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The structure of homogeneous games

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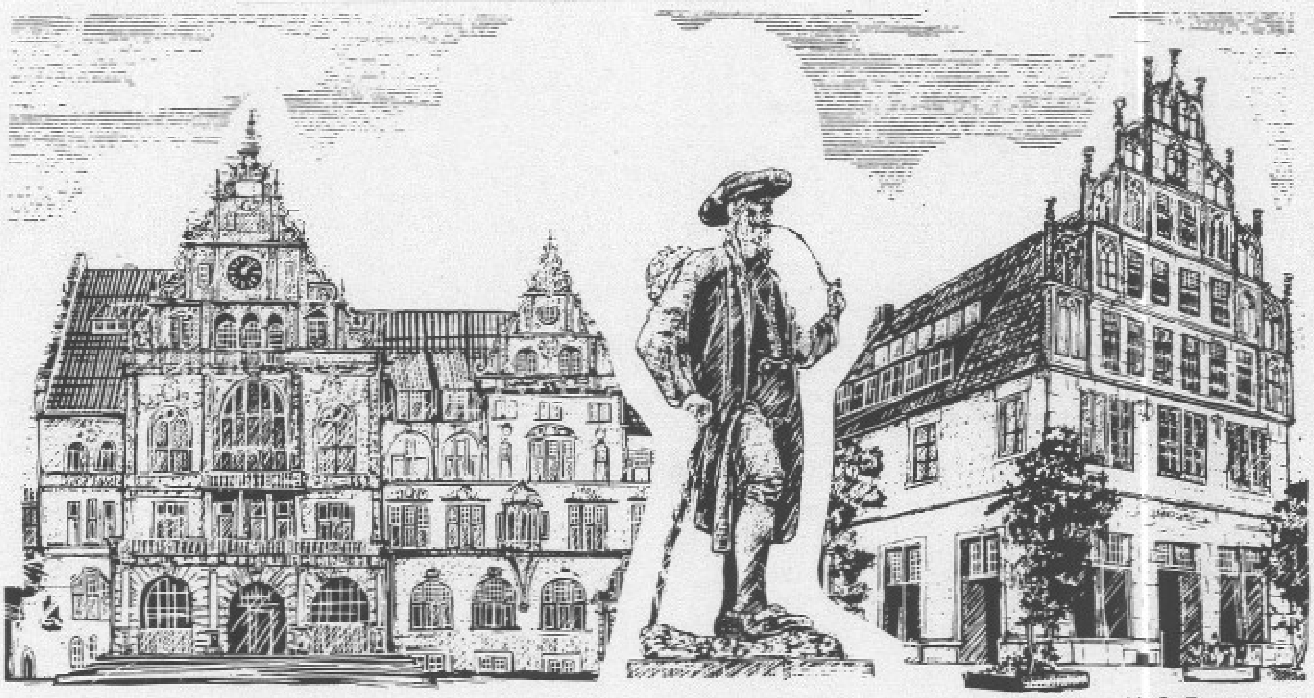

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§1 Introduction, Notations, and the Basic Lemma

Let $\Omega = \{1,\ldots,n\}$ denote the "set of players". A pair of vectors

$$(g\,;k) = (g_0,\ldots,g_r\,;\,k_0,\ldots,k_r) \in \mathbb{N}_0^{2(r+1)}$$

induces an additive set function M (a measure) on the subsets of Ω (the "coalitions") in a natural way provided

$$\sum_{i=0}^{r} k_i = n\ .$$

Indeed, put $K_\rho := \{\omega \in \Omega \mid k_{\rho-1} < \omega \leq k_\rho\}$ $(k_{-1} := -1)$ such that

$$\Omega = K_0 + K_1 + \ldots + K_r$$

is a decomposition of Ω ("+" is used instead of "$\cup$" iff the union is disjoint) and define, for $S \subseteq \Omega$,

$$(1) \qquad M(S) = \sum_{i=0}^{r} \mid S \cap K_i \mid g_i\ .$$

Thus $M : \mathfrak{P}(\Omega) \to \mathbb{N}_0$ is a mapping defined on the power set of Ω ("the coalitions"). Clearly, any $\mathbb{N}_0$-valued measure M may be represented by a suitable $(g\,;\,k) \in \mathbb{N}_0^{2\,(r+1)}$, possibly after reordering Ω.

A (_simple_) _game_ is a mapping

$$v : \mathcal{P}(\Omega) \to \{0,1\}$$

such that $v(\emptyset) = 0$ (and, in general, $v(\Omega) = 1$) The term game throughout this paper refers to simple games. A game is a _weighted majority_ if there exists a measure M and a number $\lambda \in \mathbb{N}$ such that for $S \subseteq \Omega$

$$(2) \qquad v(S) = \begin{cases} 1 & M(S) \geq \lambda \\ 0 & M(S) < \lambda \end{cases}$$

In this case, (M,λ) is called a _representation_ of v and the relation (2) is indicated by writing $v = v_\lambda^M$. If $\omega \in K_i$, then g_i is the _weight_ of ω.

Of course, a game may have various representations and we are interested in defining and computing a minimal representation for a certain class of games. Let us start by discussing the symmetry-properties of a game and its representations.

The properties of a representation are obvious: if $\omega, \eta \in \Omega$ are players with equal weight, then (M,λ) (and v) is not affected by exchanging ω and η; more precisely the permutation $\pi^{\eta,\omega} : \Omega \to \Omega$, $\pi(\eta) = \omega$, $\pi(\omega) = \eta$, $\pi(\iota) = \iota$ $(\iota \neq \eta,\omega)$ yields $\pi^{\eta,\omega} v = v$ ($\pi v(S) = v(\pi^{-1}(S))$, for any permutation π, defines the game πv).

In particular, exchanging players inside the same K_i does not affect v; we shall call the elements of K_i _fellows_ (w.r.t. (M,λ)) and i a _fellowship_.

For any game v, the symmetry group

$$\Pi^v = \{\pi : \Omega \to \Omega \mid \pi \text{ a permutation}, \ \pi v = v\}$$

describes the symmetry properties of v. Π^v decomposes Ω into transitivity domains, say

$$\Omega = T_0 + \ldots + T_t \quad .$$

If $\omega, \eta \in T_\rho$ for some ρ then there is $\pi \in \Pi^v$ such that $\pi(\omega) = \eta$ or, equivalently, η is an element of ω's orbit under Π^v. $\Theta = \{0,\ldots,t\}$ is called the set of types (of v) and $\omega, \eta \in T_\theta$ $(\theta \in \Theta)$ "belong to the same type".

If v is a weighted majority then ω and η belong to the same type if and only if $\pi^{\omega\eta} v = v$, as is easily verified.

It is also easy to verify (see OSTMANN [5]) that the representations induce an ordering of the types, i.e., if for some representation (M,λ) of v we have

$$\omega \in K_i, \quad \eta \in K_j, \quad g_i < g_j$$

i.e. $M(\omega) < M(\eta)$, then $M'(\omega) < M'(\eta)$ if ω and η belong to a different type and M', λ' is another representation of v.

It is, therefore, no loss of generality to assume that players are ordered in advance and that this ordering is provided by any representation to start out with.

We shall assume that "smaller players are recognized by smaller numbers" and that players with weights zero (if any) are first in our ordering. Thus we restrict the term "representation" as follows.

Let $\mathfrak{M}^r$ denote the set of all vectors

$$(g,k) = (g_0,g_1,\ldots,g_r \,;\, k_0,k_1,\ldots,k_r) \in \mathbb{N}_0^{2(r+1)}$$

such that

(3) $$0 = g_o \ ; \ 1 \leq g_1 \leq \ldots \leq g_r$$

(4) $$0 \leq k_o \ ; \ 1 \leq k_1, \ldots, k_r \ ; \ \sum_{i=o}^{r} k_i = n$$

is satisfied. Let $\mathfrak{M} = \bigcup_{r=1}^{\infty} \mathfrak{M}^r$. Any $M = (g,k) \in \mathfrak{M}^r$ is always interpreted as an measure on Ω. Therefore, given $M \in \mathfrak{M}^r$, the projections

(5) $$M_{i_o-1} = (g_o, \ldots, g_{i_o-1} \ ; \ k_o, \ldots, k_{i_o-1})$$

(6) $$M^c_{i_o} = (g_o, \ldots, g_{i_o-1}, \ g_{i_o} \ ; \ k_o, \ldots, k_{i_o-1}, \ k_{i_o}-c)$$

(for $1 \leq i_o \leq r$ and $1 \leq c \leq k_{i_o}$) may be interpreted as restrictions of M, that is, measures which are regarded to live on an appropriate subset of Ω. This subset is of the form

$$C = C(M^c_{i_o}) = K_o + \ldots + K_{i_o-1} + D$$

where $D \subseteq K_{i_o}$, $|D| = k_{i_o}-c$. As $g_o = 0$, the term carrier is not quite appropriate, we shall reserve this term for the vector $(k_o, \ldots, k_{i_o}-c)$.

