T_1, T_2, \check{\Lambda}_{a,\lambda,1}\, u_1, \check{\Lambda}_{a,\lambda,2}\, u_2, p)$$

$$\forall\, \Gamma = (D, d^*, T_1, T_2, u_1, u_2, p) \in \mathcal{G}_{T_1 \times T_2}.$$

If we define the mapping

$$\hat{\Lambda}_{a,\lambda} : \mathbb{R}^{T_1 \cup T_2} \to \mathbb{R}^{T_1 \cup T_2} \quad \text{by}$$

$$\hat{\Lambda}_{a,\lambda}(x) = \lambda x + a \qquad \forall\, x \in \mathbb{R}^{T_1 \cup T_2}$$

we see immediately:

$$\mathcal{U}(\Lambda^*_{a,\lambda}(\Gamma)) = \hat{\Lambda}_{a,\lambda}(\mathcal{U}(\Gamma)).$$

Splitting of types

If we describe a bargaining situation in our model every player consists of various types. One might take one type and regard this type as an aggregation of two subtypes which have the same utility functions and the same beliefs. The probability of both subtypes together equals the probability of the old type.

If $T_1 \times T_2$ is a state space and $\hat{t} \in T_1$, $\tilde{t} \notin T_1 \cup T_2$ then one can define a mapping

$$\check{\Sigma}_{\hat{t},\tilde{t}} : \mathbb{R}^{T_1 \times T_2} \to \mathbb{R}^{(T_1 \cup \{\tilde{t}\}) \times T_2} \quad \text{by}$$

$$(\check{\Sigma}_{\hat{t},\tilde{t}}(x))(t_1,\cdot) = \begin{cases} x(t_1,\cdot) & \text{if } t_1 \in T_1 \\ x(\hat{t},\cdot) & \text{if } t_1 = \tilde{t} \end{cases} \qquad \forall\, x \in \mathbb{R}^{T_1 \times T_2}$$

and a mapping

$$\hat{\Sigma}_{\hat{t},\tilde{t}} : \mathbb{R}^{T_1 \cup T_2} \to \mathbb{R}^{T_1 \cup T_2 \cup \{\tilde{t}\}} \quad \text{by}$$

$$(\hat{\Sigma}_{\hat{t},\tilde{t}}\, y)(t) = \begin{cases} y(t) \text{ if } t \in T_1 \cup T_2 \\ y(\hat{t}) \text{ if } t = \tilde{t} \end{cases} \qquad \forall\, y \in \mathbb{R}^{T_1 \cup T_2}.$$

If in addition we have $v \in (0,1)$ one can define a mapping

$$\overset{\circ}{\Sigma}_{\hat{t},\tilde{t},v} : \Delta(T_1 \times T_2) \rightarrow \Delta((T_1 \cup \{\tilde{t}\}) \times T_2) \qquad \text{by}$$

$$(\overset{\circ}{\Sigma}_{\hat{t},\tilde{t},v}\, p)\,(t_1,\cdot) = \begin{cases} p(t_1,\cdot) & \text{if } t_1 \in T_1 \backslash \{\hat{t}\} \\ v\; p(\hat{t},\cdot) & \text{if } t_1 = \hat{t} \\ (1-v)p(\hat{t},\cdot) & \text{if } t_1 = \tilde{t} \end{cases} \qquad \forall\, p \in \Delta(T_1 \times T_2).$$

Finally we define a mapping

$$\Sigma_{\hat{t},\tilde{t},v} : \mathscr{G}_{T_1 \times T_2} \rightarrow \mathscr{G}_{(T_1 \cup \{\tilde{t}\}) \times T_2} \qquad \text{by}$$

$$\Sigma_{\hat{t},\tilde{t},v}(\Gamma) = (D, d^*, T_1 \cup \{\tilde{t}\}, T_2, \check{\Sigma}_{\hat{t},\tilde{t}}\, u_1, \check{\Sigma}_{\hat{t},\tilde{t}}\, u_2, \overset{\circ}{\Sigma}_{\hat{t},\tilde{t},v}\, p)$$

$$\forall\, \Gamma = (D, d^*, T_1, T_2, u_1, u_2, p) \in \mathscr{G}_{T_1 \times T_2}.$$

If we write $\tilde{\Gamma} = \overset{\circ}{\Sigma}_{\hat{t},\tilde{t},v}\, \Gamma$ we have for the marginal probabilities

$$\tilde{q}_1(t_1) = \begin{cases} q(t_1) & \text{if } t_1 \in T_1 \backslash \{\hat{t}\} \\ v\; q(\hat{t}) & \text{if } t_1 = \hat{t} \\ (1-v)q(\hat{t}) & \text{if } t_1 = \tilde{t} \end{cases},$$

$$\tilde{q}_2 = q_2$$

and for the beliefs

$$\text{(5.1)} \qquad \tilde{p}_1(\cdot\,|t_1) = \begin{cases} p_1(\cdot\,|t_1) & \text{if } t_1 \in T_1 \\ p_1(\;|\hat{t}) & \text{if } t_1 = \tilde{t} \end{cases},$$

$$\tilde{p}_2(t_1|\cdot) = \begin{cases} p_2(t_1|\cdot) & \text{if } t_1 \in T_1 \backslash \{\hat{t}\} \\ v\; p_2(\hat{t}|\cdot) & \text{if } t_1 = \hat{t} \\ (1-v)p_2(\hat{t}|\cdot) & \text{if } t_1 = \tilde{t} \end{cases}.$$

1. Introduction

In 1950 Nash [6] formulated the two person bargaining problem and gave a solution, the Nash value, determined by a set of axioms.

Harsanyi and Selten [1] generalized the bargaining problem to the case with incomplete information and proposed a solution which is a derivation of the Nash value. In their model every player may be one type out of a given set of types. Both players know their own type and have beliefs over the frequency of the types of the opponent. The players have to agree on an allocation of expected utility for every type of both players out of a given set of possible allocations – the bargaining set.

Myerson [3] presented a modified model for a bargaining situation. Again the players have different types and beliefs. They have type dependent utility functions on a given set of possible decisions. Both have to agree on a mechanism, which is a type dependent lottery over the decision set. The players announce their type after they have agreed on a mechanism and then the (type dependent) lottery takes place. Myerson demands the mechanism to be incentive compatible, which means that it is optimal for the players to announce their true type. The set of incentive compatible mechanisms together with the beliefs and the utility functions define the bargaining set of all possible allocations of expected utility.

Myerson claims that the axiomization of Harsanyi and Selten leads to the same solution in his model. Later Myerson [5] proposes a different solution for his model.

As I tried to reformulate the axioms of Harsanyi and Selten in the model of Myerson and to derivate the Harsanyi and Selten solution some problems occurred.

First it is not obvious that it is possible to applicate the IIA axiom (independence of irrelevant alternatives). This can be treated with methods introduced by Myerson in [4]. Secondly in a key step of their proof Harsanyi and Selten use the operation 'dividing a type'. This step doesn't work if the types of the players are not independently distributed.

In chapter 3 and 4 I present the model of Myerson. The next chapter is devoted to the operations that are needed to formulate the axioms. The first problem (IIA) is treated in chapter 6 and 7. The axioms and the proof of the uniqueness of the value is presented in chapter 8. At last I discuss the problems that arise in the non–independent case.

Specially it holds:

(5.2) $$p_2(\hat{t}\,|\,\cdot\,) = \tilde{p}_2(\hat{t}\,|\,\cdot\,) + p_2(\tilde{t}\,|\,\cdot\,)$$

This operation 'splitting of types' can easily be defined for player 2 too.

Lemma 1: $\mathcal{U}(\Sigma_{\hat{t},\tilde{t},v}\,\Gamma) = \hat{\Sigma}_{\hat{t},\tilde{t}}(\,\mathcal{U}(\Gamma))$.

Proof: We write $\tilde{\Gamma} = \Sigma_{\hat{t},\tilde{t},v}\,\Gamma$.

Let $\mu \in B(\Gamma)$.

