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**Dynamic Bertrand–Edgeworth Competition  
with Entry/Exit Decisions:**

**The Case of Increasing Marginal Costs**

by

Till Requate

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H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung  
an der

Universität Bielefeld

Adresse / Address:

Universitätsstraße

4800 Bielefeld 1

Bundesrepublik Deutschland

Federal Republic of Germany

## Abstract

We consider a dynamic discrete time Bertrand–Edgeworth game with infinite horizon and discounting. The firms face increasing marginal costs and positive fixed costs. Besides setting prices they may enter or leave the market in each period. Moreover we assume that the number of potential firms is large enough that not all the firms can be active at the same time. Since with free entry there is no stationary equilibrium in pure strategies (even) in the supergame, we assume that the firms have to pay an entry cost whenever they enter or reenter the market. This makes the game a time-dependent supergame rather than a purely repeated game. Unless the entry cost or the discount factor is too low, there will be subgame perfect equilibria in this dynamic game. Our main results yield complete characterizations of the set of all stationary symmetric pure strategy equilibrium outcomes, consisting of a stationary equilibrium price and a number of active firms.

It turns out that the upper bound for equilibrium prices, i.e. the Pareto efficient outcomes with respect to the firms, depends on the size of the entry cost and the number of incumbent firms. Moreover there is an upper and lower bound for the number of firms that can be active in equilibrium. The upper (lower) bound decreases (increases) as the discount factor decreases.

### Keywords:

Bertrand Edgeworth competition, increasing marginal costs, entry costs, infinite time-dependent supergames with discounting, simple penal codes, simple strategy profiles, subgame perfect equilibria.

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# 1 Introduction

This paper is about contestable markets with an infinite discrete time horizon. We combine the idea of tacit collusion with the conjecture that high profits may attract further entrants into the market. We assume that in each period identical firms, facing increasing marginal costs and positive fixed costs, produce a homogenous commodity and engage in price competition for perfectly informed clients. With only one period this game is known as the Bertrand–Edgeworth game with U-shaped average cost curves. It is well known that this game has no Nash–equilibrium in pure strategies.

The nonexistence problem in that game has attracted much attention in the last decades. Besides the early work of EDGEWORTH [Edg25], see above all the pioneering papers by BECKMANN [Bec65] and LEVITHAN & SHUBIK [LS72] on capacity constrained firms. More recently solutions to this problem have been offered by DASGUPTA & MASKIN [DM86a,DM86b], MASKIN [Mas86] and DIXON [Dix84], who prove existence of mixed strategy equilibria under various conditions. Especially DIXON considers firms having increasing marginal costs. ALLEN & HELLWIG [AH86b,AH86a] characterize mixed strategy equilibria and show that the equilibrium prices converge in distribution to the competitive price, if the market becomes large. These models, however, all are one-shot games.

BROCK and SCHEINKMAN [BS85] and more recently LAMBSON [Lam87] investigate repeated price setting games with constant marginal costs and capacity constraints. In REQUATE (1990b) [Req90b] it has been demonstrated that the repeated Bertrand–Edgeworth game with U-shaped average cost curves has stationary equilibria in pure strategies, if the oligopoly size is not too large, and the discount factor is not too low. There, we characterized all stationary equilibrium outcomes that can be supported by optimal penal codes in pure strategies. Typically for supergames of this kind, also the monopoly price can be an equilibrium outcome if the number of firms, which has been assumed to be exogenous in that paper, is not too high, and the discount factor not too low.

However, tacit collusion may be called into question since potential new firms may be ready to enter the market as long as positive profits can be earned, unless there are any entry barriers. Indeed, the concept of monopolistic competition is based on the idea that the number of firms, operating in the market, can be endogenized by a zero-profit condition. That is, apart from integer problems, firms enter the market until all the firms make zero profits (cf. the Salop–Stiglitz model [SS77]). However, even if we allow for free entry (and exit) we do not get Nash equilibria in pure strategies in the one-shot game (see section 2). Unfortunately, in contrast to the pure oligopoly game [Req90b],

the consideration of the purely repeated version of *this* game (with free entry) is no remedy for nonexistence of pure strategy equilibria. The reason is simple: collusive outcomes are jeopardized by hit-and-run behavior of market intruders, this means, new firms could enter the market over night, undercut the incumbent firms slightly, and leave the market again in the next period before the incumbent firms are able to react. This kind of behavior could sometimes be observed at certain airlines in the U.S.. The assumption of hit-and-run behavior presupposes, however, that firms have no or only very low investment costs in order to set up a business. This is certainly not typical for most industries.

To incorporate the idea of contestability we assume in this paper that the number of potential firms is so large that not all the firms can together operate profitably in the market, even if they all charge the monopoly price. On the other hand, we assume that the firms have certain entry or investment costs to be paid if they enter or reenter the market. These entry costs will be the key for existence of stationary equilibria in pure strategies in the dynamic game. It will turn out that the higher the entry cost the higher is the maximal collusive equilibrium price. The entry cost can thus be considered as an indicator for the contestability of the market. Unless they are too low, these costs make the game a time-dependent (here: two-period-dependent) supergame (in J. FRIEDMAN's terminology) rather than a purely repeated game. For, the payoff to any firm in the current period depends not only on the price constellation of *that* period but also on the firm's own decision (being active or not) in the previous one.

By the contestability assumption, in a stationary equilibrium, there must be always some inactive firms. Hence, a stationary equilibrium will be defined by a pair, consisting of a *number* of active firms  $N$  and a *price*  $p$ , charged by the active firms in each period. The equilibrium conditions will yield us upper and lower bounds for possible stationary equilibrium prices and also bounds for possible numbers of active firms. Unfortunately, these bounds can only be given implicitly, since the equilibrium conditions can in general not be solved for  $p$  and  $N$ .

To consider potential but inactive firms as real players of the game, raises some conceptual problems. For, if we allow for the whole strategy space, also inactive firms could be activated in order to punish a possible deviator. Doing this, some equilibrium outcomes can be supported by subgame perfect strategies, which would not be achieved, if inactive firms would not engage in punishment. Therefore, we further define and characterize a set of equilibrium outcomes which do not require inactive firms to become active in order to punish other players after defection.

At the end of section 4.2, we will also consider the case where the entry costs are so high that hit-and-run will never be profitable, regardless of what price the incumbent

firms would charge.

The paper is organized as follows. In section 2 we introduce the basic assumptions and consider briefly the one-shot game. Section 3 offers the formal framework for analysing two-period-dependent supergames. We define *simple strategy profiles* for these games and recall a proposition from [Req90a] which generalizes ABREU'S main proposition from [Abr88] to this class of games. We recommend to skip this section at first reading and to continue with Section 4, where we investigate the dynamic game and offer the main results which characterize all stationary subgame perfect equilibrium outcomes under different conditions. Section 5 contains an example. For quadratic cost functions we are able to give explicit upper and lower bounds for the equilibrium prices as a function of  $N$  and the discount factor  $\delta$ , and also bounds for possible equilibrium numbers of firms. We close with some concluding remarks in section 6. The proofs of the main results are given in the appendix.

## 2 The Static Game

We consider a market for a homogeneous commodity supplied by  $n \geq 1$  identical firms and demanded by a continuum of identical consumers represented by the closed interval  $[0, 1]$ . The *technology* of a typical firm is given by its (total) cost function

$$c(q) = F + v(q) \quad (2.1)$$

with  $v(0) = 0, v' > 0, v'' > 0$ , where  $q \geq 0$  denotes the quantity produced, and  $F > 0$  is the fixed cost,  $v > 0$  is the variable cost function, which exhibits increasing marginal cost.

The *preferences* of a typical consumer are given by her demand function

$$d(p) = \begin{cases} 1 & \text{for } p \leq L \\ 0 & \text{otherwise} \end{cases} \quad (2.2)$$

where  $p \geq 0$  denotes the price to be paid for the commodity, and  $L > 0$  the consumer's *reservation price*.

The *market game* is played as follows: each firm  $i$  announces a price  $p_i$  at which it is willing to sell a certain quantity. In this case its profit is given by

$$p_i q_i - c(q_i) = p_i q_i - v(q_i) - F, \quad (2.3)$$

where  $q_i$  is the quantity sold by firm  $i$  (which will be determined precisely below). Clearly, it will not be profitable to charge a price above the reservation price  $L$ , so that

we need only consider prices in the closed interval  $[0, L] \subset \mathbb{R}$ . If a firm's price is too high, it may happen that its demand is zero and it will produce nothing. In this case its profit equals  $-F$ . Hence, each firm has the option to be not active (n.a.), which yields it a zero profit. A typical firm's strategy space can thus be written as

$$S_i := [0, L] \cup \{\text{n.a.}\},$$

where  $s_i = p_i \in [0, L]$  means that firm  $i$  is active and charges price  $p_i$ , and  $s_i = \text{n.a.}$  means that firm  $i$  is not active.<sup>1</sup> The joint strategy space is written  $S := \prod_{i=1}^n S_i$ .

The quantity sold by firm  $i$  is determined as follows. The firms announce a price and produce to order. That is, they produce as much as they can sell. At price  $p$  they are not willing to produce more than  $v'^{-1}(p)$ , since otherwise marginal cost exceeds the price. Thus  $v'^{-1}(p)$  can be considered as the self imposed capacity constraint at price  $p$ . All the customers are perfectly informed about the prices charged by the various firms and try to buy from the cheapest firm(s). If there are several customers split up equally among these. First all customers place their orders with the cheapest firm(s). These orders are fulfilled, until the cheapest firm(s)' capacity is exhausted. The remaining (unserved) customers now place their orders with the next cheapest firm(s) (again splitting up equally, if there are several). This procedure is repeated until either all the customers are served or all firms are exhausted. The *rationing scheme*<sup>2</sup> induced by this mechanism is formalized as follows: Assume (w.l.o.g. by symmetry) that the firms  $i = 1, \dots, N$  are active ( $0 \leq N \leq n$ ), charging prices  $p_1, \dots, p_N$ , and the remaining firms  $N + 1, \dots, n$  are not active. We write the strategy  $n$ -tuple with  $N$  active firms as

$$s = (s_1, \dots, s_n) = (p_1, \dots, p_N, \underbrace{\text{n.a.}, \dots, \text{n.a.}}_{n-N}) =: \vec{p}^N$$

Then the residual demand faced by firm  $i$  is<sup>3</sup>

$$D_i(\vec{p}^N) = \frac{\max\{1 - \sum_{j:p_j < p_i} v'^{-1}(p_j), 0\}}{\#\{j \in \{1, \dots, N\} : p_j = p_i\}}. \quad (2.4)$$

The quantity actually sold by firm  $i$  is

$$q_i(\vec{p}^N) = \min\{D_i(\vec{p}^N), v'^{-1}(p_i)\}. \quad (2.5)$$

<sup>1</sup>Notice that if there are no positive fixed costs, producing nothing would yield zero profits. So the difference between activity and non-activity vanishes in that case, and the extension of the strategy space by the element "n.a." would not be necessary.

<sup>2</sup>By the simple demand structure this rationing scheme coincides with efficient rationing as well as with proportional rationing.

<sup>3</sup> $\#A$  is the cardinality of the set  $A$ .

For an action vector  $s = (s_1, \dots, s_n)$  the profit of firm  $i$  is given by

$$\pi_i(s) = \begin{cases} s_i \cdot q_i(\bar{p}^N) - v(q_i(\bar{p}^N)) - F & \text{for } s_i \in [0, L] \\ 0 & \text{for } s_i = \text{n.a.} \end{cases} \quad (2.6)$$

This defines an  $n$ -player game  $G$  with strategy spaces  $S_i = [0, L] \cup \{\text{n.a.}\}$  and payoff functions  $\pi_i$  given by (2.6). If we wish to single out player  $i$ , we write  $s = (s_i, s_{-i})$  where  $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ .

Let  $AC(q) = \frac{v(q)+F}{q}$  be the average cost of producing  $q$  units of the commodity. Since  $v'' > 0$ ,  $AC$  has a unique minimum and we can define the "competitive price" by  $p_c := \min_q AC(q)$  (i.e. minimum average cost). The competitive output is defined by  $q_c := AC^{-1}(p_c)$ . Define  $m := \frac{1}{q_c}$ . For technical simplicity we make

**Assumption 1**  $m \in \mathbb{Z}$ .

Next we make the "contestability assumption", which is the crucial assumption in this model.

**Assumption 2**  $L \cdot \frac{1}{n} - v\left(\frac{1}{n}\right) - F < 0$ .

It says that the number of potential firms is large enough such that not all the  $n$  potential firms can profitably operate in the market, even if they all charge the monopoly price, which is equal to the reservation price in our model. One could also assume the number of firms to be infinite. But Assumption 2 is sufficient for our purposes.

**Assumption 3**  $L \geq p_c$ .

Assumption 3 is a necessary condition for the market to exist at all.

For simplicity we further assume

**Assumption 4**  $v'^{-1}(L) \leq 1$ .

Assumption 4 claims that the market is large enough such that a single firm's self imposed capacity constraint at the monopoly price is not greater than the whole market.<sup>4</sup>

Unless explicitly stated otherwise, Assumptions 1 – 4 will be maintained throughout this paper.

**Proposition 2.1** *If  $L > p_c$ , there is no Nash-equilibrium (in pure strategies).*

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<sup>4</sup>This is not a serious restriction but avoids tedious distinctions of several cases in the dynamic game.



**Proof:** The proof is simple and will only be sketched. By Assumption 2, not all the firms can be active in equilibrium. Further, there can be no single price equilibrium. For, at a price  $p$  higher than  $p_c$  it would pay for an inactive firm to enter the market by undercutting the incumbent firms by  $p - \epsilon$ , thus earning positive profits for  $\epsilon$  sufficiently small. If the equilibrium price equals  $p_c$ , it will pay for an active firm to charge a higher price, say  $L$ , exercising monopoly power on a market share of  $q_c$ .

At an asymmetric price configuration, there must be at least one firm with a price lower than  $L$  which exhausts its capacity, by Assumption 4. Such a firm can always increase its profit by raising its price slightly. Q.E.D.

Of course, if we assume  $L = p_c$ , we get a trivial result:

**Proposition 2.2** *If  $L = p_c$ , there is a unique equilibrium with  $p^* = p_c = L$  and  $N^* = m$ .*

**Proof:** Trivial, since  $L$  is the only price providing nonnegative profits. Q.E.D.

By the definition of the strategy space and by Assumption 2, these equilibria are similar to what in the literature is sometimes called a *free entry equilibrium*.