As a notational convention, the total mass of M is always denoted by m (indices are carried through appropriately) i.e.

$$m = \sum_{i=1}^{r} k_i \ g_i = M(\Omega)$$

(7) $$m^c_{i_o} = \sum_{i=1}^{i_o-1} k_i \ g_i + (k_{i_o} - c)g_{i_o} = M^c_{i_o}(\Omega) = M^c_{i_o}(C)$$

$$m_{i_o-1} = \sum_{i=1}^{i_o-1} k_i \ g_i = M_{i_o-1}(\Omega)$$

et.

Next, it will be necessary to compare measures (vectors (g,k)) of different length; the prerequisits for this procedure are provided by

Definition 1.1. 1. $k' \in \mathbb{N}_0^{r'+1}$ extends $k \in \mathbb{N}_0^{r+1}$ if the following holds true:

(8) $r' \geq r$;

There is $l = (l_0,\dots,l_r) \in \mathbb{N}_0^{r+1}$ such that

(9) $-1 =: l_{-1} < l_0 < l_1 < \dots < l_r$

and

(10) $\sum_{l_{\rho-1}<i\leq l_\rho} k'_i = k$

2. Let $(g,k) \in \mathfrak{M}^r$ and $(g',k') \in \mathfrak{M}^{r'}$. We shall say that (g',k') extends (g,k) if

(11) $k' \text{ ext } k$

and

(12) $g'_i = g_\rho \qquad (l_{\rho-1}+1 \leq i \leq l_\rho)$,

where l is specified by 1. If $(g,k) \in \mathfrak{M}^r$ and, for some $k' \in \mathbb{N}_0^{r'+1}$, k' ext k, then there is a unique g' sucht that (g',k') ext (g,k); let us write

$$g' = \text{ext}|_{k'} \; g$$

(extension of g w.r.t. k').

3. A <u>half ordering</u> $\precsim$ on $\mathfrak{M}$ is defined by

$$(g,k) \precsim (\tilde{g},\tilde{k})$$

if and only if

$$\tilde{k} \text{ ext } k \text{ and } \text{ext}_{\tilde{k}}\, g \leq \tilde{g} .$$

<u>Remark 1.2.</u>

1. For any $(g',k') \in \mathfrak{M}$ there is a unique minimal (w.r.t. $\precsim$) element $(g,k) \in \mathfrak{M}$ such that (g',k') ext (g,k) ("grouping fellowships of equal weight together"); g satisfies

(13)
$$0 = g_0 < g_1 < \ldots < g_r .$$

We call (g,k) the <u>reduction</u> of (g',k') and any (g,k) satisfying (13) is said to be <u>reduced</u>.

2. Clearly, whenever k' ext k , then

$$(g,k) \precsim (\text{ext}\big|_{k'}\, g,\ k').$$

3. If $\mathfrak{H} \subseteq \mathfrak{M} \times \mathbb{N}$ is a family of representations of a game v, then the term <u>minimal</u> refers to an element $(\tilde{M},\tilde{\lambda}) \in \mathfrak{H}$ such that for $(M,\lambda) \in \mathfrak{H}$, we have $\tilde{M} \precsim M$ and $\tilde{\lambda} \leq \lambda$.

4. If $M \in \mathfrak{M}^r$, then a vector $s \in \mathbb{N}_0^{r+1}$ is a <u>profile</u> (w.r.t. M) if $s \leq k$.

Profiles correspond to coalitions $S \subseteq \Omega$ such that $s_i = |\, S \cap K_i \,|$; we have

(14)
$$M(S) = \sum_{i=o}^{r} |\, S \cap K_i \,|\, g_i = \sum_{i=o}^{r} s_i\, g_i =: M(s) ,$$

and thus we shall frequently regard M as a (additive) function on profiles.

If (g',k') ext (g,k) , then a profile s' w.r.t. (g',k') corresponds naturally to a profile s (w.r.t. (g,k)) $(s_\rho = \sum_{l_{\rho-1} < i \leq l_\rho} s'_i)$ and we have

$$M'(s') = M(s) .$$

If $(g,k) \lesssim (g',k')$ then $s' \to s$ is also well defined and we have

$$M'(s') \geq M(s)$$

in this case.

The term homogeneous for a (simple) game has been introduced by VON NEUMANN and MORGENSTERN ([4]).

Let $(M,\lambda) \in \mathfrak{M} \times \mathbb{N}$. M is said to be homogeneous w.r.t. λ if

1. $M(\Omega) \geq \lambda$
2. For $S \subseteq \Omega$, $M(S) > \lambda$ there is $T \subseteq S$ such that $M(T) = \lambda$.

We write M hom λ as an abbreviation; also, M hom_0 λ means that either M hom λ or $M(\Omega) < \lambda$.

A game v is homogeneous if there exists a homogeneous representation, i.e., if there is $(M,\lambda) \in \mathfrak{M} \times \mathbb{N}$ s.t.

$$v = v_\lambda^M \text{ , M hom } \lambda.$$

Essentially, a game is described by its minimal winning coalitions (the min-win coalitions) and in a homogeneous game the min-win coalitions have exactly weight λ.

Homogeneous games are of special interest, because they allow for "nice" solution concepts. (see [6])

It is the aim of this paper to exactly describe the structure of all homogeneous representations of such a game. The construction of homogeneous games with arbitrarily prescribed weights is indicated in [9]. From this paper we take the following Lemma (Theorem 1.4., [9]), we shall refer to it as to the

<u>BASIC LEMMA</u> Let $M = (g,k)$ be reduced, $\lambda \in \mathbb{N}$, and assume

$$\lambda \leq m = M(\Omega) = \sum_{i=1}^{r} k_i \, g_i \, .$$

Then $M \text{ hom } \lambda$

if and only if there is $i_o \in \{1,\ldots,r\}$ and $c \in \mathbb{N}$, $1 \leq c \leq k_{i_o}$,

such that

(15) $$\lambda = cg_{i_o} + \sum_{i=i_o+1}^{r} k_i \, g_i$$

(16) $$M^c_{i_o} \text{ hom}_o \, g_j \quad (i_o + 1 \leq j \leq r)$$

(17) $$M_{i_o-1} \text{ hom}_o \, g_{i_o}$$

Intuitively, the Basic Lemma states that, given a homogeneous representation of a game, the measure of the largest min-win coalition (when collecting players according to rank) must exactly hit the majority level. Moreover, the remaining players - collecting their weights according to the measure $M^c_{i_o}$ – are going to play a series of "satellite games"

$$v^{(j)} = v_{g_j}^{M^c_{i_o}} \qquad i_o + 1 \leq j \leq r, \; m^c_{i_o} \geq g_j \, .$$

$$v^{(i_o)} = v_{g_{i_o}}^{M_{i_o-1}}$$

in order to replace the "large players" ($j \geq i_0 + 1$) and the medium players (i_0) - or rather, the members of the large and medium fellowships.

The fact, that [9] deals only with reduced representations may be neglected. This is verified at once by reducing and extending representations at will.

The term "largest coalition", suggested by the Basic Lemma leads to an ordering of profiles. As the largest fellows have weight g_r , it is reasonable to introduce the lexicographic order backwards on vectors (profiles) $s \in \mathbb{N}^r$; thus the last coordinate s_r is the first to be considered for the lexicographic ordering.

According to (15) (16) (17), the profile of the "lexicographically largest coalition" (the lex-max coalition) is uniquely determined by M and λ and given by

$$s^\lambda = s^\lambda_M = (0,\ldots,0, c, k_{i_0+1},\ldots,k_r)$$

Whenever M hom λ , the lex-max coalition is specified by the basic lemma (which presents i_0 and c) simultaneously with the "satellite measures" M_{i_0-1} and $M^c_{i_0}$. Any other minimal winning coalition is obtained from s^λ by successively replacing larger players by smaller ones - that is, via successively playing satellite games $v^{(j)}$.

If no $v^{(j)}$ is defined at all then all members of fellowships $\leq i_0-1$ are necessarily dummies - (cf §2), thus smaller fellows may enter a minimal winning coalition only by "replacing" larger ones via a satellite game.