We define $\tilde{\mu} \in \mathcal{M}(\tilde{\Gamma})$ by

$$\tilde{\mu}(\cdot\,|t_1, t_2) = \begin{cases} \mu(\cdot\,|t_1, t_2) & \text{if } (t_1, t_2) \in T_1 \times T_2 \\ \mu(\cdot\,|\hat{t}, t_2) & \text{if } t_1 = \tilde{t},\ t_2 \in T_2 \end{cases}.$$

Player 1 has no incentive to lie in $\tilde{\mu}$ because the beliefs of $\hat{t}$ and $\tilde{t}$ coincide. Player 2 won't lie because of [5.2] .
Therefore we have $\tilde{\mu} \in B(\tilde{\Gamma})$. One sees immediately that

$$\tilde{U}(\tilde{\mu}|\cdot\,) = \Sigma_{\hat{t},\tilde{t}}\,(U(\mu|\cdot\,)).$$

So we get:

$$\Sigma_{\hat{t},\tilde{t}}\,(\,\mathcal{U}(\Gamma)) \subset \mathcal{U}(\Sigma_{\hat{t},\tilde{t},v}\,(\Gamma)).$$

Now let $\tilde{\mu} \in B(\tilde{\Gamma})$.

In $\tilde{\Gamma}$ the types $\hat{t}$ and $\tilde{t}$ has the same beliefs and the same utility functions. Therefore we have:

$$\tilde{U}_1^*(\tilde{\mu}, \hat{t}\,|\,\tilde{t}) = \tilde{U}_1(\tilde{\mu}|\hat{t}),$$

$$\tilde{U}_1^*(\tilde{\mu}, \tilde{t}\,|\,\hat{t}) = \tilde{U}_1(\tilde{\mu}|\tilde{t}).$$

Because $\tilde{\mu}$ is incentive compatible we get:

$$\tilde{U}_1(\tilde{\mu}|\hat{t}) \geq \tilde{U}_1^*(\tilde{\mu}, \tilde{t}\,|\,\hat{t}) = \tilde{U}_1(\tilde{\mu}|\tilde{t}) \geq \tilde{U}_1(\tilde{\mu}, \hat{t}\,|\,\tilde{t}) = \tilde{U}_1(\tilde{\mu}|\hat{t}).$$

This shows that equality holds everywhere.
Therefore we have:

(5.3) $\tilde{U}_1(\tilde{\mu}|\hat{t}) = \tilde{U}_1(\tilde{\mu}|\tilde{t})$ and

$\tilde{U}_1(\tilde{\mu}, \tilde{t}|\hat{t}) = \tilde{U}_1(\tilde{\mu}|\hat{t})$.

We define a mechanism $\mu \in \mathcal{M}(\Gamma)$ by :

$$\mu(\cdot|t_1,\cdot) = \begin{cases} \tilde{\mu}(\cdot|t_1,\cdot) & \text{if } t_1 \in T_1\backslash\{\hat{t}\} \\ v\,\tilde{\mu}(\cdot|\hat{t},\cdot) + (1-v)\,\tilde{\mu}(\cdot|\tilde{t},\cdot) & \text{if } t_1 = \hat{t}. \end{cases}$$

We now show that $\mu \in B(\Gamma)$.

We have:

$$U_2^*(\mu,s_2|t_2) = \sum_{t_1 \in T_1} \sum_{d \in D} \mu(d|t_1, s_2)\, u_2(d|t_1, t_2)\, p_2(t_1|t_2)$$

$$= \sum_{t_1 \neq \hat{t}} \sum_{d} \tilde{\mu}(d|t_1,s_2)\, u_2(d|t_1, t_2)\, p_2(t_1|t_2)$$

$$+ (v\,\tilde{\mu}(d|\hat{t},s_2) + (1-v)\,\tilde{\mu}(d|\tilde{t}_1,s_2))\, u_2(d|\hat{t},t_2)\, p_2(\hat{t}|t_2)$$

$$= \sum_{t_1 \neq \hat{t}} \sum_{d} \tilde{\mu}(d|t_1,s_2)\, \tilde{u}_2(d|t_1, t_2)\, \tilde{p}_2(t_1|t_2)$$

$$+ \tilde{\mu}(d|\hat{t},s_2)\, \tilde{u}_2(d|\hat{t},t_2)\, \tilde{p}_2(\hat{t},t_2)$$

$$+ \tilde{\mu}(d|\tilde{t},s_2)\, \tilde{u}_2(d|\tilde{t},t_2)\, \tilde{p}_2(\tilde{t},t_2)$$

$$= U_2^*(\tilde{\mu}, s_2|t_2) \qquad \forall\, t_2,s_2 \in T_2,$$

because of (5.1).
Therefore player 2 has no incentive to lie in μ.

Also we have

(5.4) $\tilde{U}_2(\tilde{\mu}|\cdot) = U_2(\mu|\cdot)$.

With the equality (5.1) we get:

$$(5.5)\qquad U_1^*(\mu,s_1\,|\,t_1) = \tilde{U}_1^*(\tilde{\mu},s_1\,|\,t_1) \qquad \forall\, t_1 \in T_1,\ s_1 \in T_1\setminus\{\hat{t}\}.$$

We have for all $t_1 \in T_1$:

$$(5.6)\quad U_1^*(\mu,\hat{t}\,|\,t_1) = \sum_{t_2\in T_2}\ \sum_{d\in D} (v\,\tilde{\mu}(d\,|\,\hat{t},t_2) + (1-v)\,\tilde{\mu}(d\,|\,\tilde{t},t_2))\, u_1(d\,|\,t_1,t_2)\, p_1(t_1\,|\,t_2)$$

$$= v \sum_{t_2} \sum_{d} \tilde{\mu}(d\,|\,\hat{t},t_2)\, \tilde{u}_1(d\,|\,t_1,t_2)\, \tilde{p}_1(t_1\,|\,t_2)$$

$$+ (1-v) \sum_{t_2} \sum_{d} \tilde{\mu}(d\,|\,\tilde{t},t_2)\, \tilde{u}_1(d\,|\,t_1,t_2)\, \tilde{p}_1(t_1\,|\,t_2)$$

$$= v\, U_1^*(\tilde{\mu},\hat{t}\,|\,t_1) + (1-v)\, \tilde{U}_1^*(\tilde{\mu},\tilde{t}\,|\,t_1).$$

We get

$$U_1^*(\mu,\hat{t}\,|\,t_1) \le v\, \tilde{U}_1(\tilde{\mu}|t_1) + (1-v)\, \tilde{U}_1(\tilde{\mu}|t_1) = U_1(\mu|t_1) \qquad \forall\, t_1 \in T_1\setminus\{\hat{t}\}$$

because $\tilde{\mu}$ is incentive compatible, and equation (5.5).

(5.6) and (5.3) give :

$$(5.7)\qquad U_1(\mu|\,\hat{t}) = U_1^*(\mu,\hat{t}\,|\,\hat{t})$$

$$= v\, \tilde{U}_1^*(\tilde{\mu},\hat{t}\,|\,\hat{t}) + (1-v)\, \tilde{U}_1^*(\tilde{\mu},\tilde{t}\,|\,\hat{t})$$

$$= \tilde{U}_1(\tilde{\mu}|\,\hat{t}).$$

Incentive compatibility of $\tilde{\mu}$ and (5.5) lead to:

$$U_1^*(\mu,s_1\,|\,\hat{t}) = \tilde{U}_1^*(\tilde{\mu},s_1\,|\,\hat{t})$$

$$\le \tilde{U}_1(\tilde{\mu}|\,\hat{t}) = U_1(\mu|\,\hat{t}).$$

Therefore μ is incentive compatible. (5.4), (5.5) and (5.7) show:

$$\mathcal{U}(\Sigma_{\hat{t},\tilde{t},v}\,\Gamma) \subset \hat{\Sigma}_{\hat{t},\tilde{t}}(\,\mathcal{U}(\Gamma)).$$

#

6. Pareto optimal mechanisms

Definition: Let $\Gamma \in \mathcal{G}$.

An allocation $x \in \mathcal{U}(\Gamma)$ is called pareto optimal (in $\mathcal{U}(\Gamma)$) iff for all $x' \in \mathcal{U}(\Gamma)$ it holds:

$$x' \geq x \Rightarrow x' = x.$$

A mechanism $\mu \in B(\Gamma)$ is called pareto optimal (in $B(\Gamma)$) iff the allocation $U(\mu|\cdot)$ is pareto optimal in $\mathcal{U}(\Gamma)$.

The presentation below is inspired by Myerson [5].

In the following let $\Gamma=(D, d^*, T_1, T_2, u_1, u_2, p)$ be a fixed bargaining problem in $\mathcal{G}$.

For $\lambda \in \mathbb{R}_{++}^{T_1 \cup T_2}$ consider the following linear program $LP(\lambda)$:

$$\underset{\mu \in B(\Gamma)}{\text{maximize}} \ \langle \lambda, U(\mu|\cdot) \rangle.$$

A mechanism $\mu \in B(\Gamma)$ is pareto optimal iff there exists a $\lambda \in \mathbb{R}_{++}^{T_1 \cup T_2}$ so that μ is an optimal solution of $LP(\lambda)$.

