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ON HOMOGENEOUS WEIGHTS
FOR SIMPLE GAMES

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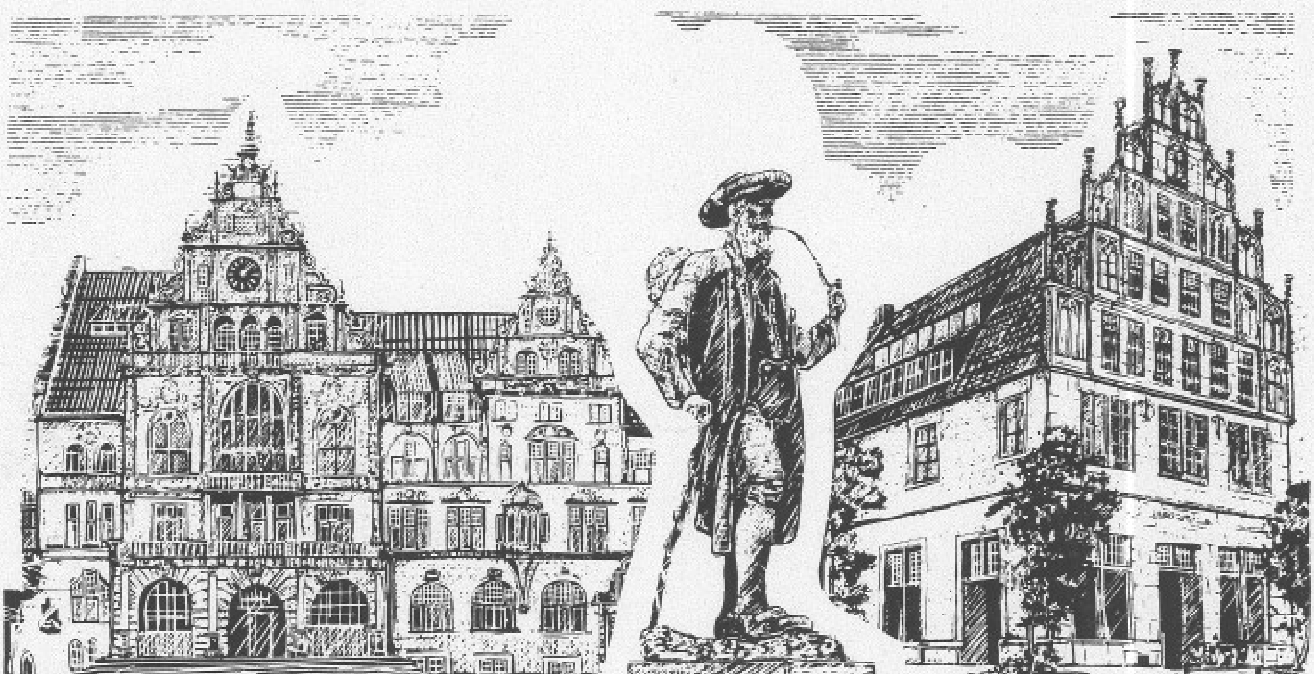

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§ 0 INTRODUCTION

Let $\Omega = \{1,\ldots,n\}$. A probability m on Ω (i.e., a vector $m = (m_\omega)_{\omega\in\Omega}$, $m \geq 0$, $\sum_{\omega\in\Omega} m_\omega = 1$, regarded as a set function on $\mathfrak{P}(\Omega)$ via $m(S) = \sum_{\omega\in S} m_\omega$ $(S \subseteq \Omega)$) is said to be *homogeneous* w.r.t. $\alpha\in(0,1)$, if, for every $S \subseteq \Omega$, $m(S) > \alpha$, there is $T \subseteq S$ such that $m(T) = \alpha$.

The notion has been introduced by von NEUMANN-MORGENSTERN [1] in the framework of Game Theory. For, if Ω represents the "players", $\mathfrak{P}(\Omega)$ the "system of coalitions", m_ω $(\omega\in\Omega)$ the "(relative) voting power" of player ω , and α the "majority level" (assuming frequently $\alpha > \frac{1}{2}$) then the function

$$v = 1_{[\alpha,1]} \circ m$$

$$v : \mathfrak{P}(\Omega) \to \{0,1\}$$

represents a "simple game" ([1] , see also SHAPLEY [8] and PELEG [2], or [4]), in the sense that $\{S \mid m(S) \geq \alpha\} = \{S \mid v(S) = 1\} \subseteq \mathfrak{P}(\Omega)$ is the system of "winning coalitions".

If "dummies get zero voting power" (see [2] [4] and the game is superadditive $(\alpha > \frac{1}{2})$ and zero-sum $(v(S) + v(S^c) = 1, S\in\mathfrak{P}(\Omega))$), then v is *uniquely represented* by m and α . On the other hand, "dummies get zero voting power" is a sufficient condition in order to derive that m is *nondegenerate* with respect to α (see [3] [4] [5] [6]) and thus, m and α are rationals.

For all practical purposes it is, therefore, sufficient to study an integer valued measure $M = (M_\omega)_{\omega\in\Omega}$, $(M_\omega \in \mathbb{N}\ (\omega\in\Omega))$ on Ω and to cosider the problem that M is *homogeneous* w.r.t. $\lambda \in \mathbb{N}$, i.e.

$$S \subseteq \Omega,\ M(S) > \lambda\ \exists\ T \subseteq S : M(T) = \lambda.$$

It is the aim of this paper to specify conditions for M and λ that are necessary and sufficient for homogeneity.

Game theoretically a list of "homogeneous pairs" (M,λ) seems to be desirable in order to enlarge our knowledge of simple games (again see SHAPLEY [8]). However, homogeneity should also be seen in connection with nondegeneracy. The latter concept has been studied in several papers by H.G. WEIDNER and the author [5], [6] see also [3] [4], and there is ample evidence that nondegeneracy is a kind of surrogate for nonatomicity (in the case that $\Omega = [0,1]$ and m a measure on the Borelian sets) (see in particular [3]). Clearly, a nonatomic measure on $[0,1]$ is "homogeneous" as well as "nondegenerate" with respect to any $\alpha \in (0,1)$ (and vice versa) and so homogeneity should also be regarded as a finite surrogate for nonatomicity. That this is more than just a superficial similarity, has been exhibited in the study of "extreme" set functions [3].

It remains to exhibit the number theoretical or combinatorical similarities and differences between both concepts. The treatment of nondegeneracy leads to the study of "(g,k)-representations" of $\lambda \in \mathbb{N}$ (see [5] and [6]) and the main result of this paper is that the same is true for homogeneity. Hence, both concepts, apart from their similarity when compared to the nonatomic case as well as in applications also grow from identical number theoretical roots.

§ 1 ON THE DISTRIBUTION OF PLAYERS IN MINIMAL WINNING COALITIONS

Let $\Omega = \{1,\dots,n\}$ and let

$$\Omega = K_1 + \dots + K_r$$

represent a decomposition of Ω into nonempty subsets K_i, $i = 1,\dots,r$, of Ω. ("+" standing for "disjoint union"). If

$$g = (g_1,\dots,g_r) \in \mathbb{N}^r \quad ,$$

then an integer-valued measure $M > 0$ on Ω is specified via

$$M(S) = \sum_{i=1}^{r} | S \cap K_i | \, g_i \; .$$

On the other hand, M specifies a decomposition of Ω and a vector $g \in \mathbb{N}^r$; intuitively, K_i represents the "players of type $i \in \{1,\dots,r\}$ " and g_i is the "weight" of players of type i .

Permutations of the "types" (i.e. of $\{1,\dots,r\}$) and of all members of a type (i.e. of some K_i) do not alter any homogeneity M might enjoy w.r.t. some $\lambda \in \mathbb{N}$, thus, if we write

$$k_i := | K_i | \in \mathbb{N}$$
$$k = (k_1,\dots,k_r) \in \mathbb{N}^r ,$$

then M and $(g,k) \in \mathbb{N}^{2r}$ determine each other up to permuations (provided of course $\sum_{i=1}^{r} k_i = n$).

Definition 1.1. An integer valued measure $M > 0$ (or a corresponding pair $(g,k) \in \mathbb{N}^{2r}$) is said to be *homogeneous* w.r.t. $\lambda \in \mathbb{N}$ (written "M hom λ" or "(g,k) hom λ") if, for $S \subseteq \Omega$, $M(S) \geq \lambda$ there is $T \subseteq S$ s.t. $M(T) = \lambda$.

(By formal reasons we admit $M(S) = \lambda$, $T = S$).

Throughout the following we shall *always* assume that

$$0 < g_1 < g_2 < \dots < g_r$$

holds true. Then $g \in \mathbb{N}^r$ induces a system of natural numbers

$$l_{ij} \qquad (i,j = 1,\dots,r;\ i < j),$$

defined by

$$l_{ij}\, g_i \leq g_j < (l_{ij} + 1) g_i ,$$

which will be used frequently without further reference.

The following lemma reduces the possibilities of "representing" a natural number λ by the weights ρ_i given the information that M hom λ.

Lemma 1.2. If M hom λ, then there is $i_0 \in \{1,\dots,r\}$ and $c \in \mathbb{N}$, $1 \leq c \leq k_{i_0}$, such that the following holds true:

(1) $$\lambda = c g_{i_0} + \sum_{i=i_0+1}^{r} k_i\, g_i \quad ;$$

(2) $$k_{i_0} \leq c + l_{i_0 i} \quad \text{for all}$$

$i \in \{i_0 + 1,\dots,r\}$ satisfying $g_{i_0} \nmid g_i$;

(3) $k_i \leq l_{ii_o}$ for all

$i \in \{1,\ldots,i_o-1\}$ satisfying $g_i \not| g_{i_o}$.

