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COMPARATIVE COOPERATIVE GAME THEORY

by

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In memory of my teacher in Japan,
Professor Ryuichi Watanabe, 1928 - 1986.

Abstract. Given two side-payment games v and w, ~~both~~ defined for the same finite player-set N, the following three welfare criteria are characterized in terms of the data v and w: (A) For every $y \in C(w)$ there exists $x \in C(v)$ such that $y \leq x$; (A′) For every $x \in C(v)$ there exists $y \in C(w)$ such that $y \leq x$; and (B) There exist $y \in C(w)$ and $x \in C(v)$ such that $y \leq x$. (Here $C(v)$ denotes the core of v.) Given two non-side-payment games v and w, sufficient conditions for the criteria (A′) and (B) are established, by observing that an ordinal convex game has a large core.

1. INTRODUCTION

In many prevalent units and aspects of the modern economy, the neoclassical market mechanism for resource allocation works only imperfectly or may even be non-existent. The economic agents in these environments cannot behave, therefore, as individualistic (noncooperative) price-takers. Alternative behavioral principles have been postulated and their consequences analyzed in the past. One of the alternatives which has been receiving increasing attention in recent years says that an agent, given his own incentive, coordinates his strategy-choice with other agents, because by doing so he and his colleagues can better serve their diverse incentives; that is, the agents play a cooperative game. Indeed, the cooperative behavioral principle has long been identified in the theory of the firm as an essential determinant of the firm activities (see, e.g., Coase (1937), Alchian and Demsetz (1972), Arrow (1974), and Ichiishi (1982, 1985)). The present paper concerns cooperative behavior.

Given one economic system in which cooperative behavior is predominant, economic theorists have successfully applied cooperative game-theoretical solution concepts. A subsequent question naturally arises: Given two economic systems, is it possible to compare the cooperative solutions of the two? The present paper is intended to take a first step towards the study of the comparative cooperative economic/game theory. Comparative study of two systems is not new (see, e.g., Sertel (1982) for a modern treatment). What is new here is comparison of two systems in terms of a cooperative solution concept.

Cooperative behavior can be modelled in various ways with diverse degrees of generality. This paper adopts two simple models -- a game in characteristic function form with side-payments (or simply, a side-payment game) and a game in characteristic function form without side-payments (or simply, a non-side-payment game) -- because of their technical tractability. By studying these models first, one can anticipate what kinds of results may be obtained in more general setups, such as games in normal form and general equilibrium models of economies with production. The cooperative solution concept analyzed here is the core, i.e., the set of payoff allocations that are feasible (via the grand coalition) and are coalitionally stable. A core payoff allocation is a coalitional analogue of a Nash equilibrium (a typical noncooperative solution concept).

Given two games in characteristic function form v and w, with or without side-payments, both defined for the same player set N, consider the following three welfare criteria: (A) For each core payoff allocation y of w, there exists a core payoff allocation x of v such that x is Pareto superior to y; (A′) For each core payoff allocation x of v, there exists a core payoff allocation y of w such that x is Pareto superior to y; and (B) There exist a core payoff allocation y of w and a core payoff allocation x of v, such that x is Pareto superior to y. The purpose of this paper is to establish conditions (in terms of the exogenous data v and w) for the criteria (A), (A′) and (B). For the side-payment case, complete characterizations of (A), (A′) and (B) will be established. The condition for the criterion (A) for the side-payment case will suggest that this criterion is rarely met under economically meaningful situations. Indeed, conditions for (A) for the non-side-payment case will be left as an open problem. On the other hand, the results for (A′) and (B) will suggest that these are likely to be satisfied under economically meaningful situations.

Specific classes of side-payment games have been paid particular attention in the past: balanced games (see, e.g., Bondareva (1962), Shapley (1967), and Schmeidler (1967)), exact games (see, e.g., Shapley (1971) and Schmeidler (1972)), and convex games (see, e.g., Shapley (1971), Rosenmüller (1971), Delbaen (1974), and Ichiishi (1981)). Balancedness characterizes nonemptiness of the core. For games with a nonempty core, one may assume without loss of generality that they are exact games (by looking at their exact envelopes). Convexity characterizes increasing returns with respect to the coalition size. Convex games are exact, and exact games are balanced. It will turn out in the present paper that convex games play important roles in the above criteria (A), (A′) and (B). In studying the criterion (A), this paper will also propose a new class of games, called here the no-gap exact games. This class is strictly smaller than the class of exact games, but is strictly larger than the class of convex games. It is dual to Sharkey's (1982) concept of the exact games with a large core; the latter plays the same role in (A′) as the no-gap exact game concept does in (A).

The convexity concept for the side-payment games has been extended to the non-side-payment games in several ways; e.g., the ordinal convexity and the cardinal convexity. It will also turn out in the present paper that ordinal convex games play important roles in the above criteria (A′) and (B).

Section 2 of this paper studies the side-payment case. As corollaries of the general characterizations of the criteria (A), (A′) and (B), the section will conclude that: (1) Given a no-gap exact game v and an exact game w, the criterion (A) holds true, if and only if $v(S) - w(S) \leq v(N) -$

$w(N)$ for every $S \in 2^N \setminus \{N\}$; (1′) Given an exact game v and a game with a large core w, the criterion (A′) holds true, if and only if $w(S) \leq v(S)$ for every $S \in 2^N \setminus \{\phi\}$; (2) Given two convex games v and w, the criterion (B) holds true, if and only if $w(S) + v(N\setminus S) \leq v(N)$ for every $S \in 2^N \setminus \{\phi\}$.