### 3 Supergame-theoretic Preliminaries

Since in our market supergame, to be defined in the next section, the payoff depends not only on the joint action tuple of the current period, but also on the own action in the last period, we offer a general framework for 2-period dependent payoffs which can be generalized to several periods with some effort. FRIEDMAN [Fri74, Fri86] calls this kind of games "time-dependent supergames".<sup>5</sup>

$I = \{1, \dots, n\}$  denotes the set of players with typical element  $i \in I$ . In each period of the game,  $S_i$  is the action space of player  $i$ , with  $s_i \in S_i$ .  $S = \prod_{i=1}^n S_i$  denotes the joint action space. The payoff function is denoted by  $\tilde{\pi}_i : S \times S \rightarrow \mathbb{R}$ ,  $i \in I$ , with  $(s', s) \mapsto \tilde{\pi}_i(s'; s)$ . It depends on the joint actions of two periods, the previous and the current one.

**Assumption 5**  $\{\tilde{\pi}(x) \mid x \in S^2\}$  is bounded.

We introduce the following notations and definitions.  $h = [h^1, \dots, h^l] \in S^l \subset H = \bigcup_{l=1}^{\infty} S^l$  is a history of length  $l \geq 1$ . The game has infinitely many periods

<sup>5</sup>We prefer the notion multi-period (2-period) dependent supergames, since the payoff depends only on the actions, not on the absolute time.

$t = -1, 0, 1, 2, \dots$ . The game starts in period  $t = 0$ .  $t = -1$  gives the initial state of the game, which is needed since  $\tilde{\pi}_i$  depends on two periods. For  $l \geq 2$  we write  $h = [h^-, h^l]$ , where  $h^- \in S^{l-1}$  is the  $(l-1)$ -history with which  $h$  begins.  $c := \{c(\tau)\}_{\tau=0}^{\infty} \in \Omega := S^{\infty}$  denotes an action path. For  $i \in I$  let  $c^i$  denote the *punishment path* in order to punish player  $i$ . We write  $C := (c^1, \dots, c^n) \in \Omega^n$ . Strategies are written  $\sigma_i \in \Sigma_i := S_i^H = \{\sigma_i : H \rightarrow S_i\}$ . Then  $\sigma := (\sigma_i)_{i \in I} \in \Sigma := \prod_{i \in I} \Sigma_i$  denotes a strategy profile.

For  $(h, \sigma) \in H \times \Sigma$  we define the path  $c[h, \sigma] \in \Omega$  by

$$\begin{aligned} c[h, \sigma](0) &= \sigma(h), \\ c[h, \sigma](t) &= \sigma(h, c(0), \dots, c(t-1)) \quad t \geq 1. \end{aligned}$$

$c[h, \sigma]$  is the path induced by strategy  $\sigma$  after history  $h$ .

For  $0 < \delta < 1$ , the discount factor, we define

$$v_i(h, c, t) := v_i(h^l, c, t) := \tilde{\pi}_i(h^l; c[h, \sigma](t)) + \sum_{\tau=1}^{\infty} \delta^{\tau} \tilde{\pi}_i(c[h, \sigma](t + \tau - 1); c[h, \sigma](t + \tau))$$

$$v_i(h, c) := v_i(h, c, 0).$$

$$\tilde{v}_i(h, \sigma) := v_i(h, c[h, \sigma]).$$

$v_i(h, c, t)$  is the value (to player  $i$ ) of the path  $c$  beginning with its  $t$ 's element after history  $h$ .  $\tilde{v}_i(h, \sigma)$  is player  $i$ 's discounted payoff if strategy  $\sigma$  is played after history  $h$ .

We write:  $\tilde{v} := (\tilde{v}_i)_{i \in I}$ ,  $v := (v_i)_{i \in I}$ . Finally,  $\Gamma(h) := (\Sigma_i, \tilde{v}_i(h, \cdot))_{i \in I}$  denotes the supergame after history  $h$  and  $\Gamma := (\Gamma(h))_{h \in H}$  denotes the whole supergame.

**Definition 3.1**  $\sigma^* \in \Sigma$  is a Nash-equilibrium of  $\Gamma(h)$ , if

$$\tilde{v}_i(h, (\sigma_i, \sigma_{-i}^*)) \leq \tilde{v}_i(h, \sigma^*) \quad \forall \sigma_i \in \Sigma_i, i \in I.$$

**Definition 3.2**  $\sigma^* \in \Sigma$  is a (subgame) perfect equilibrium of  $\Gamma$  if  $\sigma^*$  is a Nash-equilibrium of  $\Gamma(h)$ ,  $\forall h \in H$ .

**Definition 3.3** We say that  $i \in I$  deviated singly from  $\sigma \in \Sigma$  in the last period of  $h$  with  $h = [h^-, h^l]$ , if  $h_i^l \neq \sigma_i(h^-)$  and  $h_{-i}^l = \sigma_{-i}(h^-)$  ( $l \geq 2$ ).

Clearly, there is at most one  $i \in I$ , who deviated singly.

**Definition 3.4** An initial path is a mapping  $q^0 : S \rightarrow \Omega$ . For  $i \in I$ , a punishment path to punish player  $i$  is a mapping  $q^i : S_i \times S \rightarrow \Omega$ .

We write  $Q := (q^0, q^1, \dots, q^n)$ .  $q^0[s] = \{q^0[s](t)\}_{t=0}^\infty$ ,  $q^i[s'_i, s] = \{q^i[s'_i, s](t)\}_{t=0}^\infty$ .

The strategy profile  $\sigma = \sigma[Q] \in \Sigma$  induced by  $Q$  is defined by

$$\sigma(h) = c_Q[h](T_Q(h)), \quad h \in H,$$

where  $c_Q[h] \in \Omega$  (the current path prescribed by  $Q$  after  $h$ ) and  $T_Q(h) \in \{0, 1, 2, \dots\}$  (the counter of the current path) are defined as follows:

$$\left. \begin{aligned} c_Q[h] &= q^0[h] \\ T_Q(h) &= 0 \end{aligned} \right\} \text{ if } h \in S = S^1.$$

If  $h = (h^-, h^l) \in S^l$  with  $l \geq 2$ , assume inductively that  $c_Q, T_Q$  (and hence  $\sigma$ ) are already defined for  $h^- \in S^{l-1}$ , and put

$$\left. \begin{aligned} c_Q[h] &= q^i[h_i^{l-1}, h^l] \\ T_Q(h) &= 0 \end{aligned} \right\} \text{ if } i \in I \text{ deviated singly from } \sigma \text{ in the last period of } h;$$

$$\left. \begin{aligned} c_Q[h] &= c_Q[h^-] \\ T_Q(h) &= T_Q(h^-) + 1 \end{aligned} \right\} \text{ otherwise.}$$

**Definition 3.5** A strategy profile  $\sigma \in \Sigma$  is called simple if it is induced by some  $Q = (q^0, q^1, \dots, q^n)$ , where  $q^0$  is an initial path and  $q^i$  is a punishment path, for  $i \in I$ .

Loosely speaking, a simple strategy profile  $\sigma = \sigma[Q]$ ,  $Q = (q^0, q^1, \dots, q^n)$  means the following. Initially play the path  $q^0$ . As long as no player deviates from  $q^0$ , continue with  $q^0$ . If player  $i$  deviates singly from  $q^0$ , continue with the punishment path  $q^i$ . If player  $j$  deviates singly from  $q^i$ , continue with the punishment path  $q^j$ , and so on.

Simple strategy profiles will serve to characterize perfect equilibrium outcomes of the supergame. In the purely repeated case (ABREU [1988]), it suffices to consider constant mappings  $q^i$ , i.e. the punishment paths do not depend on the history. Unfortunately, in the multi-period dependent case this will not work in general and we have to consider history dependent punishment paths. Although our punishment paths  $q^i[s'_i, s]$  are not simple in the sense of ABREU, at least they do not depend on the full history but only on (at most) two preceding periods: the situation in which the deviation occurred ( $s$ ) and the deviating player's own action ( $s'_i$ ) in the period immediately before the deviation.

Let  $\sigma = \sigma[Q]$  be a simple strategy profile. We say that  $s' \in S$  can precede  $q \in S$  if  $\exists h \in H$  with  $h^l = s'$  such that  $\sigma(h) = q$ .

**Proposition 3.1** *Let  $\sigma = \sigma[Q]$  be a simple strategy profile,  $Q = (q^0, q^1, \dots, q^n)$ . Then  $\sigma$  is a (subgame) perfect equilibrium iff*

$$\begin{aligned} \forall i \in I, \forall j \in \{0, 1, \dots, n\}, \forall \tau \geq 0, \forall \bar{s} \in S, \forall \bar{s}'_j \in S_j \text{ with } j \neq 0, \\ \forall s' \in S \text{ which can precede } q := q^j[\bar{s}'_j, \bar{s}](\tau), \forall s_i \in S_i \setminus \{q_i\} : \\ \tilde{\pi}_i(s'; (s_i, q_{-i})) + \delta v_i((s_i, q_{-i}), q^i[s'_i, (s_i, q_{-i})]) \leq v_i(s', q^j[\bar{s}'_j, \bar{s}], \tau) \end{aligned} \quad (3.1)$$

where  $q^j[\bar{s}'_j, \bar{s}] = q^0[\bar{s}]$  for  $j = 0$ .

To understand the proposition, consider a situation in which after a history of the form  $(h^-, s')$  the action prescribed by the simple strategy  $\sigma(Q)$  is given by  $q := q^j[\bar{s}'_j, \bar{s}](\tau)$ . If all players follow the prescribed strategy, player  $i$ 's payoff is given by the RHS of (3.1). If player  $i$  deviates singly from  $q_i$  to  $s_i$ , and afterwards all players follow the prescribed punishment path  $q^i[s'_i, (s_i, q_{-i})]$ , then  $i$ 's payoff is given by the LHS of (3.1). Condition (3.1) says that such single-period deviations must not pay.

**Remark 3.1** *Because of the large quantity of  $\forall$  quantifiers, this proposition does not seem to be very useful, since one has to check too many inequalities. But in certain cases, (3.1) simplifies considerably, for example if the paths are piecewise constant on  $S_i \times S$  for  $i \in I$ , and on  $S$  for  $q^0$ , as it will be the case in our market supergame.*

A detailed proof of Proposition 3.1 can be found in [Req90a]. Notice that this result generalizes Proposition 1 in ABREU [Abr88] to the case of 2-period dependent supergames.<sup>6</sup>

## 4 The Dynamic Market Game

Since there is no pure strategy equilibrium in the one shot game apart from special examples, we are now interested in the question whether there is a stationary equilibrium in pure strategies in a suitable dynamic game. Since we retain Assumption 2, the answer will be "no", if we consider the purely repeated game. We therefore introduce a fixed entry cost, which will guarantee us the existence of stationary equilibria under certain conditions. Definitions and notations from Section 2 are taken for granted.

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<sup>6</sup>The idea of the proof in principal does not differ from that in ABREU's result: If infinitely many deviations are profitable, then a sufficiently large number of finitely many deviations must be profitable, since by discounting, the rest sum of profits becomes arbitrarily small. Assume there are  $n$  deviations. But the  $n - th$  deviation amounts to a single period deviation. Since this is not profitable, conforming is as least as profitable for the player than deviating. But then the  $(n - 1)th$  period deviation amounts to a single period deviation and so on until the first deviation is reached, a contradiction.

## 4.1 The Model

The dynamic market game consists of an infinite repetition of the static market game introduced in Section 2, with the one additional feature that a firm has to pay an entry cost  $K > 0$  in every period  $t$  if the firm is active in period  $t$  but has been inactive in period  $t - 1$  (see eqn. (4.2) below). This makes the payoff function 2-period dependent and yields a supergame of the form introduced in Section 3. The entry cost can be considered as an investment cost, which is necessary to start a business. In the static case, investment cost and other fixed costs can be added to one whole fixed cost. The dynamic case calls for differentiation of the fixed cost into those that have to be paid in each period if the firm is active and those to be paid only if a firm enters the market. All the assumptions about the consumers hold in each period: each customer demands exactly one unit of the homogeneous commodity in every period up to the reservation price  $L$ . The commodity is *not* durable. The customers are always perfectly informed about the prices in each period, they try to place their orders always with the cheapest firms and split up equally if there are several and they do not become biased towards certain firms. Assumption 1 – 4 also hold throughout this section.

Denote by  $\vec{s}(t)$  the action vector of period  $t$ . Let

$$\pi_i(\vec{s}(t)) = \begin{cases} s_i(t) \cdot q_i(\vec{s}(t)) - v(q_i(\vec{s}(t))) - F & \text{if } s_i(t) \in [0, L] \\ 0 & \text{if } s_i(t) = \text{n.a.} \end{cases} \quad (4.1)$$

Since we assume that a firm has to pay a fixed entry cost of size  $K$ , if it enters or reenters the market, the single period payoff function does not only depend on the joint action vector of the current period, but also on the own action in the last period. This means, we get a 2-period dependent payoff function  $\tilde{\pi}_i : S_i \times S \rightarrow \mathbb{R}$  with

$$\tilde{\pi}_i(s_i(t-1); \vec{s}(t)) = \begin{cases} \pi_i(\vec{s}(t)) - K & \forall s_i(t) \in [0, L] \text{ and } s_i(t-1) = \text{n.a.} \\ \pi_i(\vec{s}(t)) & \text{otherwise} \end{cases} \quad (4.2)$$

where  $\pi$  is defined in (4.1). The game starts in period 0.  $\hat{s}$  is the initial state of the game in period  $-1$ . If  $c = \{c(t)\}_{t=0}^{\infty} \in \Omega$  is an action path, the whole payoff over all infinitely many periods is given by

$$v_i(\hat{s}, c) := \tilde{\pi}_i(\hat{s}_i; c(0)) + \sum_{t=1}^{\infty} \delta^t \tilde{\pi}_i(c_i(t-1); c(t)) \quad \text{with } 0 < \delta < 1 \quad (4.3)$$

Since in the dynamic game, the set of active firms at the beginning of the game may differ from the set of active firms after some deviations, we cannot assume that always the first  $N$  firms are the active ones, as we have done in the stage game. We therefore

write for a price  $p \in [0, L]$ , and a subset of firms  $A \subseteq I$ :

$$(\vec{p})^A = (s_1, \dots, s_n) \quad \text{with } s_i = \begin{cases} p & \forall i \in A \\ \text{n.a.} & \forall i \in I \setminus A \end{cases}$$

Next we introduce some abbreviations, which will be used frequently in the remainder of the paper.