<u>Proof</u> Choose i_o to be maximal such that

$$\lambda \leq \sum_{i=i_o}^{r} k_i g_i$$

and, thereafter, let c be minimal such that

(4)
$$\lambda \leq cg_{i_o} + \sum_{i=i_o+1}^{r} k_i g_i$$

holds true. If, by this procedure, the $=$-sign is obtained in (4), then we are done as far as the first part of our assertions (i.e., formula (1)) is satisfied; otherwise we have

(5)
$$(c-1)g_{i_o} + \sum_{i=i_o+1}^{r} k_i g_i < \lambda < cg_{i_o} + \sum_{i=i_o+1}^{r} k_i g_i \quad .$$

The right hand side defines a set $T \subseteq \Omega$ the elements of which have at least weight g_{i_o} . But if we take weight g_{i_o} (or more) out of this set, then its measure will fall below of λ , contradicting homogeneity. This proves (1).

Next, if $(k_{i_o} - c) > l_{i_o i_1} + 1$ for some $i_1 > i_o$, $g_{i_o} \not| g_{i_1}$, then we may construct a set $S \subseteq \Omega$ such that

$$M(S) = \lambda + (l_{i_o} + 1)g_{i_o} + cg_{i_o}$$

(6)
$$+ \sum_{\substack{i=i_o+1 \\ i \neq i_1}}^{r} k_i g_i + (k_{i_1} - 1)g_{i_1}$$

$$= \lambda + (l_{i_o} + 1)g_{i_o} - g_{i_1} > \lambda$$

And as $M(S) - \lambda < g_{i_o}$, no element of this set may be removed without its measure falling below of λ , contradicting homogeneity. This verifies (2).

The meaning of this precedure is obvious: the representation

$$\lambda = cg_{i_o} + \sum_{i=i_o+1}^{r} k_i g_i$$

implies the construction of a "minimal winning coalition". If there are enough "players" of type i_o left in order to replace a player of type i_1 , i.e., if

$$(k_{i_o} - c) > l_{i_o i_1} + 1$$

$$(k_{i_o} - c)g_{i_o} > (l_{i_o i_1} + 1)g_{i_o} \geq g_{i_1}$$

then they must be capable of exactly imitating his weight (i.e. $g_{i_o} | g_{i_1}$) - otherwise homogeneity is violated.

Clearly, (3) is checked analogously. (Small players must be able to imitate players of weight g_{i_o} exactly if there are sufficiently many of them.) There is an obvious generalization to which we shall return later on.

Let $M = \sum_{i=1}^{r} | \cdot \cap K_i | g_i$ be an integer-valued measure on $\Omega = K_1 + \ldots + K_r$. Fix some $\rho \in \{1,\ldots,r\}$ and let $d \in \mathbb{N} + \{0\}$ be such that $0 \leq d < g_\rho$. As (g,k) corresponds to M, we want to consider a truncated version of M, say $M^{\rho d}$, corresponding to $((g_1,\ldots,g_\rho), (k_1,\ldots,k_{\rho-1},d))$. This may be done by specifying any $D \subseteq K_\rho$, $|D| = d$ and defining $M^{\rho d}$ by

$$M^{\rho d}(S) := \sum_{i=1}^{\rho-1} | S \cap K_i | g_i + | S \cap D | g_\rho .$$

Of course, $M^{\rho d}$ should carry an index D, however we refer to our previous remarks concerning the relations between M and (g,k). Also we accept a slight deviation from our previous viewpoint by admitting that $d = 0$.

As a further notational convenience we shall use $m = M(\Omega)$, $\tilde{m} = \tilde{M}(\Omega)$, $\hat{m} = \hat{m}(\Omega)$, $m^{\rho d} = M^{\rho d}(\Omega)$ etc. We have now

Lemma 1.3. Let M hom λ. Suppose, there is $i_0 \in \{1,\ldots,r\}$ and $a_i \in \mathbb{N}$, $1 \leq a_i \leq k_i$, $(i = i_0,\ldots,r)$ such that

$$\lambda = \sum_{i=i_0}^{r} a_i g_i .$$

Then $M^{i_0, k_{i_0}-a_{i_0}}$ hom g_i for all $i \in \{i_0,\ldots,r\}$ satisfying $m^{i_0, k_{i_0}-a_{i_0}} \geq g_i$.

<u>Proof</u> Write $\hat{M} := M^{i_o . k_{i_o} - a_{i_o}}$ and pick $i_1 \in \{i_o, \ldots, i_r\}$ such that $\hat{m} > g_{i_1}$. Assume that $\hat{M}$ hom g_{i_1} is <u>not</u> true.

Let $T \subseteq \Omega$ be a set such that

$$| T \cap K_i | = a_i \quad (i \geq i_i)$$

$$| T \cap K_i | = \emptyset \quad (i < i_i)$$

such that $M(T) = \lambda = \sum_{i=i_o}^{r} a_i \, g_i$.

Now there is $S \subseteq K_1 + \ldots + K_{i_o}$ such that

$$\hat{M}(S) > g_{i_1}, \; \hat{M}(S-\omega) < g_{i_1}$$

for every $\omega \in S$; clearly $M(S) - g_{i_1} \leq g_{i_o}$.

Next let

$$\hat{T} = T + S - \{\text{one element of } K_{i_1}\}$$

such that

$$M(\hat{T}) = \lambda + M(S) - g_{i_1}$$

and

$$0 < M(\hat{T}) - \lambda \leq g_{i_o}$$

Therefore, in order to "cut down" $\hat{T}$ to measure λ we have to remove necessarily elements of S, i.e. we find $\hat{S} \subseteq S$ s.t.

$$M\,(T + \hat{S} - \{\text{one element of } K_{i_1}\}) = \lambda$$

which amounts to $M(\hat{S}) = \lambda$, a contradiction. q.e.d.

Lemma 1.4. Let $M = \sum_{i=1}^{r} | \cdot \cap K_i | \, g_i$ be an integer valued measure on $\Omega = K_1 + \ldots + K_r$. Suppose, there is $i_o \in \{1, \ldots, r\}$ and $c \in \mathbb{N}$, $1 \leq c \leq k_{i_o}$, such that the following is satisfied:

(7) $\lambda = c\, g_{i_o} + \sum_{i=i_o+1} k_i\, g_i$;

(8) $k_{i_o} \leq c + l_{i_o i}$ for all

$i \in \{i_o+1,\ldots,r\}$ satisfying $g_{i_o} \nmid g_i$;

(9) $M^{i_o, k_{i_o}-c}$ hom g_i for all

$i \in \{i_o,\ldots,r\}$ satisfying $m^{i_o, k_{i_o}-c} \geq g_i$.

Then M hom λ holds true.

Proof For the sake of convenience, let us write

$$R_o := \{i \mid i \geq i_o + 1 \text{ , } g_{i_o} \nmid g_i\}$$

$$R^o := \{i \mid i \geq i_o + 1 \text{ , } g_{i_o} \mid g_i\}$$

Now, pick $T \subseteq \Omega$ such that $M(T) > \lambda$.

1st Step: Assume in addition that

$T \cap (K_1 +\ldots+ K_{i_o-1}) = \emptyset$. In this case

(10) $M(T) = \sum_{i=i_o}^{r} b_i\, g_i$

where $0 \leq b_i \leq k_i$ $(i = i_o,\ldots r)$.

Now, for $i \in R_o$ we have in view of (8)

$$b_{i_o} - c \leq k_{i_o} - c \leq l_{i_o i}$$

and hence

(11) $$(b_{i_o}-c)\ g_{i_o} \leq l_{i_o i}\ g_{i_o} < g_i \ .$$

As $M(T) > \lambda$, an inspection of (7) and (10) reveals that (11) implies

(12) $$b_i = k_i \qquad (i \in R_o)\ ,$$

i.e. $k_i \subseteq T \quad (i \in R_o)$. Now

$$b_{i_o} g_{i_o} + \sum_{i\in R^o} b_i g_i + \sum_{i\in R_o} b_i g_i$$

$$= M(T) > \lambda = c\ g_{i_o} + \sum_{i=i_o+1}^{r} k_i g_i$$

implies

$$b_{i_o} g_{i_o} > c\ g_{i_o} + \sum_{i\in R^o} (k_i-b_i)\ g_i$$

$$= c\ g_{i_o} + \sum_{i\in R^o} (k_i-b_i)\ l_{i_o i}\ g_{i_o}$$

or

$$b_{i_o} > c + \sum_{i\in R^o} (k_i-b_i)\ l_{i_o i} \quad .$$

Hence, there is a subset of $T \cap K_{i_o}$, say $V_o \subseteq T \cap K_{i_o}$, such that

$$|\ V_o\ | = c + \sum_{i\in R^o} (k_i-b_i)\ l_{i_o i} \quad .$$

Define

$$U := V_o + \sum_{i\in R^o} (K_i \cap T) + \sum_{i\in R_o} K_i \quad ,$$

then $U \subseteq T$ and the measure of U is computed to be

$$M(U) = \left(c + \sum_{i\in R^o} (k_i - b_i) l_{i_o i} \right) g_{i_o} +$$

$$+ \sum_{i\in R^o} b_i\, g_i + \sum_{i\in R_o} k_i\, g_i$$

$$= c\, g_{i_o} + \sum_{i\in R^o} (k_i - b_i)\, g_i + \sum_{i\in R^o} b_i\, g_i + \sum_{i\in R_o} k_i\, g_i$$

$$= \lambda \qquad ,$$

which finishes our first step.