Section 3 studies the non-side-payment case. The section will first observe that ordinal convex games have a large core, by applying two lemmas of Peleg (1986), and then conclude that: (1′) Given a non-side-payment game V and an ordinal convex game W, the criterion (A′) holds true, if $W(S) \subset V(S)$ for every $S \in 2^N \setminus \{\phi\}$; (2) Given two ordinal convex games W and V, the criterion (B) holds true, if $W(N) \subset V(N)$ and $W(S) \cap V(T) \subset W(S \cap T) \cup V(S \cap T) \cup W(S \cup T) \cup V(S \cup T)$ for all $S, T \in 2^N$. For an alternative approach to the criterion (B), see Ichiishi (1987b).

Proofs will be given in Section 4.

Throughout this paper the set of players will be assumed finite. But given the works of Schmeidler (1972) and Delbaen (1974), and also the mathematical foundations of Fan (1956), it is straightforward to generalize all the present results on the side-payment games to an arbitrary measurable space of players.

For another type of comparative cooperative game theory problem, in which effects of a change in the player-set are studied, see, e.g., Mo (1988), Alkan, Demange and Gale (1988), and Scotchmer and Wooders (1988).

2. SIDE-PAYMENT CASE

Let N denote a finite set of players, given throughout this paper, and let $\mathcal{N}$ denote the family of nonempty coalitions $2^N \setminus \{\phi\}$. A <u>side-payment game</u> (called simply a <u>game</u> in this section) is defined as a function $v : \mathcal{N} \to \mathbf{R}$; the value $v(S)$ is interpreted as the total payoff that coalition S can make independent of actions of the players outside S. For every payoff allocation $x \in \mathbf{R}^N$, define $x(S) := \sum_{j \in S} x_j$. The <u>core</u> of game v is the set $C(v) := \{x \in \mathbf{R}^N \mid x(N) \leq v(N)$, and for every $S \in \mathcal{N}$, $x(S) \geq v(S)\}$; it is the set of all payoff allocations that are feasible (via the grand coalition N) and stable (in that they cannot be improved upon by any coalition).

Let v be a game. The <u>efficiency cover of</u> v is the function $v^* : \mathcal{N} \to \mathbf{R}$ defined by:

$$v^*(S) := \max \{x(S) \mid x \in C(v)\}.$$

The <u>extended efficiency cover</u> of v is the function $v^* : \mathbf{R}^N_+ \to \mathbf{R}$ defined by

$$v^*(p) := \max \{p \cdot x \mid x \in C(v)\},$$

where $p \cdot x$ denotes the Euclidean inner product of p and x, $\sum_{j \in N} p_j x_j$. Since coalition $S \in \mathcal{N}$ is identified with its characteristic vector $\chi_S \in \mathbf{R}^N_+$, no confusion arises. The efficiency cover and the extended efficiency cover play crucial roles in the present study, so it would be useful to note:

Proposition 2.1. Let v be a side-payment game with a nonempty core, and let v^* be its extended efficiency cover. Then:

(i) $$v^*(p) = \min \left\{ \kappa v(N) - \sum_{S \in \mathcal{N}} \lambda_S v(S) \;\middle|\; \begin{array}{l} (\kappa, \lambda) \in R_+ \times R_+^{\mathcal{N}} \\ \kappa \chi_N - \sum_{S \in \mathcal{N}} \lambda_S \chi_S = p \end{array} \right\};$$

(ii) The function $v^* : R_+^N \to R$ is sublinear.

The first result (Theorem 2.2) provides a general characterization of the dominance of the core of game v over the core of another game w.

Theorem 2.2. Let v, w be side-payment games with a nonempty core, and let v^* and w^* be their extended efficiency covers respectively. Then, the following two conditions are equivalent:

(i) For every $y \in C(w)$ there exists $x \in C(v)$ such that $y \leq x$;

(ii) $w^*(p) \leq v^*(p)$, for every $p \in R_+^N$.

Corollary 2.3. Let v, w be side-payment games with a nonempty core. Then, any of the two conditions (i) and (ii) of Theorem 2.2 is satisfied, if for every $\lambda \in R_+^{\mathcal{N}}$ for which $\sum_{S \in \mathcal{N}} \lambda_S \chi_N \leq \chi_N$, it follows that $\sum_{S \in \mathcal{N}} \lambda_S (v(S) - w(S)) \leq v(N) - w(N)$.