By induction it will be seen that this situation is repeated throughout the game (§3). The satellite games provide a way to define (and prove the existence of) the minimal representation (cf [5]) and to get hold of an algorithm for obtaining this representation. Let us shortly elaborate on this topic.

The Basic Lemma permits to define (recursively) a test for homogeneity of a pair (M,λ). In addition, it provides a method of computing "all homogeneous games" via the "matrix of homogeneity" - the details may be found in [9].

Our present aim is slightly different. OSTMANN [5] proves the following

<u>Theorem</u> ("The smallest committee") For any homogeneous game there is a unique representation $(M,\lambda) \in \mathfrak{M} \times \mathbb{N}$ such that $M(\Omega)$ is minimal (and M is reduced).

Now, in this note, we shall first of all clear up the relation between the basic lemma (the satellite games) and the minimal representation à la OSTMANN, secondly provide an alternative proof for the above Theorem which, bypassing, shows that the representation is $\succsim$ - minimal, and, thirdly, lay the grounds for an algorithm which yields the minimal representation given any homogeneous representation.

This seems to be desirable since the procedure offered in [5] necessarily employs the incidence matrix of the minimal winning coalition - which means that, given a homogeneous representation first coalitions have to be tested for the minimal winning property - a procedure which is quite tedious and fastly exceeds the capacity of an ordinary computer even for medium sized games.

§2 Games with few non-dummies

In view of the Basic Lemma, the measures M_{i_0-1} and $M^c_{i_0}$ play a decisive rôle concerning the relations between smaller fellowships ($i < i_0$), the medium one ($i = i_0$) and larger fellowships ($i > i_0$).

E.g., if $m_{i_0-1} \geq g_{i_0}$, then according the the Basic Lemma, M_{i_0-1} hom g_{i_0}, thus

$$v^{(i_0)} := v^{M_{i_0-1}}_{g_{i_0}}$$

is a homogeneous game. The min-win coalitions of this game may enter those of v by replacing i_0. Call $M_{i_0-1} =: M^{(i_0)}$ the <u>satellite measure</u> of i_0. Fellowship i_0 will be called a <u>sum</u> and $v^{(i_0)}$ its <u>satellite game</u>.

If $m_{i_0-1} < g_{i_0}$, then there is no way for small fellowships to sum up their weights in order to replace i_0 (vaguely speaking at this point of the development) and i_0 is called a <u>step</u>.

Similarly, for $j \leq i_0 + 1$, the <u>satellite measure</u> is $M^c_{i_0} =: M^{(j)}$ and the <u>satellite game</u> is

$$v^{(j)} := v^{M^{(j)}}_{g_j}$$

provided $m^c_{i_0} - g_j$. If so, j is a <u>sum</u>, otherwise it is a <u>step</u>.

In §3, dummies, sums, and steps will be defined inductively as to the one of three possible <u>characters</u> of any fellowship. The present section is dealing with some degenerate cases (small fellowships are dummies and large ones constitute one type) which also form a part of the first inductive step.

In this context let us write $M = 0$ for a measure which is understood to live on K_o (thus corresponding to $(0,g_o)$), even if $K_o = \emptyset$ $(k_o = 0)$, in which case $M = 0$ may be regarded as the trivial measure on the empty set.

Remark 2.1. Let $v = v_\lambda^M$, $M = (g,k) \in \mathfrak{M}^r \times \mathbb{N}$ and $r = 1$. Essentially, there is only one weight g_1 available and

$$\lambda = c\, g_1 \qquad\qquad 1 \leq c \leq k_1 \;;$$

thus, the class of homogeneous games for $r = 1$ is specified by indicating values for g_1, k_o, k_1, and c. A typical member of this class is represented by

$$(1) \qquad (g;k;\lambda) = (0,g_1;k_o,k_1;cg_1) \; k_o + k_1 = n, \; g_1 \geq 1, \; 1 \leq c \leq k_1$$

Fellowship $i = 0$ is a dummy.

We shall call the only nondummy fellowship in this case, type 1, a step. Its satellite measure is $M^{(1)} = 0$ the total mass of which is $m^{(1)} = 0$. (Recall that $m^{(1)} = M^{(1)}(\Omega)$ is a convention).

Note that there is a unique minimal representation of any game v_λ^M given by (1), thus is

$$(2) \qquad (\bar{g};\bar{k};\bar{\lambda}) = (0,1;k_o,k_1;c)$$

Remark 2.2. Let $r \geq 2$ and consider the case where

$$m_{i_o-1} < g_{i_o} \; , \; m_{i_o}^c < g_{i_o+1}$$

It is seen at once that $s^\lambda = (0,\dots,0,c,k_{i_o+1},\dots,k_r)$ is the only profile of a minimal winning coalition; fellowships $i_o+1,\dots,r$ are inevitable while exactly c players of fellowship i_o and all members of larger fellowships (if any) are necessary and sufficient to form a minimal winning coalition. There are two subcases to be distinguished.

If $r = i_o$, $M^c_{i_o}$ is not defined and we have $\lambda = cg_r$,

$m_{i_o-1} = m_{r-1} = \sum_{i=1}^{r-1} k_i g_i < g_r$. Then, fellowships $0,1,\dots,r-1$ are <u>dummies</u>. Fellowship r is called a <u>step</u>. It's <u>satellite measure</u> is $M^{(1)} = M_{r-1}$; also $m^{(1)} = m_{r-1}$.

There is a unique minimal representation of v^M_λ given by

$$(3)\qquad (\bar{g},\bar{k},\bar{\lambda}) = (0,1;\ \sum_{i=0}^{r-1} k_i,\ k_r;\ c)$$

hence these games are members of the class described in Remark 2.1.. Obviously, there are two <u>types</u> involved in the game v.

If $i_o < r$, then we have in general (i.e. $c < k_{i_o}$) 3 types involved in the game. Fellowships $0,\dots,i_o-1$ are <u>dummies</u>, fellowship i_o is a <u>step</u>, fellowships $i_o+1,\dots,r$ are <u>steps</u> as well (and belong to one type!). The <u>satellite measures</u> are $M^{(i_o)} = M_{i_o-1}$ and $M^{(j)} = M^c_{i_o}$ $(j \geq i_o + 1)$. Again, v^M_λ has a minimal homogeneous representation given by

$$(4)\qquad (\bar{g},\bar{k},\bar{\lambda}) = (0,1,k_i-c+1;\ \sum_{i=0}^{i_o-1} k_i, k_{i_o},\ \sum_{i=i_o+1}^{r} k_i,\ c + \sum_{i=i_o+1}^{r} k_i(k_i-c+1))$$

provided $c < k_{i_o}$. For $c = k_{i_o}$, the game is again seen to be an element of the class discussed in Remark 2.1.

§3 Characters

During this section we assume that $M \in \mathfrak{M}^r$ and $\lambda \in \mathbb{N}$ satisfy M hom λ. Hence there is $i_0 \in \{1,\dots,r\}$ and c, $1 \leq c \leq k_{i_0}$ as specified by the Basic Lemma.

Definition 3.1.

1. Let $v = v_\lambda^M$; put $M^{(i_0)} := M_{i_0 - 1}$ and $M^{(j)} := M_{i_0}^c \; (j = i_0 + 1,\dots,r)$.

   $M^{(j)} \; (j = i_0 + 1,\dots,r)$ is called the satellite measure of fellowship j.

2. Let $J = J(M,\lambda) := \{j \geq i_0 \mid m^{(j)} \geq g_j\}$; if $J = \emptyset$ then r is treated by Remarks 2.1. and 2.2.

3. For $j \in J$,
   $$v^{(j)} := v_{g_j}^{M^{(j)}}$$
   is the satellite game for fellowship j.

The definitions given above refer to (M,λ). We are now going to define the term "dummy" also w.r.t. (M,λ) - however, it turns out at once that this property depends on v only.

Definition 3.2. (Inductive definition of dummies)

1. For $r = 1$, fellowship 0 is the only dummy.
2. Let $r \geq 2$. If $I = \emptyset$, then dummies are defined by Remark 2.2. Let $I \neq 0$. For $j \in I$, let $D^{(j)} = D(M^{(j)},g_j)$ denote the dummies of $(M^{(j)},g_j)$ (inductively defined) and put $D = D(M,\lambda) = \bigcap_{j \in I} D^{(j)}$ to be the dummies of (M,λ).