2nd Step: By the same argument as in the first step we may assume that

(13) $$M(T \cap (K_{i_o} + \ldots + K_r)) < \lambda \;.$$

For, if $>$ holds true, the procedure exhibited in the first step is applied to $T \cap (K_{i_o} + \ldots + K_r)$ (- and if = is the case we are already finished with our proof).

Also, there is no loss of generality in assuming that

(14) $$M(T) - \lambda \leq g_{i_o - 1} \quad ,$$

for otherwise we remove elements from $T \cap (K_1 + \ldots + K_{i_o - 1})$ until either the procedure of the first step applies or (14) becomes true.

3rd Step: It is sensible to introduce the notations

$$T_o := T \cap (K_i + \ldots + K_{i_o - 1})$$

$$T^o := T \cap (K_{i_o} + \ldots + K_r)$$

as well as

$$\hat{M} := M^{i_o, k_{i_o} - c} \quad , \quad M^o := M^{i_o,\, 0} \quad .$$

Now, we proceed as follows. First of all we have

$$\lambda < M(T) = M^o(T_o) + b_{i_o} g_{i_o} + \sum_{i=i_o+1}^{r} b_i g_i$$

(15)

$$\overset{(14)}{<} \lambda + g_{i_o}$$

and hence

$$M^o(T_o) > \lambda - b_{i_o} g_{i_o} - \sum_{i=i_o+1}^{r} b_i g_i \overset{(13)}{>} 0 \ .$$

Replacing λ via (7), we obtain from this:

$$(16) \qquad M^o(T_o) + (b_{i_o} - c)\, g_{i_o} > \sum_{i=i_o+1}^{r} (k_i - b_i)\, g_i \ .$$

The remaining two steps distinguish two cases according to whether $b_{i_o} > c$ or $b_{i_o} < c$.

4th Step: Assume first $b_{i_o} > c$.

Consider the left side of (16). As we have now $0 \leq b_{i_o} - c \leq k_{i_o} - c$, this term may be interpreted as the measure

$$M^{o}(T_o) + \hat{M}(D_o) =$$

$$= \hat{M}(T_o) + \hat{M}(D_o) = \hat{M}(T_o + D_o)$$

where D_o is a suitable subset of K_{i_o} such that $| D_o | = b_{i_o} - c$.

As $\hat{M}$ hom g_i for all $i \geq i_o$ s.t. $\hat{m} > g_i$ (our assumption reflected by (9)), we find $S \subseteq T_o + D_o$ such that

$$\hat{M}(S) = \sum_{i=i_o+1} (k_i - b_i) g_i \quad .$$

However, we know that

$$M^{o}(T_o) + (b_{i_o} - c) g_{i_o} - \sum_{i=i_o+1}^{r} (k_i - b_i) g_i < g_{i_o}$$

(cf. (15)) - i.e., when switching from $T_o + D_o$ to S we do not remove elements from $D_o \subseteq K_{i_o}$ - and hence we conclude that actually $S \supseteq D_o$.

Thus

(17) $$\hat{M}(S) = M^{o}(S \cap T_o) + (b_{i_o} - c) g_{i_o}$$

$$= \sum_{i=i_o+1}^{r} (k_i - b_i) g_i ,$$

and finally

$$M ((S \cap T_o) + T^{o})$$

$$= M^{o} (S \cap T_o) + M(T^{o})$$

$$\overset{(17)}{=} \sum_{i=i_o+1}^{r} (k_i - b_i) g_i - (b_{i_o} - c) g_{i_o} + b_{i_o} g_{i_o} + \sum_{i=i_o+1}^{r} b_i g_i$$

$$= c g_{i_o} + \sum_{i=i_o+1}^{r} k_i g_i = \lambda \quad .$$

Obviously $U := (S \cap T_o) + T^o$ is the desired subset of T having exactly measure λ .

To finish the proof we have to return to (16) and deal with the case that $c \geq b_{i_o}$.

5th Step: Indeed, if $c \geq b_{i_o}$, then (16) is rewritten:

$$M^o(T_o) > (c - b_{i_o}) g_{i_o} + \sum_{i=i_o+1}^{r} (k_i - b_i) g_i$$

Again, we use our assumption about homogeneity of $\hat{M}$ as expressed by (9), thus finding $S_o \subseteq T_o$ s.t.

$$M^o(S_o) = (c - b_{i_o}) g_{i_o} + \sum_{i=i_o+1}^{r} (k_i - b_i) g_i \quad .$$

Clearly

$$\begin{aligned} M(S_o + T^o) &= M^o(S_o) + M(T^o) \\ &= (c - b_{i_o}) g_{i_o} + \sum_{i=i_o+1}^{r} (k_i - b_i) g_i + \sum_{i=i_o}^{r} b_i g_i \\ &= \lambda \quad , \end{aligned}$$

hence $U := S_o + T^o \subseteq T$ is the desired subset of T, q.e.d.

Remark 1.5. It should now be mentioned that conditions (2) and (3) of Lemma 1.2., i.e.,

" $k_{i_o} \leq c + l_{i_o i}$ for all $i \in \{i_o+1,\dots,r\}$

satisfying $g_{i_o} \not| g_i$ "

and

" $k_i \leq l_{i i_o}$ for all $i \in \{1,\dots,i_o-1\}$

satisfying $g_i \not| g_{i_o}$ "

are in fact consequences of condition (9) of 1.4., i.e. of

" $M^{i_o, k_{i_o}-c}$ hom g_i for all $i \in \{i_o,\dots,r\}$

satisfying $m^{i_o, k_{i_o}-c} \geq g_i$ " .

Combining our results we may state the following theorem.

Theorem 1.6. Let $M = \sum_{i=1}^{r} | \cdot \cap K_i | \, g_i$ be an integer-valued measure on $\Omega = K_1 + \dots + K_r$ and let $\lambda \in \mathbb{N}$, $0 < \lambda \leq M(\Omega)$. Then

$$M \text{ hom } \lambda$$

if and only if there is $i_o \in \{1,\dots,r\}$ and $c \in \mathbb{N}$, $1 \leq c \leq k_{i_o}$, such that

(I) $$\lambda = c\, g_{i_o} + \sum_{i=i_o+1}^{r} k_i\, g_i$$

(II) $$M^{i_o, k_{i_o}-c} \text{ hom } g_i$$

for all $i \in \{i_o,\dots,r\}$ s.t. $m^{i_o, k_{i_o}-c} \geq g_i$.

In this case in addition the following conditions are satisfied

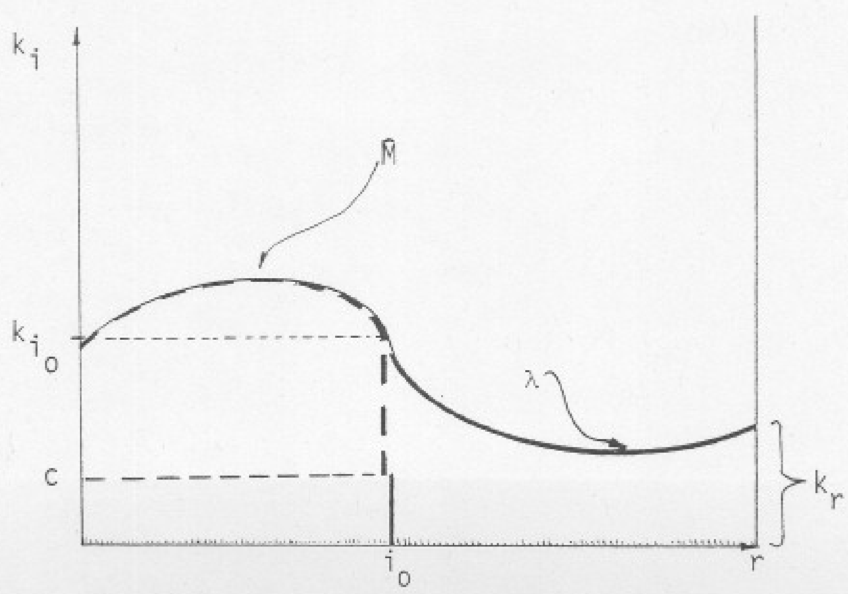
(III) $\quad k_{i_0} \leq c + l_{i_0 i}$

$$\text{for all } i \in \{i_0+1,\ldots,r\} \text{ s.t. } g_{i_0} \not\prec g_i ,$$

(IV) $\quad k_i \leq l_{i i_0}$

$$\text{for all } i \in \{1,\ldots,i_0-1\} \text{ s.t. } g_i \not\prec g_{i_0} .$$

Remark 1.7. The following interpretation of our results as stated by Theorem 1.6. is offered. $k = (k_1,\ldots,k_r)$ represents a distribution of the number of players of the various types. A representation of λ (the majority level) as indicated by (I), corresponds to a specific distribution of the players over the various types within a minimal winning coalition. Such a typical minimal winning coalition is composed by all "big" players ($i \geq i_0 + 1$) and a few players of "medium size" $(i = i_0)$, while it contains none of the small players ($i < i_0$) .