In spite of the generality of Theorem 2.2, condition (ii) therein may not be practical, because one has to go through checking the continuum of inequalities parametrized by $p \in R_+^N$. For a certain class of games, the cardinality of the set of inequalities to check reduces down to $\#\mathcal{N}$

(finite). To clarify this class, several specific games and their properties are now recalled: A game v is called exact, if for every $S \in \mathcal{N}$ there exists $x \in C(v)$ such that $x(S) = v(S)$. For a game v with a nonempty core, one may assume without loss of generality that v is exact, since Schmeidler (1972) has established that the exact envelope $\bar{\bar{v}}$ of v defined by

$$\bar{\bar{v}}(T) := \max \left\{ \sum_{S \in \mathcal{N}} \lambda_S v(S) - \kappa v(N) \;\middle|\; \begin{array}{l} (\lambda, \kappa) \in \mathbf{R}_+^{\mathcal{N}} \times \mathbf{R}_+ \\ \sum_{S \in \mathcal{N}} \lambda_S \chi_S - \kappa \chi_N = \chi_T \end{array} \right\}$$

is an exact game, that $C(\bar{\bar{v}}) = C(v)$, and that v is exact iff $v = \bar{\bar{v}}$. A game v is called convex, if for any $S, T \in \mathcal{N}$, $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$, where $v(\phi) := 0$. Convexity characterizes increasing returns with respect to the coalition size (Shapley (1971)). A game v is called additive, if $v(S) = \sum_{j \in S} v(\{j\})$ for each $S \in \mathcal{N}$. The core of an additive game v is a singleton $\{x\}$, given by $x_j = v(\{j\})$. Additive games are convex, convex games are exact (Shapley (1971); see also Delbaen (1974)), and exact games have a nonempty core.

A new class of exact games is now introduced: An exact game v is called a no-gap game, if for every $p \in \mathbf{R}_+^N$ there exists $\lambda \in \mathbf{R}_+^{\mathcal{N}}$ such that $\sum_{S \in \mathcal{N}} \lambda_S \chi_S = p$ and $\sum_{S \in \mathcal{N}} \lambda_S v^*(S) \leq v^*(p)$. (The last inequality may be replaced by equality, due to Proposition 2.1 (ii).)

Proposition 2.4. Let v be an exact game, and let v^* be its efficiency cover. Then, the following two conditions are equivalent:

(i) Game v is a no-gap game;

(ii) For every $x \in R^N$ for which $x(S) \leq v^*(S)$ for every $S \in \mathcal{N}$, there exists $y \in C(v)$ such that $x \leq y$.

Proposition 2.5. Convex games are no-gap exact games.

Theorem 2.6. Let v be an exact game, and let v^* be its efficiency cover. Suppose on the one hand that v is a no-gap game. Let w be any side-payment game with a nonempty core and let w^* be its efficiency cover. Then, the following two conditions (i) and (ii) are equivalent:

(i) For every $y \in C(w)$ there exists $x \in C(v)$ such that $y \leq x$;

(ii) $w^*(S) \leq v^*(S)$, for every $S \in \mathcal{N}$.

Conditions (i) and (ii) are implied by the following condition (iii). If, moreover, game w is exact, then any of conditions (i) and (ii) implies condition (iii):

(iii) $v(S) - w(S) \leq v(N) - w(N)$, for every $S \in 2^N \backslash \{N\}$.

Suppose on the other hand, v is not a no-gap game. Then, there exists an additive game w (say, $w(S) = \sum_{j \in S} y_j$, so that $C(w) = \{y\}$) such that $w^*(S) \leq v^*(S)$ for every $S \in \mathcal{N}$, yet $y \not\leq x$ for any $x \in C(v)$.

Remark 2.7. An example of an exact game which is not a no-gap game: Let $N = \{1,2,3,4,5\}$. Let μ, υ, π be payoff vectors given as: $\mu = (0,5,3,2,2)$, $\upsilon = (5,0,5,0,2)$, and $\pi = (5,2,4,1,0)$. Define an exact game v by $v(S) := \min \{\mu(S), \upsilon(S), \pi(S)\}$ for every $S \in \mathcal{N}$. It will be proved in Section 4 that this game does not satisfy condition (ii) of Proposition 2.4. □

Remark 2.8. An example of a no-gap exact game which is not convex: Let $N = \{1,2,3,4\}$. Let $\mu = (4,2,4,2)$, and $\upsilon = (3,4,1,4)$, and define an exact game v by $v(S) := \min \{\mu(S), \upsilon(S)\}$ for every $S \in \mathcal{N}$. This example is due to Schmeidler (1972), who showed that it is not convex. It will be proved in Section 4 that this game satisfies the no-gap condition. □

Remark 2.9. There are two factors which cause the dominance of $C(v)$ over $\overline{C(w)}$;[2] they are the best captured when game v is no-gap exact and w is exact. One factor requires that the grand coalition is more efficient in game v than in game w, i.e., $v(N) - w(N)$ (the right-hand side of the inequality of condition (iii) of Theorem 2.6) is suitably big to guarantee the inequality. The other factor requires that the blocking power of each coalition S in game v is not overly big compared with that in game w, i.e., $v(S) - w(S)$ (the left-hand side) is suitably small to guarantee the inequality. □

The welfare criterion dual to the dominance criterion is now studied. Let v be a game with a nonempty core. The <u>extended exact envelope</u> of v is the function $\bar{\bar{v}} : R_+^N \to R$ defined by $\bar{\bar{v}}(p) := \min \{p \cdot x \mid x \in C(v)\}$. It is easy to check:

$$\bar{\bar{v}}(p) = v(N) - v^*(\chi_N - p), \quad \text{for every } p \in R^N \text{ such that } 0 \le p \le \chi_N.$$

(This identity is false for a non-balanced game.) In view of this identity, the following result which is dual to Theorem 2.2 can analogously be established:

Theorem 2.10. *Let* v, w *be side-payment games with a nonempty core, and let* $\bar{\bar{v}}$ *and* $\bar{\bar{w}}$ *be their extended exact envelopes respectively. Then, the following two conditions are equivalent:*

(i) *For every* $x \in C(v)$ *there exists* $y \in C(w)$ *such that* $y \leq x$;

(ii) $\bar{\bar{w}}(p) \leq \bar{\bar{v}}(p)$ *for every* $p \in R^N_+$.