Define the sets

$$A_i = \{\alpha_i \in \mathbb{R}_+^{T_i^2} \mid \alpha_i(t|t) = 0 \ \forall t \in T_i\}, \qquad i=1,2$$

and

$$A = \{\alpha \in \mathbb{R}_+^{T_1^2 \cup T_2^2} \mid \alpha|_{T_i^2} \in A_i \ \text{ for } i=1,2\}.$$

For i =1,2 define the mapping

$$\bar{W}_i : \mathbb{R}^{T_i} \times \mathbb{R}_{++}^{T_1 \cup T_2} \times A \rightarrow \mathbb{R}^{T_i} \text{ by}$$

$$(\bar{W}_i(x,\lambda,\alpha))\,(t_i) = (\lambda(t_i) + \sum_{s_i \in T_i} \alpha(s_i|t_i))\, x\,(t_i) - \sum_{s_i \in A_i} \alpha(t_i|s_i)\, x\,(s_i)$$

$$\forall\, t_i \in T_i, \quad \forall\, x \in \mathbb{R}^{T_i}, \alpha \in A, \lambda \in \mathbb{R}_{++}^{T_1 \cup T_2}.$$

$\bar{W}_i$ is linear and in Lemma 1 of [4] Myerson proves that the mappings $\bar{W}_i(\cdot\,,\lambda,\alpha) : \mathbb{R}^{T_i} \to \mathbb{R}^{T_i}$ are bijective for all $\alpha \in A$, $\lambda \in \mathbb{R}_{++}^{T_1 \cup T_2}$.

For i =1,2 define the mapping

$$W_i : \mathbb{R}^{T_1 \times T_2} \times \mathbb{R}_{++}^{T_1 \cup T_2} \times A \to \mathbb{R}^{T_1 \times T_2} \quad \text{by}$$

$$(W_i(x,\lambda,\alpha))\,(t_1,t_2) := (\lambda(t_1) + \sum_{s_i \in T_i} \alpha(s_i|t_i))\, x\,(t_i,t_{-i}) - \sum_{s_i \in T_i} \alpha(t_i|s_i)\, x\,(s_i,t_{-i})$$

For all $\alpha \in A$, $\lambda \in \mathbb{R}_{++}^{T_1 \cup T_2}$, i =1,2 the mappings

$$W_i(\cdot\,,\lambda,\alpha) : \mathbb{R}^{T_1 \times T_2} \longrightarrow \mathbb{R}^{T_1 \times T_2}$$

are linear and bijective because the mappings $\bar{W}_i(\cdot\,,\lambda,\alpha)$ are bijective and linear.

Define for i =1,2, the mappings

$$V_i : D \times T_1 \times T_2 \times \mathbb{R}_{++}^{T_1 \cup T_2} \times A \to \mathbb{R} \text{ by}$$

$$V_i(d,\, t_1,\, t_2,\, \lambda,\, \alpha) := W_i(p_i \cdot u_i(d|\cdot,\cdot),\lambda,\alpha)\,(t_1,t_2)$$

$$\forall\, t_1 \in T_1,\, t_2 \in T_2,\, d \in D,\, \alpha \in A,\, \lambda \in \mathbb{R}_{++}^{T_1 \cup T_2}.$$

Because of our assumption

$$p(t_1,t_2) > 0 \;\; \forall\, (t_1,t_2) \in T_1 \times T_2$$

the mappings $u_i \mapsto W_i(p_i \cdot u_i(d|\cdot,\cdot),\lambda,\alpha) = V_i(d,\cdot\,,\cdot\,,\lambda,\alpha)$ is bijective for all $\alpha \in A$, $\lambda \in \mathbb{R}_{++}^{T_1 \cup T_2}$, i =1,2.

We have for $i = 1,2$, $\alpha \in A$, $\lambda \in \mathbb{R}_{++}^{T_1 \cup T_2}$:

$$(6.1) \qquad \sum_{t_{-i} \in T_{-i}} V_i(d,t_1,t_2,\lambda,\alpha) = \bar{W}_i(U_i(d\,|\cdot),\lambda,\alpha))\,(t_i) \qquad \forall\, t_i \in T_i.$$

Define for $t_1 \in T_1$, $t_2 \in T_2$, $\lambda \in \mathbb{R}_{++}^{T_1 \cup T_2}$, $\alpha \in A$:

$$v\,(t_1,\, t_2,\, \lambda,\, \alpha) = \max_{d\ \in\ D}\ \sum_{i=1}^{2} V_i(d,\, t_1,\, t_2,\, \lambda,\, \alpha).$$

In [5] Myerson showed that $\mu \in B(\Gamma)$ is an optimal solution of LP(λ) iff there exists an $\alpha \in A$ so that

$$(6.2) \qquad \text{i)} \quad U(\mu\,|\,t) > U(\mu,s\,|\,t) \Rightarrow \alpha(s\,|\,t) = 0 \qquad \forall (s,t) \in T_1^2 \cup T_2^2, \text{ and}$$

$$\text{ii)} \quad \mu(d\,|\,t_1,t_2) > 0 \Rightarrow \sum_{i=1}^{2} V_i(d,t_1,t_2,\lambda,\alpha) = v\,(t_1,t_2,\lambda,\alpha)$$

$$\forall\, t_1 \in T_1,\ t_2 \in T_2,\ d \in D.$$

The value for LP(λ) then is:

$$\sum_{(t_1,t_2) \in T_1 \times T_2} v\,(t_1,t_2,\lambda,\alpha).$$

7. Extension

Definition: A bargaining problem $\tilde{\Gamma}=(\tilde{D}, \tilde{d}^*, \tilde{T}_1, \tilde{T}_2, \tilde{u}_1, \tilde{u}_2, \tilde{p}) \in \mathcal{G}$

is called an extension of a given bargaining problem $\Gamma=(D, d^*, T_1, T_2, u_1, u_2, p) \in \mathcal{G}$

iff $\tilde{T}_1 = T_1$, $\tilde{T}_2 = T_2$, $\tilde{p} = p$, $\tilde{d}^* = d^*$, $\tilde{D} \supseteq D$ and $\tilde{u}_i|_{D \times T_1 \times T_2} = u_i$, $i = 1,2$.

Now let $\Gamma=(D, d^*, T_1, T_2, u_1, u_2, p) \in \mathcal{G}$, $\mu \in B(\Gamma)$ be pareto optimal and an optimal solution of LP(λ). Let α be chosen, so that (6.2) is fulfilled.

All this is fixed in the following and the arguments λ and α are suppressed where no confusion is to be expected.

Lemma 2: There exists an extension $\tilde{\Gamma}$ of Γ so that:

$$\mathcal{U}_r(\tilde{\Gamma}) = \{x \in \mathbb{R}^{T_1 \cup T_2} \mid x \geq U(d^*|\cdot) \text{ and } \langle x, \lambda \rangle \leq \langle U(\mu|\cdot), \lambda \rangle\}.$$

Proof: Because $\mathcal{U}(\Lambda^*_{a,\beta}\Gamma) = \hat{\Lambda}_{a,\beta}(\mathcal{U}(\Gamma))$ for all $a \in \mathbb{R}^{T_1 \cup T_2}$, $\beta \in \mathbb{R}_{++}^{T_1 \cup T_2}$ we might restrict ourself to the case $U(d^*|\cdot) = 0$.

We extend the decision set D by the set

$$\bar{D} = \{d_t \mid t \in T_1 \cup T_2\}.$$

The utility of these decisions have to be defined. The extended utility function will also be called u.

Let $w = \langle U(\mu|\cdot), \lambda \rangle$.

We claim for all $\hat{t} \in T_1 \cup T_2$:

C1 $\quad U(d_{\hat{t}}|t) = 0 \qquad \forall\, t \in (T_1 \cup T_2) \setminus \{\hat{t}\}$

C2 $\quad U(d_{\hat{t}} \mid \hat{t}) = \frac{w}{\lambda(\hat{t})}$

C3 $\quad \sum_{i=1}^{2} V_i(d_{\hat{t}}, t_1, t_2) = v\,(t_1,t_2) \qquad\qquad \forall\,(t_1,t_2) \in T_1 \times T_2.$

The convex combinations of the dictatorial mechanisms induced by the $d_{\hat{t}}$'s and d^* generate the right set.

Because $\max \{ \sum_{i=1}^{2} V_i(d,t_1,t_2) \mid d \in D \cup \bar{D}\} = v\,(t_1,t_2)$ the proof is closed if the claim is fulfilled.