However, there are different types of minimal coalitions feasible: Small players may be able to eliminate a big player as follows. $\widehat{M}$ represents the distribution of types over the remaining medium sized players and the small players. Now, whenever a group of small players has sufficiently much weight in order to exceed the weight of one big player i_1 (that is $\hat{m} > g_{i_1}$), then the small players are capable of imitating the big one (because $\widehat{M}$ hom g_i), that is they may choose a subcoalition of exactly the weight g_{i_1} and they may replace one player of typ i_1 in every minimal winning coalition of the original type. More formally the fact that $\widehat{M}$ is homogeneous with respect to g_{i_1} $(i_1 > i_o)$ implies further representations of λ , for instance

$$\lambda = c\, g_{i_o} + \sum_{\substack{i=i_o+1 \\ i \neq i_1}}^{r} k_i\, g_i + (k_{i_1} - 1)\, g_{i_1} + \widehat{M}(S)$$

where $\quad S \subseteq K_1 + \ldots + K_{i_o}$ and $|\, S \cap K_{i_o} | \leq k_{i_o} - c$.

Of course the condition represented by (II) induces further conditions for the distribution of players over the types. For instance condition (IV) may be seen as follows. The relation means that the total weight of all players of type i ("small players") does not exceed the weight of one player of medium size if the small players are not capable of imitating the latter one. Similarly condition (III), i.e.,

$$(k_{i_o} - c)\, g_{i_o} \leq l_{i_o i}\, g_{i_o} < g_i \quad (i > i_o\, ,\ g_{i_o} \nmid g_i)$$

is readily interpreted as follows: Those players of medium size which are not represented within a typical minimal coalition of the above mentioned type must not have a total weight exceeding the weight of a big player if they are not capable of imitating him.

§ 2 INTERVALLS OF HOMOGENEITY

A vector $(g,k) \in \mathbb{N}^{2r}$ and an integer valued measue M on $\Omega = K_1+\ldots+K_r$ are closely related objects, thus (g,k) determines a certain subset of natural numbers λ s. t. M hom λ.

For $(g,k) \in \mathbb{N}^{2r}$ and $i_0 \in \{1,\ldots,r\}$, $c \in \{1,\ldots,k_{i_0}\}$, recall the definition of $\hat{M} = M^{i_0, k_{i_0}-c}$; the abbreviation $\hat{M}$ is used whenever i_0 and c are specified. Let us write

(1)
$$I^r_{i_0} = I^r_{i_0}(g,k) = \{cg_{i_0} + \sum_{i=i_0+1}^{r} \;\Big|\; 1 \leq c \leq k_{i_0}, \quad \hat{m} > g_i \text{ implies } \hat{M} \text{ hom } g_i \text{ for } i \in \{i_0,\ldots,r\}\}.$$

Then the first part of Theorem 1.6 is reformulated as follows.

Theorem 2.1 M hom λ if and only if

$$\lambda \in \bigcup_{i_0=1}^{r} I^r_{i_0} .$$

Loosely speaking, we regard each $I^r_{i_0}$ as an "intervall" of natural numbers λ with the property M hom λ.

It is easily seen that the $I^r_{i_0}$ in some sense are indeed "intervalls". More exactly, for any (g,k) and $i_0 \in \{1,\ldots,r\}$ there is

$c^r_{i_o} = c^r_{i_o}(g,k) \in \mathbb{N} \cup \{\infty\}$ such that

$$(\cdot) \qquad I^r_{i_o} = \{cg_{i_o} + \sum_{i=i_o+1}^{r} k_i\, g_i \mid c^r_{i_o} \leq c \leq k_{i_o}\}.$$

For, if $I^r_{i_o} = \emptyset$ we put conveniently $c^r_{i_o} = \infty$ and if

M hom $\lambda^c_{i_o} := cg_{i_o} + \sum_{i=i_o+1}^{r} k_i\, g_i$, then (3) reveals at once that

M hom $\lambda^{c+1}_{i_i}$ provided $c < k_{i_o}$.

<u>Hence, the set of numbers</u> $(c^r_{i_o})_{i_o=1,\ldots,r}$ <u>completely describes the intervalls of homogeneity with respect to a given</u> (g,k).

Theorem 1.6 in fact offers an inductive procedure for computing these "intervalls of homogeneity". The present section is intended to provide more insight in the first induction steps (r = 2,3) as well as into the nature of the general $I^r_{i_o}$ as computed by the various I^ρ_i $(\rho < r,\ i \in \{1,\ldots,\rho\})$.

Let $(g,k) \in \mathbb{N}^{2r}$ and, for $R \subseteq \{1,\ldots,r\}$ let $[g_i | i \in R]$ be the intersection of $\mathbb{N}$ and the ideal spanned by g_i $(i \in R)$ (i.e., the positive multiples of g.c.d. $(g_i)_{i \in R}$). Furthermore, for $\rho \in \{1,\ldots,r\}$

$$(2) \qquad J_\rho := \{R \subseteq \{1,\ldots,\rho-1\} \mid \sum_{i \in R} k_i g_i > g_\rho,\ g_\rho \notin [g_i \mid i \in R]\}$$

Thus, if $R \subseteq J_\rho$ then players of types $i \in R$ may muster enough strength to exceed the weight of a player of type ρ, however, they cannot exactly imitate him. For convenience, we write $i \in J_\rho$ instead of $\{i\} \subseteq J_\rho$ and if $R \subseteq J_\rho$ we shall sometimes say that R disturbs ρ.

Now returning to our intervalls of homogeneity, it is not hard to see (by Theorem 1.6) that

$$(3) \qquad I^r_{i_o} = \begin{cases} \emptyset & (J_{i_o} \neq \emptyset) \\ \{cg_{i_o} + \sum_{i=i_o+1}^{r} \mid l_{i_o} \leq c \leq k_{i_o}, & \\ \hat{m} > g_i \text{ implies } \hat{M} \text{ hom } g_i \ (i \in \{i_o,\ldots,r\})\} & (J_{i_o} = \emptyset) \end{cases}$$

where

$$(4) \quad l_{i_o} = \begin{cases} \max \{k_{i_o} - l_{i_o i} \mid i \in \{i_o+1,\ldots,r\},\ i_o \in J_i\} \text{ if } i_o \in \bigcup_{i>i_o} J_i \\ 1 \quad \text{otherwiese} \end{cases}$$

such that in particular $l_r = 1$.

For $r = 2$ we have obviously

$$I_2^2 = \begin{cases} \emptyset & (1 \in J_2) \\ \{cg_2 \mid 1 \leq c \leq k_2,\ \hat{m} > g_2 \text{ implies } \hat{M} \text{ hom } g_2\} & (1 \notin J_2) \end{cases}$$

Here, $\hat{M}$ corresponds to $((g_1,g_2),(k_1,k_2-c))$ and for $1 \notin J_2$ it turns out that $\hat{M}$ hom g_2 is always satisfied. Hence

$$I_2^2 = \begin{cases} \emptyset & (1 \in J_2) \\ \{cg_2 \mid 1 \leq c \leq k_2\} & (1 \notin J_2) . \end{cases} \tag{5}$$

Similarly,

$$I_1^2 = \{cg_1+k_2g_2 \mid l_1 \leq c \leq k_1,\ \hat{m} > g_i \text{ implies } \hat{M} \text{ hom } g_i \text{ for } i=1,2\} .$$

Here $\hat{M}$ corresponds to $((g_1),(k_1-c))$ such that $\hat{M}$ hom g_1 is always satisfied while $\hat{M}$ hom g_2 means $g_1 \mid g_2$. Thus

$$I_1^2 = \{cg_1+k_2g_2 \mid l_1 \leq c \leq k_1,\ (k_1-c)g_1 > g_2 \text{ implies } g_1 \mid g_2\}$$

$$= \begin{cases} \{cg_1+k_2g_2 \mid l_1 \leq c \leq k_1\} & \text{if } g_1 \mid g_2 \text{ or } k_1g_1 \leq g_2 \\ \{cg_1+k_2g_2 \mid l_1 \leq c \leq k_1,\ cg_1 < k_1g_1-g_2\} & \text{otherwiese} \end{cases}$$

$$= \begin{cases} \{cg_1+k_2g_2 \mid l_1 \leq c \leq k_1\} & (1 \notin J_2) \\ \{cg_1+k_2g_2 \mid l_1 \leq c \leq k_1,\ cg_1 < k_1g_1-g_2\} & (1 \in J_2) \end{cases} \tag{6}$$

$$= \begin{cases} \{cg_1+k_2g_2 \mid 1 \le c \le k_1\} & (1 \notin J_2) \\ \{cg_1+k_2g_2 \mid k_1-l_{12} \le c \le k_1\} & (1 \in J_2) \end{cases}$$

Corollary 2.2 Let $r = 2$ and M correspond to $((g_1,g_2),(k_1,k_2))$. Then M hom λ if and only if

(7) $$\lambda \in \begin{cases} \{cg_1+k_2g_2 \mid k_1-l_{12} \le c \le k_1\} & (1 \in J_2) \\ \{cg_2 \mid 1 \le c \le k_2\} \cup \{cg_1+k_2g_2 \mid 1 \le c \le k_1\} & (1 \notin J_2) \end{cases}$$