Sharkey (1982) defined a game with a large core; a game v *has a large core*, if for every $z \in R^N$ for which $v(S) \leq z(S)$ for every $S \in \mathcal{N}$, there exists $x \in C(v)$ such that $x \leq z$. He showed that convex games have large cores, but not conversely. In view of Proposition 2.4, the exact games with a large core play precisely the same role in the dual result to Theorem 2.6 (as the no-gap exact games do in Theorem 2.6):

Theorem 2.11. *Let* v *be an exact game, and* w *be an exact game with a large core. Then, the following two conditions are equivalent:*

(i) *For every* $x \in C(v)$ *there exists* $y \in C(w)$ *such that* $y \leq x$;

(ii) $w(S) \leq v(S)$ *for every* $S \in \mathcal{N}$.

The final results reported in this section are characterizations of the *weak dominance* of $C(v)$ over $C(w)$. Extend an exact game w to R^N_+ by:

$$w(p) := \min \{p \cdot x \mid x \in C(w)\}.$$

Theorem 2.12. *Let* v *be a side-payment game with a nonempty core, and let* w *be an exact game. Then, the following two conditions are equivalent:*

(i) <u>There exist</u> $y \in C(w)$ <u>and</u> $x \in C(v)$ <u>such that</u> $y \leq x$;

(ii) $w(p) \leq v^*(p)$ <u>for every</u> $p \in R_+^N$.

Corollary 2.13. <u>Let</u> v, w <u>be two side-payment games</u>. <u>Then, the following two conditions are equivalent</u>.

(i) <u>There exist</u> $y \in C(w)$ <u>and</u> $x \in C(v)$ <u>such that</u> $y \leq x$;

(ii) <u>For any</u> $\lambda, \mu \in R_+^{\mathcal{N}}$ <u>for which</u> $\sum_{S \in \mathcal{N}} (\lambda_S + \mu_S)\chi_S = \chi_N$, <u>and for any</u> $\alpha \in R$ <u>for which</u> $\max_{j \in N} \sum_{S \ni j} \lambda_S \leq \alpha \leq 1$, <u>it follows that</u> $\sum_{S \in \mathcal{N}} (\lambda_S v(S) + \mu_S w(S)) \leq \alpha v(N) + (1-\alpha) w(N)$.

Corollary 2.14. <u>Let</u> v, w <u>be two convex games</u>. <u>Then, the following two conditions are equivalent</u>:

(i) <u>There exist</u> $y \in C(w)$ <u>and</u> $x \in C(v)$ <u>such that</u> $y \leq x$;

(ii) $w(S) + v(N \setminus S) \leq v(N)$, <u>for every</u> $S \in \mathcal{N}$.

Remark 2.15. Game v is said to <u>dominate</u> game w, if $w(S) \leq v(S)$ for all $S \in \mathcal{N}$. Dominance of game v over w may not result in weak dominance of the core C(v) over C(w); there is an abundance of such examples. Shift of a social system from w to a dominating system v may, therefore, result in "exploitation" of somebody in the society in the sense that for any $y \in C(w)$ and any $x \in C(v)$ there exists $j \in N$ such that $y_j > x_j$. If both game w and a dominating game v satisfy increasing returns (so that condition (ii) of Corollary 2.14 is obviously satisfied), however, shift from w to v always keeps the possibility of "no exploitation." □

3. NON-SIDE-PAYMENT CASE

A <u>non-side-payment game</u> (called simply a <u>game</u> in this section) is defined as a correspondence (a set-valued map) $V : 2^N \to \mathbf{R}^N$ such that $V(\phi) = \phi$ and such that for each $S \in \mathcal{N}$ the set $V(S)$ is a cylinder, i.e., $[u, v \in \mathbf{R}^N, \ \forall i \in S: \ u_i = v_i]$ implies $[u \in V(S)$ if and only if $v \in V(S)]$. The set $V(S)$, or rather its projection to $\mathbf{R}^S$, is interpreted as the set of all utility allocations attainable in coalition S. The games studied in this section are assumed to satisfy the following regularity conditions (1) - (3). Define $b \in \mathbf{R}^N$ by $b_j := \sup \{u_j \in \mathbf{R} \mid u \in V(\{j\})\}$.

$$V(S) - \mathbf{R}^N_+ = V(S), \quad \text{for every} \quad S \in \mathcal{N}; \tag{1}$$

There exists $M \in \mathbf{R}$ such that for every $S \in \mathcal{N}$,
$[u \in V(S), \ u \geq b]$ implies $[u_i < M$ for every $i \in S]$; (2)

$$V(N) \text{ is closed in } \mathbf{R}^N. \tag{3}$$

A game V is called <u>ordinal convex</u>, if for any $S, T \in \mathcal{N}$, $V(S) \cap V(T) \subset V(S \cap T) \cup V(S \cup T)$; it reduces to a convex game for the side-payment case.

