Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

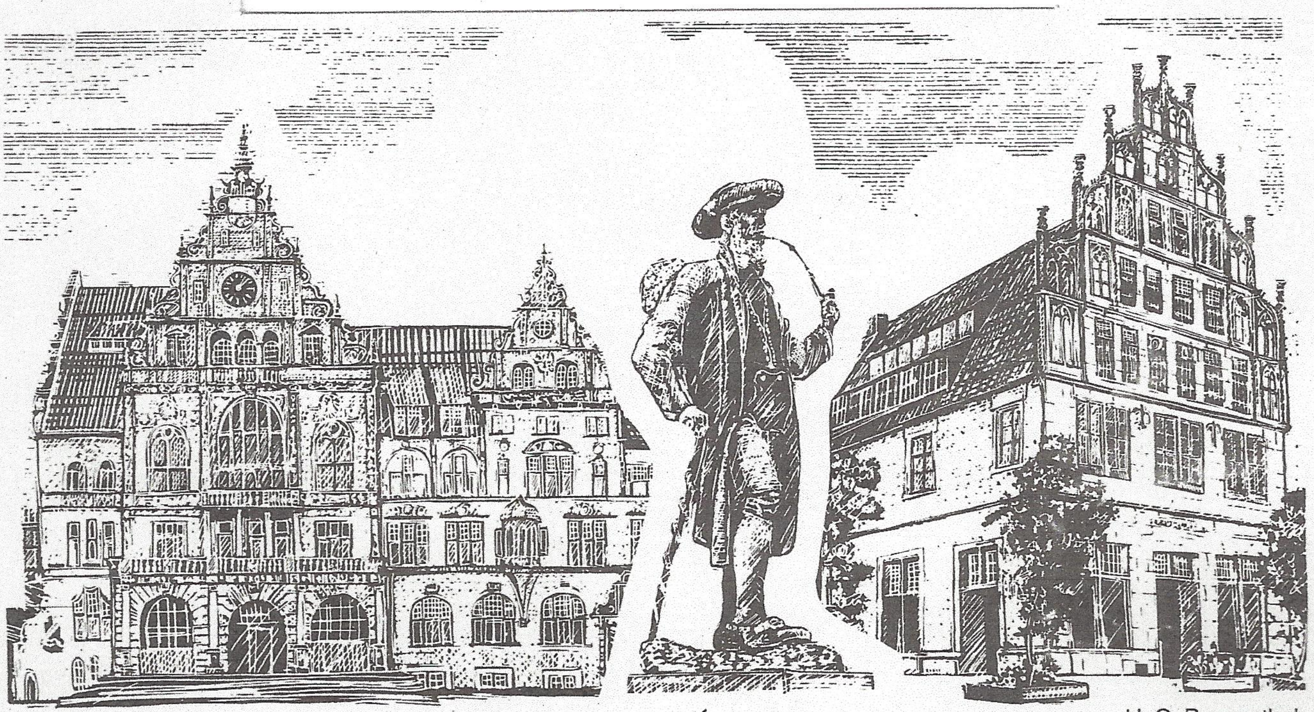
Nr. 185

Domination, Core and Solution
(A short survey of Russian results)

von

Olga Bondareva

April 1990



H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

Domination, Core and Solution
(A short survey of russian results) [1]

1. Introduction

This paper is a short survey of russian papers in cooperative game theory. We do not pretend this survey to be complete. It consists of

1) the main results as obtained by Bondareva, Vilkov, Kulakovskaja, Naumova and Sokolina and represented in their survey in Russian (1977).,
2) some unpublished results of Bondareva and Kulakovskaja.

All results are connected with the topic of domination, core and within the concept of the von Neumann–Morgenstern solution, other topics are omitted. All theorems are given without proofs.

2. The Abstract Domination, Core, Solution

Consider an arbitrary set A and a binary relation dom $\subseteq A \times A$, called domination. We shall write x dom y if $(x,y) \in$ dom, $\text{dom}(x) = \{y \in A : x \text{ dom } y\}$, $\text{dom}(B) = \bigcup_{x \in B} \text{dom}(x)$.

Let $x \subseteq A$, then the core C and the solution V are defined as follows

(1) $\quad C(X) = X - \text{dom}(X)$

(2) $\quad V(X) = X - \text{dom}(V(X))$.

The solution was defined by von Neumann and Morgenstern (1944). Usually the term v.N–M–solution is employed. For convenience, we shall just refer to it as "the solution".