In other words, the intervalls of homogeneity are described by

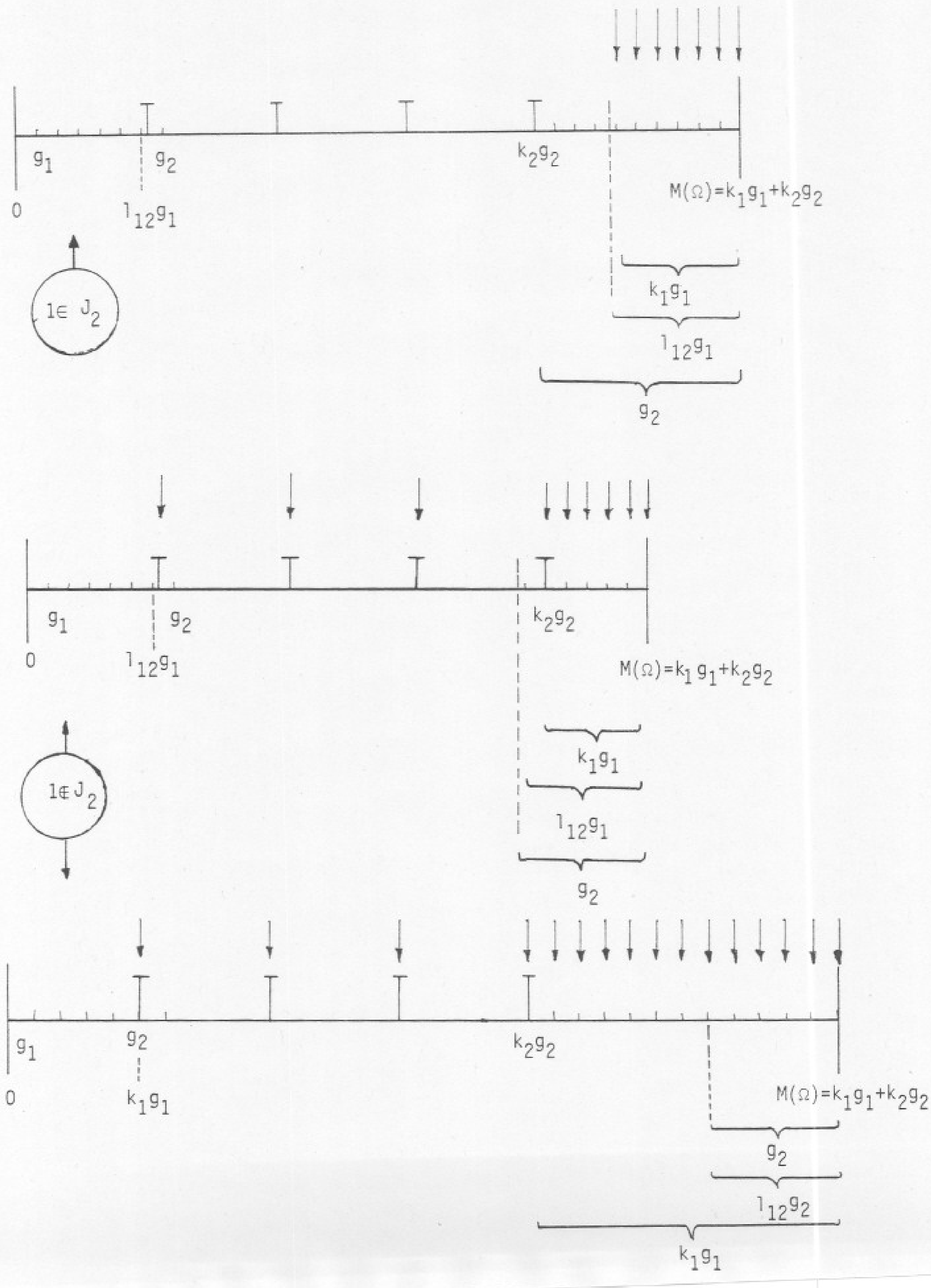
$$c^2_2 = \begin{cases} \infty & 1 \in J_2 \\ 1 & 1 \notin J_2 \end{cases}$$

$$c^2_1 = l_1$$

Fig. 2 represents the Intervalls of homogeneity for $r = 2$.

The case $r = 3$ requires a few preparations. It is important to note that simple divisibility properties as expressed by Corollary 2.2 will not suffice to tackle the general shape of the intervalls of homogeneity. Rather, it is the theory of "(g,k)-representations as developed in [ ] that yields further results. The reader is referred to this paper as we shall draw on it for the following presentation.

Fig. 2


g1
g2
k2g2
0
l12g1
M(Ω)=k1g1+k2g2
1∈ J2
k1g1
l12g1
g2
g1
g2
k2g2
0
l12g1
M(Ω)=k1 g1+k2g2
1∈ J2
k1g1
l12g1
g2
g1
g2
k2g2
0
k1g1
M(Ω)=k1g1+k2g2
g2
l12g2
k1g1

Let g_1, $g_2 \in \mathbb{N}$ with g.c.d. $(g_1,g_2) =: d_2$ and let

(8) $$N_2 := (\frac{g_1}{d_2} - 1)\ (g_2-d_2).$$

For $\lambda \in \mathbb{N}$ define

(9) $$C := C(\lambda) := \{c\in\mathbb{N} \mid c \geq 0,\ \lambda-cg_2 \geq 0,\ g_1 \mid \lambda-cg_2\}$$

Clearly $C = \emptyset$ if $\lambda \not\equiv 0 \mod d_2$. Assume therefore

$$\lambda \equiv 0 \mod d_2 .$$

Next put

$$\tilde{g}_i := \frac{g_i}{d_2}\ (i=1,2)\ ;\ \tilde{\lambda} := \frac{\lambda}{d_2}$$

According to a well known theorem of elementary number theory (e.g. SCHOLZ-SCHOENEBERG, [7] Th 27, p.41) the congruence

$$x\tilde{g}_2 \equiv \tilde{\lambda} \mod \tilde{g}_1$$

has a <u>unique</u> solution (mod $\tilde{g}_1$), i.e., there is a mapping

(10) $$\kappa : \{\lambda \in \mathbb{N} \mid \lambda \equiv 0 \mod d_2\} \to \{0,\ldots,\tilde{g}_1-1\}$$

such that

$$\kappa(\lambda)\ \tilde{g}_2 \equiv \tilde{\lambda} \mod \tilde{g}_1 .$$

Clearly, κ depends only on the equivalence class mod g_1 of λ, i.e.

if $\lambda' \equiv 0 \bmod d_2$ and $\lambda' \equiv \lambda \bmod g_1$, then $\tilde{\lambda}' \equiv \tilde{\lambda} \bmod \tilde{g}_1$ and $K(\lambda') = K(\lambda)$.

Next, define a mapping

(12) $$\tau : \{\lambda \in \mathbb{N} \mid \lambda \equiv 0 \bmod d_2\} \to \mathbb{N} \cup \{0\}$$

by

(13) $$\tau(\lambda) = \max \{t \in \mathbb{Z} \mid \lambda - (K(\lambda) + t\tilde{g}_1)g_2 \geq 0\} .$$

Then we have the following lemma.

Lemma 2.3. Let $g_1, g_2 \in \mathbb{N}$, g.c.d. $(g_1,g_2) =: d_2$.

Let $\lambda \in \mathbb{N}$ and let N_2, $C(\lambda)$, $K(\lambda)$, and $\tau(\lambda)$ be given by (8), (9), (10), (13).

For $\lambda \equiv 0 \bmod d_2$ and $\lambda \geq N_2$ we have

$$C = C(\lambda) = \{K(\lambda), K(\lambda) + \tilde{g}_1, \ldots, K(\lambda) + \tau(\lambda)\tilde{g}_1\}$$

<u>PROOF</u> *1st Step:* For $\lambda \geq N_2$ there is a "(g_1,g_2)-representation" according to Lemma 3.1. and Theorem 3.2. of [5] .
In fact the bound N_{12} was already known to SYLVESTER, but we want to draw on [5].

Thus $C \neq 0$. Let us show that there is $\underline{c}$ and $\bar{\tau}$ s.t.

$$C = \{\underline{c}, \underline{c} + \tilde{g}_1, \ldots, \underline{c} + \bar{\tau}\, \tilde{g}_1\}.$$

To this end, it suffices to show that $c \in C$ implies $g_1 / \lambda - (c \pm \tilde{g}_1)g_2$.

But as $c \in C$ we have

$$\lambda = c\, g_2 + dg_1 \qquad (d \geq 0)$$

$$\lambda = (c \pm \tilde{g}_1)g_2 + (d \mp \tilde{g}_2)\, g_1$$

and hence $g_1 / \lambda - (c \pm \tilde{g}_1)g_2$; this completes our first step.

2nd Step: Inspection of Lemma 3.1. of [5] shows the following.

As $\tilde{\lambda} \geq (\tilde{g}_1 - 1)\ (\tilde{g}_2 - 1)$, there is κ, $\kappa \equiv 0 \bmod d_2$, $0 \leq \kappa \leq \tilde{g}_1 - 1$, such that

$$\tilde{\lambda} \equiv \kappa\, \tilde{g}_2 \bmod \tilde{g}_1 \ ,\ \tilde{\lambda} - \kappa\, \tilde{g}_2 \geq 0$$

Obviously $\kappa = K(\lambda)$ (because of the uniqueness of the solution of the above mentioned congruence, Lemma 3.1. of [5] just claims that <u>in addition</u> $\tilde{\lambda} - \kappa\, \tilde{g}_2 \geq 0$). This means of course

$$\tilde{\lambda} = \kappa\, \tilde{g}_2 + d\, \tilde{g}_1 \qquad d \geq 0$$

$$\lambda = \kappa\, g_2 + d\, g_1$$

and hence $\kappa = K(\lambda) \in C(\lambda)$.

3rd Step: Using again $\kappa := K(\lambda)$ we have for $t \in \mathbb{Z}$

$$\lambda = (\kappa + t\, \tilde{g}_i)g_2 + (d - t\, \tilde{g}_2)\, g_1 \ .$$

Clearly, for $0 \leq t \leq \tau = \tau(\lambda)$ we have $\kappa + t\, \tilde{g}_1 \geq 0$ and $(d - t\, \tilde{g}_2) \geq 0$ (by definition of $\tau(\lambda)$ while $\kappa - \tilde{g}_1 < 0$ and $\lambda - (\kappa + (\tau + 1)\, \tilde{g}_1)g_2 < 0$, again by definition of $K(\cdot)$ and $\tau(\cdot)$.