<u>Lemma 3.3.</u> D is independent of the representation. More precisely:

1. $j \in D = D(M,\lambda)$ iff each $\omega \in K_j$ is a v-Dummy in the ordinary sense (see e.g. [8], CH III, SEC. 2),

2. $T_o = \sum_{j\in D} K_j$, $\mathring{k}_o = \sum_{j\in D} k_j$ where $\Omega = T_o + \ldots + T_t$ is the decomposition of Ω into types and $\mathring{k} = (\mathring{k}_1,\ldots,\mathring{k}_t) = (|T_o|,\ldots,|T_t|)$ the distribution of players over the types (cf. §1).

3. If (M', λ') is a further homogeneous representation of v we may (after suitable extension (cf §1) assume that $M, M' \in \mathfrak{M}^r$. Then

$$D(M,\lambda) = D(M', \lambda').$$

<u>Proof:</u> It suffices to check that the dummies given by Definition 3.2. are exactly the ones of v in the ordinary sense (ignoring the difference between player and fellowships. This is performed by induction.

1. For $r = 1$ or $I = \emptyset$ our statement is obvious.

2. For $r \geq 0$ and $I \neq \emptyset$ our statement is obvious for $j \geq i_o$.

   Let $\bar{i} \leq i_o-1$. Suppose $\bar{i} \notin \bigcup_{j\in I} D^{(j)}$, say $\bar{i} \notin D^{(\bar{j})}$;

   Clearly, $g_{\bar{i}} > 0$, i.e., $\bar{i} > 0$. By induction $\omega \in K_{\bar{i}}$ is not a dummy in $v^{(\bar{j})}$ (in the ordinary sense). Assume $\bar{j} \geq i_o + 1$ ($\bar{j} = i_o$ runs analogously). There is a profile

$$s^o = (s^o_1,\ldots,s^o_{i_o})$$

   of a min-win coalition in $v^{(\bar{j})}$ such that $s^o_{\bar{i}} > 0$ and

$$M^{(\bar{j})} (s^o) = M^c_{i_o} (s^o) = g_{\bar{j}} .$$

Note that $s_{i_o}^o \leq k_{i_o} - c$.

The profile

$$\bar{s} = (s_1^o,\ldots, s_{i_o-1}^i, s_{i_o}^o + c, k_{i_o+1},\ldots,k_{\bar{j}} - 1,\ldots,k_r)$$

reflects a min-win coalition in Ω since

$$M(\bar{s}) = \lambda$$

and as $\bar{s}_{\bar{i}} = s_{\bar{i}}^o > 0$, $\omega \in k_{\bar{i}}$ is not a v-Dummy, which completes the first part of the proof.

On the other hand, suppose that $\bar{i} \leq i_o-1$ and $\omega \in K_{\bar{i}}$ is not a v-dummy; again $g_{\bar{i}} > 0$ is necessarily true. Let $\tilde{s}$ be the profile of a min-win coalition (in Ω) s.t. $\tilde{s}_{\bar{i}} > 0$. Since

$$\tilde{s} \neq s^\lambda = (0,\ldots,0, c, k_{i_o+1},\ldots,k_r)$$
$$M(\tilde{s}) = \lambda$$

we have

$$0 = M(\tilde{s}-s^\lambda) = \sum_{i<i_o} \tilde{s}_i g_i + (\tilde{s}_{i_o}-c)g_{i_o} - \sum_{i>i_o} (k_i-\tilde{s}_i)g_i \text{ .}$$

Assume $\tilde{s}_{i_o} \geq c$ ($\tilde{s}_{i_o} < c$ is treated analogously replacing $M_{i_o}^c$ by M_{i_o-1}). Then, putting $\tilde{s}^o = (\tilde{s}_1,\ldots,\tilde{s}_{i_o}-c,0\ldots0)$

(1) $M_{i_o}^c(s^o) = M_{i_o}^c(\tilde{s}) = cg_{i_o} + \sum_{i>i_o} (k_i-\tilde{s}_i)\, g_i > 0$.

Verbally: as mass may be removed from the min-win coalition corresponding to s^λ only in lumps of at least size g_{i_o} , there must be small and medium sized fellows in $\tilde{S}$ in order to play a satellite game $v^{(j)}$.

In view of the basic lemma, $M^c_{i_0}$ hom g_j $(j \geq i_0)$, thus (1) suggests that s^0 may be decomposed into profiles s^{jl} s.t.

$$(2) \qquad M^c_{i_0}(s^{jl}) = g_j \qquad \begin{array}{l}(j = r_0,\ldots,r;) \\ l = 1\ldots(k_j - \tilde{s}_j); \\ (l = 1,\ldots,c \text{ for } j = i_0)\end{array}$$

As $s^0_{\bar{i}} > 0$, at least one of the $s^{jl}_{\bar{i}} > 0$, thus (2) shows that $\omega \in K_{\bar{i}}$ is a member of a min-win coalition in some v^j. By induction, $i \notin D^{(j)}$ and by Definition 3.2.2., $i \notin D$, q.e.d.

<u>Definition 3.4.</u> (Inductive definition of satellite measures)
Let $v = v^M_\lambda$ and $\lambda = \lambda^c_{i_0}$.

For $i \notin D = D(M,\lambda)$, the satellite measure $M^{(i)}$ w.r.t. (M,λ) is defined as follows.

1. For $r = 1$, $M^{(1)} := 0$ (and $m^{(1)} = 0$), in accordance with Remark 2.1.

2. Let $r \geq 2$ and $j \geq i_0$.
   For $j = i_0$, $M^{(j)} := M_{i_0 - 1}$;
   for $j > i_0$, $M^{(j)} := M^c_{i_0}$;

   in accordance with Definition 3.1.

3. Let $r \geq 2$ and $i < i_0$.
   Assuming that $i \notin D$, we know that $J = J(M,\lambda) \neq \emptyset$ (Definition 3.2. and Remark 2.2.). Let

$$J^i := \{j \geq i_0 \mid i \notin D^j\} \neq \emptyset$$

For $j \in J^i$, let $M^{(i,j)}$ be the satellite measure of i w.r.t. $(M^{(j)}, g_j)$ (which is defined by induction). Pick $j_o \in J^i$ such that

$$m^{(i,j_o)} = \max \{m^{(i,j)} \mid j \in J^i\}$$

and put

$$M^{(i)} := M^{(i,j_o)} \quad .$$

<u>Remark 3.5.</u> 1. $M^{(i)}$ is a projection of M and corresponds to a certain $(0,g_1,\ldots,g_i;\ k_o,k_1,\ldots,k_{i_1} - d) \in \mathfrak{M}^{i_1}$

where $i_1 < i$. In addition, we have

(3) $$M^{(i)} \ \hom_o g_i \ .$$

To see this, observe that it is true for $j \geq i_o$ by the Basic Lemma and follows for $i < i_o$ at once by induction. Let us write

(4) $$C^{(i)} := (k_o,k_1,\ldots,k_{i_1} - d)$$

(the <u>"carrier"</u> of $M^{(i)}$) .

2. In particular, $M^{(i,j)}$ corresponds to a certain

$$(g_o,\ldots,g_{i(j)};\ k_o,\ldots,k_{i(j)}) \in \mathfrak{M}^{i(j)}$$

(this is used within the above mentioned induction proof). Hence j_o in step 4. of Definition 3.4. is in fact choosen such that $M^{(i,j_o)}$ is the largest or lexicographically largest (backwards!) measure or vector. We may, therefore, (regarding $M^{(i,j)}$ as a measure or vector) also write

(5) $$M^{(i)} = M^{(i,j_o)} \geq M^{(i,j)} \quad (j \in I^i)$$

or

(6) $$M^{(i)} = \text{lex max}\ \{M^{(i,j)} \mid j \in I^i\}$$

Definition 3.6. 1. Let $v = v_\lambda^M$ and $i \notin D = D(M,\lambda)$.
If $m^{(i)} \geq g_i$, then i is called a sum.
If $m^{(i)} < g_i$, then i is a step.

2. If i is a sum, then

$$v^{(i)} := v_{g_i}^{M^{(i)}}$$

is satellite game of i (and $(M^{(i)}, g_i)$ is a hom representation of $v^{(i)}$. The lexicographically largest profile of a min win coalition (in $v^{(i)}$)

$$s^{(i)} := s_{g_i}^{M^{(i)}}$$

is said to represent the substitutes of i , $s^{(i)}$ has the shape

(7) $$s^{(i)} = (0,\ldots,e,\ k_{i_2+1},\ldots,k_{i_1-1},\ k_{i_1} - d)$$

and satisfies, of course

(8) $$M^{(i)}\ (s^{(i)}) = g_i$$

3. If i is a step, then the carrier of $M^{(i)}$

(9) $$s^{(i)} := C^{(i)} = (k_1,\ldots,k_{i_1} - d)$$

is said to represent the pseudo substitutes of i; clearly we have in this case:

(10) $$M^{(i)}\ (s^{(i)}) < g_i\ .$$

Thus, a fellowship must have one of three characters. If it is a dummy, he plays no essential rôle in the game; it's satellite measure is not defined at all. If it is a step, then it's satellite measure is well defined; but as it is too small, there is no satellite game. If it is a sum, then it's satellite measure is sufficiently large, the smaller players can combine their weights in order to play the satellite game and the substitutes are the largest (lex) coalition to replace it in a min win coalition of v.