3. A Solution of Abstract Domination, Continuity

Von Neumann and Morgenstern (1944) proved that for dom to have a solution for each $X \subset A$, a necessary condition is dom to be acyclic. A sufficient condition is dom to be strictly acyclic. Let A be a subset of a metric space (separable for simplicity). We say that dom is preserved in the limit if dom is transitive and for each sequence $(x_n)_{n \in \mathbb{N}}$ $x_n \xrightarrow[\mathbb{N}]{} x$, satisfying x_n dom x_{n-1} $(n \in \mathbb{N})$, it follows that x dom x_{n_0} for some n_0.

From the results of Kulakovskaja (1976) it is following the result.

[1] The author is grateful to Prof.Dr.J.Rosenmüller for providing an opportunity to write this paper and to Mrs.K.Fairfield for preparing the text.

Theorem 1:
dom has a solution on each compact $X \subset A$ if and only if it is preserved in the limit.

A relation dom is called strongly continuous (see Bondareva (1978)) on A if for x_n, y_n, $x, y \in A$ $(n \in \mathbb{N})$, $x_n \xrightarrow[\mathbb{N}]{} x$, $y_n \xrightarrow[\mathbb{N}]{} y$

1) x_n dom y_n $(n \in \mathbb{N})$ implies x dom y provided $x \neq y$

and

2) x dom y implies x_n dom y_n for all n exceeding some n_0.

Theorem 2: (Bondareva 1978)

The core as a function $C : 2^A \rightarrow 2^A$ is continuous in Hausdorf matrix if and only if dom is strongly continuous.

Because strongly continuous relations exist only in 1–dimensional spaces (see Bondareva (1987)), a finite approximation of the core is impossible in general n–dimensional spaces with $n > 1$. For special dom it is possible (see below).

4. The relation dom in games

Consider the following examples of dom.

1) A classical cooperative game $<I,v>$, $I = \{1,2,\ldots,n\}$, $v : 2^I \rightarrow \mathbb{R}^1$

$$A = \{x \in \mathbb{R}^n, x_i \geq v(\{i\}), x(I) = v(I)\} \quad (x(S) = \sum_{i \in S} x_i)$$

dom $= \bigcup_{S \subset I} \mathrm{dom}_S$ where

$x \; \mathrm{dom}_S \; y \longleftrightarrow$ 1) $x_i > y_i$, $i \in S$

2) $x(S) \leq v(S)$.

Bondareva and Naumova (1971) considered a slightly different dom defined via a condition

1) $x_i \geq y_i$, $(i \in S)$, $x(S) > y(S)$.

A second example

2) A game without side payments <I,V,A>, where V is a function $V : 2^I \to \mathbb{R}^n$,
A is a compact subset of $\mathbb{R}^n$,
$\text{dom} = \cup_{S \subset I} \text{dom}_S$, $x \text{ dom}_S Y \longleftrightarrow$ 1) $x_i > Y_i$, $i \in S$
2) $x \in V(S)$.

3) A game defined by Morovov (1973) is similar to a game of Lucas in partition function form. This is a special case of games without side payments. Let τ be a partition of I. Denote

$$A(\tau) = \{x \in \mathbb{R}^n : x_i \geq v(\{i\}), x(B) = v(B), B \in T\},$$

$$V(S) = \cup_{\tau \in B} A(\tau), H = V(I) = \cup_\tau A(\tau)$$

and define dom as in 2).

4) A game with an infinite set of players I, <I, Σ, v>, where Σ is an algebra of coalitions, v a real nonnegative function defined on Σ satisfying the conditions $v(\emptyset) = 0$, $v(I) < \infty$. A is the set of all additive functions x on Σ with $x(I) = v(I)$.

There is Owen's definition of dom in such a game, but there are no results about this domination. Kulakovskaja (1976) considered the dom defined with the aid of a special class Σ' of coalitions S with $v(S) > 0$ for dom', but dom_S is the same as in 1). For this dom Kulakovskaja received a sufficient condition for core to be a solution.

5. An approximation of game with domination

Consider an arbitrary game with dom as a game without side payment <I,V,A> and let A be a compact, set in $\mathbb{R}^n$. Let A_ϵ be an ϵ–net for A in $\mathbb{R}^n$. The net A_ϵ is named a convenient net if

1) $d(A_\epsilon, A) \leq \epsilon$, where d is Hausdorf distance,

2) $A_\epsilon \cap V(S)$ is an ϵ–net for $V(S)$.

Such nets always exist, the condition 1 always holds for $A_\epsilon \subset A$, from this condition $A_\epsilon \xrightarrow[\epsilon \to 0]{} A$ in Hausdorf metric d.

Consider δ–dom: x δ–dom y $\longleftrightarrow$ S exists

1) $x_i - y_i \geq \delta, i \in S,$

2) $x \in V(S)$

Let $C_\delta(A_\epsilon)$ is a core for δ–dom on A_ϵ.