Thus, in view of the 1st step, $\kappa + t\,\tilde{g}_1 \in C \quad 0 \leq t \leq \tau$,
$\kappa - \tilde{g}_1 \notin C$, $\kappa + (\tau + 1)\,\tilde{g}_1 \notin C$; q.e.d.

The notations N_2, d_2, $\tilde{g}_i$, $C(\cdot)$, $K(\cdot)$, $\tau(\cdot)$ will be used in the following theorem which completely describes the case $r = 3$.

This case is treated under the additional assumption $g_3 \geq N_2$, which enables us to compute a closed formula for the numbers c_i^3 $(i = 1,2,3)$. It is, however, only the third case of the theorem below hich uses this requirement; the subsequent remarks are meant to enlighten this procedure.

Theorem 2.4 Let $r = 3$ and $g_3 \geq N_2$. Then

$$c_1^3 = l_1$$

$$c_2^3 = \begin{cases} \infty & 1 \in J_2 \\ k_2 - \left[\dfrac{g_3 - k_1 g_1}{g_2}\right] & 1 \notin J_2,\ 12 \in J_2 \\ k_2 - \max\left(\left[\dfrac{g_3 - k_1 g_1}{g_2}\right], \kappa + \tau\tilde{g}_1\right) & 1 \notin J_2,\ 12 \notin J_3,\ 2 \in J_3 \\ 1 & 1 \notin J_2,\ 12 \notin J_3,\ 2 \notin J_3 \end{cases}$$

$$c_3^3 \begin{cases} \infty & \text{if } k_1 g_1 + k_2 g_2 > g_3 \text{ and } (g_1, g_2; k_1, k_2) \text{ not hom } g_3 \\ 1 & \text{otherwise} \end{cases}$$

where $\kappa = K(g_3)$, $\tau = \tau(g_3)$.

<u>PROOF</u> We shall only compute c_2^3 ; the remaining cases are treated analogously.

By (3) we have:

$$(15) \qquad I_2^3 = \begin{cases} \emptyset & 1 \in J_2 \\ \{cg_2+k_3g_3 \mid l_2 \le c \le k_2,\ \hat{m} > g_i \\ \qquad \text{implies } \hat{M} \text{ hom } g_i \ (i=2,3)\} & 1 \notin J_2 \end{cases}$$

Here $\hat{M}$ corresponds to $((g_1,g_2),(k_1,k_2-c))$.

The condition "$\hat{M}$ hom g_i" may therefore be decided according to Corollary 2.2, where each $I^2_{\cdot}$ now will depend on c. More precisely we observe that $\hat{M}$ hom g_i if and only if

$$(16) \qquad g_i \in \begin{cases} \{dg_1+(k_2-c)g_2 \mid k_1-l_{12} \le d \le k_1\} & 1 \in J_2 \\ \{dg_2 \mid 1 \le d \le k_2-c\} \cup \\ \qquad \cup\ \{dg_1+(k_2-c)g_2 \mid 1 \le d \le k_1\} & 1 \notin J_2 \end{cases}$$

- and in view of (15) only the case $1 \notin J_2$ matters in (16). Now a straightforward computation shows that for $1 \notin J_2$ "$\hat{m}$ g_2 implies $\hat{M}$ hom g_2" is *always* correct. Hence we conclude that

$$(17) \qquad I_2^3 = \begin{cases} \emptyset & 1 \in J_2 \\ \{cg_2+k_3g_3 \mid l_2 \le c \le k_2, \\ \ g_1k_1+g_2(k_2-c) > g_3 \text{ implies} \\ \ g_3 \in \{dg_2 \mid 1 \le d \le k_2-c\} \cup \{dg_1+(k_2-c)g_2 \mid 1 \le d \le k_1\} \\ & (1 \notin J_2) \end{cases}$$

Now, if $12 \in J_3$, then g_3 will never be an element of one of the sets indicated in (17). Hence we have

$$k_2 - c \leq \frac{g_3 - k_1 g_1}{g_2}$$

for all c such that $cg_1 + k_2 g_2 \in I_2^3$. The *minimal* value of all these ($= c_2^3$) is the maximum of the lower boundaries thus obtained, i.e.,

$$c_2^3 = \max \left(l_2,\ k_2 - \left[\frac{g_3 - k_1 g_1}{g_2} \right] \right).$$

But as

$$l_2 = \begin{cases} k_2 - l_{2\,3} & \cdots \\ 1 & \cdots \end{cases} \leq k_2 - \left[\frac{g_3 - k_1 g_1}{g_2} \right] ,$$

our assertion follows in this case. (Note that $k_1 g_1 + k_2 g_2 > g_3$ implies $k_2 > [\ldots]$.)

Consider now the case $1 \in J_2$, $12 \notin J_3$, $2 \in J_3$. Because of $2 \in J_3$ the first set in (17) is to be disregarded. Moreover observe that $l_2 = k_2 - l_{2\,3}$. Hence

$$I_2^3 = \{cg_2 + k_3 g_3 \mid 0 \leq k_2 - c \leq l_{2\,3},$$

$$k_2 - c > \frac{g_3 - k_1 g_1}{g_2} \quad \text{implies} \quad g_3 \in \{dg_1 + (k_2 - c) g_2 \mid 1 \leq d \leq k_1\} .$$

As $k_2 g_2 > g_3$ and $12 \notin J_3$ we have $g_3 \in [g_1, g_2]$ and, using now Lemma 2.3., we may continue by

$$I_2^3 = \{cg_2 + k_3g_3 \mid 0 \leq k_2 - c \leq l_{2\,3}\ ,$$

(18) $$k_2 - c > \frac{g_3-k_1g_1}{g_2} \quad \text{implies} \quad k_2 - c \in \{\kappa,\dots,\kappa + \tau\,\tilde{g}_1\},$$

$$\frac{g_3-k_1g_1}{g_2} \leq k_2 - c \leq \frac{g_3-g_1}{g_2}\} \ .$$

In order to obtain c_2^3 we have to compute "max k_2-c" over the set as indicated by (18). To this end observe that, by definition of κ and τ, we have

(19) $$g_3 = \kappa g_2\ r\ (\tau\tilde{g}_2+r)g_1$$

with suitable $r \in \mathbb{N}$, $1 \leq r \leq \tilde{g}_2-1$ (as $g_2 \nmid g_3$!), thus

(20) $$g_3 = (\kappa + \tau\tilde{g}_1)g_2 + r\ g_1\ .$$

This implies

(21) $$l_{23} = \left[\frac{g_3}{g_2}\right] \geq \left[\frac{g_3-g_1}{g_2}\right] = \kappa + \tau g_1 + \left[\frac{(r-1)g_1}{\tilde{g}_2}\right] \geq \kappa + \tau\tilde{g}_1$$

From this it follows at once that the last condition to be imposed on $k_2 - c$ in (18) may be omitted at once. Moreover it is seen that "max k_2-c" in (18) is given by

$$\max\left(\left[\frac{g_3-k_1g_1}{g_2}\right],\ \kappa + \tau\tilde{g}_1\right),$$

which finishes this case.

It remains to treat the case $1 \notin J_2$, $12 \notin J_3$, $2 \notin J_3$. Here, it suffices to check that $c = 1$ is feasible in (17). Observe that $l_2 = 1$. We have to show that

(22) $$k_1 g_1 + (k_2-1)g_2 > g_3 \text{ implies } g_3 \in \{\ldots\} \cup \{\ldots\} .$$

(cf. (17)). Now, if $k_2 g_2 > g_3$ then $g_2 \mid g_3$, thus $g_3 = dg_2$, $d < k_2$.

On the other hand, if $k_2 g_2 \leq g_3$, then the condition of (22) implies $k_1 g_1 > g_2$ and hence $g_1 \mid g_2$. As $g_3 \in [g_1, g_2]$ we have also $g_1 \mid g_3$ and $g_3 \in \{\ldots\} \cup \{\ldots\}$ follows at once.

q.e.d.

Remark 2.5 1. If $g_2 \nmid g_3$, then

$$l_{23} \geq \left[\frac{g_3 - g_1}{g_2}\right] \geq \kappa + \tau\tilde{g}_1$$

holds always true. For, by definition of $\kappa = K(g_3)$ and $\tau = \tau(g_3)$ we have

$$\kappa g_2 \equiv g_3 \bmod g_1 ,$$

i.e.

$$g_3 = \kappa g_2 + mg_1 = \kappa g_2 + (\tau\tilde{g}_2 + r)g_1$$

$$= (\kappa + \tau\tilde{g}_1)g_2 + rg_1$$

with suitable m and r (τ is "maximal with this property"). In particular, it follows that $1 \leq r \leq \tilde{g}_2 - 1$ (by definition of τ and $g_2 \nmid g_3$). Thus

(23)
$$l_{23} \geq \left[\frac{g_3-g_1}{g_2}\right] = \left[\kappa + \tau\tilde{g}_1 + \frac{(r-1)g_1}{g_2}\right]$$
$$= \left[\kappa + \tau\tilde{g}_1 + \frac{(r-1)\tilde{g}_1}{\tilde{g}_2}\right] \geq \kappa + \tau\tilde{g}_1$$

2. Consider the case that $k_1 g_1 < g_2$.

Because of

$$l_{23} \geq \left[\frac{g_3-k_1g_1}{g_2}\right] \geq \left[\frac{g_3-g_2}{g_2}\right] = l_{23} - 1$$

the bounds c_2^3 as specified for the second and the third case ($1 \notin J_2$, $12 \subseteq J_2$ and $1 \notin J_2$, $12 \notin J_3$, $2 \in J_3$, that is,) differ at all only if

(24)
$$l_{23} = \left[\frac{g_3-g_1}{g_2}\right] = \kappa + \tau\tilde{g}_1 > \left[\frac{g_3-k_1g_1}{g_2}\right] = \left[\frac{g_3-g_2}{g_2}\right] = l_{23} - 1$$

holds true. Thus, only if (24) is satisfied the requirement "$12 \in J_3$" yields an additional homogeneous pair M, λ.