The <u>core</u> of game V is the set $C(V) := \{u \in \mathbf{R}^N \mid u \in V(N)$, and it is not true that there exist $S \in \mathcal{N}$ and $v \in V(S)$ such that $u_i < v_i$ for all $i \in S\}$. It is the set of all feasible and stable utility allocations.

Theorem 3.1. Let V be a non-side-payment game, and let W be an ordinal convex game that satisfies (1) - (3). Then, the following condition (ii) implies (i):

(i) For every $x \in C(V)$ there exists $y \in C(W)$ such that $y \leq x$;

(ii) $W(S) \subset V(S)$ for every $S \in \mathcal{N}$.

Theorem 3.2. Let V, W be two ordinal convex games, each of which satisfies (1) - (3). Then, the following condition (ii) implies (i):

(i) There exist $y \in C(W)$ and $x \in C(V)$ such that $y \leq x$;

(ii) $W(N) \subset V(N)$, and for any $S, T \in \mathcal{N}$, $V(S) \cap W(T) \subset V(S \cap T) \cup W(S \cap T) \cup V(S \cup T) \cup W(S \cup T)$.

Proofs of the above Theorems 3.1 and 3.2 are based on the following extension (Theorem 3.3) of a theorem of Sharkey (1982), which is of interest in its own right. A game V is said to have a large core, if for any $z \in R^N$ for which it is not true that there exist $S \in \mathcal{N}$ and $u \in V(S)$ such that $z_i < u_i$ for every $i \in S$, there exists $x \in C(V)$ such that $x \leq z$.

Theorem 3.3. Any ordinal convex game that satisfies (1) - (3) has a large core.

4. PROOFS

Let v be a side-payment game with a nonempty core $C(v)$. Define

$A(v) := C(v) - \mathbf{R}^N_+$; and

$B(v) := \{x \in \mathbf{R}^N | \forall\ S \in \mathcal{N}:\ \chi_S \cdot x \leq v^*(S)\}$.

By the duality theorem for linear programming problems (see, e.g., Ichiishi (1983, Ch. 5, Exercise 4, p. 115)),

$$\forall p \geq 0: \quad \max\ \{p \cdot x \mid x \in B(v)\}$$
$$= \min\ \{\Sigma\lambda_S v^*(S) \mid \lambda \geq 0,\ \ \Sigma\lambda_S\chi_S = p\}. \tag{4}$$

<u>Proof of Proposition 2.1</u>. The number $v^*(p)$ is the optimal value of the linear programming problem,

Maximize $p \cdot x$,

subject to $\chi_S \cdot x \geq v(S)$ for every $S \in \mathcal{N}$,

and $-\chi_N \cdot x \geq -v(N)$.

So the duality theorem establishes assertion (i). Assertion (ii) is straightforward. Q.E.D.

<u>Proof of Theorem 2.2</u>. Condition (i) is equivalent to: $C(w) \subset A(v)$. This last condition is shown to be equivalent to condition (ii), by applying an elementary version of the separation theorem (see, e.g., Ichiishi (1983, Theorem 1.5.1, p. 18)) to the closed, convex, comprehensive set $A(v)$. (Here, a subset A of $\mathbf{R}^N$ is called comprehensive if $A = A - \mathbf{R}^N_+$.) Q.E.D.

Proof of Corollary 2.3. Obvious.

Proof of Proposition 2.4. The no-gap condition (i) means:

$$\forall p \geq 0 : \min \{\Sigma\lambda_S v^*(S) \mid \lambda \geq 0, \Sigma\lambda_S \chi_S = p\} \leq v^*(p).$$

By identity (4), this condition becomes:

$$\forall p \geq 0 : \max \{p\cdot x \mid x \in B(v)\} \leq \max \{p\cdot x \mid x \in A(v)\},$$

which in turn is equivalent to: $B(v) \subset A(v)$, by a separation theorem. This last condition is precisely condition (ii). Q.E.D.

Proof of Proposition 2.5. Let v be a convex game. Choose any $p \in R_+^N$, and let $\{\mu_i, T_i\}_{i=1}^s$ be the canonical form of p, viz.,

$$T_1 \supsetneqq T_2 \supsetneqq \cdots \supsetneqq T_s,$$

$$\mu_i > 0, \quad i = 1,\ldots,s, \quad \text{and}$$

$$\sum_{i=1}^{s} \mu_i \chi_{T_i} = p.$$

By Shapley (1971), there exists $x \in C(v)$ such that $x(N\backslash T_i) = v(N\backslash T_i)$ for all i, or equivalently such that

$$x(T_i) = v(N) - v(N\backslash T_i) = v^*(T_i), \quad i = 1,\ldots,s.$$

Then,

$$\sum_{i=1}^{s} \mu_i v^*(T_i) = \sum_{i=1}^{s} \mu_i x(T_i) = p\cdot x \leq v^*(p).$$ Q.E.D.

Proof of Theorem 2.6. Suppose v is a no-gap, exact game. Due to Theorem 2.2, one only needs to show that (ii) implies (i). Suppose there exists $y \in C(w)$ such that $y \not\lesssim x$ for any $x \in C(v)$. Then, $y \notin A(v)$, so (by Proposition 2.4 (ii)) $y \notin B(v)$. Consequently, there exists $S \in \mathcal{N}$ such that $w^*(S) \geq y(S) > v^*(S)$, which contradicts (ii). Assertions about condition (iii) are straightforward.