Note that fellows of $j \geq i_0$ are inevitable players: they all show up in any min win coalition. On the other hand, the only way smaller fellows $(i < i_0)$ may enter a min win coalition is via replacing successively sums by playing satellite games. This will become obvious during the later development.

Remark 3.7. 1. Fellowship $i < i_0$ is a sum (w.r.t. (M,λ) if and only if it is a sum w.r.t. $(M^{(j)}, g_j)$ for at least one $j \geq i_0$.

2. Let $i < i_0$ be a sum and let $j_0 \geq i_0$ be such that

$$M(i) = M(i,j_0)$$

Then i is a sum w.r.t. $(M^{(j_0)}, g_{j_0})$

3. In this case clearly $s^{(i,j_0)} = s^i$ (with obvious notation) i.e., the substitutes of i w.r.t. (M,λ) and w.r.t. $(M^{(j_0)}, g_{j_0})$ are the same. (Remark 3.5.2.)

Our first aim is to show that (like dummies) the characters "sum" and "step" are awarded by the game v and not by the representation (M,λ). The next Theorem however, shows somewhat more.

Theorem 3.8. Let $v = v_\lambda^M$ and let i be a sum (w.r.t. (M,λ)). Suppose i' is of the same type as i (w.r.t. v). Then $g_i = g_{i'}$ (i.e., if the representation is reduced, $i = i'$) and, of course, i' is a sum as well.

Proof: For $r = 1$ and for $r \geq 2$, $I = \emptyset$, there is nothing to be proved as there are no sums (Remarks 2.1., 2.2.). Assume therefore, $r \geq 2$ and $I \neq \emptyset$.

Next observe that $i \geq i_0$ implies $i' \geq i_0$ and vice versa for otherwise it is easily verified that i and i' cannot be of the same type.

1st CASE: $i, i' \geq i_0$.

Suppose $i < i'$. As i is a sum, one player of fellowship i may be replaced by his substitutes in order to change the profile of the lex max coalition; thus there is a min win coalition with profile

$$\begin{aligned}\tilde{s} &= (s_1,\ldots,s_{i_0},\ldots,k_i - 1,\ldots,k_r)\\ &= s^\lambda - e^i g_i + s^{(i)}\end{aligned}$$

(say, the tacit assumption $i > i_0$ made by writing down (11) is unimportant). Here, e^i is the i'th coordinate vector and $s^{(i)}$ the lex max profile in $(M^{(i)}, g_i)$, i.e.

$$s^{(i)} := (s_1,\ldots, s_{i_0} - c).$$

However, the profile

$$\hat{s} := \tilde{s} + e^i g_i - e^{i'} g_{i'}$$

has to total weight $M(\hat{s}) < \lambda$, thus i and i' are different types if $g_{i'} < g_i$, a contradiction.

On the other hand, if $i > i'$, then

$$m^{(i')} = m^c_{i_o} = m^{(i)} \geq g_i > g_{i'} \, ,$$

thus i' is a sum. We may then repeat the above argument, exchanging i and i'. This settles the first case.

2nd CASE: $i, i' < i_o$. As i is a sum (w.r.t. (M,λ)) he is a sum w.r.t. some $(M^{(j)}, g_j)$ $(j \geq i_o)$. By induction, i' is a sum w.r.t. the same $(M^{(j)}, g_j)$ and $g_i' = g_i$. By definition, i' is a sum w.r.t. (M,λ), q.e.d.

Remark 3.9.

1. Whatever the representation of a homogeneous game v, sums of the same type have the same weight.

2. Dummies, steps, and sums are defined w.r.t. v. For a precise version of this statement, modify the statements of Lemma 3.3. accordingly. In particular, the types are classified to belong to one of the three characters.

3. For any representations (M,λ) of v, $T = T\,(M,\lambda)$ denotes the set of steps and $\Sigma = \Sigma\,(M,\lambda)$ the set of sums; thus

$$\{1,\ldots,r\} = D + T + \Sigma \; ;$$

if (M', λ') is a further representation, that has the same length (possibly after extension of both of them), then $T(M,\lambda) = T(M',\lambda')$ etc.

Note that $T \neq \emptyset$ while D and Σ may be empty.

<u>Lemma 3.10.</u> Let $v = v_\lambda^M$ and $i, i' < i_o$. Then i and i' belong to the same type (w.r.t. v) if and only if they belong to the same type w.r.t. each $v^{(j)}$ $(j \in J)$.

<u>Proof:</u> For $r = 1$ or $J = \emptyset$, nothing has to be proved: both, i and i' are dummies (Remarks 2.1., 2.2.). We may therefore, assume that $r \geq 2$ and $J \neq \emptyset$ for the remaining part of the proof.

<u>1st STEP:</u> Assume that i and i' belong to the same type w.r. to each $v^{(j)}$ $(j \in J)$.

Let $s \in \mathbb{N}_o^r$ be the profile of a min win coalition such that $s_i > 0$ and $s_{i'} < k_{i'}$; we have to show that

$$s' = s - e^i + e^{i'}$$

is also a min win coalition (e^i denoting the i'th basis vector).

Now, as $s_i > 0$ clearly $s \neq s^\lambda$, therefore either $s_{i_o} \geq c$ and

$$M_{i_o}^c \, (s_1, \ldots, s_{i_o} - c) \geq g_{i_o+1}$$

or otherwise

$$M_{i_o-1} \, (s_1, \ldots, s_{i_o-1}) \geq g_{i_o} \;;$$

we treat the first alternative as the second is delt with analogously.

Put $s^o := (s_1, \ldots, s_{i_o-1}, s_{i_o} - c) \in \mathbb{N}_o^{i_o}$ and observe that

$$\text{(11)} \qquad M_{i_o}^c \, (s^o) = \lambda - c g_{i_o} - \sum_{l \geq i_o+1} s_l g_l = \sum_{l \geq i_o+1} (k_l - s_l) \, g_l$$

As $M^c_{i_o}$ hom g_j $(j \geq i_o+1)$, it is seen that s^o may be decomposed into a sum of profiles $(\in \mathbb{N}_o^{i_o})$ each of which has exactly the weight g_j for some $j \geq i_o+1$. (Only indices $j \in J$ are involved!) The corresponding coalitions are min win in $v^{(j)}$ and at least one of these - say $\tilde{\mathring{s}}$ - has the property that

$$\tilde{\mathring{s}}_i > 0 \ , \ \tilde{\mathring{s}}_{i'} < k_{i'} \ .$$

As i, i' belong to the same type w.r.t. $v^{(j)}$, $\mathring{s}$ does not change its weight whenever we replace one member of weight i by one member of weight i', i.e.,

$$\hat{\mathring{s}} = \tilde{\mathring{s}} - e^i + e^{i'}$$

Therefore, adding up all the min win profiles of the $v^{(j)}$ again yields a profile $\mathring{s}'$ such that $\mathring{s}'$ has the same weight as $\mathring{s}$ and

$$\mathring{s}' = \mathring{s} - e^i + e^{i'}$$

But

$$M^c_{i_o} \ (\mathring{s}') = \sum_{l \geq i_o+1} (k_l - s_l) \ g_l$$

in view of (11) is fastly rewritten to mean

$$M^c_{i_o} \ (\mathring{s}') + M \ (0,\ldots,0,c,s_{i_o+1},\ldots,s_r) = \lambda$$

i.e.

$$M \ (s + e^i - e^{i'}) = \lambda$$

which completes the first step.

<u>2^nd STEP:</u> Assume now, that i and i' belong to the same type as v is concerned. Consider a min win profile for some $v^{(j)}$ $(j \in I)$, say $\overset{o}{s}$, such that

$$\overset{o}{s}_i > 0 \ , \ \overset{o}{s}_{i'} < k_{i'} \ .$$

Then

$$s = (\overset{o}{s} + ce^{i_o}, k_{i_o+1}, \ldots, k_j - 1, \ldots, k_r)$$

is a min win profile for v (assuming tacitly $j \geq i_o+1$, which is unimportant).