If  $N$  firms are active ( $i \in A$ ) and charge the same price  $p$ , they will get a profit of

$$\pi_i(\vec{p}^N) := \pi_i((\vec{p})^A) := p \cdot \min\left\{\frac{1}{N}, v'^{-1}(p)\right\} - v\left(\min\left\{\frac{1}{N}, v'^{-1}(p)\right\}\right) - F \quad \text{for } |A| = N.$$

The highest profit that can be earned at price  $p$  (that is, at which price equals marginal cost) is denoted by

$$\pi^*(p) = p \cdot v'^{-1}(p) - v(v'^{-1}(p)) - F.$$

Note that under Assumption 4

$$\sup_{p_i < p^*} \pi(p_i, \vec{p}_{-i}^N) = \pi^*(p),$$

this is the upper bound for profits a firm can earn by undercutting a symmetric price outcome  $\vec{p}^N$ .

If the consisting price is small, it might be profitable to deviate to a higher price, and to exercise monopoly power on  $1 - (N - 1)v'^{-1}(p)$  many customers, those who are possibly unserved by the remaining firms that charge  $p$  and that are not willing to sell more than  $v'^{-1}(p)$ . We therefore define

$$\pi^L(\vec{p}^N) := L \cdot \max\{0, 1 - (N - 1)v'^{-1}(p)\} - v(\max\{0, 1 - (N - 1)v'^{-1}(p)\}) - F.$$

Further we define

$$\pi_e^L(\vec{p}^N) := \pi^L(\vec{p}^{N+1}) := L \cdot \max\{0, 1 - N \cdot v'^{-1}(p)\} - v(\max\{0, 1 - N \cdot v'^{-1}(p)\}) - F,$$

which is the profit an entrant earns, if she tries to exercise monopoly power on an unserved market share, if there is any.

**Lemma 4.1**  $\pi^*(p)$  is strictly increasing in  $p$ ,  $\frac{d\pi}{dp}(p) = v'^{-1}(p)$ .

**Lemma 4.2**  $\frac{d^2\pi}{(dp)^2}(p) = \frac{1}{v''(v'^{-1}(p))} > 0$ .

**Proofs:** See [Req90b].

Define

$$p^{ubs} := \max\{p \mid \pi^*(p) \leq K\}.$$

This is the absolute upper bound for equilibrium prices, since at prices lower than  $p^{uabs}$ , firms cannot cover the entry fixed cost  $K$  in only one period, even if they sell the optimal output of  $v'^{-1}(p)$ . Therefore,  $p^{uabs}$  is the lower bound for all prices that make possible or encourage hit-and-run. Clearly, a price higher than  $p^{uabs}$  cannot be an equilibrium price, since at those prices, it is always profitable for inactive firms to enter the market once by undercutting, making a positive profit net entry cost, and leaving the market again in the next period.

By Lemma 4.2,  $\pi^*(p)$  increases faster than linearly in  $p$ , whereas  $\pi(\bar{p}^N)$  increases linearly in  $p$ . Hence, and since  $\pi^*(p)$  and  $\pi(\bar{p}^N)$  are continuous in  $p$ , we define for all  $N$  for which  $\exists p$  with  $\pi^*(p) \leq \frac{1}{1-\delta} \pi(\bar{p}^N)$ :

$$\tilde{p}^u(N) := \max \left\{ p \mid \pi^*(p) \leq \frac{1}{1-\delta} \pi(\bar{p}^N) \right\}. \quad (4.4)$$

Further let

$$p^u(N) := \min \{ p^{uabs}, \tilde{p}^u(N) \}.$$

Finally we define  $\tilde{K} := (1-\delta)K$  and  $\tilde{F} := F + \tilde{K}$ , where  $\tilde{K}$  is that part of the entry cost that has to be paid in each period, if a firm stays in the market from the beginning of the game and distributes the entry cost equally over all periods. In this case, the fixed cost in each period adds to  $\tilde{F} = F + \tilde{K}$ . Hence we define also

$$\widetilde{AC}(q) := \frac{v(q) + \tilde{F}}{q}, \quad \tilde{p}_c := \min_q \widetilde{AC}(q), \quad \tilde{q}_c := v'^{-1}(\tilde{p}_c) \quad \text{and} \quad \tilde{m} := \frac{1}{\tilde{q}_c}.$$

**Definition 4.1**  $(p, A^0)$  is called a stationary (quasi-symmetric) equilibrium outcome (st.e.o.) iff there is a subgame perfect equilibrium strategy  $\sigma$ , and a history  $h$  such that  $c[h, \sigma](t) =: s^t(h, \sigma) \quad \forall t \geq 0$ .

That is,  $(p, A^0)$  is a stationary equilibrium outcome if the firms  $i \in A^0$  charge  $p$  in each period and the firms  $i \in I \setminus A^0$  stay "not-active" forever. The equilibrium outcome is called quasi-symmetric, because it is symmetric with respect to the active firms. Since  $|A^0|$ , the number of active firms, often is the interesting variable, we sometimes talk of  $(p, N)$  as of an equilibrium outcome, if  $N = |A^0|$ .

## 4.2 The Main Results

We are now ready to state our first result, which gives a full characterization of the stationary quasi-symmetric equilibrium outcomes.

Define the region

$$R_1 := \{ (p, N) \in [0, L] \times \mathbb{N} \mid (p, N) \text{ satisfies (4.5) - (4.8) given below} \}$$

$$p \leq p^{u_{abs}} \quad (4.5)$$

$$\pi_c^L(\vec{p}^N) \leq K \quad (4.6)$$

$$\pi^*(p) \leq \frac{1}{1-\delta} \pi(\vec{p}^N) \quad (4.7)$$

$$\pi^L(\vec{p}^N) \leq \frac{1}{1-\delta} \pi(\vec{p}^N) \quad (4.8)$$

Since the equilibrium conditions are not linear in  $p$  and  $N$ , we cannot solve for prices and numbers of firms in general. For quadratic cost functions, the inequalities (4.5) — (4.8) are solvable for  $p$ . But for (4.6) and (4.8) this is tedious even in this special case.<sup>7</sup> (See section 5.)

**Theorem 4.1** *Let  $p^* \in [0, L]$  and  $A^* \subset I$  be given. Set  $N^* := |A^*|$ . Assume*

$$p^{u_{abs}} < L, \quad (4.9)$$

$$(p^{u_{abs}}, m) \in R_1, \quad (4.10)$$

and

$$\delta \geq \max \left\{ \frac{K}{[p^{u_{abs}} - p_c]q_c + K}, \frac{L - p_c}{L + p^{u_{abs}} - 2p_c} \right\}. \quad (4.11)$$

- a) *If  $\hat{s}_i \neq n.a.$   $\forall i \in A^*$ , then  $(p^*, A^*)$  is a stationary subgame perfect quasi-symmetric equilibrium outcome if and only if  $(p^*, N^*) \in R_1$ .*
- b) *If  $\exists i \in A^*$  such that  $\hat{s}_i = n.a.$ , then  $(p^*, A^*)$  is a stationary subgame perfect quasi-symmetric equilibrium outcome if and only if  $(p^*, N^*) \in R_1$  and additionally satisfies*

$$p^* \geq \widetilde{AC} \left( \frac{1}{N^*} \right) \quad \text{if} \quad N^* > \widetilde{m}, \quad (4.12)$$

$$p^* \geq \tilde{p}_c \quad \text{if} \quad N^* \leq \widetilde{m}. \quad (4.13)$$

To understand the theorem let us start with the assumptions. (4.9) guarantees that the highest possible equilibrium price is below the monopoly price. In Theorem 4.3 and 4.4 we will relax this assumption. But (4.9) is the more interesting case. (4.10) guarantees that  $(p^{u_{abs}}, m)$  satisfies the equilibrium conditions. Notice that (4.10) already guarantees existence of an equilibrium at all. As we will see in the proof, this

<sup>7</sup>A simple sufficient condition for (4.6) and (4.8) to hold is  $p \geq v' \left( \frac{1}{N-1} \right)$ , since then even  $N-1$  firms can serve (are willing to serve) the whole market. Hence a firm that charges a higher price has no customers. Since  $v' \left( \frac{1}{N-1} \right) > v' \left( \frac{1}{N} \right)$ , also an intruder cannot capture any customers by charging a higher price than the incumbent firms' one.



is necessary, since equilibria with  $N < m$  can only survive if there are equilibria with  $N \geq m$ . (4.11) is sufficient that the equilibrium strategy, to be constructed in the proof, will work. It is not necessary for existence of an equilibrium.

The equilibrium conditions (4.5) and (4.6) prevent inactive firms from hit-and-run. If (4.5) is satisfied, undercutting and running away does not pay, as mentioned above. By (4.6) charging a higher price does not pay. (4.7) and (4.8) are sufficient and necessary to prevent *active* firms from deviating to a lower price, or to a higher price, respectively. The LHSs of (4.7) and (4.8) are the suprema of profits earned by undercutting, or charging a higher price, respectively. The RHS of (4.7) and (4.8) is the discounted sum of profits earned if no firm deviates from the initial equilibrium path. This leads to the supposition that firms can be always held down to zero after any deviation, by a subgame perfect equilibrium strategy. Indeed this is the fact, and for  $N > m$  this is not so much of a problem. The punishment strategies work as follows. If an *active* firm deviates by charging a price different from  $p^*$ , all the active firms charge a low price for a certain number of periods, yielding negative profits per period, and charge the highest possible equilibrium price ( $p^u(N)$ ) afterwards forever, such that the overall discounted sum of profits earned after deviation is zero. Notice that this kind of punishment has stick-and-carrot character (cf. [Abr86,Lam87], see also [Req90b] for the purely repeated version of this game). For  $N \leq m$  this kind of strategy does not work in this way. That is, the (up to now) active firms cannot hold the deviator down to zero by their own. They rather need help by some of the (up to now) inactive firms, that is, a certain number of these firms has to enter the market, filling the number of active firms up to the size of  $m$ . All these from now on active firms drive the deviator out of the market by charging  $p_c$  for one period, and going to  $p^{uab}$  after that forever, unless another firm (including the former deviator) deviates another time.

*Intruders*, that is deviators from "n.a.", will always be driven out of the market again. If  $N \geq m$ , this will be done by the  $N$  originally active firms. If  $N < m$ , help is needed again by some of the inactive firms, which have to enter the market. Since this strategy can only work if there is a pair  $(p, m)$  that already forms a stationary equilibrium, and since entry must yield nonnegative profits for those former inactive firms that are supposed to enter the market in order to punish the deviator, we require (4.10) and (4.11) to hold. The exact proof is given in the appendix. In part b) of the theorem, (4.12) and (4.13) guarantee nonnegative profits for firms that enter the market in period 0.

The punishment strategy, just described informally, thus presupposes a kind of contract between the  $N$  active firms and at least  $m - N$  inactive firms. This is unsatisfactory and not very much plausible. Therefore, it seems to be reasonable to concentrate on

those equilibria, which do not require (former) inactive firms to engage in punishment.

**Definition 4.2**  $(p, A)$  is a reasonable equilibrium outcome (r.e.o.), if for any deviation to an action  $\neq n.a.$  there is a subgame perfect equilibrium strategy which does not require a new firm to become active. Formally:

$(p, A)$  is a r.e.o. if there is a subgame perfect strategy  $\sigma$ , s.t.  $\sigma(\hat{s}) = (\bar{p})^A$  and  $\forall h \in H$ , with  $l \geq 2$  for which  $\exists i \in I$  s.t. "i deviated singly in  $h^l$ " and  $n.a. \neq h_i^l \neq \sigma_i(h^-)$ , we have  $\sigma_j(h) = n.a. \forall j$  with  $h_j^l = n.a.$ .

Why do we exclude deviations that amount in exit (deviations to "n.a.")? Now, to require inactive firms to *never* engage in punishment is a bit too strong. Since there is the option to be not active, firms could leave the market by and by until only few firms are left that exercise monopoly power. Of course, there is no incentive to leave the market, when playing the initial equilibrium path. But there is a myopic incentive to leave the market during a punishment path, when negative profits have to be earned for some periods. In this case, entry sometimes has to be required on a subgame perfect equilibrium path, that is, the leaving firm has to be substituted by a former inactive firm.

Before characterizing the set of *reasonable equilibrium outcomes* some auxiliary results and definitions are necessary:

**Lemma 4.3** Let  $S'_i = [0, L] \forall i \in I$  (there is no option to be inactive for a moment). If  $L > v' \left( \frac{1}{N} \right)$  (which is satisfied under Assumption 4),

i) there is a unique  $p^{\text{pun}} \in S'_i$  such that

$$\pi_i(L, \vec{p}_{-i}^{\text{pun}}) = \sup_{p_i < p^0} \pi_i(p_i, \vec{p}_{-i}^{\text{pun}}), \quad \text{where } \vec{p}_{-i}^{\text{pun}} = \underbrace{(p^{\text{pun}}, \dots, p^{\text{pun}})}_{n-1} \quad (4.14)$$

ii)  $\arg \max_{p_i \in S'_i} \pi_i(p_i, \vec{p}_{-i}^{\text{pun}})$  exists,

iii)  $v'^{-1} \left( \frac{1}{N} \right) < p^{\text{pun}} < v'^{-1} \left( \frac{1}{N-1} \right)$ .

**Proof:** See [Req90b].

Lemma 4.3 claims that there is a price  $p^{\text{pun}}$  such that the best response against  $\vec{p}_{-i}^{\text{pun}}$  is to charge the monopoly price. Actually the player who is to be punished is (almost) indifferent between charging the monopoly price and undercutting  $p^{\text{pun}}$  slightly.

**Lemma 4.4** For all  $p_{-i} \in S'_{-i} := [0, L]^{N-1}$  we have

$$\sup_{p_i \in S'_i} \pi_i(p_i, \vec{p}_{-i}^{\text{pun}}) \leq \sup_{p_i \in S'_i} \pi_i(p_i, p_{-i})$$

**Proof:** See [Req90b].

Lemma 4.4 claims that  $\vec{p}_{-i}^{pun}$  minimizes  $\sup_{p_i \in S_i} \pi_i(p_i, \cdot)$  among all punishment vectors  $\vec{p}_{-i} \in S^{N-1}$ .