Suppose v is not a no-gap game. Then, (by Proposition 2.4 (ii)) there exists $y \in B(v) \backslash A(v)$. Define an additive game w by $w(\{j\}) = y_j$. Game w satisfies all the required properties. Q.E.D.

Proof That the Example of Remark 2.7 Does Not Satisfy Condition (ii) of Proposition 2.6. Consider $x := (2,5,3,1,0) \in R^N$. A game w is exact, iff there exists a family of additive games $\{\mu^i \mid i \in I\}$ satisfying $\mu^i(N) = \mu^j(N)$ for all $i, j \in I$, such that $w(S) = \min \{\mu^i(S) \mid i \in I\}$ for every $S \in \mathcal{N}$. In this case, $\mu^i \in C(w)$ for all $i \in I$, and $w^*(S) = \max \{\mu^i(S) \mid i \in I\}$ for all $S \in \mathcal{N}$. Given this general fact, it is routine to verify in the present example, $x(S) \leq v^*(S)$ for every $S \in \mathcal{N}$, so $x \in B(v)$. Notice:

$$
\begin{aligned}
x_2 + x_3 &= 8 = v^*(\{2,3\}) \\
x_1 + x_2 + x_4 &= 8 = v^*(\{1,2,4\}) \\
x_1 + x_2 + x_5 &= 7 = v^*(\{1,2,5\}) \\
x_1 + x_2 + x_3 + x_4 + x_5 &= 11 < v^*(\{1,2,3,4,5\}).
\end{aligned}
$$

Therefore, there exists no $y \geq x$ satisfying $y(S) \leq v^*(S)$ for $S = \{2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$ and $y(N) = v^*(N)$. So, $x \notin A(v)$. Q.E.D.

Proof That the Example of Remark 2.8 Is a No-Gap Game. It suffices to show:

$$\forall p \geq 0: \exists \ \lambda^p \in R_+^{\mathcal{N}}: \exists \ x^p \in C(v):$$

$$\Sigma \lambda_S^p \chi_S = p, \quad \text{and} \quad \Sigma \lambda_S^p v^*(S) = p \cdot x^p. \tag{5}$$

Due to the symmetry in players 2 and 4 in this example, one may assume without loss of generality $p_2 \leq p_4$. One may also assume $p_m := \min \{p_j \mid j \in N\} = 0$. Indeed, suppose $p_m > 0$. Then, define $q \in R_+^N$ by: $q_j := p_j - p_m$ for every j. Since $q_m = 0$, there exist $\lambda^q \in R_+^{\mathcal{N}}$ and $x^q \in C(v)$ satisfying condition (5) for q. Notice $\lambda_N^q = 0$. Define $\lambda^p \in R_+^{\mathcal{N}}$ by: $\lambda_S^p := \lambda_S^q$ if $S \neq N$; and $\lambda_N^p := p_m$. Then, (λ^p, x^q) satisfies condition (5) for the given p. Now, condition (5) is verified for various cases according to the order among $\{p_1, p_2, p_3, p_4\}$. If $0 = p_1 \leq p_2 \leq p_3 \leq p_4$, for example, define λ^p and x^p by: $\lambda_{\{4\}}^p := p_4 - p_3$; $\lambda_{\{3,4\}}^p := p_3 - p_2$; $\lambda_{\{2,3,4\}}^p := p_2$; $\lambda_S^p := 0$ for all other S; $x^p := (3,3,2,4)$. It is routine to verify that λ^p, x^p are the vectors required for (5). One can verify (5) analogously for the other cases, although it is tedious. Q.E.D.

Proofs of Theorems 2.10 and 2.11. Analogous to the proofs of Theorems 2.2 and 2.6. Q.E.D.

Proof of Theorem 2.12. The negation of condition (i) says $A(v) \cap C(w) = \phi$, which is equivalent to : $\exists\, p \in R^N_+\setminus\{0\}$: $\max p\cdot A(v) < \min p\cdot C(w)$. The last condition is the negation of condition (ii), since w is exact. Q.E.D.

Proof of Corollary 2.13. Notice first that condition (ii) implies that both w and v are balanced. By substituting the exact envelope formula for $w(p)$ and the extended efficiency cover formula for $v^*(p)$ in condition (ii) of Theorem 2.12, condition (i) of Corollary 2.13 is equivalent to:

$$(\forall\, p \in R^N_+):$$

$$(\forall\, (\lambda, \alpha) \in R^{\mathcal{N}}_+ \times R_+ : \quad -\Sigma\, \lambda_S \chi_S + \alpha\, \chi_N = p) :$$

$$(\forall\, (\mu, \beta) \in R^{\mathcal{N}}_+ \times R_+ : \quad \Sigma\, \mu_S \chi_S - \beta\, \chi_N = p) :$$

$$\Sigma\, \mu_S w(S) - \beta\, w(N) \le -\Sigma\, \lambda_S v(S) + \alpha v(N).$$

It is routine to verify that this last condition is equivalent to condition (ii) of Corollary 2.13. Q.E.D.