Theorem 3: (Bondareva (1976))

If for some sequence $\{\epsilon\} \to 0$: $C_{2\epsilon}(A_\epsilon) \neq \emptyset$ then $\lim_{\epsilon\to 0} C_{2\epsilon}(A_\epsilon) = C(A)$ and if $C(A) \neq \emptyset$ then $\{A_\epsilon\}_{\epsilon\to 0}$ exists such that $\lim_{\epsilon\to 0} C(A_\epsilon) = C(A)$.

Corollary:

The core of an acyclic game (with acyclic dom) is nonempty.

For solution it is possible to prove only that each solution is a limit of a sequence of solutions of finite games on ϵ–nets (see Bondareva (1976)).

6. A structure of domination, acyclic game, Lucas game without solution

Consider $U(\sigma_1, \sigma_2)$ a subset of A where dom is possible via a set of coalition σ_2 and impossible via σ_1. For classical games

(3) $$U(\sigma_1,\sigma_2) = \{x \in A : x(S) \geq v(S),\ S \in \sigma_1,\ x(T) < v(T),\ T \in \sigma_2\}.$$

If $\sigma_1 \cup \sigma_2 = 2^I - I - \emptyset$ then the sets $U(\sigma_1, \sigma_2)$ constitute a partition of A.

Put X S–dom Y, X, Y $\subset$ A if $x \in X$ and $y \in Y$ exist such that x dom y.

Name a set $B \in U(\sigma_1, \sigma_2)$ a pyramidal set of 1 sort if $\cap \sigma_2 = \bigcap_{S\in\sigma_2} S \neq \emptyset$. Name B a pyramidal set of 2 sorts if $B = B_1 \cup \ldots \cup B_k$, $B_i \cap B_j = \emptyset$, $B_i \in U(\sigma_1^i, \sigma_2^i)$, $\bigcap_i \cap \sigma_2^i \neq \emptyset$ and there exists an infinite cycle for dom:

$$\{x_n\} : x_{nk+i} \text{ dom } x_{nk+i-1},\ x_{nk+i} \in B_i.$$

Note that there exists such i(B) that x dom y implies $x_i > y_i$.

Theorem 4: (Bondareva (1975))
An acyclic game has an acyclic structure, i.e. there is partition A into a system of pyramidal sets A_α acyclic via S–dom.

The pyramidal component exists in the Lucas game without solution.

7. Coverings (balanced collections of sets) and the core

A collection $\lambda = \{\lambda_S \geq 0\}_{S \subset I}$ is named a (balanced) covering iff $\sum_{S \subset I} \lambda_S \chi_S = \chi_I$ (see Bondareva (1963)). Name a system $\sigma = \{S\}$ of coalition independent if the system of incident vector χ_S is linearly independent. The covering λ is named reduced one if the system $\Theta(\lambda) = \{S : \lambda_S > 0\}$ is independent. $\Theta(\lambda)$ is named the balanced collection of sets.

Theorem 5:
The core C of game <I,s> is nonempty if and only if for each reduced covering

$$\lambda : \sum_{S \subset I} \lambda_S v(S) \leq v(I).$$

In above described structure of A (see (3)) the core is the set $U(\Sigma,\emptyset)$ where Σ is the system of all non–empty coalitions except I. To receive a condition of $U(\sigma_1,\sigma_2) \neq \emptyset$ is needed to extend the notion of the covering.

8. Extensive coverings (definition)

Name a collection $\lambda = \{\lambda_S\}_{S \subseteq I}$ an extensive covering iff $\sum_{S \subseteq I} \lambda_S \chi_S = 0$.

Denote $\Theta_1(\lambda) = \{S : \lambda_S > 0, S \neq I\}$, $\Theta_2(\lambda) = \{S : \lambda_S < 0, S \neq I\}$, $\Theta_i'(\lambda) = \{T : T = I–S, S \in \Theta_i(\lambda)\}$, i = 1,2.

The extensive covering λ is named the reduced one if the system $\Theta_1(\lambda) \cup \Theta_2'(\lambda)$ is linear independent. This definition is correct because if $\Theta_1(\lambda) \cup \Theta_2'(\lambda)$ is independent then $\Theta_1'(\lambda) \cup \Theta_2(\lambda)$ is also independent. Denote $\Lambda = \{\lambda\}$ the set of all extensive covering for n-player game, Λ^* – the set of all reduced ones. The set Λ^* is finite.