In the Lemmata 4.3 and 4.4,  $N$  was considered as fixed. But of course, the value of  $p^{pun}$  depends on  $N$ , the number of active firms. In the following Lemmata we, therefore, write  $p^{pun}(N)$  instead of  $p^{pun}$ .

**Lemma 4.5** For all  $N \geq 2$ ,  $p^{pun}(N)$  is decreasing in  $N$ .

**Proof:** Clear from Lemma 4.3.iii) and Lemma 4.2. Q.E.D.

**Lemma 4.6**  $\pi^*(p^{pun}(N))$  is decreasing in  $N$ .

**Proof:** Clear from Lemmata 4.1 and 4.5. Q.E.D.

The next result characterizes the set of stationary equilibrium outcomes, if nonactive firms cannot be employed to punish a deviator, that is, the set of *reasonable st.e.o.*'s.

Define the region

$$R_2 := \{(p, N) \in [0, L] \times \mathbb{N} \mid (p, N) \text{ satisfies (4.15) - (4.16) given below } \}$$

$$\pi^*(p) \leq K - \max \left\{ 0, \frac{\delta}{1-\delta} \pi^*(p^{pun}(N+1)) \right\} \quad (4.15)$$

$$\pi^*(p) \leq \frac{1}{1-\delta} [\pi(\vec{p}^N) - \max \{ 0, \delta \pi^*(p^{pun}(N)) \}] \quad (4.16)$$

We make the following assumption about  $R_2$ :

**Assumption 6** For all  $N < m$  for which there is  $p$  with  $(p, N) \in R_2$ , there is  $p'$  such that  $(N+1, p') \in R_2$ .

We will explain this assumption below.

**Theorem 4.2** Let  $p^* \in [0, L]$  and  $A^* \subset I$  be given. Set  $N^* := |A^*|$ . Assume that Assumption 6 holds. Assume again

$$p^{u_{abs}} < L, \quad (4.17)$$

$$(p^{u_{abs}}, m) \in R_2, \quad (4.18)$$

and

$$\delta \geq \frac{L - p_c}{L + p^{u_{abs}} - 2p_c}. \quad (4.19)$$

- a) If  $\hat{s}_i \neq n.a. \forall i \in A^*$ , then  $(p^*, A^*)$  is a stationary subgame perfect quasi-symmetric outcome of a reasonable equilibrium if and only if  $(p^*, N) \in R_2$ .
- b) If  $\exists i \in A^*$  such that  $\hat{s}_i = n.a.$ , then  $(p^*, N)$  must additionally satisfy (4.12) and (4.13).

Assumption 6 has been made by the following reason. If an inactive firm has deviated unilaterally from the original path of a *reasonable* equilibrium by entering the market, it cannot be driven out again by the incumbent firms only, as long as  $N < m$ . Hence, the game has to go on with the original number of firms plus one. But the new path must end with another possible equilibrium path, that is, there must be an outcome  $(p, N + 1) \in R_2$ . In other words, equilibria with  $N < m$  only exist if there are equilibria with  $N \geq m$ . Thus, Assumption 6 guarantees the existence of a subgame perfect continuation path with  $N + 1$  firms after an intruder happened to enter the market, as long as  $N < m$ . Note that Assumption 6 also implies that  $R_2$  is not empty. The proof is given in the appendix, but we give a sketch of the equilibrium strategies constructed in the proof. For  $N > m$ , the strategy is defined in the same way as in the proof of Theorem 4.1. For  $N \leq m$ , an *active* firm that has deviated will not be driven out but will rather be punished symmetrically by the active firms. However, it can only be held down to  $\frac{\delta}{1-\delta}\pi^*(p^{un}(N)) > 0$  by the remaining  $N - 1$  active firms. If an *inactive* firm deviates by entering the market and  $N$  is not smaller than  $m$ , the intruder can be held down to zero and, therefore, can be driven out of the market, like in the proof of Theorem 4.1. If  $N < m$ , it can be held down only to  $\frac{\delta}{1-\delta}\pi^*(p^{un}(N + 1)) > 0$ , that is, the intruder cannot be driven out by the  $N$  incumbent firms. Thus, by every deviation, caused by an intruder, the number of active firms increases, until  $N = m$  is reached.

**Remark 4.1** *If we require Assumption 6 to hold for all  $N \leq m$  instead of  $N < m$ , we need not assume (4.18) and (4.19). Assumption (4.19) guarantees that for  $N = m$ , it does not pay to deviate to a higher price on the path that drives out an intruder. If Assumption 6 holds also for  $N = m$ , the penal code could be modified by starting to drive out intruders only for  $N \geq m + 1$ .*

The next lemma is useful for the proof of Theorem 4.2.

**Lemma 4.7** *For all  $N \geq 2$ , prices not greater than  $p^{un}(N)$  cannot be stationary prices in a reasonable equilibrium.*

**Proof:** Same as the proof of Lemma 3.5 in [Req90b], since the number of firms is taken as fixed, and there are no inactive firms in that paper.

**Corollary 4.1** *For any stationary price  $p$  in a reasonable equilibrium we have*

$$i) \quad \pi^L(\bar{p}^N) < \pi^*(p),$$

$$ii) \quad \pi_e^L(\bar{p}^N) < \pi^*(p).$$

**Proof:** i) If  $p$  is the outcome of a *reasonable* equilibrium, then  $p > p^{pun}(N)$  by Lemma 4.7. But then  $\pi^*(p) > \pi^L(\bar{p}^N)$ , by definition of  $p^{pun}(N)$ . ii) Clearly,  $\pi_e^L(\bar{p}^N) = \pi^L(\bar{p}^{N+1}) \leq \pi^L(\bar{p}^N)$ . Q.E.D.

In both Theorems, 4.1 and 4.2, the stationary equilibrium prices can never be greater than  $p^{uabs}$  since otherwise hit-and-run would be encouraged. By its definition,  $p^{uabs}$  is increasing in  $K$ . In other words, the lower the entry cost the more competitive is the market. On the other hand, there is no stationary equilibrium in pure strategies if  $K$  is too low. For, if  $K$  is close to zero,  $p^{uabs}$  is close to  $p_c$ . For prices too close to  $p_c$ , however, it becomes profitable to deviate to the monopoly price since the remaining incumbent firms cannot serve the whole market if they charge a price close to the competitive price, that is, (4.8) would be violated. We summarize these arguments in the following

**Proposition 4.1** *For  $K = 0$ , the dynamic game has no stationary equilibrium.*

**Corollary 4.2** *The purely repeated game of section 2 has no stationary equilibrium.*

Up to now we assumed to have relatively small entry costs ( $p^{uabs} < L$ ), or what is the same, sufficiently long periods such that the entry costs could be covered within one period, if the price were sufficiently high. We may ask now what will happen, if we assume that  $L \leq p^{uabs}$  instead of  $L > p^{uabs}$ . Restricting the analysis to *all* possible stationary symmetric outcomes we get a somewhat surprising result.

Define the region

$$R_3 := \{(p, N) \in [0, L] \times \mathbb{N} \mid (p, N) \text{ satisfies (4.20) - (4.21) given below } \}$$

$$\pi^*(p) \leq \frac{1}{1-\delta} \pi(\bar{p}^N) \quad (4.20)$$

$$\pi^L(\bar{p}^N) \leq \frac{1}{1-\delta} \pi(\bar{p}^N) \quad (4.21)$$

**Theorem 4.3** *Let  $p^* \in [0, L]$  and  $A^* \subset I$  be given. Set  $N^* := |A^*|$ . Assume*

$$p^{uabs} \geq L, \quad (4.22)$$

$$(L, m) \in R_3, \quad (4.23)$$

and

$$\delta \geq \frac{K}{[L - p_c]q_c + K} . \quad (4.24)$$

- a) If  $\hat{s}_i \neq n.a. \forall i \in A^*$ , then  $(p^*, A^*)$  is a stationary subgame perfect quasi-symmetric equilibrium outcome if and only if  $(p^*, N^*) \in R_3$ .
- b) If  $\exists i \in A^*$  such that  $\hat{s}_i = n.a.$ , then  $(p^*, N)$  must additionally satisfy (4.12) and (4.13).

This result is in so far surprising since it admits equilibria which leave some customers unserved. As illustrated in Figure 3, also outcomes with  $N \cdot v'^{-1}(L) < 1$  can be supported by an equilibrium strategy. That means, there are inefficient outcomes even at the monopoly price since more firms could operate in the market without doing harm to the incumbent firms which are sold out.

Since  $p^{uabs} \geq L$ , the incumbent firms are not jeopardized by hit-and-run. Of course, also in this case the incumbent firms cannot punish a deviator by their own. Similar to Theorem 4.1, equilibria with  $N < m$  can only be supported by strategies that employ former inactive firms to punish the deviator. In the appendix we call these strategies "drive-out-and-substitute-strategies".

If one does not like these strategies — and indeed they look rather artificial — we should look for "reasonable" equilibrium outcomes again, that is for those that do not rely upon inactive firms to engage in punishment.

Define the region

$$R_4 := \{(p, N) \in [0, L] \times \mathbb{N} \mid (p, N) \text{ satisfies (4.25) - (4.26) given below} \}$$

$$\pi^*(p) \leq K - \max \left\{ 0, \frac{\delta}{1 - \delta} \pi^*(p^{pun}(N + 1)) \right\} \quad (4.25)$$

$$\pi^*(p) \leq \frac{1}{1 - \delta} \left[ \pi(\bar{p}^N) - \max \{ 0, \delta \pi^*(p^{pun}(N)) \} \right] \quad (4.26)$$

Similar to Assumption 6 we make

**Assumption 7** For all  $N < m$  for which there is  $p$  with  $(p, N) \in R_4$ , there is  $p'$  such that  $(N + 1, p') \in R_4$ .

**Theorem 4.4** Let  $p^* \in [0, L]$  and  $A^* \subset I$  be given: Set  $N := |A^*|$ . Assume that Assumption 7 holds and assume

$$p^{uabs} \geq L , \quad (4.27)$$

$$(L, m) \in R_4 , \quad (4.28)$$

and

$$\delta \geq \frac{1}{2}. \quad (4.29)$$

- a) If  $\hat{s}_i \neq n.a. \forall i \in A^*$ , then  $(p^*, A^*)$  is a stationary subgame perfect quasi-symmetric outcome of a reasonable equilibrium if and only if  $(p^*, N) \in R_4$ .
- b) If  $\exists i \in A^*$  such that  $\hat{s}_i = n.a.$ , then  $(p^*, N)$  must additionally satisfy (4.12) and (4.13).

### 4.3 Non-uniqueness of Equilibrium, Equilibrium Selection and Stability

As a typical feature of most supergames, also our Bertrand–Edgeworth market game exhibits a multiplicity (a continuum) of equilibrium outcomes. That is, there is a finite number of firms that can be active in a stationary equilibrium and for each number of active firms there is a continuum (a closed interval) of possible equilibrium prices, or there is no equilibrium at all. Due to the multiplicity of equilibria it has often been argued that players will or have to *agree* on a certain equilibrium. Since a fixed action or outcome path can be supported by different strategies, an agreement is necessary, anyway. The most prominent candidates for equilibrium outcomes to be agreed on are certainly the Pareto–optimal outcomes. Necessary for an outcome  $(p^*, N^*)$  to be Pareto efficient is that  $p^* = p^u(N^*)$ . For  $R_1$  the Pareto–optimal outcomes coincide with all  $(p^u(N), N) \in R_1$ .

For  $R_2$ , which is the set of equilibrium outcomes when there is no need for inactive firms to engage in punishment, this may be different. With a little abuse of notation we define for all  $N$  for which  $\exists p$  s.t.  $(p, N) \in R_2$ :

$$p^u(N) := \max\{p \mid \text{s.t. } (p, N) \in R_2\}.$$

Then, for  $N \geq m$ , all  $(p^u(N), N) \in R_2$  are Pareto–efficient. For  $N < m$ ,  $(p^u(N), N)$  may not be Pareto–optimal for all  $N$ , since the highest price that still deters entry may be so low, that it would be better for the incumbent firms to let new firms enter the market until the "optimal" number of firms (the number that maximizes profits at the highest possible equilibrium price) is reached (see Fig. 2).

The same reasonings apply to  $R_4$ , the equilibrium set of Theorem 4.4, when hit–and–run is never profitable at any price (i.e.  $p^{u_{abs}} \geq L$ ), and inactive firms do never engage in punishment.

Theorem 4.3 also relates to the case  $p^{u_{abs}} \geq L$ . However, inactive firms *may* engage in punishment. Since hit–and–run does not pay at any price, the conditions (4.20) and

(4.21) allow also for outcomes  $(L, N)$  with  $N < \frac{1}{v^{j-1}(L)}$ . For these outcomes not all the customers are served. If  $N < \lfloor \frac{1}{v^{j-1}(L)} \rfloor$ , more firms could operate in the market without taking customers from the incumbent firms. Thus, the set of Pareto-efficient equilibrium outcomes of  $R_3$  consists of  $\{(p, N) \mid p = L, N \geq \lfloor \frac{1}{v^{j-1}(L)} \rfloor\}$ .

### Stability

Regardless of restricting to *reasonable* equilibria or not, for all regions  $R_1 - R_4$  equilibrium outcomes with  $N < m$  can only survive, if there is at least an equilibrium path with  $N = m$ . That is, for  $N < m$ , the number of active firms increases by every intruder until  $N = m$  is reached, since an intruder cannot be held down to zero by the incumbent firms only. One could say that equilibria with  $N \leq m$  are not stable in a certain sense. In [Req90b], we defined the concepts of *number-* and *set-stability*. To recall these concepts informally, an equilibrium outcome is called *number-stable* if the game can go on with the same number of active firms, also after deviations. An equilibrium outcome is called *set-stable* if the game can go on with the same *set* of active firms, unless a firm happened the market voluntarily. Due to these concepts, all equilibrium outcomes with  $N < m$  are neither *set-* nor *number-stable*. For  $N = m$ , the equilibrium outcomes  $\in R_1 \setminus R_2$  are *number-* but not *set-stable*, whereas the equilibrium outcomes  $\in R_2 \cap \{(p, N) \mid N = m\}$  are *number-* and *set-stable*. For  $N \geq m + 1$ , all equilibrium outcomes are *number-* and *set-stable*.