Returning to the notation of (23), this means in view of

$$g_3 = (\kappa + \tau\tilde{g}_1)g_2 + rg_1; \; l_{23} = \kappa + \tau\tilde{g}_1 + \frac{r\tilde{g}_1}{\tilde{g}_2}$$

that we have to take care for

(25)
$$r\tilde{g}_1 \leq \tilde{g}_2 - 1 , \quad r \leq \frac{\tilde{g}_2-1}{\tilde{g}_1} .$$

E.g., for $r = 1$ we have (25) guaranteed in advance. What are the smallest "games" (M,λ) induced by this consideration?

Clearly we must have $k_1 \geq 2$. If we start with $g_1 = 2$ then $g_2 > k_1 g_1$, i.e. $g_2 = 5$. Now, for $\kappa = 1$ and $r = 1$ we have

$$g_3 = (1+2\,\tau)\,5 + 2$$

i.e. $g_3 = 7$ for $\tau = 0$. Because of $k_2 g_2 > g_3$ we have at least $k_2 \geq 2$. Hence

$$(g,k) = ((2,5,7),\ (2,2,k_3))$$

or

$$M = (2,2;\ 5,5;\ 7,\dots,7)\ .$$

The important λ is induced by

$$\bar{c}_2^3 = k_2 - (\kappa + \tau\tilde{g}_1) = k_2 - 1$$

i.e.

$$\begin{aligned}\lambda &= (k_2-1)g_2 + k_3 g_3 \\ &= \quad 1\ g_2 + k_3 g_3 \\ &= \quad 5 \quad + k_3 .\ 7\ .\end{aligned}$$

Thus, the "typical" minimal winning coalition is formed by all big players (weight g_3) and all but one medium player. The remaining players may well replace the members of such a coalition, but because of (2,2; 5) hom 7 and $2 + 2 < 5$, homogeneity is not disturbed.

The next game is given by

$$(g,k) = ((2,5,11), (2,3,k_3))$$

or

$$M = (2,2; \underbrace{5,\dots,5}_{\geq 3}; 11,\dots,11) .$$

Here

$$c_2^3 = k_2 - (1+2) = k_2 - 3$$

$$\lambda = (k_2-3)g_2 + k_3g_3$$

$$= (k_2-3) \cdot 5 + k_3 \cdot 11 .$$

3. Consider now the case that $k_1g_1 \geq g_2$ and $g_1 \mid g_2$. We have now

$$d_2 = g_1, \tilde{g}_1 = 1, r\tilde{g}_1 = r < \tilde{g}_2 - 1 ,$$

hence

$$l_{23} = \kappa + \tau\tilde{g}_1$$

$$\left[\frac{g_3-k_1g_1}{g_2}\right] \leq \left[\frac{g_3-g_2}{g_2}\right] = l_{23} - 1 .$$

Thus, the requirement $12 \notin J_3$ may yield a considerable improvement ("more homogeneous λ's") when changing from the second case to third one.

We are now going to treat the general case by offering a recursive formula for the computation of a $c_{i_0}^r$.

Observe that $I_{i_0}^r = \emptyset$ if $J_{i_0} \neq \emptyset$ and hence trivially $c_{i_0}^r = \infty$ in this case. Next, assume $J_{i_0} = \emptyset$ such that

$$I^r_{i_o} = \bigcap_{j=i_o}^{r} \{cg_{i_o} + \sum_{i=i_o+1}^{r} k_i\, g_i \mid l_{i_o} \leq c \leq k_{i_o} ,$$

$$\hat{m} > g_j \text{ implies } \hat{M} \text{ hom } g_j\ (j=i_o,\dots,r)\}$$

where $\hat{M}$ corresponds to $((g_1,\dots,g_{i_o}), (k_1,\dots,k_{i_o}-c))$.

For $j = i_o$ the requirement $\hat{m}$ hom g_j is tantamount to $((g_1,\dots,g_{i_o}-1), (k_1,\dots,k_{i_o}-1))$ hom g_{i_o} . Therefore, if we introduce

$$^j c^r_{i_o} := \min \{c \mid l_{i_o} \leq c \leq k_{i_o} ,$$

$$\hat{m} > g_j \text{ implies } \hat{M} \text{ hom } g_j\} ,$$

then, clearly

$$(26) \qquad c^r_{i_o} = \begin{cases} \infty & J_{i_o} \neq \emptyset \\ \infty & J_{i_o} = \emptyset \text{ and } \\ & (g_1,\dots,g_{i_o}-1)(k_1,\dots,k_{i_o}-1) \text{ not hom } g_{i_o} \\ \max\limits_{j=i_o+1}^{r} {}^j c^r_{i_o} & \text{otherwise} \end{cases}$$

Now, for the new quantities we have

$$^j c^r_{i_o} = \min \{c \mid l_{i_o} \leq c \leq k_{i_o} ,\ \hat{m} > g_j \text{ implies}$$

$$g_j \in \{dg_{i_o} \mid c^{r-1}_{i_o} \leq d \leq k_{i_o}-c\} \cup$$

$$(27) \qquad \cup\ \{dg_{i_o-1} + (k_{i_o}-c)g_{i_o} \mid c^{r-1}_{i_o-1} \leq d \leq k_{i_o-1}\} \cup$$

$$\cup\ \dots$$

$$\vdots$$

$$\cup\ \{dg_1 + \sum_{l=2}^{i_o-1} k_i g_i + (k_{i_o}-c)g_{i_o} \mid c^{r-1}_1 \leq d \leq k_1\} .$$

Here, the quantities $c_{\cdot}^{r-1}$ depend on $g_1,\ldots,g_{i_o},k_1,\ldots,k_{i_o}$ only, the notation $c_{\cdot}^{i_o}$ would be appropriate as well.

Now, if $\{1,\ldots,i_o\} \in J_j$ then all sets listed in (27) are empty, thus

$$^j c_{i_o}^r = \infty \qquad \{1,\ldots,i_o\} \in J_j .$$

For $\{1,\ldots,i_o\} \notin J_j$, choose i_1 such that

$$\begin{aligned} &\{1,\ldots,i_o\} \notin J_j \\ &\{2,\ldots,i_o\} \notin J_j \\ &\quad\vdots \\ &\{i_1,\ldots,i_o\} \notin J_j \\ &\{i_1+1,\ldots,i_o\} \in J_j \quad , \end{aligned}$$

i.e.

(28) $$i_1 = \max \{i \mid \{i,\ldots,i_o\} \notin J_j,\ 1 \leq i \leq i_o\} ,$$

(such that in particular $i_o \notin J_j$ if $i_1 = i_o$).

Then, in formula (27) all sets vanish up to and including the one that starts with $\{dg_{i_1+1} + \ldots\}$. <u>If $i_1 < i_o$</u> , this means

$$^j c_{i_o}^r = \min \{c \mid l_{i_o} \leq c \leq k_{i_o},\ \hat{m} > g_j \text{ implies}$$
$$g_j \in \{dg_{i_1} + \sum_{i=i_1+1}^{i_o-1} \ldots + (k_{i_o}-c)g_{i_o} \mid c_{i_o}^{r-1} \leq d \leq k_{i_1}\}$$

$$\cup \ldots$$

(29) $$\vdots$$

$$\cup \{dg_1 + \sum_{l=1}^{i_o-1} \ldots + (k_{i_o}-c)g_{i_o} \mid c_1^{r-1} \le d \le k_1\}\}$$

$$= \min_{\underline{i}=1}^{i_1} \min \{c \mid l_{i_o} \le c \le k_{i_o},\ \hat{m} > g_j \text{ implies}$$

$$g_j \in \{dg_{\underline{i}} + \sum_{i=\underline{i}+1}^{i_o-1} \ldots + (k_{i_o}-c)g_{i_o} \mid c_{\underline{i}}^{r-1} \le d \le k_{\underline{i}}\}\}$$

The last min again is simplified by

$$\min\{\ \} = \max\,(l_{i_o},\ \min\,[\min\{c \mid \hat{m} \le g_j\}\ ,$$

$$\min\ \{c \mid \hat{m} > g_j \text{ and } g_j \in \{dg_{\underline{i}} + \ldots|\ldots\}\}]\ )\ .$$

Here, "$\hat{m} > g_j$" may be cancelled as it follows from $g_j \in \{\ldots\}$. Therefore, if we put

(30) $$\,^{j}_{\underline{i}}c^{r}_{i_o} := \min\{c \in \mathbb{N} \mid g_j \in \{dg_j + \sum_{i=\underline{i}+1}^{i_o-1} k_i g_i +$$

$$+ (k_{i_o}-c)g_{i_o} \mid c_{\underline{i}}^{r-1} \le d \le k_{\underline{i}}\}\}\ ,$$

it turns out that

$$jc^r_{i_o} = \begin{cases} \infty & \text{if } \{1,\ldots,i_o\} \in J_j \\ \max\,(l_{i_o},\ \min\,[k_{i_o} - \dfrac{g_j - \sum_{i=1}^{i_o-1} k_i g_i}{g_{i_o}}\ ,\ \min_{\underline{\imath}=1}^{i_1}\ {}^{j}_{\underline{\imath}}c^{r}_{i_o}\,] & \\ & \text{if } \{1,\ldots,i_o\} \notin J_j \text{ and } i_1 \text{ is given by (28).} \end{cases}$$ (31)