9. Extensive coverings and exact bounds for the core

Consider the problem to find the exact bounds for core-imputations as a linear problem $\{\underline{P}_i\}$ and $\{\overline{P}_i\}$:

$$\begin{cases} x(S) \geq v(S) \\ x(I) = v(I) \\ \min x_i \end{cases} \qquad (\underline{P}_i)$$

$$\begin{cases} x(S) \geq v(S) \\ x(I) = v(I) \\ \max x_i \end{cases} \qquad (\overline{P}_i)$$

The application of the theory of duality generates extensive covering of a special kind. Name a collection $\overline{\lambda}^i = \{\overline{\lambda}^i_S \geq 0\}_{S \subsetneq I}$ an upper i-covering if $\sum_{S \subset I} \overline{\lambda}^i_S \chi_S - \overline{\lambda}^i_I \chi_I = \chi_{\{i\}}$ and name $\underline{\lambda}^i = \{\underline{\lambda}^i_S \geq 0\}_{S \subsetneq I}$ a lower i-covering if $\underline{\lambda}_I \chi_I - \sum_{S \subset I} \underline{\lambda}^i_S \chi_S = \chi_{\{i\}}$. Note that $\overline{\lambda}^i$, $\underline{\lambda}^i$ are extensive coverings with the systems $\Theta_1(\overline{\lambda}^i) = \{S : S \neq \{i\}\}$ and $\Theta_2(\overline{\lambda}^i) = \{\{i\}\}$ and $\Theta_1(\underline{\lambda}^i) = \{\{i\}\}$, $\Theta_2(\underline{\lambda}^i) = \{S : S \neq \{i\}\}$.

Denote the sets of all reduced upper and lover i-coverings by $\overline{\Lambda}^*_i$ and $\underline{\Lambda}^*_i$.

Theorem 6:

$$\min_{x \in C} x_i = \max_{\overline{\lambda}^i \in \overline{\Lambda}^*_i} \left(\sum_{S \subset I} \overline{\lambda}^i_S v(S) - \overline{\lambda}_I v(I) \right)$$

$$\max_{x \in C} x_i = \min_{\underline{\lambda}^i \in \underline{\Lambda}^*_i} \left(\underline{\lambda}_I v(I) - \sum_{S \subset I} \underline{\lambda}^i_S v(S) \right)$$

The bounds considering by Tijs and Driessen are also expressed with the aid of i-coverings.

10. A new necessary and sufficient condition for a strong stable core existence

Core is named a stable one if it is also a solution. Name core a strong stable one if it has the following properties. Let $x \notin C$ then for each S: $x(S) < v(S)$ there exists $y \in C$: $y \text{ dom}_S x$.

Denote CV the set of games with the core stable property and $\overline{CV}$ the set of games with the strong core stable property. The property for core to be stable depends on the structure of the game, i.e., from emptiness or nonemptiness of sets $U(\sigma_1, \sigma_2)$.

Name an extensive covering $\lambda \in \Lambda$ generating by system (σ_1, σ_2) of coalition S, where $\{\{1\}, \{2\},\ldots,\{n\}\} \subset \sigma_1$ if $\Theta_1(\lambda) \in \sigma_1$ and $\Theta_2(\lambda) \in \sigma_2$ (definition of $\Theta_i(\lambda)$ see in section 8).

Theorem 7: (Bondareva (1984))

$U(\sigma_1, \sigma_2) \neq \emptyset$ if and only if each $\lambda \in \Lambda^*$ generated by system (σ_1, σ_2) satisfies the condition $\sum_{S \subseteq I} \lambda_S v(S) \leq 0$, and unequality is strong if $\Theta_2(\lambda) \neq \emptyset$.

Name a coalition S active, if there exists $x \in C : x(S) = v(S)$.

Theorem 8:

The coalition S is active if and only if for each $\lambda \in \Lambda^* : \Theta_2(\lambda) = \{S\}$ satisfies the condition $\sum_{T \subset I} \lambda_T v(T) - \lambda_S v(S) \leq 0$.

Define S–covering as a collection $\lambda^S = \{\lambda_T^S\}$ satisfied the condition $\Sigma_T^S \chi_T \leq \chi_S$. As above put $\Theta_1(\lambda^S) = \{T : \lambda_T^S < 0, T \neq I\}$. Define $E(\lambda^S) = \Sigma \lambda_T^S v(S)$. Name λ^S reduced one if $\Theta(\lambda^S) = \{T : \lambda_S \neq 0\}$ is independent.

Theorem 9:

A game <I,v> $\in \overline{CV}$ (has a strong stable core) if and only if for each system σ of active coalitions such that $U(\sigma) \neq \emptyset$ and each $S \in \sigma$ and each reduced S–covering λ^S such that $\Theta_1(\lambda^S) = \{S\}$ there exists a reduced S–covering μ^S with $\Theta_1(\mu^S) \subset \sigma$ such that

(5) $$\Sigma \lambda_T^S \chi_T = \Sigma \mu_T^S \chi_T$$

and

(6) $$\Sigma \lambda_T^S v(S) \geq \Sigma \mu_T^S v(T).$$

From condition (5) it is following that $\nu = \{\{\lambda_T^S\}, \{-\mu_T^S\}\}$ is an extensive covering for which $\Sigma \nu_S v(S) \geq 0$. So the necessary and sufficient condition for core to be a strong solution is expressed in a term of extensive coverings, but not reduced ones.