Now we fix $\hat{t} \in T_1 \cup T_2$ and show that the equations C1, C2, C3 have a solution.

Let $\hat{t} \in T_1$ (without restriction).

The claims C1, C2 can be transformed equivalently by application of the bijections $\bar{W}_i$, $i = 1,2$.

We get with the help of (6.1)

C1' $\quad \sum_{t_1 \in T_1} V_2(d_{\hat{t}}, t_1, t_2) = 0 \qquad\qquad \forall\ t_2 \in T_2$

and

C2a' $\quad \sum_{t_2 \in T_2} V_1(d_{\hat{t}}, t_1, t_2) = 0 - \sum_{s_1 \in T_1} \alpha(t_1 \mid s_1)\, U_1(d_{\hat{t}} \mid s_1) = -\alpha(t_1 \mid \hat{t})\, \frac{w}{\lambda(\hat{t})}$

$\forall\, t_1 \in T_1 \setminus \{\hat{t}\},$

C2b' $\quad \sum_{t_2 \in T_2} V_1(d_{\hat{t}}, \hat{t}, t_2) = (\lambda(\hat{t}) + \sum_{s_1 \in T_1} \alpha(s_1 \mid \hat{t}))\, \frac{w}{\lambda(\hat{t})}.$

If we find a solution $(V_1(d_{\hat{t}},\cdot,\cdot), V_2(d_{\hat{t}},\cdot,\cdot))$ for the equations C1', C2a', C2b', C3 we have a solution $(u_1(d_{\hat{t}} \mid \cdot,\cdot), u_2(d_{\hat{t}} \mid \cdot,\cdot))$ for C1, C2, C3 because $p(t_1,t_2) > 0$ for all $(t_1,t_2) \in T_1 \times T_2$.

From C3 it follows

$$\sum_{(t_1,t_2)\in T_1 \cup T_2} \sum_{j=1}^{2} V_j(d_{\hat{t}}, t_1,t_2) = \sum_{(t_1,t_2)} v(t_1,t_2) = w$$

and together with C1', C2a' we have

$$w = \sum_{(t_1,t_2)\in T_1 \times T_2} \sum_{j=1}^{2} V_j(d_{\hat{t}}, t_1,t_2) = \sum_{(t_1,t_2)} V_1(d_{\hat{t}},t_1,t_2)$$

$$= \sum_{t_1 \in T_1} \sum_{t_2 \in T_2} V_1(d_{\hat{t}}, t_1,t_2)$$

$$= \sum_{t_2 \in T_2} V_1(d_{\hat{t}}, \hat{t},t_2) - \sum_{t_1 \in T_1} \alpha(t_1 | \hat{t}) \frac{w}{\lambda(\hat{t})}.$$

In the last equation we used the fact $\alpha(\hat{t} | \hat{t}) = 0$.

Therefore the equation C2b' follows from the equations C3, C1', C2a'.

We have shown:
For all $\hat{t} \in T_1$ the following system of equations is equivalent to the three equations C1, C2, C3.

D1 $\quad \sum_{t_1 \in T_1} V_2(d_{\hat{t}},t_1,t_2) = 0 \qquad \forall\, t_2 \in T_2,$

D2 $\quad \sum_{t_2 \in T_2} V_1(d_{\hat{t}},t_1,t_2) = -\alpha(t_1 | \hat{t}) \frac{w}{\lambda(\hat{t})} \qquad \forall\, t_1 \in T_1 \backslash \{\hat{t}\},$

D3 $\quad \sum_{i=1}^{2} V_i(d_{\hat{t}},t_1,t_2) = v(t_1,t_2) \qquad \forall\, (t_1,t_2) \in T_1 \times T_2.$

Now we show that this system has a solution.

1. case $|T_1| = 1, \quad T_1 = \{\hat{t}\}.$

According to D1 we choose

$$V_2(d_{\hat{t}}, \hat{t}, t_2) = 0 \qquad \forall\, t_2 \in T_2$$

and according to D3 we choose

$$V_1(d_{\hat{t}}, t_1, t_2) = v\,(t_1, t_2).$$

Equation D2 is empty.

2. case $|T_1| > 1.$

To fulfill D2 we choose

$$V_1(d_{\hat{t}}, t_1, t_2) = -\alpha(t_1 | \hat{t})\, \frac{w}{\lambda(\hat{t})} \cdot p_1(t_2 | \hat{t}) \qquad \forall (t_1, t_2) \in (T_1 \setminus \{\hat{t}\}) \times T_2.$$

D3 determines $V_2(d_{\hat{t}}, t_1, t_2)$ for all $(t_1, t_2) \in (T_1 \setminus \{\hat{t}\}) \times T_2$.

We compute $V_2(d_{\hat{t}}, \hat{t}, t_2)$ with D1 and finally $V_1(d_{\hat{t}}, \hat{t}, t_2)$ with D3.

The choice of $V_1(d_{\hat{t}}, \cdot, \cdot)$ corresponds to the utility

$$u_1(d_{\hat{t}}, t_1, \cdot) = \begin{cases} 0 & \text{for } t_1 \in T_1 \setminus \{\hat{t}\} \\ \dfrac{w}{\lambda(\hat{t})} & \text{for } t_1 = \hat{t}\,. \end{cases}$$

#

8. The axioms

Now we want to characterize a solution for the bargaining problem, which is a function that determines how the expected utility should be allocated in every bargaining problem.

Harsanyi and Selten [1] gave a set of axioms which uniquely determined a solution in their model.

One step of the proof cannot be carried over to our model. We drop one axiom and have to restrict to the case of independently distributed types.

Let $\mathscr{L}$ be the set of all functions $L : \mathscr{G}^I \to \bigcup_{\Gamma \in \mathscr{G}^I} \mathscr{U}(\Gamma)$ with $L(\Gamma) \in \mathscr{U}(\Gamma)$ for all $\Gamma \in \mathscr{G}^I$.

Our solutions $L \in \mathscr{L}$ should satisfy the following 8 axioms.

For all bargaining problems

$$\Gamma=(D,d^*,T_1,T_2,u_1,u_2,p) \in \mathscr{G}^I$$

we claim:

<u>Axiom 1:</u> Let $\phi_D : D \to \tilde{D}$, $\phi_1 : T_1 \to \tilde{T}_1$, $\phi_2 : T_2 \to \tilde{T}_2$ be bijections of decision sets resp. type sets. Then L fulfills:

$$L(\phi^*(\Gamma)) = \hat{\phi}(L(\Gamma)).$$

<u>Axiom 2:</u> $L(\psi(\Gamma)) = L(\Gamma)$.

<u>Axiom 3:</u> For a bargaining problem $\tilde{\Gamma} \in \mathscr{G}_p$ with $\mathscr{U}_r(\tilde{\Gamma}) = \mathscr{U}_r(\Gamma)$

L fulfills: $L(\tilde{\Gamma}) = L(\Gamma)$.

<u>Axiom 4:</u> If $(a,\lambda) \in \mathbb{R}^{T_1 \cup T_2} \times \mathbb{R}_{++}^{T_1 \cup T_2}$ it follows

$$L(\Lambda^*_{a,\lambda}(\Gamma)) = \hat{\Lambda}_{a,\lambda}(L(\Gamma)).$$

<u>Axiom 5:</u> $L(\Gamma)$ is pareto optimal in $\mathscr{U}(\Gamma)$.

Axiom 6: $(L(\Gamma))(t) > \mathcal{U}(d^*|t) \qquad \forall t \in T_1 \cup T_2.$

Axiom 7: If $\tilde{\Gamma}$ is an extension of Γ and if $L(\tilde{\Gamma}) \in \mathcal{U}(\Gamma)$ then it holds:

$$L(\Gamma) = L(\tilde{\Gamma}).$$

Axiom 8: If $\hat{t} \in T_1$, $\tilde{t} \notin T_1 \cup T_2$ and $v \in (0,1)$ then if follows:

$$L(\Sigma_{\hat{t},\tilde{t},v}(\Gamma)) = \hat{\Sigma}_{\hat{t},\tilde{t}}(L(\Gamma)).$$

We now define the Harsanyi & Selten value L^*.

If $\Gamma \in \mathcal{G}$ then the set $\mathcal{U}_r(\Gamma)$ is convex and

$$\{x \in \mathcal{U}(\Gamma) \mid x(t) > U(d^*|t) \quad \forall t \in T_1 \cup T_2\} \neq \emptyset$$

For all $b \in \mathbb{R}_{++}^{T_1 \cup T_2}$ the function

$$\Pi_{b,\Gamma} : \mathbb{R}^{T_1 \cup T_2} \to \mathbb{R} \quad \text{defined by}$$

$$\Pi_{b,\Gamma}(x) = \prod_{t \in T_1 \cup T_2} (x(t) - U(d^*|t))^{b(t)} \quad \forall x \in \mathbb{R}^{T_1 \cup T_2}$$

is strictly concave.