### Possible equilibrium numbers of active firms

Clearly, any number  $N$  for which there is a price satisfying the equilibrium conditions of the related regions  $R_1 - R_4$  is possible. Thus, we can endogenize the possible number of firms that can be active in equilibrium only up to a certain degree by giving upper and lower bounds. By the nonlinearity of the equilibrium conditions these bounds can only be given implicitly.

Notice that for the equilibrium regions  $R_1 - R_4$ , the inequalities (4.7), (4.15), (4.21) and (4.25) for  $N > m$  all take the form

$$\pi^*(p) \leq \frac{1}{1 - \delta} \pi(\bar{p}^N) . \quad (4.1)$$

Now, the following lemma holds:

**Lemma 4.8** *Let  $N > m$ . If  $p$  satisfies (4.1), then  $p > AC\left(\frac{1}{N}\right)$ .*

**Proof:** See [Req90b].



By this lemma the lower bound for equilibrium prices must be greater than the average cost, when firms have an equal market share of  $\frac{1}{N}$ . Hence they must earn positive profits in equilibrium. This in turn means that the greatest number of firms that can be active together in equilibrium is at most as high as the "zero-profit-number" (the number  $N$  that satisfies  $L \cdot \frac{1}{N} - v\left(\frac{1}{N}\right) - F = 0$ ). For discount factors sufficiently bounded away from one the greatest number of firms for which there is a price  $p$  satisfying (4.1) will be strictly smaller than the zero-profit-number. Notice that the oligopoly argument, often used in static models, that firms will enter the market as long as profits are positive, is no longer valid in a dynamic model. This calls also into question the method of endogenizing the number of firms in the supergame model of BROCK and SCHEINKMAN [BS85], who assume that firms will enter the market as long as profits at the collusive outcome are positive.

About the lower bounds we could simply say the same: it is the lowest number of firms for which there is a price satisfying the conditions of the related regions. If the presuppositions of Theorem 4.3 are satisfied, this number is even zero. But such an equilibrium is unlikely to be agreed on. At least the lowest number for which a Pareto optimal outcome is possible seems to be a better prediction. In the long run, however, set-stable equilibrium outcomes seem to be likeliest. Among those the minimum number of firms is  $m$  or  $m + 1$ .

## 5 An example (Quadratic Cost Functions)

Let  $C(q) = \frac{\alpha}{2}q^2 - F$ . This yields  $v'^{-1}(p) = \frac{p}{\alpha}$ , hence  $p_c = \sqrt{2F\alpha}$ ,  $q_c = \sqrt{\frac{2F}{\alpha}}$ ,  $\pi^*(p) = \frac{p^2}{2\alpha} - F$ , and for  $v'^{-1}(p) > \frac{1}{N}$  we get

$$\pi(\vec{p}^N) = p \frac{1}{N} - \frac{\alpha}{2N^2} - F$$

Consider first the region  $R_1$ :

$$\pi^*(p) \leq K \Leftrightarrow p \leq \sqrt{2\alpha(F + K)} \quad (5.1)$$

$$\pi_e^L(\vec{p}^N) \leq K \Leftrightarrow p \geq \frac{1}{N} \left[ \alpha - L + \sqrt{L^2 - 2\alpha(F + K)} \right] \quad (5.2)$$

Condition (4.7), that is,  $\pi^*(p) \leq \frac{1}{1-\delta} \pi(\vec{p}^N)$ , yields

$$p \leq \frac{\alpha}{(1-\delta)N} \left[ 1 + \sqrt{\delta [1 - (1-\delta)(q_c N)^2]} \right] \quad (5.3)$$

and

$$p \geq \frac{\alpha}{(1-\delta)N} \left[ 1 - \sqrt{\delta [1 - (1-\delta)(q_c N)^2]} \right]. \quad (5.4)$$

Finally (4.8), that is,  $\pi_e^L(\vec{p}^N) \leq \frac{1}{1-\delta}\pi(\vec{p}^N)$ , yields the following terrible expression

$$p \geq \max \left\{ \alpha - L + \frac{1}{(1-\delta)N(N-1)^2} [-\alpha + \sqrt{(1-\delta)(N-1)\{L^2(1-\delta)N^2(N-1) - \alpha[2N(\alpha-L) - \alpha(N-1) - 2\delta FN^2(N-1)]\} + \alpha^2}], \right. \\ \left. \frac{1}{(1-\delta)(N-1) + \frac{1}{N-1}} \left[ \alpha - L + \sqrt{L^2 + \delta\alpha[2(F-L) + \alpha]} - \frac{\alpha}{(N-1)^2} \left[ \alpha - 2L - \frac{2\delta}{1-\delta}F \right] \right] \right\} \quad (5.5)$$

The argument of the square roots of (5.3) and (5.4) is not negative iff

$$N \leq \frac{1}{q_c \sqrt{1-\delta}} = \frac{1}{\sqrt{1-\delta}} m$$

The upper bound of firms that can be active in a stationary equilibrium is then given by

$$N \leq \min \left\{ \frac{\sqrt{\alpha(F+K)}}{\sqrt{2[F+(1-\delta)K]}} \left[ 1 + \sqrt{\delta \frac{K}{F+K}} \right], \frac{1}{\sqrt{1-\delta}} m \right\}. \quad (5.6)$$

The first term in brackets is achieved by the intersection of  $p(N) = \frac{\alpha}{(1-\delta)N} \left[ 1 - \sqrt{\delta[1 - (1-\delta)(q_c N)^2]} \right]$  and  $p = p^{abs} = \sqrt{2\alpha(F+K)}$ . Notice that for  $\delta \rightarrow 1$ ,  $\frac{1}{\sqrt{1-\delta}}m$  goes to infinity, whereas the first term in brackets converges to  $\frac{\sqrt{\alpha(F+K)}}{\sqrt{2F}} \left[ 1 + \sqrt{\frac{K}{F+K}} \right]$ . But  $N \leq \frac{\sqrt{\alpha(F+K)}}{\sqrt{2F}} \left[ 1 + \sqrt{\frac{K}{F+K}} \right]$  is equivalent to  $p^{abs} \leq AC\left(\frac{1}{N}\right)$ . Similarly, the lower bound for  $p$  converges to  $AC\left(\frac{1}{N}\right)$ . In other words, all "strictly" individually rational prices ( $p > AC\left(\frac{1}{N}\right)$ ) of a symmetric outcome are stationary equilibrium prices, if  $\delta$  is sufficiently close to one, which is consistent with the folk theorem (cf. FUDENBERG, MASKIN (1986) [FM86]).

The region  $R_2$  is given by (4.15) and (4.16). For  $N > m$ , (4.16) leads to (5.3) and (5.4). For  $N \leq m$  it leads to

$$p \leq \frac{\alpha}{(1-\delta)N} \left[ 1 + \sqrt{\delta \left[ 1 - (1-\delta) \left( \frac{N \cdot p^{pun}}{\alpha} \right)^2 \right]} \right] \quad (5.7)$$

and

$$p \geq \frac{\alpha}{(1-\delta)N} \left[ 1 - \sqrt{\delta \left[ 1 - (1-\delta) \left( \frac{N \cdot p^{pun}}{\alpha} \right)^2 \right]} \right] \quad (5.8)$$

where

$$p^{pun}(N) = (1/(N-1)^2 + 1) \cdot \left[ (\alpha - L)(N-1) + \sqrt{L^2(N-1)^2 + \alpha(2L - \alpha)} \right] \quad (5.9)$$

(4.16) takes the form (5.1) for  $N \geq m$ . For  $N < m$  we get

$$p \leq \sqrt{\left[ K + \frac{1}{1-\delta}F \right] 2\alpha - \frac{\delta}{1-\delta} [p^{pun}(N+1)]^2} \quad (5.10)$$

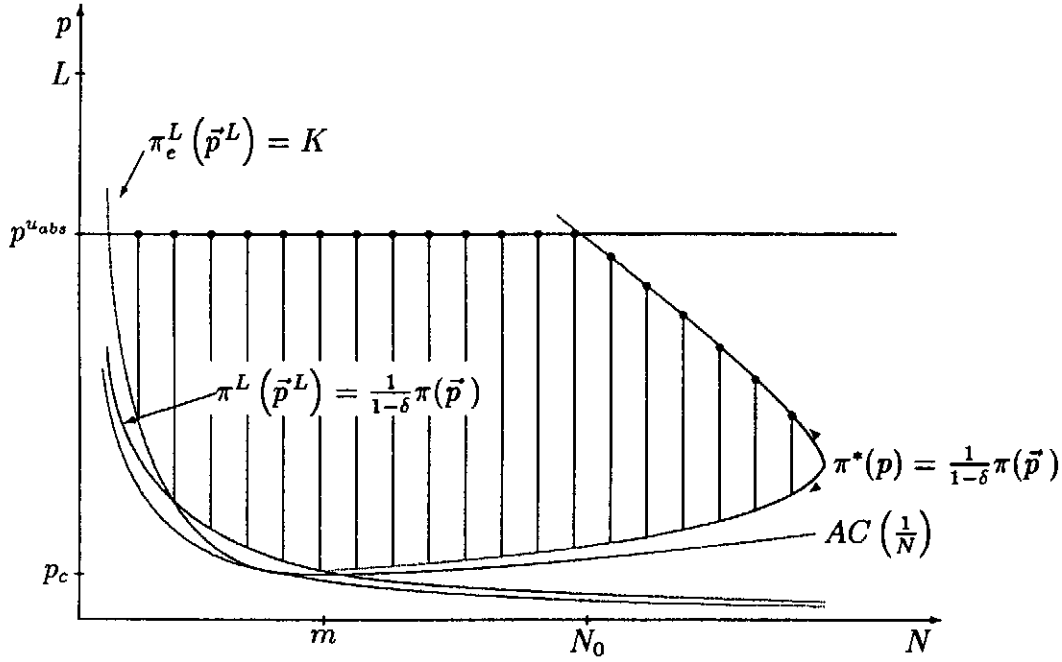


Figure 1: The equilibrium region  $R_1$  is represented by the vertical lines. In this example, the upper bound for prices decreases in  $N$  from  $N_0$  on. The black dots represent the Pareto optimal collusive outcomes. Notice that all the constraints are binding.

For the region  $R_3$ , the inequality (4.20) takes the form (5.3) and (5.4), whereas (4.21) takes the form (5.5).

For the region  $R_4$ , the inequality (4.25) takes the form (5.1) for  $N \geq m$ , and (5.10) for  $N < m$ . (4.26) takes the form (5.3) and (5.4) for  $N > m$  and (5.7), and (5.8) for  $N \leq m$ .

Figures 2 – 4 illustrate typical outfits of the regions  $R_1 - R_4$ .

## 6 Concluding Remarks

It was the aim of this paper to analyse the classical Bertrand–Edgeworth market game with increasing marginal costs under consideration of entry and exit decisions in a dynamic framework. If we assume there to be always some firms that threaten to enter the market, which seems to be much more realistic than the assumption of pure oligopoly, the situation turns out to become unforeseen complicated. The results of Theorems 4.1 and 4.3 show how large the set of equilibrium outcomes can be if we allow for the whole space of equilibrium strategies, which includes strategies that employ former inactive firms in order to punish a deviator. The proof of these results suggests that not all the strategies are equally plausible, which calls for a restriction

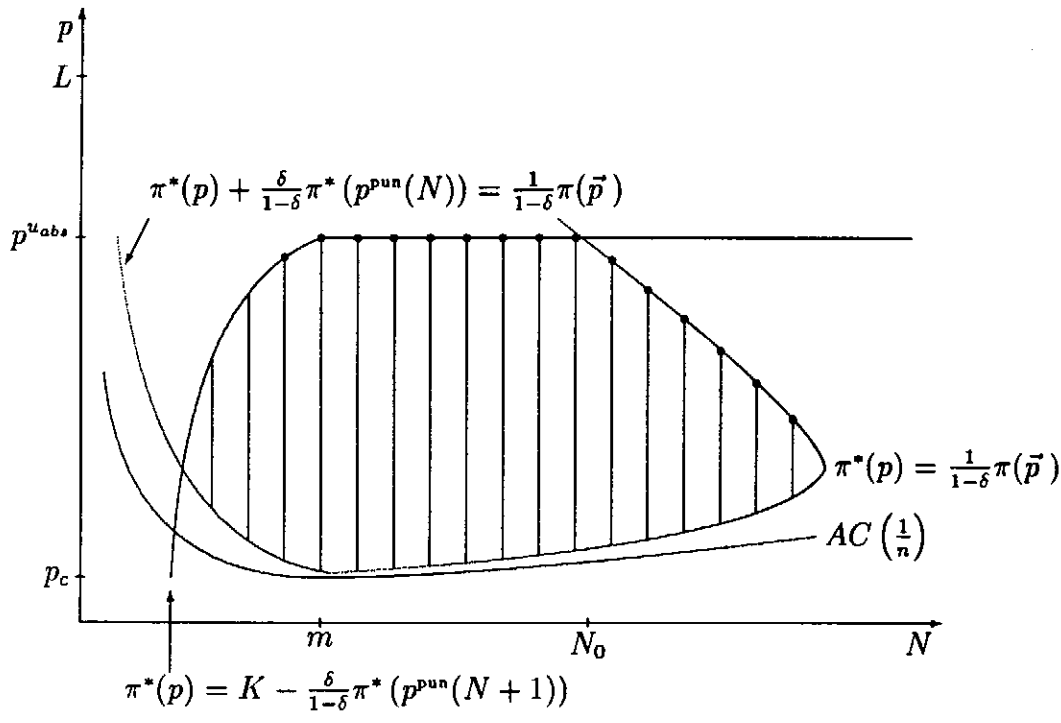


Figure 2: The region  $R_2$ . The black dots denote the Pareto optimal collusive outcomes. At the very left one, the incumbent firms face the highest profits.