Proof of Corollary 2.14. It suffices to show that condition (ii) of Corollary 2.14 implies condition (ii) of Theorem 2.12. The former is equivalent to:

$$w(S) \le v^*(S), \quad \text{for every } S \in \mathcal{N}.$$

Choose any $p \geq 0$, and let $\{\mu_i, T_i\}^S_{i=1}$ be the canonical form of p (see Proof of Proposition 2.5 for the definition). Then,

$$w(p) = \Sigma\, \mu_i\, w(T_i), \text{ and}$$

$$v^*(p) = \Sigma\, \mu_i\, v^*(T_i).$$

Condition (ii) of Theorem 2.12 now follows. Q.E.D.

Given a subset X of $\mathbb{R}^N$, denote by $\bar{X}$ (by $\mathring{X}$, resp.) the closure (the interior, resp.) of X in $\mathbb{R}^N$. In the proofs of the theorems for the non-side-payment case, the following identity under condition (1) will simplify the notation:

$$\mathring{V}(S) = \{x \in V(S) \mid \exists\, u \in V(S) : \forall\, i \in S : x_i < u_i\}.$$

<u>Derivation of Theorem 3.1 from Theorem 3.3</u>. Choose any $x \in C(V)$. Then, $x \notin \mathring{V}(S)$ for any $S \in \mathcal{N}$, so $x \notin \mathring{W}(S)$ for any $S \in \mathcal{N}$ by condition (ii). Since game W has a large core, there exists $y \in C(W)$ such that $y \leq x$. Q.E.D.

<u>Derivation of Theorem 3.2 from Theorem 3.3</u>. Define the non-side-payment game U by: $U(S) := V(S) \cup W(S)$, for every $S \in \mathcal{N}$. Under condition (ii), it is routine to verify that game U is ordinal convex, satisfies (1) - (3), and that $U(N) = V(N)$. By Peleg (1982), $C(U) \neq \phi$. Choose any $x \in C(U)$. On the one hand, $x \in C(V)$. On the other hand, $x \notin \mathring{U}(S)$ so that $x \notin \mathring{W}(S)$ for any $S \in \mathcal{N}$, which guarantees (by the large core property of W) the existence of $y \in C(W)$ such that $y \leq x$. Q.E.D.

It remains to prove Theorem 3.3. For any $x \in \mathbb{R}^N$ and any $S \subset N$, set $x^S := (x_i)_{i \in S} \in \mathbb{R}^S$. Given any non-side-payment game $V : 2^N \to \mathbb{R}^N$, and any $T \in \mathcal{N} \setminus \{N\}$, the associated <u>subgame</u> $V_T : 2^T \to \mathbb{R}^T$ is defined by: $V_T(\phi) :=$

ϕ, $V_T(S) := \{x^T \in R^T \mid x \in V(S)\}$ if $S \subset T$ and $\phi \neq S \neq T$, and $V_T(T) :=$ the closure of $\{x^T \in R^T \mid x \in V(T)\}$ in R^T. Set $M := N \setminus T$, and choose any $z^T \in V_T(T)$. The associated reduced game $V^*_M : 2^M \to R^M$ is defined by: $V^*_M(\phi) := \phi$, $V^*_M(S) := \bigcup_{R \subset T} \{x^M \in R^M \mid x \in V(S \cup R)$, and $\forall\, i \in T : x_i > z^T_i\}$ if $S \subset M$ and $\phi \neq S \neq M$, and $V^*_M(M) := \{x^M \in R^M \mid (x^M, z^T) \in V(N)\}$. If a game satisfies (1) - (3), then any of its subgames and reduced games satisfies (1) - (3). The following two lemmas are due to Peleg (1986, pp. 85-86). (The second of these lemmas, Lemma 4.2, is stated slightly differently from the corresponding lemma of Peleg (1986, Lemma 2.7), but there is no difference in the proofs.)

Lemma 4.1 (Peleg 1986). Let $V : 2^N \to R^N$ be an ordinal convex game that satisfies (1) - (3). Let $T \subset N$, $\phi \neq T \neq N$, let $z^T \in V_T(T)$, and let $M := N \setminus T$. If $z^T \in C(V_T)$ and for $R \subset T$, $R \neq T$, $z^T \notin \overline{V_T(R)}$, then V^*_M is ordinal convex.

Lemma 4.2 (Peleg 1986). Under the assumptions of Lemma 4.1, choose any $y^M \in C(V^*_M)$ and any $h \in R^M_+ \setminus \{0\}$. Let $t^* := \max\{t \in R_+ \mid (y^M + t\,h, z^T) \in V(N)\}$, and set $x := (y^M + t^*\,h, z^T)$. Then $x \in C(V)$.

Proof of Theorem 3.3. By induction on #N. The theorem is trivial for a one-person game. Choose any integer $n \geq 2$. Assume that the theorem is true for all k-person games with $k \leq n - 1$. Let N be a player set such that $\#N = n$, and let $V : 2^N \to R^N$ be an ordinal convex game that satisfies (1) - (3). Choose any $x \in R^N \setminus \bigcup_{S \in \mathcal{N}} \mathring{V}(S)$. Then there exist $z \leq x$ and $T \in \mathcal{N}$ such that $z \notin \bigcup_{S \in \mathcal{N}} \mathring{V}(S)$, but $z \in \overline{V(T)}$. If $z \in \overline{V(N)}$ (= $V(N)$), then $z \in C(V)$, so there is nothing to prove. Assume, therefore,

$$z \notin V(N). \tag{6}$$

Without loss of generality, T can be assumed to be a minimal set having the property that $z \in \overline{V(T)}$. Then,

$z^T \in C(V_T)$,

$z^R \notin \overline{V(R)}$ for all $R \subset T$, $R \neq T$, and

$\phi \neq T \neq N$, in view of (6).