Therefore

$$s' = s - e^i + e^{i'}$$

has the same weight $M(s') = \lambda$ which implies that

$$\overset{o}{s}' = \overset{o}{s} - e^i + e^{i'}$$

has the same weight

$$M^c_{i_o} (\overset{o}{s}') = g_j \ , \qquad \text{q.e.d.}$$

<u>Corollary 3.11.</u> Steps i and i' belong to the same type if and only if they have the same satellite measure (i.e., the same pseudo substitutes).

<u>Proof:</u> For $r = 1$ or $J = \emptyset$ the statement is obviously true. Let $r \geq 2$ and $J \neq \emptyset$. Again, for $i, i' \geq i_o$ or $i \geq i_o > i'$, the corollary is verified at once. It remains to consider the case that i and i' both satisfy $< i_o$; this is treated by an inductive argument.

As i and i' are steps, they are steps or dummies in any v^j $(j \in J)$ and this character they share <u>simultaneously</u> by Lemma 3.10.

For any $j \in J^i$, i and i' are steps in v^j and by induction they have the same satellite measure, i.e.

$$M^{(i,j)} = M^{(i',j)} \qquad (j \in J^i = J^{i'})$$

By Definition 3.4. it follows immediately that $M^{(i)} = M^{(i')}$, q.e.d.

Remark 3.12. Let us shortly clear up the connection to OSTMANN's [5] presentation. No proofs will be offered.

1. In [5], characters are awarded to players, not to types (not to speak of fellowships).

2. In both versions, dummies are identical.

3. In [5], a sum and a step may belong to the same type. If a type is called "step" whenever it has a member (a player) who is step, then both versions are identical.

4. It is not hard to see that "steps rule their followers", i.e., if i is a step, $1 < i$ and $s_1 > 0$ for some min win profile s, then $s_i = k_i$.

5. The typical "final step" is i_o (and its inductive counterparts). this means exactly $M^{(i)} = (k_i,\dots,k_{i-1})$, $m^{(i)} < g_i$.

6. If the smallest nondummy is the only step, then all weight of larger fellowships are multiples of g_i. Thus, there is a homogeneous representation $(\bar{M},\bar{\lambda})$ s.t. $\bar{g}_1 = 1$, $\bar{g}_o = 0$. This representation is minimal (after reduction). This follows in our present framework by means of the homogeneity of any satellite measure $M^{(i)}$ w.r.t. g_i.

7. If v is (homogeneous), constant sum and superadditive, then there is only one step, the smallest nondummy (see [5]). Therefore, given some normalization (say $g_1 = 1$ or $M(\Omega) = 1$) and the requirement $g_0 = 0$ (dummies get zero weight), the representation is unique. This is a well known fact; however, Theorem 3.8. provides a further proof. (another one being given in [5], the standard proofs are to be found in [4],[7],[9].

Lemma 3.13. Let $v = v_\lambda^M$ and let $i \notin D$ be a nondummy fellowship. If $i < j$ then ($j \notin D$ and) $M^{(i)} \leq M^{(j)}$.

Proof: For $j \geq i_0$ the statement is obvious. For $j < i_0$ we have also $i < i_0$ and we may proceed by induction.

Indeed, whenever $j^* \geq i_0$ and $M^{(i,j^*)}$ denotes the satellite measure of i w.r.t. $(M^{(j^*)}, g_{j^*})$ (i.e. $M^{(j^*)} = M^c_{i_0}$ or $M^{(j^*)} = M_{i_0-1}$ respectively), then we may by induction assume that

$$M^{(i,j^*)} \leq M^{(j,j^*)}$$

holds true whenever both terms are defined, i.e. whenever i (and, consequently j) is not a dummy "in v^{j^*}". Clearly (see Definition 3.4.3.)

$$J^i \subseteq J^j$$

and hence

$$M^{(i)} = \underset{j^* \in I^i}{\text{lex max}}\, M^{(i,j^*)} \leq \underset{j^* \in I^j}{\text{lex max}}\, M^{(i,j^*)} = M^{(j)}, \qquad \text{q.e.d.}$$

Lemma 3.14. Let $v = v_\lambda^M$ and let $i < j$ be fellowships such that i is not a dummy in $v^{(j)}$. Also, let $M^{(i,j)}$ denote the satellite measure of i w.r.t. $(M^{(j)}, g_j)$. Then

$$M^{(i,j)} \leq M^{(i)} \quad .$$

<u>Proof:</u> For $j \geq i_o$ we have necessarily $i < i_o$ and, by Definition 3.4.:

$$M^{(i,j)} \leq \operatorname*{lex\,max}_{j^* \in J^i} M^{(i,j^*)} = M^{(i)}$$

Now let $j < i_o$. Observing that $M^{(i)}$ is defined w.r.t. (M,λ) write $M^{(i)} = M^{(i)}\ (M,\lambda)$ for the moment, such that

$$M^{(i,j)} = M^{(i)}\ (M^{(j)}, g_j)$$

and

$$M^{(i,j,l)} := M^{(i)}\ (M^{(j)}\ (M^{(l)}, g_l), g_j)$$

whenever all measures are defined. Choose $j_o \geq i_o$ such that

$$M^{(j)} = M^{(j)}\ (M^{(j_o)}, g_{j_o}) = M^{(j,j_o)}$$

Then

$$M^{(i,j)} = M^{(i)}\ (M^{(j)}, g_j) = M^{(i,j,j_o)}$$

Consider $(M^{(j_o)}, g_{j_o})$; by induction we have

$$M^{(i,j,j_o)} \leq M^{(i,j_o)}$$

and hence

$$M^{(i,j)} = M^{(i,j,j_o)} \leq M^{(i,j_o)}$$

$$\leq \operatorname*{lex\,max}_{j^* \in I^i} M^{(i,j\ )} = M^{(i)} ,$$ q.e.d.

<u>Theorem 3.15.</u> Let $v = v_\lambda^M$, $i < j$ and assume that i is a sum in $v^{(j)}$. Then i is a sum in v.

<u>Proof:</u> By applying (8) in Definition 3.6.1. to $(M^{(j)}, g_j)$ we have

(12) $$M^{(i,j)}\ (s^{(i,j)}) = M^{(j)}\ (s^{i,j}) = g_i$$

and hence

$$m^{(i)} \geq M^{(i)}\ (s^{(i,j)} = M^{(i,j)}\ (s^{(i,j)}) = g_i ,$$

this shows $i \in \Sigma$.

§4 The minimal representation

This section is devoted to the question of finding all (homogeneous) representations of a homogeneous game $v = v_{\lambda}^{M}$ and in particular the minimal one. The construction of the minimal (homogeneous representation slightly differs from the procedure offered in [5], the proofs in the present framework basically refer to inductive arguments and not to a lexicographic procedure as in [5].

Remark 4.1. Let $s \neq s^\lambda$ be a min-win profile. If we put

$$s^o := (s_o, s_1, \ldots, s_{i_o-1}, (s_{i_o}-c)^+)$$

$$s^{o-} := (s_o, s_1, \ldots, s_{i_o-1})$$

Then $\overset{o}{s}$ is a profile of a coalition "suitable" for v^j $(j > i_o)$ and, similarly $\overset{o-}{s}$ is suitable for v^{i_o}.

Now, as $\lambda = c + \sum_{j>i_o} k_i g_i$ and $M(s) = \lambda$,

$$M^c_{i_o}(\overset{o}{s}) = (c-s_{i_o})^+ \, g_{i_o} + \sum_{j>i_o} (k_j - s_j)\, g_j$$

(1)

$$M_{i_o-1}(\overset{o}{s}) = (c-s_{i_o})\, g_{i_o} + \sum_{j>i_o} (k_j - s_j)\, g_j$$

Consider the case $c > s_{i_o}$. We have $M_{i_o-1}(s^{o-}) \geq (c-s_{i_o})\, g_{i_o} \geq g_{i_o}$

and hence (by the Basic Lemma) $\overset{o-}{s}$ contains $(c-s_{i_o})$ subcoalitions of

weight $M_{i_o-1}(\cdot) = g_{i_o}$. If it so happens that $\sum_{j>i_o} (k_j - s_j)\, g_j > 0$,

then $\overset{o}{s}$ contains in addition $(k_j - s_j)$ coalitions of weight $M^c_{i_o}(\cdot) = g_j$.

By this procedure $\overset{o}{s}$ is completely exhausted.