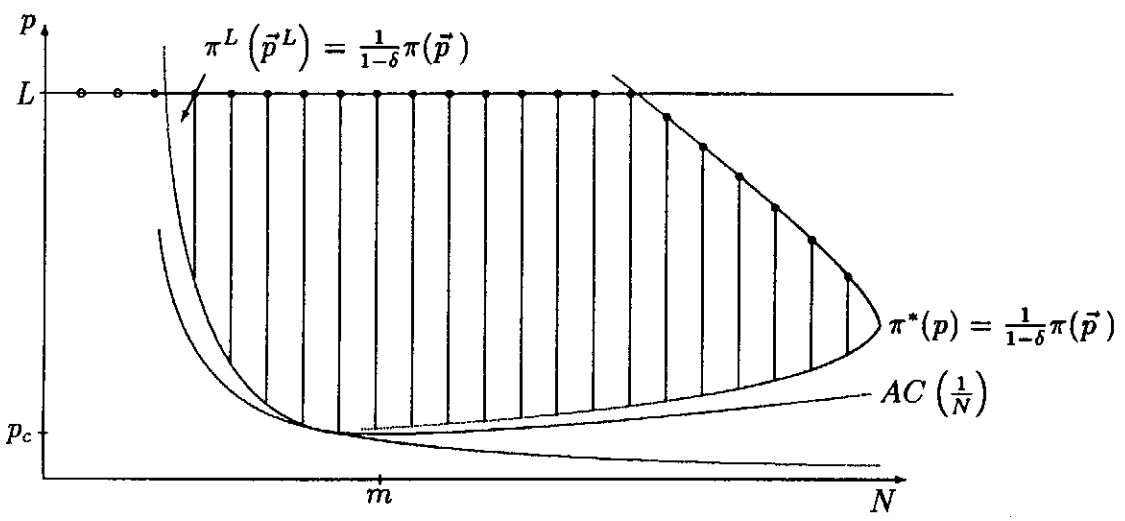


Figure 3: The region  $R_3$ . The black dots represent the Pareto optimal collusive outcomes. The white dots on the very left belong to  $R_3$ , but are not Pareto efficient.

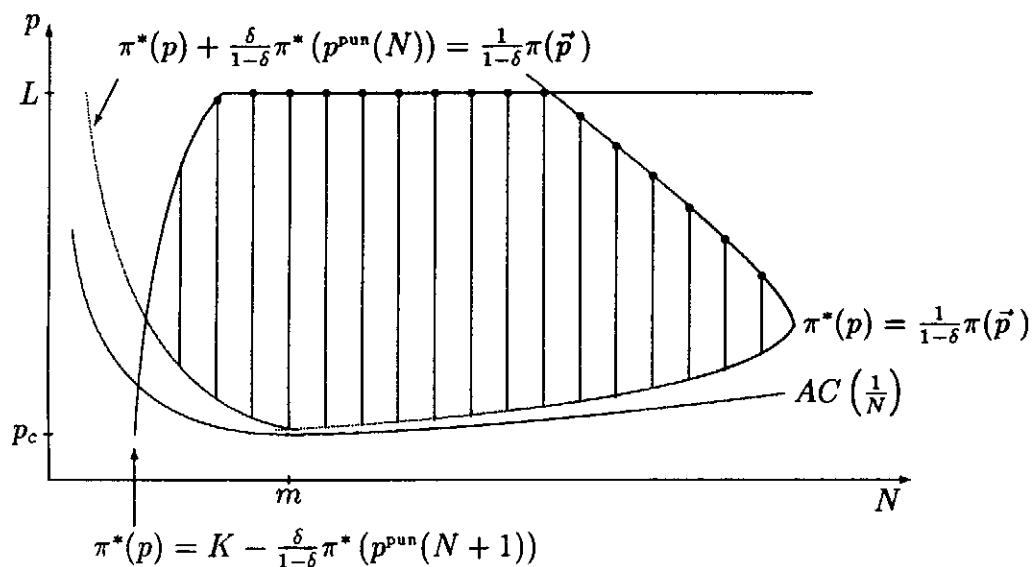


Figure 4: The region  $R_4$ . The black dots denote the Pareto optimal collusive outcomes.

to a subset of equilibria, which we called "reasonable".

Our results also call into question the traditional method of endogenizing the number of firms in oligopolies, that is, to assume that firms will enter the market as long as positive profits are earned by the incumbent firms (cf. [BS85]). In the classical Bertrand–Edgeworth model, however, there is no pure strategy equilibrium even in the repeated game if we allow for free entry. By introducing entry costs, as it is done in this paper, we *do* get (stationary) equilibria, with *several* possible numbers of firms which can be active in equilibrium. This does not need to be a disadvantage. The number of active firms, the market will end up with in equilibrium, may heavily depend on the prehistory. If we assume that the firms will enter the market by and by, and the incumbent firms will agree on a *reasonable* equilibrium outcome, it can be an advantage that there is room for several possible numbers of firms that can collude in equilibrium. By Lemma 4.8, the lower bound for equilibrium prices is always greater than the average cost of the produced and sold quantity, if  $N$  firms charge the same price and share the market fairly. Since the lower and upper bounds for equilibrium prices yield upper bounds for possible equilibrium numbers of firms, we conclude from Lemma 4.8 that the number of firms given by the zero-profit condition *cannot* be an equilibrium number of firms if the discount factor  $\delta$  is sufficiently bounded away from one. Notice that the smaller  $\delta$ , the greater the lower bound for equilibrium prices and the smaller the greatest possible equilibrium number of firms. On the other hand, one can show, that the set of equilibrium outcomes converges to the whole set of individually rational outcomes when  $\delta$  goes to unity. This is in accordance with the folk theorem

on purely repeated games (cf. [FM86]). In many applications, however, the discount factor is bounded away from one.

For the sake of simplicity and shortness, we confined ourselves to (quasi-) symmetric equilibrium outcomes, that is, all the active firms charge the same price in equilibrium and also on the punishment paths. It is not much of a problem to characterize also the asymmetric stationary equilibrium outcomes.

Of course, the paper includes some restrictive assumptions: consumers have identical preferences and even the individual demand structure is very simple, also the firms are assumed to have identical technologies. The problem becomes certainly even more complicated if we allow for individual elastic demand or different reservation prices, in which case different rationing schemes have to be considered. This paper may also be considered as a first step towards a theory of long term monopolistic competition, since it contains two features of that kind of market form: the U-shaped average cost curve and the number of firms, which is endogeneously determined. What is missing is the issue of differentiated commodities. However, a lot of conceptual difficulties will arise since the dimension of the consumers' consumption space may vary if firms enter or leave the market!

## A Appendix

### A.1 Proof of Theorem 4.1

In the following we will often write  $\vec{p}$  instead of  $\vec{p}^N = (\underbrace{p, \dots, p}_N, \underbrace{\text{n.a.}, \dots, \text{n.a.}}_{n-N})$ , especially this is done for  $p = p^{\text{pun}}(N)$  and  $p = p^u(N)$ . For the active firms of a quasi-symmetric outcome  $\vec{p}$ , we will omit the subscript  $i$ , if not necessary, and write  $\pi$  instead of  $\pi_i$ .

**Claim A.1**  $\forall \delta > 0, \forall N > m$ , with  $\exists p$ , s.t.  $(p, N) \in R_1$ , there is  $T_0 \in \mathbb{N}$  such that:

$$\sum_{t=0}^{T_0-1} \delta^t \pi(\vec{p}^{\text{pun}}(N)) + \sum_{t=T_0}^{\infty} \delta^t \pi(\vec{p}^u(N)) < 0 \quad (\text{A.1})$$

**Proof:** Since  $N > m \in \mathbb{Z}$ , we have  $N - 1 \geq m = \frac{1}{q_c}$ . Hence  $\frac{1}{N-1} \leq q_c$  yielding  $v'(\frac{1}{N-1}) \leq v'(q_c) = p_c$ . By Lemma 4.3.iii) we have  $p^{\text{pun}}(N) < v'(\frac{1}{N-1}) \leq p_c$ . This yields  $\pi(\vec{p}^{\text{pun}}(N)) < 0$ . Since the second term of (A.1) becomes arbitrarily small for  $T_0$  sufficiently large, the claim holds obviously. Q.E.D.

By Claim A.1 we can define for  $N > m$ :

$$T_0 := T_0(N) := \min \left\{ T \mid \sum_{t=0}^{T-1} \delta^t \pi(\bar{p}^{pun}(N)) + \sum_{t=T}^{\infty} \delta^t \pi(\bar{p}^u(N)) \leq 0 \right\} \quad (\text{A.2})$$

By Lemma 4.8,  $p^u(N) > AC\left(\frac{1}{N}\right)$  holds for all  $N$  for which  $p^u(N)$  exists. Hence  $T_0 \geq 1$ . Further we define the punishment action  $p^l(N)$  of the "last punishment period"<sup>8</sup> by

$$\sum_{t=0}^{T_0-2} \delta^t \pi(\bar{p}^{pun}(N)) + \delta^{T_0-1} \pi(\bar{p}^l(N)) + \sum_{t=T_0}^{\infty} \delta^t \pi(\bar{p}^u(N)) = 0 \quad (\text{A.3})$$

Since  $\pi(\bar{p})$  is continuous and increasing in  $p$  there exists a price  $p^l(N)$  with  $p^{pun}(N) \leq p^l(N) < p^u(N)$  solving (A.3).

### Step 1: Construction of the Equilibrium Strategy

Let

$$A(h) := \{i \in I \mid \sigma_i(h) \neq \text{n.a.}\}$$

be the set of players (firms) who are supposed by the strategy to be active after history  $h$ . Let

$$Z := \{(i, A) \in I \times 2^I \mid i \in A\}$$

$$\bar{Z} := \{(i, A) \in I \times 2^I \mid i \notin A\}$$

Now we define 6 generic paths.

**Definition A.1** The initial path  $c^0 : [0, L] \times 2^I \rightarrow \Omega$  is defined by

$$c^0[p, A](\tau) = (\bar{p})^A \quad \forall \tau \geq 0 \quad (\text{A.4})$$

**Definition A.2** The Quasi-Symmetric-Punishment-path  $c^{QSP} : 2^I \rightarrow \Omega$  is defined by

$$c^{QSP}[A](\tau) = \begin{cases} (\bar{p}^{pun}(N))^A & \text{for } 0 \leq \tau < T_0 \\ (\bar{p}^l(N))^A & \text{for } \tau = T_0 \\ (\bar{p}^u(N))^A & \text{for } \tau > T_0 \end{cases} \quad (\text{A.5})$$

where  $N = |A|$ , and  $T_0 = T_0(N)$  and  $p^l(N)$  are defined by (A.2) and (A.3).

<sup>8</sup>This construction has been borrowed from Lambson (1987).

**Definition A.3** The Drive-Out-and-Substitute-path  $c^{DOS} : Z \rightarrow \Omega$  is defined by

$$c^{DOS}[i, A](\tau) = \begin{cases} (\vec{p}_c)^{B'} & \text{for } \tau = 0 \\ (\vec{p}^{u_{\text{obs}}})^{B'} & \text{for } \tau > 0 \end{cases} \quad (\text{A.6})$$

with  $B' = [A \setminus \{i\}] \cup \min^{(m-N+1)}[I \setminus A]$ ,

where  $\min^{(k)}[A]$  denotes the first  $k$  smallest elements<sup>9</sup> of  $A$ .

**Definition A.4** The Drive-Out-Intruders-path  $c^{DOI} : \bar{Z} \rightarrow \Omega$  is defined by

$$c^{DOI}[i, A](\tau) = \begin{cases} (\vec{p}_c)^{B''} & \text{for } \tau = 0 \\ (\vec{p}^u(N))^{B''} & \text{for } \tau > 0 \end{cases} \quad (\text{A.7})$$

with  $B'' = A \cup \min^{(m-N)}[I \setminus (A \cup \{i\})]$ . Notice that  $B'' = A$  for  $N \geq m$ .

**Definition A.5** The Substitute-After-Exit-path  $c^{SAE} : Z \rightarrow \Omega$  is defined by

$$c^{SAE}[i, A](\tau) = (\vec{p}^u(N))^{B'} \quad \text{for } \tau \geq 0 \quad (\text{A.8})$$

with  $B'$  defined as in Definition A.3.

**Definition A.6** The No-Entry-after-Exit-path  $c^{NEE} : Z \rightarrow \Omega$  is defined by

$$c^{NEE}[i, A](\tau) = (\vec{p}^u(N))^{B'''} \quad \text{for } \tau \geq 0 \quad (\text{A.9})$$

with  $B''' = [A \setminus \{i\}]$ .

Let  $Y := \{0, QSP, DOS, DOI, SAE, NEE\}$  and  $C = \{c^k\}_{k \in Y}$ .

Now let

$$q : H \rightarrow \Omega \quad \text{with} \quad q[h] = \{q[h](\tau)\}_{\tau=0}^{\infty}$$

and

$$T : H \times C \rightarrow \mathbb{N}$$

Then the strategy is defined by

$$\sigma(h) := q[h](T(h, c^k)) \quad (\text{A.10})$$

where  $q$  and  $T$  are defined as follows:

<sup>9</sup>If  $A$  is a finite linear ordered set,  $\min^{(k)}[A]$  is defined inductively as follows:  $\min^{(k)}[A] = \{\} \forall k \leq 0$ ,  $\min^1[A] := \min\{i \in A\} =: a_1$ . Let  $a_k = \min\{i \in A \setminus \{a_1, \dots, a_{k-1}\}\}$ . Then  $\min^{(k)}[A] := \{a_1, \dots, a_k\}$ .



If  $h \in S$ , then

$$\begin{cases} q[h] = c^0[p^*, A^*] \\ T(h, c^0) = 0 \end{cases}$$

If  $\nexists i \in I$ , such that "i deviated singly" in the last period of  $h$ , then

$$\begin{cases} q[h] = q[h^-] \\ T(h, c^k) = T(h^-, c^k) + 1 \quad \forall k \in Y \end{cases}$$

We write for short:  $A = A(h^-)$  for the set of players that is supposed to be active in  $h^l$ .