There is a following sequence from Theorem 9. Consider all extensive coverings λ with no more than 2n–1 coalitions in $\Theta(\lambda)$. The corresponding conditions $\Sigma \lambda_S v(S) = 0$ define the partition of the space of n–person games into the regions. The union of some such regions is $\overline{VC}$.

Theorem 9 has been proved by Bondareva and Kulakovskaja. This theorem gives the more simple condition than conditions of Kulakovskaja (1971).

11. Necessary conditions for core to be a solution

There are some sets $U(\sigma_1, \sigma_2)$ undominated by core.

For example: Let $S_1 \cup S_2 = I$ $|S_1| = |S_2| = n-1$ and $\sigma_1 = \{T : T \subset S_1 \cap S_2\}$, then if $x \in U(\sigma_1, \sigma_2)$, $\sigma_2 = \{S_1, S_2\}$ and y dom_S x. Put $S_1 = I - \{1\}$, $S_2 = I - \{2\}$, then S must contain 1 or 2 because of $x(T) \geq v(T)$, $T \subset S_1 \cap S_2 = I - \{1,2\}$.

So y dom_S x implies $y_1 > x_1$ or $y_2 > x_2$, therefore $y(S_2) < x(S_2) < v(S_2)$ or $y(S_1) < x(S_1) < v(S_1)$, and $y \notin C$. Then for C to be a solution necessary $U(\sigma_1, \sigma_2) = \emptyset$.

Name a coalition essential one if the core of game projection is nonempty. Note that game with nonempty core is equivalent to game with all essential coalitions.

Theorem 10: (Bondareva (1984))

The necessary condition for the core of game <I,v> to be a solution is

$$v(I-\{i\})+v(I-\{j\})-v(I-\{i,j\}) \leq v(I)$$

if $I-\{i,j\}$ is essential or

$$v(I-\{i\}+v(I-\{i\})+\Sigma \lambda_S v(S) \leq v(I),$$

where $\{\lambda_S \geq o\}_{S\subset I-\{i,j\}}$ is a covering of the set $I-\{i,j\}$, if $I-\{i,j\}$ is unessential.

The well known condition

$$v(I-\{i\})+v(I-\{i\}) \leq v(I) \text{ fir } I=\{1,2,3\}$$

follows from theorem 11.

12. Sufficient condition for core to be stable

There are many conditions for games to have a stable core. The most general ones were received by Kulakovskaja (1969) as the conditions for undominated by core $U(\sigma_1, \sigma_2)$ to be empty.

Consider a cover of I or $\sigma = \{S\} : \cup \sigma = I$, a cover σ is a minimal one if for each $S_0 \in \sigma$: $\cup \sigma - \{S_0\} \neq I$.

Theorem 11: (Kulakovskaja (1969))

If for any minimal cover σ in a game <I,v> : $\sum_{S\in\sigma} v(S) \leq v(I)$ then the game has a stable core.

Note that the extensive covering λ corresponds to cover σ by the following trivial manner: $\lambda_S = 1$, $S \in \sigma$, $\lambda_I = -1$, $\lambda_{\{i\}} = 1 - \sum_{i\in S} \lambda_S$, and λ is reduced one if core is minimal.

The conditions of Djubin (1973):

$$v(S) \leq \frac{v(S)}{n-|S|+1},$$

of Gillies (1959):

$$v(S) \leq \frac{1}{n}$$

and of Bondareva (1963):

$$v(S) \leq \frac{1}{r}$$

(r is a rang of incident matrix) are the sequences of these conditions.

13. The core in game without side payments

There are some well known sufficient conditions for games without side payments to have a nonempty core (Skarf conditions, Billera conditions). Vilkov (1973) received the necessary and sufficient conditions of core nonemptiness for games with polyhedral sets V(S).

Vilkov and Kulakovskaja (1975) received for this game the necessary and sufficient condition for core–stability. This condition is very complicated. Vilkov (1977) defined the class of the game without side payment with the stable core.
A game <I, V, H> is named a slight convex game if

1) $V(S) \cap V(\{i\}) \subseteq V(S \cup \{i\}), \qquad S \subset I$

2) $V(S \cup \{i\}) \cap V(S \cup \{j\}) \subseteq V(S \cup \{i,j\}) \cup V(S) \cup (V(\{i\}) \cap V(\{j\}))$

and game also satisfies of some condition of nondegeneracy.