Therefore the maximization problem

$$\max_{x \in \mathcal{U}_r(\Gamma)} \Pi_{b,\Gamma}(x)$$

has a unique solution.

We define $L^*(\Gamma)$ to be the solution of

$$\max_{x \in \mathcal{U}_r(\Gamma)} \Pi_{q,\Gamma}(x)$$

where q denotes the marginal probabilities.

Theorem 1: L^* satisfies the axioms.

Proof: See Harsanyi & Selten [1].

Theorem 2: L* is the only solution that satisfies all axioms.

We will prove several lemmas which together show the theorem.

Let $T_1 \times T_2$ be a state space and $p \in \Delta^I(T_1 \times T_2)$.

A bargaining problem $\Gamma \in \mathcal{G}_p$ is called norm bargaining problem to p iff

$$\mathcal{U}_r(\Gamma) = \{x \in \mathbb{R}_+^{T_1 \cup T_2} \mid \sum_{t \in T_1 \cup T_2} x(t) \leq 2\}.$$

Let $\mathcal{G}_p^N$ denote the set of those bargaining problems.

There exist norm bargaining problems for every distribution p. They can be constructed by methods used in the proof of Lemma 2.

In the following let $L \in \mathcal{L}$ be a solution that satisfies all 8 axioms.

Let $\Gamma, \Gamma' \in \mathcal{G}_p^N$.

Because of axiom 3 we have

$$L(\Gamma) = L(\Gamma')$$

Therefore we may define $e_p \in \mathbb{R}^{T_1 \cup T_2}$ by $e_p := L(\Gamma)$ where $\Gamma \in \mathcal{G}_p^N$.

Lemma 3: Let $T_1 \times T_2$ be a state space and $p \in \Delta^I(T_1 \times T_2)$.

Then it holds:

a) $e_p \in \mathbb{R}_{++}^{T_1 \cup T_2}$.

b) $\sum_{t \in T_1 \cup T_2} e_p(t) = 2.$

c) If $\Gamma \in \mathcal{G}_p$ then $L(\Gamma)$ is the solution of the maximization problem:

$$\max_{x \in \mathcal{U}_r(\Gamma)} \prod_{t \in T_1 \cup T_2} (x(t) - U(d^* \mid t))^{e_p(t)}$$

<u>Proof:</u> a) and b) are immediate consequences of the axioms 6) and 5).
For $\Gamma \in \mathcal{G}_p$ define φ_Γ: $\mathcal{U}_r(\Gamma) \to \mathbb{R}$ by

$$\varphi_\Gamma(x) = \sum_{t \in T_1 \cup T_2} e_p(t)\, \ln(x(t) - U(d^* | t)) \qquad \forall\, x \in \mathcal{U}_r(\Gamma).$$

Then the maximization problem in the lemma has the same solution as the maximization problem

$$\max_{x \in \mathcal{U}_r(\Gamma)} \varphi_\Gamma(x).$$

We call the solution $L^e(\Gamma)$.

Now we prove $L^e(\Gamma) = L(\Gamma) \qquad \forall\, \Gamma \in \mathcal{G}_p$.

First consider the case $\Gamma \in \mathcal{G}_p^N$.
Solving the maximization problem

$$\max \varphi_\Gamma(x) \text{ under the constraint}$$

$$\sum_{t \in T_1 \cup T_2} x(t) = 2$$

yields the solution $x = e$.

Now consider the case of a linear bargaining problem $\Gamma \in \mathcal{G}_p$, that means:

There exist $\lambda \in \mathbb{R}_{++}^{T_1 \cup T_2}$ and $w \in \mathbb{R}$ so that

$$\mathcal{U}_r(\Gamma) = \{x \in \mathbb{R}^{T_1 \cup T_2} \mid \langle x, \lambda \rangle \le w \text{ and } x \ge U(d^* | \cdot)\}$$

These bargaining problems can be derived from a norm bargaining problem by a linear utility transformation. A simple computation shows together with axiom 4 that

$$L^e(\Gamma) = L(\Gamma).$$

Now let $\Gamma \in \mathcal{G}_p$ be arbitrary.

Define $\lambda := \dfrac{e(\cdot)}{L^e(\Gamma)(\cdot) - U(d^*|\cdot)} \in \mathbb{R}_{++}^{T_1 \cup T_2}$

and $\quad w := \varphi_\Gamma(L^e(\Gamma))$.

λ is the gradient of φ_Γ in the optimal point $L^e(\Gamma)$.

φ_Γ is strictly concave and therefore

$$\mathscr{U}_\Gamma(\Gamma) \subset X := \{x \in \mathbb{R}^{T_1 \cup T_2} \mid \langle x,\lambda \rangle \leq w \text{ and } x \geq U(d^*|\cdot)\}.$$

(See Harsanyi & Selten [1] for a proof.)

A simple computation shows that the maximization problem

$$\max_{x \in X} \varphi_\Gamma(x)$$

also has the solution $L^e(\Gamma)$.

Lemma 2 shows that there exists an extension $\tilde{\Gamma}$ of Γ so that $\mathscr{U}_\Gamma(\tilde{\Gamma}) = X$.
$\tilde{\Gamma}$ is a linear bargaining problem. Therefore we have $L(\tilde{\Gamma}) = L^e(\tilde{\Gamma})$.
Because $L^e(\tilde{\Gamma}) = L^e(\Gamma) \in \mathscr{U}(\Gamma)$ we can apply axiom 7 to close the proof.

#

<u>Lemma 4:</u> Let $T_1 \times T_2$ be a state space and $p \in \Delta^I(T_1 \times T_2)$.
If $\hat{t} \in T_1$, $\tilde{t} \notin T_1 \cup T_2$, $v \in (0,1)$ and $\tilde{p} = \overset{\circ}{\Sigma}_{\hat{t},\tilde{t},v}\, p$ then it holds:

$$e_p(t) = \begin{cases} e_{\tilde{p}}(t) & , \ \forall\, t \in (T_1 \cup T_2)\backslash\{\hat{t}\} \\ e_{\tilde{p}}(\hat{t}) + e_{\tilde{p}}(\tilde{t}) \ , & t = \hat{t} \end{cases} .$$

<u>Proof:</u> Let $\Gamma \in \mathscr{G}_p^N$ and $\tilde{\Gamma} := \Sigma_{\hat{t},\tilde{t},v}\, \Gamma$.

Then axiom 8 yields

$$L(\tilde{\Gamma}) = \hat{\Sigma}_{\hat{t},\tilde{t}}\, (L(\Gamma)) = \hat{\Sigma}_{\hat{t},\tilde{t}}\, (e_p).$$

Lemma 3 yields

$$L(\tilde{\Gamma}) = L^e(\tilde{\Gamma}).$$

We know by Lemma 1 that

$$\mathcal{U}_r(\tilde{\Gamma}) = \Sigma_{\hat{t},\tilde{t}} (\mathcal{U}_r(\Gamma))$$

$$= \{x \in \mathbb{R}^{T_1 \cup T_2 \cup \{\tilde{t}\}} \mid \sum_{t \in T_1 \cup T_2} x(t) = 2, x(\tilde{t}) = x(\hat{t})\}$$

A standard computation shows

$$(L^e(\tilde{\Gamma})(t) = \begin{cases} e_{\tilde{p}}(t) & , \quad t \in (T_1 \cup T_2)\setminus\{\hat{t}\} \\ e_{\tilde{p}}(\hat{t}) + e_{\tilde{p}}(\tilde{t}), & t \in \{\hat{t}, \tilde{t}\} \end{cases}$$

A comparison with $\hat{\Sigma}_{\hat{t},\tilde{t}}\ e_p$ closes the proof.