Again appealing to the Basic Lemma we argue that the exhaustion procedure can be arranged in a way such that largest fellows are taken first. Thus $\overset{o}{s}$ decomposes

(2) $$\overset{o}{s} = (\overbrace{s_1, s_2,}^{s^j} \quad \overbrace{\ldots}^{s^{j^*}} \quad \overbrace{s_{i_o-1}, (s_{i_o}-c)^+}^{s^{j'}})$$

Here, s^j is a min win profile "in v^j", i.e. $M^j(s^j) = g_j$ for some

$j \geq i_o$ etc.

The case $c \leq s_{i_o}$ is handled accordingly. We shall refer to (2) as to a "canonical decomposition" of $\overset{o}{s}$. Note that the ordering of the s^j is arbitrary due to the various homogeneity properties of measures $M^{(j)}$ involved.

<u>Remark 4.2.</u> Let $s \neq s^\lambda$ be a min win profile and let l be the first index such that $s_l \neq 0$. Also let i be the first index larger than l s.t. $s_i < k_i$. Generically, s has the shape

(3) $$s = (0,\ldots,0,c,k_{l+1},\ldots,k_{i-1}, s_i, \otimes \ldots \otimes)$$

where $c > 0$ and $s_i < k_i$ (of course the 0's and k_q's could not appear). i is called the <u>smallest dropout</u> in s.

<u>Lemma 4.3.</u> ("The substitution lemma") Let i be the smallest dropout of a min win profile s. Then $M^{(i)}(s) \geq g_i$. In particular, i is a sum and $s^{(i)} \leq s$.

<u>Proof:</u> Let $\overset{o}{s}$ be defined as in 4.1. Consider the case that $i > i_o$. As $s \neq s^\lambda$ and $s_i < k_i$, it is seen at once that $M^i(\overset{o}{s}) \geq g_i$. The shape of s (consider the coordinates $i' \leq i$) implies the assertion of the Lemma. The case $i = i_o$ is handled analogously.

Therefore it remains to handle the case $i < i_o$ which is done by induction. If $M^{(j)}(0,\ldots, c, k_{l+1},\ldots,k_{i-1}, s_i,0,\ldots,0) \geq g_j$ then apply the Basic Lemma in order to construct

$$s^j = (0,\ldots,0,\ldots, c', k_{q+1},\ldots,k_{i-1}, s_i,0\ldots0)$$

which is a min win coalition for v^j, i.e.

$$M^{(j)}(s^j) = g_j \text{ , } q \geq 1 \text{ ,}$$

satisfying $s^j \leq s$. If, for all s^j, we have $M^{(j)}(\quad) < g_j$, then a canonical decomposition of $\overset{o}{s}$ (cf. Remark 4.2)

$$\overset{o}{s} = (0,\ldots,0,\overbrace{c,\ k_{l+1},\ldots}^{s^{j'}},\overbrace{k_{i-1},s_i,\ldots}^{s^j})$$

serves to the same purpose.

Thus it is possible to construct a min win profile s^j s.t. i is the smallest dropout in s^j (and hence $s^j \neq s^{(j)}$). By induction, i is a sum "in v^j" and $M^{(i,j)}(s^j) \geq g_i$. Hence

$$M^{(i)}(s) = \max_{j^* \in J} M^{(i,j^*)}(s) \geq M^{(i,j)}(s) \geq g_i$$

q.e.d.

<u>Remark 4.4.</u> If i is the smallest dropout of a min win profile s, then

(4) $$s^+ = s - s^i + e^i$$

is a min win profile as well. (4) shows that, on the other hand, s is obtained from s^+ by inserting i's substitutes for one fellow of fellowship i; let us call this procedure a <u>substitution</u>.

Since s^λ is the only profile that has no smallest dropout, we infer that <u>any min win profile is obtained from s^λ by finitely many substitutions</u>.

<u>Lemma 4.5.</u> ("Pseudo substitution lemma")

Let s be a min win coalition and let l be the first coordinate such that $s_l > 0$, i.e.

$$s = (0,\ldots,0,\ s_l,\ \boxtimes \ldots s_i \ldots \boxtimes\ )$$

Then

$$C^{(l)} \geq (k_0,\ldots,k_{l-1},\ 0,\ldots,0)$$

and for $i > l$

$$C^{(i)} \geq (k_0,\ldots,k_{l-1},k_l-s_l,\ 0,\ldots,0).$$

Proof: Observe that we have necessarily $l \leq i_0$; thus, for $i \geq i_0$ and $l = i_0$ the statement of the lemma is obvious. Assume, therefore, $l < i_0$, and proceed by induction. Again, $i \geq i_0$ is trivial, thus let $i < i_0$.

Consider a canonical decomposition of $\overset{0}{s}$ (cf. 4.1.), say

$$\overset{0}{s} = (0,\underbrace{\ldots,0,\ s_l,\ldots}_{s^{j'}},\underbrace{s_i,\ldots}_{s^{j}} \ldots)$$

There is some s^j, $M^{(j)}(s^j) = g_j$ such $s_i^j > 0$ ($i_0 \leq j \leq r$). The first nonvanishing coordinate of s^j, say l' satisfies $l \leq l' \leq i$ and $s_{l'}^j > 0$.

Now i is no dummy in $v^{(j)}$ and by induction hypothesis we conclude that

$$C^{(i,j)} \geq (k_0,\ldots,k_{l'}-s_{l'}^j,\ 0,\ldots,0)$$
$$\geq (k_0,\ldots,k_l-s_l,\ 0,\ldots,0)$$

(in case $l' < i$, say; the other cases are treated analogously). Hence

$$C^{(i)} = \max_{j\in J} C^{(i,j)} \geq (k_0,\ldots,k_l-s_l,\ 0,\ldots,0)$$

This settles the proof of the lemma. Note that $C^{(i)} = s^{(i)}$ if i is a step. Thus, if i is a step, it follows that

$$s^{(i)} \geq (k_0,\ldots,k_l-s_l) .$$

Remark 4.6. The term "maximal loosing profile" is supposed to be self explaining. Let s be maximal loosing and let i be the first coordinate such that $s_i < k_i$. Then s has generically the shape

$$s = (k_0,\dots,k_{i-1},\ s_i,\ \boxtimes \dots \boxtimes)$$

Clearly

$$(k_0,\dots,k_{i-1},\ s_i+1,\ \boxtimes \dots \boxtimes)$$

is winning, so by the basic lemma, we find

$$t = (0,\dots,0,\ c,\ k_{l+1},\dots,k_{i-1},\ s_i+1,\ \boxtimes \dots \boxtimes)$$

(typically), a min win coalition. Let

$$\tilde{s} := (k_0,\dots,k_{l-1}, k_l - c,\ 0 \dots 0)\ ,$$

by the pseudo substitution lemma 5.6., applied to t we have

$$s^{(i)} \geq \tilde{s}$$

We have thus

$$(5) \qquad s = t + \tilde{s} - e^i\ ,\ \tilde{s} \leq s^{(i)}$$

That is, the maximal loosing coalition s is obtained from a min-win coalition t by replacing a step by a subcoalition which is at least as weak as the coalition of pseudo substitutes.

Now, alternatively let i be a sum (it cannot be a dummy!). Then obviously $s \geq s^{(i)}$. Throw out the substitutes and put in player i. The profile

$$s^+ = s - s^{(i)} + e^i$$

has the same weight as s, thus it is loosing and its first coordinate l such that $s_l^+ < k_l$ is smaller then i.

We may repeat this procedure until we find the first coordinate which is smaller than k_* to be a step - and then repeat the procedure indicated above.

It follows that there is a subset I of sums such that

(6) $$s = \sum_{i \in I} (s^{(i)} - e^i) + t + \tilde{s} - e^k$$

where t is minimal winning, k is a step and $\tilde{s} \leq s^{(k)}$.

Thus, any maximal loosing coalition s is obtained from a minimal winning coalition t in the following way: replace a finite number of sums by their substitutes and replace a step by a coalition which is at least as weak as its pseudo substitute.

Definition 4.7. Let $(M,\lambda), (M',\lambda') \in \mathfrak{M}^r \times \mathbb{N}$ <u>such that $k = k'$</u>. Assume that M hom λ such that $v = v_\lambda^M$ is a homogeneous game. For $i \notin D = D(M,\lambda)$, let $C^{(i)} = (k_1,\ldots,k_{i_1}-d)$ be the carrier of $M^{(i)}$ (cf. 3.5.) and <u>define</u> a family

$$M'^{(i)} \qquad (i \notin D)$$

by

$$M'^{(i)} = M'\big|_{C^{(i)}}$$

The restriction on $C^{(i)}$, or, equivalently, the projection on the first coordinates).