Theorem 12: (Vilkov (1977))

Each slight convex game has a nonempty stable core.

14. The core of a game with a countable set of players

Consider a game $<I, \Sigma, v>$ with a countable I (see Sec.4), here $\Sigma = 2^I$. Naumova (1971) investigated the following problem. Let M be the set of countable additive measures μ on Σ such that $\mu(S) \geq v(S)$, $S \subset I$, $S \neq I$, the countable core of the game is $C_M = C \cap M$. Put $t(v) = \inf_{\mu \in M} \mu(I)$. If $t(v) > v(I)$ then $C_M = \emptyset$, if $t(v) < v(I)$ then $C_M \neq \emptyset$, if $t(v) = v(I)$ it is shown that both is possible.

In Naumova (1971) the algorithm for t(v) approximation is given.

The necessary and sufficient conditions are given for $t(v) < \infty$.

15. Solutions with discriminations

A solution V discriminates coalition S if for each $x \in V : x(S) = \text{const}$. If $S = \{i\}$ then player is discriminated. If $x_{i_1} = a_{i_1}$ and $x_{i_2} = a_{i_2}$ for each $x \in V$ then V is named a double discriminatory solution.

Socolina (1974, 1985, 1986) investigated the solutions discriminating two person coalitions {1,2}. Put $M = I - \{1,2\}$, $P_M = \{x_M \mid x \in P\}$ where x_M is the projection x on M. Consider solutions lying in the set $x_1 + x_2 = a$. It is proved that each such solution is represented in a form

(7) $$V = \{x \mid x_M \in V_M, x_1 \in [\alpha(x_M), \gamma(x_M)], x_2 = a - x_1\}.$$

Therefore to each $x_M \in V_M$ there corresponds in V some segment $[\alpha(x_M), \gamma(x_M)]$ passed through by the component x_1.

In Socolina (1985) the necessary and sufficient condition is found for the set of the form (7) to be a solution.

In Socolina (1986) the solution of the form (7) is constructed in which $\alpha(x_M) \equiv \alpha$ and $\gamma(x_M) \equiv \gamma$ for all $x_M \in V_M$. The necessary and sufficient conditions are received for existence double discriminatory solutions.

The double discriminatory solutions for five person games are investigated in Arakeljan (1973).

16. Solutions for special classes of games

Bondareva (1969) investigated a solution with discrimination for special class games named (n–1)–game, where $v(S) \neq 0$ iff $|S| \geq n-1$.

Theorem 13: (Bondareva (1969) (1))

The necessary and sufficient condition for (n–1) games to have a discriminatory solution

$$V_k(\alpha) = \{x \in A : x_k = \alpha\} \qquad (\alpha < 1)$$

is

$$v(I-\{k\} > 2V(I) - \frac{1}{n-1} \sum_{i \neq k} v(I-\{i\})$$

and

$$v(I-\{k\}) \geq \alpha > 2V(I) - \frac{1}{n-1} \sum_{i \neq k} v(I-\{i\}).$$

For simple games the sufficient condition is

(8) $$1 \geq \alpha > \frac{n-2}{n-1}.$$

The known result of von Neumann–Morgenstern (1944) $\alpha > \frac{1}{2}$ for 3–person game follows from (8).

For (n–1) games the core is stable iff

$$v(I-\{i\} + v(I-\{k\}) \leq v(I)$$

(see Bondareva (1969) (1)).

Bondareva (1972) investigates the following class of monotonic games in 0–1–reduced form (v(I) = 1).

Put

$$v_t(S) = \max\,[\,0, (v_0(S) - t(S)) \,/\, (1-t(I))]$$

where

$$a = (a_1,\ldots,a_n) : a_i \leq \min\,[\,v_0(S \cup \{i\}) - v_0(S)]\,,\ a(I) \leq 1.$$

Theorem 16:

If the game $<I,v_0>$ has a stable core then each game $<I, v_t>$, $t \in [0,1]$ has a stable core.

Menshikova (1976) received the necessary and sufficient condition for symmetric games to have a stable core.

Naumova (1972) constructs solutions for a class of simple games with a countable set of players.

17. Solutions for all four–person games

Kulakovskaja (1979) and Bondareva (1979) constructed a solution for arbitrary four - person games with nonempty core and for games from some class with empty core. For example:

Theorem 15: (Kulakovskaja (1979))

The necessary and sufficient conditions for a four–person game has a stable core are:

(9) $v(\{i,j\}) + v(\{i,k\}) + v(\{i,l\}) \leq v(I)$

(10) $v(\{i,j,k\}) - v(\{i,l\}) \leq v(I)$

(11) $v(\{i,j,k\}) - v(\{i,j,l\}) - v(\{i,j\} \leq v(I)$

$\{i,j,k,l\} = \{1,2,3,4\} = I.$

Note that all conditions (9), (10), (11) are generated by extensive coverings. For example,

(9) $\chi_{\{i,j\}} + \chi_{\{i,k\}} + \chi_{\{i,l\}} - 2\chi_{\{i\}} - \chi_I = 0.$

Naumova (1979) constructs a solution for all other classes of four person games, so the following theorem is proved.