#

Lemma 5: There exists a function $\varphi : (0,1] \to \mathbb{R}$, so that for all state spaces and all distributions $p_1 \in \Delta(T_1)$, $p_2 \in \Delta(T_2)$ and for $i = 1,2$ it holds:

$$e_{p_1 \otimes p_2}(t_i) = \varphi(p_i(t_i)) \qquad \forall\, t_i \in T_i.$$

Proof: First we want to define φ.
Let $T_1 = \{\bar{a}_1, \bar{a}_2\}$, $T_2 = \{\bar{b}\}$ be type spaces and $r \in (0,1)$.
The distributions $p_1 \in \Delta(T_1)$ and $p_2 \in \Delta(T_2)$ are defined by:

$$p_2(\bar{b}) = 1, \qquad p_1(\bar{a}_1) = r, \qquad p_1(\bar{a}_2) = 1 - r.$$

Define $\varphi(r)$ by

$$\varphi(r) = e_{p_1 \otimes p_2}(\bar{a}_1),$$

and define $\quad \varphi(1) := 1.$

Now let $T_1 \times T_2$ be any state space and $p_1 \in \Delta(T_1)$, $p_2 \in \Delta(T_2)$.

In view of axiom 2 it suffices to prove the lemma for player 1.

Let $b \in T_2$ and define $T_2' := \{b\}$ and p_2 to be the only element in $\Delta(T_2')$. We can derive the distribution $p = p_1 \otimes p_2$ by iterated 'splitting of types (of player 2)' from $p_1 \otimes p_2'$.

The preceding lemma shows

$$e_{p_1 \otimes p_2'}(t_1) = e_{p_1 \otimes p_2}(t_1) \qquad \forall\, t_1 \in T_1.$$

We can restrict ourself to the case $T_2 = T_2' = \{b\}$, $p_2 = p_2'$.

Now take any $a_1 \in T_1$.

1. case: $\quad T_1 = \{a_1\}$

This is the case with complete information and it is clear that

$$e_{p_1 \otimes p_2}(a_1) = 1 = \varphi(1) = \varphi(p_1(a_1)).$$

2. case: $\quad |T_1| > 1.$

Take $a_2 \in T_1$, $a_2 \neq a_1$, and define $T_1' = \{a_1, a_2\}$ and $p_1' \in \Delta(T_1')$ by

$$p_1'(a_1) = p_1(a_1) \;,\; p_1'(a_2) = 1 - p(a_1).$$

We can derive the distribution $p_1 \otimes p_2$ by iterated splitting of type a_2 from the distribution $p_1' \otimes p_2$. The preceding lemma shows

$$e_{p_1' \otimes p_2}(a_1) = e_{p_1 \otimes p_2}(a_1).$$

Because of axiom 1 we have

$$e_{p_1' \otimes p_2}(a_1) = \varphi(p_1'(a_1))$$

and that closes the proof.

#

<u>Lemma 6:</u> $\varphi(p) = p \qquad \forall\, p \in (0,1]$

<u>Proof:</u> We know $\varphi(1) = 1$ and $\varphi(p) > 0 \qquad \forall\, p \in (0,1]$ (see lemma 3a)).
Next we show

$$\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2) \qquad \forall\, r_1, r_2 > 0,\ r_1 + r_2 \leq 1.$$

1. case: $\quad r_1 + r_2 = 1.$

If you split the case without uncertainty you get the result from Lemma 4.

2. case: $\quad r_1 + r_2 < 1.$

Take the type sets

$$T_1 = \{a_1, a_2\},\ T_2 = \{b\}$$

and the distributions

$$p_1 \in \Delta(T_1),\ p_2 \in \Delta(T_2) \text{ defined by}$$

$$p_1(a_1) = 1 - r_1 - r_2,\ p_1(a_2) = r_1 + r_2,\ p_2(b) = 1.$$

Now split a_2 and you get the result from lemma 4.

φ is nonnegative and additive. Therefore φ is monotone.

Because $\varphi(1) = 1$ and φ is additive we know

$$\varphi(p) = p \qquad \forall\, p \in (0,1] \cap \mathbb{Q}.$$

The monotonicity closes the proof.

#

The four lemmas together show that $L = L^*$.

9. Some problems with the non–independent case

In their paper Harsanyi and Selten didn't restrict to the case that the types are independently distributed.

They defined two other operations which enable them to prove the uniqueness of the value with the help of an additional axiom. In Myerson's model these operations and the axiom bear some problems.

Harsanyi and Selten model a bargaining problem as a pair (X,p) where p is a probability distribution on a state space $T_1 \times T_2$ and $X \subset \mathbb{R}^{T_1 \cup T_2}$ is the bargaining set representing the possible allocations of expected utility for the types of both players.

Given two bargaining problems (X,p'), (X,p") with identical bargaining set they define the mixture of these bargaining problems to be the bargaining problem (X,p) with $p = \frac{1}{2}(p + p')$.

We try to carry over this operation to our model.

Let $\Gamma' = (D,d^*,T_1,T_2,u_1,u_2,p')$ and $\Gamma'' = (D,d^*,T_1,T_2,u_1,u_2,p'')$ be two bargaining problems which only differ in the probability distributions. Then we can define the mixture of these bargaining problems to be the bargaining problem $\Gamma=(D,d^*,T_1,T_2,u_1,u_2,p)$ with distribution $p = \frac{1}{2}(p' + p'')$.

Harsanyi and Selten demand the following axiom to hold for the solution L:
Let (X,p'), (X,p") be two bargaining problems and (X,p) be the mixture of them.

If $\quad L((X,p')) = L((X,p'')) \qquad$ then $\qquad L((X,p)) = L((X,p'))$.

This axiom cannot be transferred to our model. It is not clear that Γ' and Γ'' (defined as above) do have identical bargaining sets. Even if we demand this as a necessary condition for the axiom it might be the case that the bargaining set of the mixture differs from that of Γ' and Γ''.

Example:

Let $D:=\{d^*,d_1,d_2\}$, $T_1=\{a_1,a_2\}$, $T_2:=\{b_1,b_2\}$

and define u_1 and u_2 by the following schedule

(u_1,u_2)	d^*	d_1	d_2
a_1,b_1	0,0	1,1	0,0
a_1,b_2	0,0	0,1	1,0
a_2,b_1	0,0	1,0	0,1
a_2,b_2	0,0	0,0	1,1

Three bargaining problems are defined by

$$\Gamma\ =(D,d^*,T_1,T_2,u_1,u_2,p)$$
$$\Gamma'\ =(D,d^*,T_1,T_2,u_1,u_2,p')$$
$$\Gamma''=(D,d^*,T_1,T_2,u_1,u_2,p'')$$

with $p\ (\cdot,\cdot)=\frac{1}{4}$,

$p'(\cdot,b_1)=\frac{1}{6}$, $\qquad p'(\cdot,b_2)=\frac{1}{3}$

$p''(\cdot,b_1)=\frac{1}{3}$, $\qquad p''(\cdot,b_2)=\frac{1}{6}$.

We will now prove

$$\mathcal{U}(\Gamma')=\mathcal{U}(\Gamma'').$$

If we define the three bijections

$$\phi_D:D\rightarrow D,\qquad \phi_1:T_1\rightarrow T_1,\qquad \phi_2:T_2\rightarrow T_2$$

by

$$\phi_D(d^*)=d^*,\ \phi_D(d_1)=d_2,\ \phi_D(d_2)=d_1$$

$$\phi_1(a_1)=a_2,\ \phi_1(a_2)=a_1,$$

$$\phi_2(b_1)=b_2,\ \phi_2(b_2)=b_1,$$

then it holds for $i=1,2$:

i) $u_i(\cdot\,|\phi_i(\cdot),\cdot) = u_i(\cdot\,|\cdot,\cdot)$,

ii) $p_i'(\cdot\,|\phi_i(\cdot)) = p_i'(\cdot\,|\cdot)$,

$p_i''(\cdot\,|\phi_i(\cdot)) = p_i''(\cdot\,|\cdot)$,

iii) $u_i(\phi_D(\cdot)|\phi_1(\cdot),\phi_2(\cdot)) = u_i(\cdot\,|\cdot,\cdot)$,

iv) $p_i'(\phi_{-i}(\cdot)|\phi_i(\cdot)) = p_i''(\cdot\,|\cdot)$.

Let $\mu' \in B(\Gamma')$.
Then incentive compatibility i) and ii) imply for $i = 1,2$, $s_i, t_i \in T_i$:

$$\begin{aligned} U_i'(\mu'|t_i) &= U_i^{*\prime}(\mu', t_i|s_i) \leq U_i'(\mu'|s_i) \\ &= U_i^{*\prime}(\mu', s_i|t_i) \leq U_i'(\mu'|t_i) \end{aligned}$$

Therefore we have equality everywhere.