If  $\exists i \in I$ , such that "i deviated singly" in the last period of  $h$ , then

$$q[h] = \begin{cases} c^{QSP}[A] & \text{if } i \in A \quad \text{and } h_i^l \neq \text{n.a.} \quad |A| \geq m + 1 \\ c^{DOS}[i, A] & \text{if } i \in A \quad \text{and } h_i^l \neq \text{n.a.} \quad |A| \leq m \\ c^{DOI}[A] & \text{if } i \in I \setminus A \\ c^{SAE}[i, A] & \text{if } i \in A \quad \text{and } h_i^l = \text{n.a.} \quad |A| \leq m \\ c^{NEE}[i, A] & \text{if } i \in A \quad \text{and } h_i^l = \text{n.a.} \quad |A| > m \end{cases}$$

$$T(h, c^k) = 0 \quad \forall k \in Y \setminus \{0\}$$

**Comment:** In words, the strategy works as follows. At the beginning, in period zero of the game, when the history consists only of the initial state of the game,  $\hat{s}$ , the initial path  $c^0$  will be started. The counter of the path  $T(\hat{s}, c^0)$  will be set equal to zero. If the game is on any path  $c^k$ ,  $k \in \{0, QSP, DOS, DOI, SAE, NEE\}$ , maybe on the initial one or on any punishment path, and if no player has deviated singly in the last period of the history  $h$ , the path  $c^k$  will be continued. In particular, the counter  $T(h, c^k)$  will be increased by 1. If any player has deviated singly in the last period of the history  $h$ , one of the punishment paths  $c^k$  will be started and the counter  $T(h, c^k)$  of this path will be set equal to zero. The choice of the path depends on the actions from which one to which one the player has deviated and also on the joint action tuple of the last period, in particular on  $|A(h^-)|$ : If an *active* firm deviates by charging a price different from the one described by the current path and if  $|A(h^-)| > m$ , the path  $c^{QSP}$  will be started. That is, all the active firms, including the deviator herself, punish the deviator symmetrically (we call the path *quasi-symmetric* since it is symmetric with respect to the active firms). Notice that all the active firms earn the same profit if they play  $c^{QSP}$ .  $c^{QSP}$  is as heavy as possible also for the *punishing* firms. If  $|A(h^-)| \leq m$ ,  $c^{DOS}$  will be started, that is,  $m - N + 1$  inactive firms  $\in I \setminus A(h^-)$  enter the market and the deviator will be driven out by the  $N - 1$  former and the  $m - N + 1$  new active firm. Notice that in this case, the *punishing* firms earn more than zero, unless  $\delta \leq \frac{K}{[p^{n_{aba}} - p_c]q_c + K}$ , whereas

the firm that is to be punished gets zero. If an *inactive* firm enters the market, the *active* firms drive it out again by playing the path  $c^{DOI}$  if  $N \geq m$ . For  $N < m$ , again  $m - N$  new firms enter the market to drive out the deviator. Hence  $c^{DOI}$  is similar to  $c^{DOS}$  if  $N < m$ . If a firm leaves the market without being told to by the strategy, the path  $c^{SAE}$  will be started, if  $|A(h^-)| \leq m$ , and  $c^{NEE}$  if  $|A(h^-)| > m$ . (Notice that there may be a one-shot incentive to leave the market at the beginning of the path  $c^{QSP}$  if  $\tau < T_0$  and  $p^{pun}(N)$  will have to be charged.)  $c^{SAE}$  is similar to  $c^{DOS}$  apart from the fact that it starts immediately with  $p^u(N)$ . If  $|A(h^-)| > m$ , the deviator (who has gone out of the market) will not be substituted. Several deviations during one period will be ignored.

## Step 2: Computing the values of the paths

Let  $\Sigma^p \in \Sigma$  be the set of perfect strategy profiles of  $\Gamma$ . Let  $\Omega^p(h) = \{q(h, \sigma) \mid \sigma \in \Sigma^p\}$  be the set of paths generated by perfect equilibrium strategies after history  $h$ . We call this the set of *perfect equilibrium paths*.

**Definition A.7** A profile of punishment paths  $(q^1, \dots, q^n)$  is called optimal if  $\forall h \in H$ :

$$q^i \in \Omega^p(h) \quad \text{and} \quad v_i(h, q^i) = \min\{\tilde{v}_i(h, \sigma) \mid \sigma \in \Sigma^p\} .$$

Since each firm can guarantee itself zero in each subgame by being not active forever, we have to show that  $v_i(h, c^k) \geq 0 \forall h \in H$  and  $\forall k \in Y$ . Moreover, we will show that  $v_j(h, c^k) = 0$  if  $j$  has deviated singly in  $h^l$ , that is, the punishment paths are optimal.

1.)  $v_i(\hat{s}, c^0)$ :

$$v_i(\hat{s}, c^0(p^*, A^*)) = 0 \quad \forall i \in I \setminus A^* \quad (\text{A.11})$$

$$= \frac{1}{1-\delta} \pi_i(\vec{p}^*) \quad \forall i \in A^* \quad \text{with} \quad \hat{s}_i \neq \text{n.a.} \quad (\text{A.12})$$

$$= \frac{1}{1-\delta} \pi_i(\vec{p}^*) - K \quad \forall i \in A^* \quad \text{with} \quad \hat{s}_i = \text{n.a.} \quad (\text{A.13})$$

Since  $v'(\frac{1}{N}) < AC(\frac{1}{N})$  if  $N > m$ , and  $v'(\frac{1}{N}) \geq AC(\frac{1}{N})$  if  $N \leq m$ , to show that the RHS of (A.12) is nonnegative it suffices to show that  $p^* \geq AC(\frac{1}{N})$  for  $N > m$ , and  $p^* \geq p_c$  for  $N \leq m$ . The first one holds by (4.7) and Lemma 4.8. Now suppose  $N \leq m$  and  $p^* < p_c$ . Then  $Nv'^{-1}(p^*) < Nv'^{-1}(p_c) = Nq_c \leq mq_c = 1$ . But then  $\pi^L(\vec{p}^*) > 0$ , whereas  $\frac{1}{1-\delta} \pi(\vec{p}^*) < 0$  by  $p^* < p_c$ , contradicting (4.8).

The RHS of (A.13) to be nonnegative is equivalent to  $p^* \geq \widetilde{AC}(\frac{1}{N})$  if  $v'^{-1}(p^*) \geq \frac{1}{N}$  and equivalent to  $p^* \geq \tilde{p}_c := \min_q \widetilde{AC}(q)$  if  $v'^{-1}(p^*) \leq \frac{1}{N}$ . Since again  $v'(\frac{1}{N}) < \widetilde{AC}(\frac{1}{N})$  if  $N > \tilde{m}$ , and  $v'(\frac{1}{N}) \geq \widetilde{AC}(\frac{1}{N})$  if  $N \leq \tilde{m}$ , this holds by (4.12) and (4.13).

Assume now that "j has deviated singly in  $h^l$ ".

Therefore, in 2.) - 5.) all the statements about  $v_i(h, c^k)$  with  $k \in Y \setminus \{0\}$  are meant to hold for all  $h$  with "j has deviated singly in  $h^l$ ". Hence, for simplicity we relax  $h$  as an argument of  $v_i$  and write  $v_i(h, c^k) = v_i(c^k) \quad \forall k \in Y \setminus \{0\}$ . Furthermore, we will again write for short  $A := A(h^-)$ .

2.)  $v_i(c^{QSP})$ : By construction we have

$$v_i(c^{QSP}[A]) = 0 \quad \forall i \in I \quad (\text{especially for } i = j) \quad (\text{A.14})$$

3.) Clearly

$$v_i(c^k) = 0 \begin{cases} \forall i \in I \setminus B' & \text{for } k = DOS, \\ \forall i \in I \setminus B'' & \text{for } k = DOI, \\ \forall i \in I \setminus B' & \text{for } k = SAE, \\ \forall i \in I \setminus B''' & \text{for } k = NEE. \end{cases} \quad (\text{A.15})$$

4.)  $v_i(c^{DOS})$ :

$$v_i(c^{DOS}[j, A]) = \pi_i((\vec{p}_c)^{B'}) + \frac{\delta}{1-\delta} \pi_i((\vec{p}^{u_{abs}})^{B'}) \quad \forall i \in A \setminus \{j\} \quad (\text{A.16})$$

$$= \pi_i((\vec{p}_c)^{B'}) + \frac{\delta}{1-\delta} \pi_i((\vec{p}^{u_{abs}})^{B'}) - K \quad (\text{A.17})$$

$$\forall i \in \min^{(m-N+1)}[I \setminus A]$$

By construction,  $|B'| = m$ . Hence  $\pi_i((\vec{p}_c)^{B'}) = 0$ , and the RHS of (A.16) is not less than zero. The RHS of (A.17) is not less than zero if  $\frac{\delta}{1-\delta} \pi_i((\vec{p}^{u_{abs}})^{B'}) - K \geq 0$ , which is equivalent to  $\delta \geq \frac{K}{[\vec{p}^{u_{abs}} - \vec{p}_c]_{q_c} + K}$ , which holds by (4.19).

5.)  $v_i(c^{DOI})$ :

$$v_i(c^{DOI}[A]) = \pi_i((\vec{p}_c)^{B''}) + \frac{\delta}{1-\delta} \pi_i((\vec{p}^u)^{B''}) \quad \forall i \in A \quad (\text{A.18})$$

$$= \pi_i((\vec{p}_c)^{B''}) + \frac{\delta}{1-\delta} \pi_i((\vec{p}^u)^{B''}) - K \quad \forall i \in \min^{(m-N)}[I \setminus A] \quad (\text{A.19})$$

If  $|A| \leq m$ , the same arguments hold as for  $v_i(c^{DOS})$ . Notice that in this case  $p^u(N) = p^{u_{abs}}$ . If  $|A| > m$ ,  $\min^{(m-N)}[I \setminus (A \cup \{j\})]$  is empty. Hence it remains to show that the RHS of (A.18) is nonnegative. By (4.7) we know that

$$\pi^*(p^u(N)) \leq \frac{1}{1-\delta} \pi(\vec{p}^u(N)) \quad (\text{A.20})$$

$$\Leftrightarrow \pi(\vec{p}^u(N)) - \pi^*(p^u(N)) + \frac{\delta}{1-\delta} \pi(\vec{p}^u(N)) \geq 0$$

Hence, the RHS of (A.18) is nonnegative if

$$\begin{aligned} \pi_i(\bar{p}_c^N) &\geq \pi(\bar{p}^u(N)) - \pi^*(p^u(N)) \quad \text{for } N \geq m \\ \Leftrightarrow \pi^*(p^u(N)) &\geq [p^u(N) - p_c] \frac{1}{N} \end{aligned} \quad (\text{A.21})$$

Applying Lemma 4.8 to (A.20) we get:  $p^u(N) \geq p_c$  for  $N \geq m$ . Moreover  $\frac{d\pi^*}{dp}(p^u(N)) = v'^{-1}(p^u(N)) \geq q_c$  for  $N \geq m$ . On the other hand,  $\frac{1}{N} \leq q_c$  for  $N \geq m$ . Hence the LHS of (A.21) increases faster than the RHS in  $p^u(N)$  and both sides are zero for  $p^u(N) = p_c$ . This establishes (A.21).

6.)  $v_i(c^{SAE})$ :

$$\begin{aligned} v_i(c^{SAE}[j, A]) &= \frac{1}{1-\delta} \pi_i((\bar{p}^{u_{aba}})^{B'}) - K \quad \forall i \in \min^{(m-N+1)}[I \setminus A] \\ &\geq v_i(c^{DOS}) \geq 0 \quad \forall i \in \min^{(m-N+1)}[I \setminus A] \end{aligned}$$

Hence also  $v_i(c^{SAE}[j, A]) = \frac{1}{1-\delta} \pi_i((\bar{p}^{u_{aba}})^{B'}) \geq 0 \quad \forall i \in A \setminus \{j\}$ .

7.)  $v_i(c^{NEE})$ :

$$\begin{aligned} v_i(c^{NEE}[j, A]) &= \frac{1}{1-\delta} \pi_i(p^u(|A| - 1)) \quad \forall i \in A \setminus \{j\} \\ &\geq \frac{1}{1-\delta} \pi_i(p^u(|A|)) \geq \pi_i((\bar{p}^*)^A) \geq 0 \quad \forall |A| > m \end{aligned}$$

Thus we have shown

$$v_i(h, c^k) \geq 0 \quad \forall i \in I \quad \forall k \in \{0, QSP, DOS, DOI, SAE, NEE\} \quad (\text{A.22})$$

and  $h \in H$  such that  $c^k$  is to be played after  $h$ .

Since for each path the prices charged during that path are not decreasing in  $\tau$ , we get also

$$v_i(h, c^k, \tau) \geq v_i(h, c^k) \geq 0 \quad (\text{A.23})$$

$$\forall \tau \geq 0 \quad \forall i \in I \quad \forall k \in \{0, QSP, DOS, DOI, SAE, NEE\}$$

with  $h^l = c^k(\tau - 1)$  for  $\tau \geq 1$  and  $h$  such that  $c^k$  is to be played after  $h$ , otherwise.

By (A.14) and (A.15), the strategy is even optimal since it holds the deviator always down to her security level.

### Step 3: Perfectness of $\sigma$

In Section 3 we defined  $q^j$  as the punishment path for player  $j$ . Set  $q^j = q \quad \forall j \in I$ . That is, all the players will be punished by the same paths. By construction,  $q[h]$

only depends on  $h_j^{l-1}$  and  $h^l$ , if  $j$  has deviated singly in  $h^l$ . Therefore, we can apply Proposition 3.1 in order to check, whether  $\sigma$  is a perfect equilibrium. The number of inequalities in (3.1) reduces considerably since by construction,  $q[h]$  is piecewise constant on the set of histories. Since  $\tilde{\pi}_i$  depends only on the *own* action rather than the joint action tuple in the previous period, we will write  $v_i(h_i^l, c^k(h))$  instead of  $v_i(h^l, c^k(h))$ .

A) Observe first that it does never pay for any  $i \in A(h)$  and  $\forall h \in H, \forall k \in Y$  to deviate to  $s_i = \text{n.a.}$ , since

$$\forall \tau \geq 0, \quad \forall k \in Y, \quad \forall c_i^k(\tau) \neq \text{n.a.}, \quad \forall h_i^l \in S_i \quad \text{with} \quad h_i^l = c_i^k(\tau - 1) \quad \text{for} \quad \tau \geq 1$$

and  $\forall h^l$  that can precede  $c^k$  if  $\tau = 0$

$$\tilde{\pi}_i(h_i^l; (\text{n.a.}, c_{-i}^k(\tau))) + \delta v_i(\text{n.a.}, c^{SAE}(i, A(h))) = 0 \stackrel{(A.23)}{\leq} v_i(h_i^l, c^k, \tau) \quad (\text{A.24})$$

if  $|A| \leq m$  and

$$\tilde{\pi}_i(h_i^l; (\text{n.a.}, c_{-i}^k(\tau))) + \delta v_i(\text{n.a.}, c^{NEE}(i, A(h))) = 0 \stackrel{(A.23)}{\leq} v_i(h_i^l, c^k, \tau) \quad (\text{A.25})$$

if  $|A| > m$ .