Theorem 16: (Bondareva, Kulakovskaja, Naumova (1978))

Each four person game has a solution.

18. Compound games

Consider games without payments

$$G_1 = \langle I_1, V_1 H_1 \rangle$$

and

$$G_2 = \langle I_2, V_2, H_2 \rangle.$$

Name G = <I,V,H> a composition of G_1 and G_2 if

1) $I_1 \cap I_2 = \emptyset, \quad I = I_1 \cup I_2$

2) $V(S) \subset V_1(S \cap I_1) \times V_2(S \cap I_2)$ if $S \cap I_i \neq \emptyset$,

$V(S) = V_i(S) \times \mathbb{R}^j$ if $S \subseteq I_i$, $\{i,j\} = \{1,2\}$

3) a) $H \subseteq H_1 \times H_2$ and protection of H on $\mathbb{R}^{I_i}$ is H_i

or b) $H \supseteq H_1 \times H_2$.

Vilkov (see Bondareva, Vilkov, Kulakovskaja, Naumova and Socolina (1976)) proved the following results.

Theorem 17:

If G is a composition of G_1 and G_2 satisfied 1, 2, and 3a then G has a solution if and only if $V = V_1 \times V_2$ where V_i is a solution of G_i.

Theorem 18:

If G is a composition of G_1 and G_2 satisfied 1,2, and 3b, then G has a nonempty core if and only if each game G_i, i = 1,2 has a nonempty core.

19. Compound solutions of noncompound games

Consider a game $\Gamma = <I,v>$ and let $I = M \cup N$, $M \cap N = \emptyset$. Consider families of games

$$\Gamma_M(\alpha) = <M,\alpha> \text{ and } \Gamma_N(\beta) = <N,\beta>$$

denote $V_m(\alpha)$ a solution of $\Gamma_M(\alpha)$ and $V_N(\beta)$ a solution of $\Gamma_N(\beta)$.

Name the functions α, β connected, define $\alpha \sim \beta$, if

1) $\alpha(M) + \beta(N) = v(I)$

2) $\alpha(S) + \beta(T) \geq v(S \cup T)$, $S \subset M$, $T \cap N$.

Construct the compound solution $V(\alpha_0) = V_M(\alpha) \tilde{X} V_N(\beta)$ as follows:

For each x the projections

$$x_M \in V_M(\alpha) \text{ and } x_N \in V_N(\beta) \text{ where } \alpha(M) = \alpha_0,\ \alpha \sim \beta \text{ and}$$

$$v(S) \leq \alpha(S) \leq \max\{v(S \cup T) - v(T),\ T \subset N\}$$

$$v(T) \leq \beta(T) \leq \max\{v(S \cup T) - v(S),\ S \subset M\}$$

for each $S \subset M$, $T \subset N$.

Theorem 18: (Bondareva (1969) (2))

If in game Γ there exists a partition $I = M \cup N$ such that

1) $v(M) + v(N) \geq v(I)$

2) games $\Gamma_M(\alpha)$ and $\Gamma_N(\beta)$ have a strong stable property ($\Gamma_M(\alpha)$, $\Gamma_N(\beta) \in \overline{CV}$) for all α, β above defined.

Then $V(\alpha_0)$ is a solution of Γ for each $\alpha_0 : v(I) - v(N) \leq \alpha_0 \leq v(M)$.

Note that the core of Γ below is empty. Socolina and Bondareva (1971) used the similar construction to receive a solution for games from al class games with nonempty core.

20. Axiomatization of the core and von Neumann-Morgenstern solutions as functions of non-fuzzy and fuzzy choice

In this section it will be proved that not only the core but also the von Neumann–Morgenstern solution is determined with the aid of all known axioms of choice. Axioms for core and for solution will have been extended to a fuzzy choice. The conditions for the fuzzy core to be a solution will be given (see Bondareva 1988).

Let A be an infinite set of alternatives for choice; μ, μ', V, $\lambda \in [0,1]^A$ are fuzzy sets (see, for example, [I]: the characteristic function $\chi_x \in [0,1]^A$ represents the non-fuzzy set $X \subset A$. We abolish the uniqueness condition of choice function, i.e., the choice function will be $\overline{C} : [0,1]^A \to 2^{[0,1]^A}$.