Set $M := N \setminus T$. Consider the reduced game V^*_M given z^T; it is ordinal convex by Lemma 4.1.

One now claims that $z^M \notin \mathring{V}^*_M(S)$ for any $S \subset M$. Indeed, suppose the contrary; i.e., $\exists\, S \subset M : z^M \in \mathring{V}^*_M(S)$. Then, there exists $y^M \in V^*_M(S)$ such that $y^M_i > z^M_i$ for all $i \in M$. If $S \neq M$, then there exist $R \subset T$ and y^T such that $(y^M, y^T) \in V(S \cup R)$ and $y^T_i > z^T_i$ for all $i \in T$, so that $(z^M, z^T) \in \mathring{V}(S \cup R)$ -- a contradiction. If $S = M$, then $(y^M, z^T) \in V(N)$, so $(z^M, z^T) \in V(N)$ -- a contradiction of (6). The claim is thus proved.

By the inductive hypothesis, there exists $y^M \in C(V^*_M)$ such that $y^M \leq z^M \leq x^M$. If $y^M = x^M$, then $y^M = z^M$, so $(z^M, z^T) \in V(N)$ -- a contradiction of (6). Therefore, $h := x^M - y^M \in R^M_+ \backslash \{0\}$. Choose t^* as in Lemma 4.2. Then $(y^M + t^* h, z^T) \in C(V)$. It remains to show that $y^M + t^* h \leq x^M$, or equivalently that $t^* \leq 1$. Suppose $t^* > 1$. Then $y^M + t^* h \geq x^M$. In view of $(y^M + t^* h, z^T) \in V(N)$, it follows that $(x^M, z^T) \in V(N)$. Then, $(z^M, z^T) \in V(N)$ -- a contradiction of (6). Q.E.D.

FOOTNOTE

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REFERENCES

Alchian, A. A., and Demsetz, H. (1972) Production, information costs, and economic organization. American Economic Review 62: 777-795.

Alkan, A., Demange, G., and Gale, D. (1988) Rawlsian justice and fair allocation of indivisible goods (Abstract). mimeo, April.

Arrow, K. J. (1974) The limits of organization. Norton. New York.

Bondareva, O. N. (1962) Teoriia iadra v igre n lits. Vestnik Leningrad Univ. Math 13: 141-142.

Coase, R. H. (1937) The nature of the firm. Economica 4: 386-405.

Delbaen, F. (1974) Convex games and extreme points. Journal of Mathematical Analysis and Applications 45: 210-233.

Fan, K. (1956) On systems of linear inequalities. In Kuhn, H. W. and Tucker, A. W. (eds) Linear inequalities and related systems, pp. 99-156. Princeton University Press. Princeton.

Ichiishi, T. (1981) Super-modularity: Applications to convex games and to the greedy algorithm for LP. Journal of Economic Theory 25: 283-286.

Ichiishi, T. (1982) Management versus ownership, I. International Economic Review 23: 323-336.

Ichiishi, T. (1983) Game theory for economic analysis. Academic Press. New York.

Ichiishi, T. (1985) Management versus ownership, II. European Economic Review 27: 115-138.

Ichiishi, T. (1987a) A note on welfare comparison of two side-payment games. CORE Discussion Paper No. 8741, CORE, Université Catholique de Louvain, August.

Ichiishi, T. (1987b) Weak dominance of cores. Working Paper No. 87-05, Department of Economics, The Ohio State University, August.

Mo, J.-P. (1988) Entry and structures of interest groups in assignment games. Journal of Economic Theory

Peleg, B. (1982) Convex effectivity functions. Research Memorandum No. 46, Center for Research in Mathematical Economics and Game Theory, The Hebrew University of Jerusalem, April.

Peleg, B. (1986) A proof that the core of an ordinal convex game is a von Neumann-Morgenstern solution. Mathematical Social Sciences 11: 83-87.

Rosenmüller, J. (1971) On core and value. Operations Research Verfahren **9**: 84-101.

Schmeidler, D. (1967) On balanced games with infinitely many players," Research Program in Game Theory and Mathematical Economics, Research Memorandum No. 28, Department of Mathematics, The Hebrew University of Jerusalem.

Schmeidler, D. (1972) Cores of exact games, I. Journal of Mathematical Analysis and Applications **40**: 214-225.

Scotchmer, S., and Wooders, M. H. (1988) Monotonicity in games that exhaust gains to scale. mimeo, March.

Sertel, M. R. (1982) Workers and incentives. North-Holland. Amsterdam.

Shapley, L. S. (1967) On balanced sets and cores. Naval Research Logistics Quarterly **14**: 453-460.

Shapley, L. S. (1971) Cores of convex games. International Journal of Game Theory **1**: 11-26.

Sharkey, W. W. (1982) Cooperative games with large cores. International Journal of Game Theory 11: 175-182.