(M',λ') is said to be <u>compatible</u> with (M,λ) if the following conditions are satisfied:

(7) $g_i' \geq m'^{(i)} + 1 \qquad i \in T = T(M,\lambda)$

(8) $g_i' = M'^{(i)}(s^i) \qquad i \in \Sigma = \Sigma(M,\lambda)$

(9) $\lambda' = M'(s^\lambda)$

Theorem 4.8.

1. If (M',λ') is compatible with (M,λ), then (M',λ') is a homogeneous representation of v, i.e.

$$v = v_{\lambda}^{M} = v_{\lambda'}^{M'}$$

2. Any two homogeneous representations are compatible with each other.

3. There is a unique minimal (homogeneous) representation of any homogeneous v; this is obtained by requiring an equation in any inequality (7). (and awarding dummies the weight 0).

4. The minimal representation may be computed by starting with the smallest nondummy and proceeding according to (7), (8), and (9).

Two proofs are being offered. The first one is relying on the results of this section (and thus rests in the spirit of [5]. The second one uses only Lemma 4.5. and is purely inductive through the subgames.

Note that we do not take care of non homogeneous representation. The minimal homogeneous representation is minimal according to [5].

First Proof: 1st Step: If $M(s) = \lambda$ then $M'(s) = \lambda$.
Clearly, $M'(s^{\lambda}) = \lambda'$ by (9).
If s is min win then, by Lemma 4.3.,

$$s = \sum_{i \in I} s^{(i)} - e^{i} + s^{\lambda}$$

where $I \subseteq \Sigma$. Using (8) we find

$$M'(s) = \sum_{i \in I} (M'(s^{(i)}) - g_i') + M'(s^{\lambda}) = M'(s^{\lambda}) = \lambda' \; .$$

<u>2nd Step:</u> If $M(s) < \lambda$ then $M'(s) < \lambda'$.

As s is maximal loosing. Pick I, k, t, and $\tilde{s}$ as in Remark 4.7, formula (6), $i \in I$ is sum, k is step, t is min win and $\tilde{s} \leq s^{(k)}$. By the 2st Step $M'(t) = \lambda'$. Using (7) and (8) we find:

$$\begin{aligned} M'(s) &= \sum_{i \in I} (M'(s^{(i)}) - M'(e^i) + M'(t) + M'(\tilde{s}) - g'_k \\ &= \lambda' + M'(\tilde{s}) - g'_k \\ &\leq \lambda' + M'(s^{(k)}) - g'_k < \lambda' \end{aligned}$$

This proves that $v = v^{M'}_{\lambda'}$.

The second part follows from 3.8., 3.9., and 3.11.

The third and fourth part of our Theorem is now obvious. q.e.d.

Now to the <u>Second Proof:</u> Put $M^{r+1} := M$, $g_{r+1} := \lambda$ and $v^{(r+1)} := v = v^M_\lambda$. Similarly for the quantities M', λ'. $r+1$ is formally called a "sum". Suppose $\underline{i}$ is the first non-dummy fellowship; we are going to show by induction

(10) "If $i \in \Sigma$, then $v^i = v^{M'(i)}_{g'_i}$".

Now, for $i = \underline{i}$ there is nothing to show because $\underline{i}$ is a step.

Therefore, fix some j, $\underline{i} < j \leq r+1$ and assume that (10) is true for all $i < j$. We shall show that (10) hold true for j. We proceed by two steps assuming that j is a sum.

<u>1st Step:</u> Let us check for $i < j$:

a) If $i \in \Sigma^j$ then

$$v_{g_i}^{M^{(i,j)}} = v_{g_i'}^{M'^{(i,j)}} .$$

b) If $i \in T^j$ then $m'^{(i,j)} + 1 \leq g_i'$.

Here, $M'^{(i,j)}$ denotes the projection of M' on $C^{(i,j)}$ etc.

Now, as for statement a), we know that $i \in \Sigma^j$ and thus $i \in \Sigma$ and

(11) $$M^{(i)} \geq M^{(i,j)}$$

Moreover, using our induction hypothesis

(12) $$M^{(i)}(s) = g_i \text{ iff } M'^{(i)}(s) = g_i'$$

We want to show

(13) $$M^{(i,j)}(s) = g_i \text{ iff } M'^{(i,j)}(s) = g_i' ,$$

which is equivalent to

(14) $$M^{(i)}(s \wedge C^{(i,j)}) = g_i \text{ iff } M'^{(i)}(s \wedge C^{(i,j)}) = g_i'$$

in view of (11) and the projection properties of M and M'. Clearly, (14) follows from (12) and a) is checked.

As for statement b), let $i \in T^j$.

If $i \in T$, then

$$m'^{(i,j)} \leq m'^{(i)} \leq g_i' - 1$$

in view of (7) of Definition 4.7.

If $i \in \Sigma$, then $v^i = v_{g_i'}^{M'(i)}$ by induction hypothesis, thus

$$g_i > m^{ij} = M^{(i,j)}\,(C^{(i,j)}) = M^{(i)}(C^{(i,j)})$$

implies

$$g_i' > M'^{(i)}(C^{(i,j)}) = m'^{(i,j)} ,$$

hence we are through with b).

<u>2nd Step:</u> In view of the first step we may not only assume that $(M'^{(i)}, g_i')$ represents $v^{(i)}$ for $i < j$, but also that $(M'^{(i,j)}, g_i')$ represents $v^{ij} = v_{g_i}^{M^{(i,j)}}$ for $i < j$ whenever $i \in \Sigma^j$.

Therefore we may, as a technicallity, omit the index j (this arguing so to speak, our case for $r \to r+1$) and instead of (10) show that

(15) If $v^i = v_{g_i'}^{M'(i)}$ hold strue for $i \in \Sigma$ then $v = v_{\lambda'}^{M'}$.

Again, two statements have to be chucked, namely

c) If $M(s) = \lambda$ then $M'(s) = \lambda'$

and

d) If $M(s) < \lambda$ then $M'(s) < \lambda'$.

Let us start out with c):

If s is a min win coalition and $s = s^{\lambda}$ then, nothing has to be proved as $M'(s) = \lambda'$ follows from 4.7.

Assume $s \neq s^\lambda$; an inspection of $\overset{o}{s}$ (cf. 4.1) teaches: if $s_j < k_j$ for some $j \geq i_o$, then $M^j(\overset{o}{s}) \geq (k_j - s_j)\, g_j$, i.e. $\overset{o}{s}$ contains (in view of the Basis Lemma) $k_j - s_j$ profils of $M^{(j)}$-measure g_j . These are min win coalitions of $v^{(j)}$ (j must be a sum!) and by (15), v^j is represented by $(M'^{(j)}, g_j')$. Thus any of the min coalitions of $v^{(j)}$ mentioned above has M'-measure g_j'! This way it is seen that $\overset{o}{s}$ decomposes such that its total M'-measure is

$$M'(\overset{o}{s}) = (c - s_{i_o})^+ g'_{i_o} + \sum_{j > i_o} (k_j - s_j)\, g_j' \quad .$$

Consequently

$$M'(s) = M'(\overset{o}{s}) + \min (s_{i_o}, c)\, g'_{i_o} + \sum_{j > i_o} s_j\, g_j' = M'(s^\lambda) = \lambda'$$

which finishes statement c).

Finally, as to statement d), assume (w.l.o.g.) that s is maximal loosing.
Let i be the first coordinate such that $s_i < k_i$, thus

$$s = (k_o, \ldots, k_i, s_i, \emptyset \ldots \emptyset)$$

By means of the Basis Lemma we construct a min-win coalition

$$t = (0, \ldots, 0, c, k_{l+1}, \ldots, k_i, s_i + 1, \emptyset \ldots \emptyset) \; .$$

Put $s^* := s - t + e^i = (k_o, \ldots, k_{l+1}, k_{l-i})$ (say, in case $i > l$, $i = l$ is treated analogously). In view of Lemma 4.5., $s^* \leq C^{(i)}$, thus

$$M^{(i)}(s^*) = M(s^*) < g_i$$

because t is winning and s is loosing.

Observing (15) and (7) we have

$$M'^{i}(s^*) < g_i' \qquad (\text{if } i \in \Sigma)$$

$$M'^{i}(s^*) \leq m'^{i} < g_i' \qquad (\text{if } i \in T)$$

As $M'(t) = \lambda'$ by statement c), we have

$$\begin{aligned} M'(s) &= M'(t) + M'(s^*) - M'(e^i) \\ &= \lambda' + M'(s^*) - g_i' \\ &< \lambda' \end{aligned}$$

q.e.d.

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