B) Secondly, it does never pay for any  $i \in I \setminus A(h)$  and  $\forall h \in H$  with  $h_i^l = \text{n.a.}$  to deviate by entering the market since

$$\forall s_i \neq \text{n.a.}, \quad \forall \tau \geq 0, \quad \forall k \in Y,$$

$$\begin{aligned} \tilde{\pi}_i(\text{n.a.}; (s_i, c_{-i}^k(\tau))) + \delta v_i(s_i, c^{DOI}(A(h))) &\leq \sup_{s_i \in S_i} \tilde{\pi}_i(\text{n.a.}; (s_i, c_{-i}^k(\tau))) + 0 \\ &\leq \max \{ \pi^*(p^{u_{abs}}), \pi_e^L(p^u(|A|)) \} - K \\ &\leq 0 \end{aligned} \quad (\text{A.26})$$

The second inequality holds due to the fact that for any path  $k \neq 0$ , we have  $|A(h)| \geq m$  and for all  $i \in A(h), \tau \geq 0, k \neq 0$ , we have  $c_i^k(\tau) \geq p_c$ . Hence, entering by charging a higher price than  $c_i^k(\tau)$  yields no demand. The last inequality holds due to (4.5) and (4.6).

By A) and B), in the remainder of the proof, it is sufficient to show that it does not pay for an active firm (active in the previous period) to deviate (in the current period) from any path  $c^k, k \in Y$ , by charging a price different from the action prescribed by the strategy. Notice that this includes also deviations from  $\text{n.a.}$ , in the case that a firm has to leave the market, if the path  $c^{DOS}$  is to be played. This case is not yet covered by B! Deviations from  $c_i^k(\tau) \neq \text{n.a.}$  will be punished by  $c^{QSP}$  or  $c^{DOS}$ , deviations from

$c_i^k(\tau) = \text{n.a.}$  by  $c^{DOI}$ . Furthermore, it is sufficient to consider the supremum of payoffs resulting from one-shot deviations. Hence (3.1) takes the form

$$\begin{aligned} \sup_{s_i \in \bar{S}_i} \pi_i(s_i, c_{-i}^k(\tau)) + \underbrace{\delta v_i(s_i, c^r)}_{=0} &\leq v_i(h_i^l, c^k, \tau) \\ \Leftrightarrow \sup_{s_i \in \bar{S}_i} \pi_i(s_i, c_{-i}^k(\tau)) &\leq v_i(h_i^l, c^k, \tau) \end{aligned} \quad (\text{A.27})$$

$$\forall k \in Y, \quad \forall \tau \geq 0, \quad \forall h_i^l \neq \text{n.a.} \quad \text{and } r = \begin{cases} QSP & \text{if } c_i^k(\tau) \neq \text{n.a.} \quad |A| > m \\ DOS & \text{if } c_i^k(\tau) \neq \text{n.a.} \quad |A| \leq m \\ DOI & \text{if } c_i^k(\tau) = \text{n.a.} \end{cases}$$

C) In [Req90b] we have employed the path  $c^{QSP}$  for the purely repeated game and we have already shown there that it does not pay to deviate from  $c^{QSP} = c^{QSP}[A]$ .

D) Deviations from  $c^{DOS}$  and  $c^{DOI}$  never pay for player  $j$  (the original deviator who caused  $c^{DOS}$  or  $c^{DOI}$ ), since if firm  $j$  stays in the market, there will be  $N + 1 > m$  firms in the market. Hence at least  $N$  firms can serve the whole market even at price  $p_c$  (the first punishment action of  $c^{DOS}$  and  $c^{DOI}$ ). Thus, deviation to a higher price yields no customers. Undercutting  $p_c$  yields also negative profits. It remains to show that deviating does not pay for the punishing firms, either.

case a)  $|A| = m$  (for  $c^{DOS}$ ,  $|A|$  equals  $m$ ). subcase i)  $\tau = 0$ . In this case the best response is to charge  $L$ . Hence (3.1) takes the form

$$\begin{aligned} Lq_c - v(q_c) - F &\leq 0 + \frac{\delta}{1 - \delta} [p^{u_{abs}} \cdot q_c - v(q_c) - F] \\ \Leftrightarrow [L - p_c]q_c &\leq \frac{\delta}{1 - \delta} [p^{u_{abs}} - p_c]q_c \\ \Leftrightarrow \delta &\geq \frac{L - p_c}{L + p^{u_{abs}} - 2p_c} \end{aligned}$$

which holds by (4.19).

ii)  $\tau > 0$ . Charging a higher price does not pay for  $\tau > 0$ , if it does not pay for  $\tau = 0$ . Undercutting  $p^{u_{abs}}$  does not pay due to (4.7) and (4.10).

case b)  $|A| > m$ . In this case, charging a higher price yields no customers and, therefore, negative profits. Undercutting  $p_c$  yields also negative profits. Undercutting  $p^u(N)$  has already been discussed.

E) Deviating from  $c^{SAE}$  and  $c^{NEE}$  does not pay by the same arguments as for  $c^{DOS}$  and  $c^{DOI}$  for  $\tau > 0$ .

F) It remains to show that it does not pay to deviate from the initial path  $c^0$ . Actually this is trivial since (4.5) – (4.8) (plus (4.12) and (4.13) in case b) of the Theorem) are the

corresponding equilibrium conditions: **case a)** Entering the market by undercutting does not pay by (4.5) since  $\pi_i^*(p^*) \leq \pi_i^*(p^u(N)) \leq K \forall i, N$ . **case b)** Entering the market by charging a higher price does not pay by (4.6). **case c)** Undercutting does not pay for an active firm due to (4.7). **case d)** Charging the monopoly price does not pay for an inactive firm by (4.8).

### Necessity:

It is easy to see that (4.5) – (4.8) are also necessary, since otherwise deviating *does* pay, if all the firms  $\in A^*$  are incumbent at  $t = 0$ . If a firm  $i \in A^*$  has to enter the market in period 0, that is, if  $\hat{s}_i = \text{n.a.}$ , (4.12) and (4.13) are necessary.

## A.2 Proof of Theorem 4.2

We elaborate this proof only on the parts that differ from the proof of Theorem 4.1.

With a little abuse of notation,  $\forall N$  for which  $\exists p$  s.t.  $(p, N) \in R_2$  we write again  $p^u(N) := \max\{p \mid \text{s.t. } (p, N) \in R_2\}$ .

**Claim A.2**  $\forall \delta > 0, \forall N$  for which  $\exists p$  s.t.  $(p, N) \in R_2, \exists T_0 \in \mathbb{N}$  such that:

$$\sum_{t=0}^{T_0-1} \delta^t \pi(\bar{p}^{\text{pun}}(N)) + \sum_{t=T_0}^{\infty} \delta^t \pi(\bar{p}^u(N)) < \max\left\{0, \frac{1}{1-\delta} \pi^*(p^{\text{pun}}(N))\right\} \quad (\text{A.28})$$

**Proof:** For  $N > m$ , see Claim A.1. So let  $N \leq m$ . Since  $v'(\frac{1}{N}) < p^{\text{pun}}(N)$ , we get  $v'^{-1}(p^{\text{pun}}(N)) > \frac{1}{N}$ . Hence  $\pi^*(p^{\text{pun}}(N)) > \pi(\bar{p}^{\text{pun}}(N))$ . The rest is obvious. Q.E.D.

So we set

$$T'_0 := T'_0(N) := \min\{T_0 \mid T_0 \text{ satisfies (A.28)}\} \quad (\text{A.29})$$

Again, we define the punishment action  $p^l(N)$  of the "last punishment period" by

$$\begin{aligned} \sum_{t=0}^{T_0-2} \delta^t \pi(\bar{p}^{\text{pun}}(N)) + \delta^{T_0-1} \pi(\bar{p}^l(N)) + \sum_{t=T_0}^{\infty} \delta^t \pi(\bar{p}^u(N)) \\ = \max\left\{0, \frac{1}{1-\delta} \pi^*(p^{\text{pun}}(N))\right\} \quad (\text{A.30}) \end{aligned}$$

Now we construct some more punishment paths:

**Definition A.8** The QSP'-path  $c^{\text{QSP}'}$  :  $2^I \rightarrow \Omega$  is defined by

$$c^{\text{QSP}'}[A](\tau) = \begin{cases} (\bar{p}^{\text{pun}}(N))^A & \text{for } 0 \leq \tau < T'_0 \\ (\bar{p}^l(N))^A & \text{for } \tau = T'_0 \\ (\bar{p}^u(N))^A & \text{for } \tau > T'_0 \end{cases} \quad (\text{A.31})$$

where  $T'_0 = T'_0(N)$  and  $p^l = p^l(N)$  are defined by (A.29) and (A.30), and  $N = |A|$ .

**Definition A.9** The Punish-Intruders-path  $c^{PI} : \bar{Z} \rightarrow \Omega$  is defined by

$$c^{PI}[i, A](\tau) = \begin{cases} (\bar{p}^{pun}(N+1))^{B''''} & \text{for } 0 \leq \tau < T'_0(N+1) \\ (\bar{p}^l(N+1))^{B''''} & \text{for } \tau = T'_0(N+1) \\ (\bar{p}^u(N+1))^{B''''} & \text{for } \tau > T'_0(N+1) \end{cases} \quad (\text{A.32})$$

where  $B'''' = A \cup \{i\}$ .

$q' : H \rightarrow \Omega$  is defined like  $q$  in the proof of Theorem 4.1, apart from the case, when a player deviated. If  $\exists i \in I$ , such that "i deviated singly in  $h^l$ ", then

$$q'[h] = \left\{ \begin{array}{ll} c^{QSP'}[i, A] & \text{if } i \in A \text{ and } h_i^l \neq \text{n.a.} \\ c^{DOI}[A] & \text{if } i \in I \setminus A \text{ and } |A| \geq m \\ c^{PI}[A] & \text{if } i \in I \setminus A \text{ and } |A| < m \\ c^{SAE}[i, A] & \text{if } i \in A \text{ and } h_i^l = \text{n.a. } |A| \leq m \\ c^{NEE}[i, A] & \text{if } i \in A \text{ and } h_i^l = \text{n.a. } |A| > m \end{array} \right\} \quad (\text{A.33})$$

$$T(h, c^k) = 0 \quad \forall k \in Y \setminus \{0\} \quad (\text{A.34})$$

where  $c^{DOS}$ ,  $c^{SAE}$  and  $c^{NEE}$  are defined by Definitions A.4 - A.6.

**Comment:** Since we consider "reasonable equilibria", firms cannot be driven out of the market with the help of former inactive firms. Hence, also for  $N \leq m$  active deviators will be punished symmetrically by the path  $c^{QSP'}$ . For  $N < m$  intruders will also be punished symmetrically by the path  $c^{PI}$ , holding it down to  $\frac{1}{1-\delta} \pi^*(p^{pun}(N+1))_i 0$ .

### The values of the paths

It is easy to check that the values of the additional paths  $c^{QSP'}$  and  $c^{PI}$  are nonnegative by constuction.

### Perfection of the Strategy

A) First we show that it does not pay to deviate from  $c^0$ . Undercutting does not pay by (4.15) for an intruder and by (4.16) for incumbent firms. By Corollary 4.1, also to charge a higher price does never pay. By Lemma 4.8,  $p^* \geq AC\left(\frac{1}{N}\right)$  for  $N > m$ . By Lemma 4.7,  $p^* > p^{pun}(N) > v'\left(\frac{1}{N}\right) \geq AC\left(\frac{1}{N}\right)$  for  $N \leq m$ . Hence it does not pay to leave the market.



B) In [Req90b] we employed the path  $c^{QSP'}$  for the purely repeated game. Deviating from this path has been demonstrated to be unprofitable for an incumbent firm, in that paper. *Entry* by undercutting does not pay for  $\tau > T'_0$  by (4.15). Also, it does not pay to undercut for  $\tau \leq T'_0$  since  $p^{pnn}(N), p^l(N)$  are not greater than  $p^u(N)$ . Since the incumbent firms have excess capacity at  $\tilde{p}^{pnn}(N)$ , that is for  $\tau < T'_0$ , they do so for  $\tau \geq T'_0$ . Hence charging the same or a higher price as the incumbent firms do, does not pay for an entrant, either.

C) Deviations from  $c^{PI}$ . For  $\tau \leq T'_0(N+1)$  deviation does not pay by the same reasons as for  $c^{QSP'}$ . For  $\tau > T'_0(N+1)$  we have to show that

$$\pi_i^*(p^u(N+1)) + \frac{\delta}{1-\delta} \pi_i^*(p^{pnn}(N+1)) \leq \frac{1}{1-\delta} \pi_i(\tilde{p}^u(N+1)) \quad (\text{A.35})$$

However, by Assumption 6,  $\forall N < m \exists p$  s.t.  $(p, (N+1)) \in R_2$ .

D) Deviating from  $c^{DOI}$  by charging a lower price does not pay by the same arguments as in Appendix A.1. Charging a higher price never pays for  $|A| \geq m+1$  since there is no demand. For  $|A| = m$ , it does not pay by (4.19).

**Necessity** is trivial, since if (4.15) is not satisfied, entry and playing the best response forever is profitable. If (4.16) is violated, it will pay for an incumbent firm to undercut and to play the best response forever. This completes the proof of Theorem 4.2.

### A.3 Appendix: Further Proofs

**Proof of Theorem 4.3:** Substitute  $p^{uab}$  by  $L$  and substitute  $p^u(N)$  by  $p^{uL}(N) := \min\{L, \tilde{p}^u(N)\}$ , where  $\tilde{p}^u(N)$  is defined by (4.4). Then follow the lines of Appendix A.1. (4.24) guarantees that  $v_i(c^{DOS}) \geq 0$  for  $i \in \min^{(m-N+1)}[I \setminus A]$ . It remains to show that (4.24) implies that deviation to a higher price does not pay if  $|A| = m$  and  $c^{DOI}$  is employed. This is the case if

$$L \cdot q_c - v(q_c) - F \leq 0 + \frac{\delta}{1-\delta} [L \cdot q_c - v(q_c) - F] \Leftrightarrow \delta \geq \frac{1}{2} \quad (\text{A.36})$$

But by (4.24) we have  $\delta \geq \frac{K}{[L-p_c]q_c+K} > \frac{K}{\pi^*(L)+K} > \frac{K}{K+K} = \frac{1}{2}$ .

**Proof of Theorem 4.4:** Follow the lines of Appendix A.2 and take notice of (A.36).

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