We now define a mechanism $\mu'' \in \mathcal{M}(\Gamma'')$ by

$$\mu''(\cdot\,|\cdot,\cdot) = \mu'(\phi_D(\cdot)|\phi_1(\cdot),\phi_2(\cdot)).$$

For $i = 1,2$, $s_i, t_i \in T_i$ we get with the help of iii) and iv):

$$U_i^{*\prime\prime}(\mu'', s_i|t_i) = \sum_{t_{-i}} \sum_{d} \mu''(d|s_i, t_{-i})\, u_i(d|t_i, t_{-i})\, p_i''(t_{-i}|t_i)$$

$$= \sum_{t_{-i}} \sum_{d} \mu'(\phi_D(d)|\phi_i(s_i), \phi_{-i}(t_{-i}))\, u_i(\phi_D(d)|\phi_i(t_i), \phi_i(t_{-i})\, p_i'(\phi_{-i}(t_{-i})|\phi_i(t_i))$$

$$= U^{*}{}_i'(\mu', \phi_i(s_i)|\phi_i(t_i)) = U_i^{*\prime}(\mu', s_i|t_i).$$

Therefore $\quad \mu'' \in B(\Gamma'')$
and $\quad U''(\mu''|\cdot) = U'(\mu'|\cdot)$,
which shows

$$\mathcal{U}(\Gamma'') \supset \mathcal{U}(\Gamma').$$

In the same way you can prove the converse implication.

Let the mechanism $\mu \in \mathscr{M}(\Gamma)$ be defined by

$$\mu(d_1 | a_1, \cdot) = 1,$$

$$\mu(d_2 | a_2, \cdot) = 1.$$

μ is incentive compatible because $p_1(\cdot | \cdot) = \frac{1}{2}$.

μ leads to the allocation

$$U_1(\mu | a_1) = U_1(\mu | a_2) = \tfrac{1}{2},$$

$$U_2(\mu | b_1) = U_2(\mu | b_2) = 1.$$

This allocation is not an element of $\mathscr{U}(\Gamma')$:

Suppose we have a mechanism $\mu' \in B(\Gamma')$ which generates that allocation. Then μ is determined uniquely by the utility for player 2.

We must have:

$$\mu'(d_1 | a_1, \cdot) = 1 \quad \text{and}$$

$$\mu'(d_2 | a_2, \cdot) = 1.$$

But the type a_1 will not tell the truth in this mechanism because $p_1'(b_1 | a_1) = \frac{1}{3} < p_1'(b_2 | a_1) = \frac{2}{3}$. So we get a contradiction.

Γ' and Γ'' have the same bargaining sets and $p = \frac{1}{2}(p' + p'')$. But Γ has a different bargaining set. #

Harsanyi and Selten also define an operation 'dividing a type' which generalizes the operation 'splitting a type'.

Let (X,p) be a bargaining problem:
Given a type $\hat{t} \in T_1$ and a function $w \in (0,1)^{T_2}$ this operation divides $\hat{t}$ into two subtypes. One gets a new bargaining problem $(\tilde{X},\tilde{p})$ with bargaining set
$\tilde{X} = \Sigma_{\hat{t},\tilde{t}} X \subset \mathbb{R}^{T_1 \cup \{\tilde{t}\} \cup T_2}$ and probability distribution $\tilde{p} \in \Delta((T_1 \cup \{\tilde{t}\}) \times T_2)$ which equals p on $(T_1 \backslash \{\hat{t}\}) \times T_2$ and is defined on $\{\hat{t},\tilde{t}\} \times T_2$ by

$$\tilde{p}(\hat{t},\cdot) = w(\cdot)\, p(\hat{t},\cdot),$$

$$\tilde{p}(\tilde{t},\cdot) = (1 - w(\cdot))\, p(\hat{t},\cdot).$$

With the help of their mixing axiom Harsanyi and Selten prove that their value splits up if you divide a type in the same manner as if you split the type. With this result they prove an analogue of Lemma 4 for the operation 'dividing a type'.

Now I try to transfer this operation to our model.

Let $T_1 \times T_2$ be a state space and $\hat{t} \in T_1$, $\tilde{t} \notin T_1 \cup T_2$, $w \in (0,1)^{T_2}$.

Define the mapping

$$\overset{\circ}{\Sigma}'_{\hat{t},\tilde{t},w} : \Delta(T_1 \times T_2) \rightarrow \Delta((T_1 \cup \{\tilde{t}\}) \times T_2)$$

by

$$(\overset{\circ}{\Sigma}'_{\hat{t},\tilde{t},w}\, p)(t_1,\cdot) = \begin{cases} p(t_1,\cdot) & \text{for } t_1 \in T_1 \backslash \{\hat{t}\} \\ w(\cdot)\, p(\hat{t},\cdot) & \text{for } t_1 = \hat{t} \\ (1 - w(\cdot))\, p(\hat{t},\cdot) & \text{for } t_1 = \tilde{t} \end{cases} \quad ,\forall\, p \in \Delta(T_1 \times T_2),$$

and the mapping $\Sigma'_{\hat{t},\tilde{t},w} : \mathscr{G}_{T_1 \times T_2} \rightarrow \mathscr{G}_{(T_1 \cup \{\tilde{t}\}) \times T_2}$ by

$$\Sigma'_{\hat{t},\tilde{t},w}\, \Gamma = (D, d^*, T_1 \cup \{\tilde{t}\}, T_2, \check{\Sigma}_{\hat{t},\tilde{t}}\, u_1, \check{\Sigma}_{\hat{t},\tilde{t}}\, u_2, \overset{\circ}{\Sigma}'_{\hat{t},\tilde{t},w}\, p)$$

$$\forall\, \Gamma = (D, d^*, T_1, T_2, u_1, u_2, p) \in \mathscr{G}_{T_1 \times T_2}.$$

The types $\hat{t}, \tilde{t}$ have the same utility functions but different beliefs in general.

<u>Lemma 7:</u> Let $\tilde{\Gamma} = \Sigma'_{\hat{t},\tilde{t},w}\, \Gamma$.

Then $\tilde{p}_1(\cdot \mid \hat{t}) = \tilde{p}_1(\cdot \mid \tilde{t})$ iff $w(t_2) = w(t_2')$ for all $t_2, t_2' \in T_2$.

Proof: For all $(t_1,t_2) \in (T_1 \cup \{\tilde{t}\}) \times T_2$ it holds

$$\tilde{p}_1(t_2|t_1) = \frac{\tilde{p}(t_1,t_2)}{\sum_{t_2 \in T_2} \tilde{p}(t_1,t_2)}$$

$$= \frac{\tilde{p}(t_1,t_2)}{\tilde{q}_1(t_1)} \ ,$$

where $\tilde{q}_1$ is the marginal distribution on $T_1 \cup \{\tilde{t}\}$.

Therefore

$$\tilde{p}_1(t_2|\hat{t}) = \frac{w(t_2)\ p(\hat{t},t_2)}{\tilde{q}_1(\hat{t})}$$

and

$$\tilde{p}_1(t_2|\tilde{t}) = \frac{(1 - w(t_2))\ p(\hat{t},t_2)}{\tilde{q}_1\ (\tilde{t})} \ .$$

If $\tilde{p}_1(\cdot\ |\hat{t}) = \tilde{p}_1(\cdot\ |\tilde{t})$ we get for all $t_2 \in T_2$:

$$w(t_2) = \frac{\tilde{q}_1(\hat{t})}{\tilde{q}_1(\tilde{t}) + \tilde{q}_1(\hat{t})}$$

$$= \frac{\tilde{q}_1(\hat{t})}{q_1(\hat{t})} \ .$$

The right side is independent of t_2 therefore w is constant.

The other direction is trivial.

#

Because the beliefs differ we will have

$$\mathcal{U}\,(\Sigma'_{\hat{t},\tilde{t},w}\ \Gamma) \neq \hat{\Sigma}_{\hat{t},\tilde{t}}\ \ \mathcal{U}(\Gamma)$$

in general.