If $i_1 = i_o$, formula (31) holds true as well with the additional definition of ${}^{j}_{i_o}c^{r}_{i_o}$ beeing supplied by

(32)
$$ {}^{j}_{i_o}c^{r}_{i_o} := \begin{cases} \min\{c \in \mathbb{N} \mid g_j \in \{dg_{i_o} \mid c^{r-1}_{i_o} \le d \le k_{i_o} - c\}\} & (i_1 = i_o) \\ \text{undefined} & (i_1 < i_o) \end{cases} $$

This last quantity is readily computed as follows. We have (assuming $i_1 = i_o$)

$$ \{c \in \mathbb{N} \mid g_j \in \{dg_{i_o} \mid c^{r-1}_{i_o} \le d \le k_{i_o} - c\}\} $$

$$ = \begin{cases} \emptyset & g_{i_o} \nmid g_j \\ \{c \in \mathbb{N} \mid c^{r-1}_{i_o} \le \frac{g_j}{g_{i_o}} \le k_{i_o} - c\} & g_{i_o} \mid g_j \end{cases} $$

$$ = \begin{cases} \emptyset & g_{i_o} \nmid g_j \text{ , } k_{i_o} g_{i_o} > g_j \\ \{c \in \mathbb{N} \mid c^{r-1}_{i_o} \le \frac{g_j}{g_{i_o}} \le k_{i_o} - c\} & \text{otherwise} \end{cases} $$

$$ = \{c \in \mathbb{N} \mid c^{r-1}_{i_o} \le \frac{g_j}{g_{i_o}} \le k_{i_o} - c\} $$

as $i_1 = i_o$, i.e. $i_o \notin J_j$. Thus, the "min" over this set is obivously

(33)
$$ {}^{j}_{i_o}c^{r}_{i_o} = \begin{cases} k_{i_o} - l_{i_o j} & \text{if } k_{i_o} \ge l_{i_o j} \ge c^{r-1}_{i_o} \\ \infty & \text{otherwiese} \end{cases} $$

provided $i_1 = i_o$. It remains to deal with the general ${}^{j}_{\underline{i}}c^{r}_{i_o}$ for $\underline{i} < i_i$ (no matter whether $i_1 = i_o$ or not.)

Now, by (30) we have trivially

$$(34) \qquad {}^{j}_{\underline{i}}c^{r}_{i_o} = \infty \qquad \text{if} \quad \sum_{i=\underline{i}}^{i_o-1} k_i g_i < g_j$$

while $\sum_{i=\underline{i}}^{i_o-1} k_i g_i \geq g_j$ implies $g_j \in [g_{\underline{i}},\ldots,g_{i_o}]$ in view of $\{\underline{i},\ldots,i_o\} \notin J_j$. Consider the ideal $[g_{\underline{i}}, g_{i_o}]$. With respect to this ideal a function $\kappa = \kappa_{\underline{i}}^{i_o}$ is defined by Lemma 2.3 as well as a function $\tau = \tau_{\underline{i}}^{i_o}$. Put

$$(35) \qquad \begin{aligned} {}^{j}\kappa_{\underline{i}}^{i_o} &:= \kappa_{\underline{i}}^{i_o}\,(g_j - \sum_{i=\underline{i}+1}^{i_o-1} k_i g_i) \\ {}^{j}\tau_{\underline{i}}^{i_o} &:= \tau_{\underline{i}}^{i_o}\,(g_j - \sum_{i=\underline{i}+1}^{i_o-1} k_i g_i) \ . \end{aligned}$$

Using this quantity, the set indicated by (30) is developed as follows:

$$\{c \in \mathbb{N} \mid g_j \quad \{\ldots|\ldots\}\} =$$

$$\{c \in \mathbb{N} \mid g_j - \sum_{i=\underline{i}+1}^{i_o-1} k_i g_i - (k_{i_o}-c) g_{i_o} \geq 0 \ ,$$

$$(36) \qquad g_{\underline{i}} \mid g_j - \sum_{\ldots} \cdots - (k_{i_o}-c) g_{i_o},\ c_{\underline{i}}^{r-1} \leq \frac{g_j - \sum_{\ldots} \cdots - (k_{i_o}-c) g_{i_o}}{g_{\underline{i}}} \leq k_{\underline{i}}\}$$

$$= \{c \in \mathbb{N} \mid k_{i_o} - c \in C(g_j - \underset{\cdots}{\Sigma} \ldots),\ c_{\underline{i}}^{r-1} \le \frac{g_j - \underset{\cdots}{\Sigma} \ldots - (k_{i_o} - c) g_{i_o}}{g_{\underline{i}}} \le k_{\underline{i}}\}$$

$$= \begin{cases} \emptyset & g_j - \underset{\cdots}{\Sigma} \cdots \notin [g_{\underline{i}}, g_{i_o}] \\ \{c \in \mathbb{N} \mid k_{i_o} - c \in \{{}^j\kappa_{\underline{i}}^{i_o}, \ldots, {}^j\kappa_{\underline{i}}^{i_o} + {}^j\tau_{\underline{i}}^{i_o}\, \tilde{g}_{\underline{i}}^{i_o}\}\ , & \\ \dfrac{g_j - \underset{\cdots}{\Sigma} \ldots - k_{\underline{i}} g_{\underline{i}}}{g_{i_o}} \le k_{i_o} - c \le \dfrac{g_j - \underset{\cdots}{\Sigma} \ldots - c_{\underline{i}}^{r-1}\, g_{\underline{i}}}{g_{i_o}}\} & \text{otherwise} \end{cases}$$

where $\tilde{g}_{\underline{i}}^{i_o} := \dfrac{g_{\underline{i}}}{g.c.d.(g_{\underline{i}}, g_{i_o})}$, as suggested by Lemma 2.3. Note that we have to assume that

(37) $$g_j - \sum_{i=\underline{i}+1}^{i_o-1} k_i g_i \ge N_2(g_{\underline{i}}, g_{i_o}) \quad \text{with suitably adopted notation}$$

for N_2 (cf. Lemma 2.3). In other words, it is deduced from (30 (34) that

(38) $${}^j_{\underline{i}}c^r_{i_o} = \begin{cases} \infty & g_j > \sum_{i=\underline{i}}^{i_o-1} k_i g_i \quad \text{or} \quad g_j \notin [g_{\underline{i}}, g_{i_o}] \\ k_{i_o} - \max\,\{{}^j\kappa_{\underline{i}}^{i_o} + t\,\tilde{g}_{\underline{i}}^{i_o} \mid 0 \le t \le {}^j\tau_{\underline{i}}^{i_o}\ , & \\ \dfrac{g_j - \sum_{i=\underline{i}}^{i_o-1} k_i g_i}{g_{i_o}} \le {}^j\kappa_{\underline{i}}^{i_o} + t\tilde{g}_{\underline{i}}^{i_o} \le \dfrac{g_j - \sum_{i=\underline{i}+1}^{i_o-1} k_i g_i - c_{\underline{i}}^{r-1}\, g_{\underline{i}}}{g_{i_o}} & \text{otherwise} \end{cases}$$

where the last expression is understood to be ∞ if $\{\ \}$ is empty.

Combining we have the following

Theorem 2.5 Let $(g,k) \in \mathbb{N}^{2r}$. For $r = 2,3$ the quantities $c^r_{i_o}$ ($i_o \in \{1,\ldots,r\}$) are given by Corollary 2.2. and Theorem 2.4. respectively. For $r > 3$ they are recursively defined by (26) (31) (33) and (38), provided (37) is satisfied whenever, for some $\underline{i} \geq i_1$ (i_1 beeing defined by (28)) and some $j > i_o$ we have $g_j < \sum_{i=\underline{i}}^{i_o-1} k_i g_i$ and $g_j \in [g_{\underline{i}}, g_{i.}]$.

LITERATURE

[1] v. NEUMANN, J., and MORGENSTERN, O.: Theory of Games and Economic Behavior. Princeton University Press, Princeton, N.J. (1944, 1953).

[2] PELEG, B.: On Weights of Constant Sum Majority Games. SIAM Journal of Applied Mathematics 16. (1968). 527-532.

[3] ROSENMÜLLER, J.: Extreme Games and their solutions. Lecture Notes in Economics and Mathematical Systems 145. Springer Verlag, Heidelberg. (1977).

[4] ROSENMÜLLER, J.: The Theory of Games and Markets. North Holland Publishing Company, Amsterdam. (1981).

[5] ROSENMÜLLER, J., and WEIDNER, H.-G.: A Class of Extreme Convex Set Functions With Finite Carrier. Advances in Mathematics 10. (1973). 1-38.

[6] ROSENMÜLLER, J., and WEIDNER, H.-G.: Extreme Convex Set Functions With Finite Carrier: General Theory. Discrete Mathematics 10. (1974). 343-382.

[7] SCHOLZ, A., and SCHOENEBERG, B.: Einführung in die Zahlentheorie. De Gruyter Verlag, Berlin. (1961).

[8] SHAPLEY, L.S.: Simple Games. An Outline of the Descriptive Theory. Behavioral Science 7. (1962). 59-66.

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