<u>Example:</u> Let $\Gamma=(D,d^*,u_1,u_2,T_1,T_2,p) \in \mathcal{G}$ be defined by

$$D=\{d^*,d_1\},\ T_1=\{a_1\},\ T_2=\{b_1,b_2\}$$

$$u_1(d^*|\cdot,\cdot)=u_2(d^*|\cdot,\cdot)=0$$

$$u_1(d_1|a_1,b_1)=1,\ u_1(d_1|a_1,b_2)=2$$

$$u_2(d_1|\cdot,\cdot)=1.$$

$$p(\cdot,\cdot)=\tfrac{1}{2}.$$

We get

$$\mathcal{U}(\Gamma)=\{\,x\in\mathbb{R}^{T_1\cup T_2}\,|\,\exists\,\lambda\in[0,1] \text{ so that } x(a_1)=\lambda\cdot\tfrac{3}{2},\ x(b_1)=x(b_2)=\lambda\}.$$

Let $w\in\mathbb{R}^{T_2}$ be defined by

$$w(b_1)=\tfrac{1}{3},\ w(b_2)=\tfrac{2}{3},$$

and $$\tilde{\Gamma}:=\Sigma_{a_1,a_2,w}\,\Gamma.$$

We have

$$\mathcal{U}(\tilde{\Gamma})=\{\,x\in\mathbb{R}^{\tilde{T}_1\cup\tilde{T}_2}\,|\,\exists\,\lambda\in[0,1] \text{ so that } x(a_1)=\lambda\cdot\tfrac{5}{3},\ x(a_2)=\lambda\cdot\tfrac{4}{3},\ x(b_1)=x(b_2)=\lambda\}.$$

#

Of course, this makes it impossible to prove (or demand)

(9.1) $$L(\Sigma'_{\hat{t},\tilde{t},w}\,\Gamma)=\hat{\Sigma}'_{\hat{t},\tilde{t}}\,L(\Gamma), \qquad \forall\,\Gamma\in\mathcal{G}_{T_1\times T_2},\ w\in(0,1)^{T_2},$$

if L obeys all axioms and a mixing axiom, like Harsanyi and Selten have done.

But to follow the proof of Harsanyi and Selten it is necessary to prove an analogue of lemma 4 for the operation dividing a type.

We need equation (9.1) for a class of bargaining problems including some norm bargaining problems.

It is a problem of further research to define such a class in such a way that the restriction to these bargaining problems is not too artificial.

10. Further discussion

There is another problem with this presentation which is more fundamental than the problems with the non–independent case.

A solution L is a function which determines an allocation of expected utility for every bargaining problem. In Myerson's model one is interested not only in this outcome but also in the mechanism that implements it. Of course, there exists one mechanism that yields the allocation but the mechanism is not unique.

There are also some problems with axiom 3 in this connection. Axiom 3 says that the solution does not depend on allocations that are not individual rational and whenever two bargaining problems have the same bargaining set the value is the same. But the shape of the bargaining set is determined by different kind of constraints. A mechanism might be pareto optimal because there are no lotteries which give a higher utility of one cannot improve the mechanism because of the incentive constraints. Therefore in two bargaining problems the whole setup might be very different but the bargaining set and therefore the value is the same. On the other hand the mechanisms implementing the value might be very different again.

At last I want to present an example that shows that the operation 'dividing a type' also bears some problems in connection with the discussion above.

Example: Let the bargaining problem

$\Gamma=(D,d^*,T_1,T_2,u_1,u_2,p)$ be defined by

$T_1=\{a_1,a_2\}$, $T_2=\{b_1,b_2\}$,

$D=\{d^*,d_1,d_2,d_3,d_4,d_5\}$, $p=\frac{1}{4}$

and let u be determined by the following schedule

u_1, u_2	d^*	d_1	d_2	d_3	d_4	d_5
a_1, a_2	0, 0	2, −1	0, 1	−1, 2	1, 0	3, −2
a_1, b_2	0, 0	2, 1	0, 1	1, 0	−1, 2	1, 0
a_2, b_1	0, 0	0, 1	2, −1	−1, 2	1, 0	−1, 2
a_2, b_2	0, 0	0, 1	2, −1	1, 0	−1, 2	1, 0

First we will see that Γ is a norm bargaining problem.

The convex combinations of the dictatorial mechanisms induced by the decisions $d^*, d_1, \ldots, d_5$ generate the set of allocations

$$\{ x \in \mathbb{R}_+^{T_1 \cup T_2} \mid \sum_{t \in T_1 \cup T_2} x(t) \leq 2 \}.$$

On the other hand look at the linear program LP(λ) for $\lambda = 1$:

$$\max_{\mu \in B_r(\Gamma)} \langle U(\mu|\cdot), 1 \rangle.$$

For $\alpha = 0$ you get

$$\sum_{i=1}^{2} V_i(d,t_1,t_2,\lambda,\alpha) = \tfrac{1}{2} \qquad \forall \; d \in D \setminus \{d^*\}.$$

Therefore the value of the linear program is 2. This proves

$$\mathcal{U}_r(\Gamma) = \{ x \in \mathbb{R}_+^{T_1 \cup T_2} \mid \sum_{t \in T_1 \cup T_2} x(t) \leq 2 \}.$$

The Harsanyi–Selten value L^* selects the allocation

$$L^*(\Gamma) = \tfrac{1}{2}.$$

The dictatorial mechanisms u_1 and u_2 defined by

$$u_1 = \tfrac{1}{4}(d_1 + d_2 + d_3 + d_4)$$

and

$$u_2 = \tfrac{1}{4}(d_5 + d_2 + d_3 + d_4)$$

implement this allocation.

Now we divide type a_2. Let $w \in \mathbb{R}^{T_2}$ be defined by

$$w(b_1) = \frac{1}{4}, \; w(b_2) = \frac{3}{4},$$

and define

$$\tilde{\Gamma} := \Sigma^{\prime}_{t_1, t_2, w}(\Gamma).$$

Then the distribution $\tilde{p}$ and the marginals $\tilde{q}$ are given by the schedule

$\tilde{p}$	b_1	b_1	$\tilde{q}_1$
a_1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
a_2	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{4}$
a_3	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{1}{4}$
$\tilde{q}_2$	$\frac{1}{2}$	$\frac{1}{2}$	

The beliefs are given by

p_1, p_2	b_1	b_2
a_1	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$
a_2	$\frac{1}{4}, \frac{1}{8}$	$\frac{3}{4}, \frac{3}{8}$
a_3	$\frac{3}{4}, \frac{3}{8}$	$\frac{1}{4}, \frac{1}{8}$

Now look at the maximization problem

$$\max_{\mu \in B_{\Gamma}(\tilde{\Gamma})} \langle \tilde{U}(\mu | \;), \tilde{q} \rangle .$$

If you choose the dual variable $\tilde{\alpha}=0$ you get

$$\sum_{i=1}^{2} \tilde{V}_i(d,t_1,t_2,\tilde{q},\tilde{\alpha})$$

$$=\tilde{q}_1(t_1)\,\frac{\tilde{p}(t_1,t_2)}{\tilde{q}_1(t_1)}\,\tilde{u}_1(d\,|\,t_1,t_2) + \tilde{q}_2(t_2)\,\frac{\tilde{p}(t_1,t_2)}{\tilde{q}_2(t_2)}\,\tilde{u}_2(d\,|\,t_1,t_2)$$

$$=\tilde{p}(t_1,t_2)\,(\tilde{u}_1(d\,|\,t_1,t_2) + \tilde{u}_2(d\,|\,t_1,t_2))$$

$$=\tilde{p}(t_1,t_2) \qquad \forall\,(t_1,t_2)\in\tilde{T}_1\times\tilde{T}_2\,,\ \forall\, d\in D\setminus\{d^*\}.$$

Therefore all dictatorial mechanisms are pareto–optimal and

$$\mathcal{U}_\Gamma(\tilde{\Gamma})\subset\{\,x\in\mathbb{R}_+^{\tilde{T}_1\cup\tilde{T}_2}\mid\langle x,\tilde{q}\rangle\leq 1\,\}=:X.$$

The maximization problem

$$\max_{x\,\in\,X}\ \sum_{t\in\tilde{T}_1\in\tilde{T}_2}\ \tilde{q}(t)\,\ln x(t)$$

has the solution

$$x^*=\tfrac{1}{2}.$$

This allocation can be implemented by the mechanism

$$\tilde{\mu}_1:=\tfrac{1}{4}\,(d_1+d_2+d_3+d_4)$$

and is an element of $\mathcal{U}_\Gamma(\tilde{\Gamma})$ therefore.

With IIA we get

$$L^*(\tilde{\Gamma})=x^*.$$

One may say that $\tilde{\mu}_1$ is canonically derivated from μ_1 by 'splitting the type a_2'. But if you look at the derivation

$$\tilde{\mu}_2 := \frac{1}{4}(d_5 + d_2 + d_3 + d_4)$$

of μ_2 you see that

$$\tilde{U}(\tilde{\mu}_2 | t) = \begin{cases} \frac{1}{2} & \text{if } t \in \{a_1, b_1, b_2\} \\ \frac{1}{2} - \frac{1}{8} & \text{if } t = a_2 \\ \frac{1}{2} + \frac{1}{8} & \text{if } t = a_3 \end{cases}$$

This shows again that there are problems with the value if it is not implemented uniquely.

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