Denote as earlier $C : [0,1]^A \to [0,1]^A$; any element of $\overline{C}$ is also denoted by C. A value of the membership function $C(\mu)$ is denoted $C(\mu)(x)$; $C(\chi_X)(x) = C(X)(x)$. Always assume that $C(\mu) \subset \mu$ and $C(\mu\, \chi_{\{x\}})(x) = \mu(x)$, $x \in A$, where $\mu\nu(x) = \mu(x)\, \nu(x)$.

Because all axioms of choice are expressed with the aid of the operations "$\cup$", "$\cap$" and "$\subset$", then it is possible to translate them into the fuzzy sets language. Use the "sup" and "inf" as "$\cup$" and "$\cap$" but remember that all results are able expressed in another form of $\cup$ and $\cap$. In any case $\mu' \subset \mu \longleftrightarrow \mu' \le \mu \longleftrightarrow \mu'(x) \le \mu(x)$, $x \in A$.

Non–uniqueness of choice will be marked with NU, non–fuzzy form will be marked NF, fuzzy – F.

All following axioms in NF and not NU form are well known.

N (heritage condition)

NF: $C(X') \supset C(X) \cap X'$, $X' \subset X$;

F: $C(\mu')(x) \ge \min\,(C(\mu)(x), \mu'(x))$, $\mu' \le \mu$

C: (direct Condorcet condition)

NF, NU: If $x \in X$ belong to any $C(\{x,y\}) \in \overline{C}(\{x,y\})$ for all $y \in X$, then x belong to any $C(X)$, if $\overline{C}(X) \ne \emptyset$.

F, NU: $C(\mu)\,(x) \ge \inf\limits_{y \in A} C(\mu\, \chi_{\{x,y\}})(x)$, $C(\mu) \in \overline{C}(\mu) \ne \emptyset$

$C(\mu\chi_{\{x,y\}}) \in \overline{C}(\mu\chi_{\{x,y\}})$

F: $C(\mu)(x) \ge \inf\limits_{y \in A} C(\mu\chi_{\{x,y\}})(x)$

CB (choice conservation)

NF,NU: If for some $C(X) \in \overline{C}(X) : X' \subset C(X) \subset X$ then any $C(X') \in \overline{C}(X') : C(X') = X'$;

F, NU: If $\mu' \le C(\mu) \le \mu$ for some $C(\mu) \in \overline{C}(\mu)$ then any $C(\mu') = \mu$

O (rejection condition or independence of irrelevant alternative)

NF: If $C(X) \subset X' \subset X$, then $C(X') = C(X)$

F, NU: If any $C(\mu) \leq \mu' \leq \mu$, then $C(\mu) \in \overline{C}(\mu)$.

Let R be the antireflexive fuzzy relation on A with the membership function

$$\mu_R : A \times A \to [0,1],\ \mu_R(x, x) = 0;$$

denote x R y iff $\mu_R(x, y) = 1$.

Put R(x) the R–dominion of x, i.e.,

NF: $R(x) = \{y \in A : x R y\}$;

F: $R(x)(y) = \mu_R(x,y),\ y \in A$.

Denote NF: $R(X) = \cup_{x \in X} R(x)$.

F: $R(X)(y) = \sup_{x \in X} \mu_R(x,y)$.

The core C_R of relation R on X is defined:

(12) NF: $C_R(X) = X - R(X)$

(13) F: $C_R(X)(x) = 1 - \sup_{y \in X} \mu_R(y,x)$

The von Neumann–Morgenstern solution (NMS) is:

(14) NF: $V_R(X) = X - R(V_R(X))$

It is impossible to define fuzzy NMS with (14) because $R(\mu)$ for a fuzzy μ is not defined but FNMS will be defined below with the axioms.

The following theorem describes the connection between the external definition of a choice function with (12) or (13) or (14) and internal definition with the axioms above.

Theorem 19:

The core of the fuzzy relation R as a choice function $C_R : A \to [0,1]^A$ satisfies the condition HF and CF.

These axioms are also a sufficient condition for core in the following sense.

Theorem 20:

If the choice function $C : [0,1]^A \to [0,1]^A$ satisfies the condition HF and CF then $C(\mu)(x) = \inf_{y \in A} C(\mu\chi_{\{x,y\}})(x)$ and $C(X) = C_R(X)$, $X \in 2^A$, where $R : \mu_R(x,y) = 1 - C(\chi_{\{x,y\}})(y)$ and for fuzzy μ:

$$C(\mu)(x) \geq \min(\mu(x), 1 - \sup_{y : \mu(y) > 0} \mu_R(y,x))$$

Theorem 21:

The von Neumann–Morgenstern solution of NF relation R as a choice function $V_R : 2^A \to 2^{2^A}$ satisfies the conditions CNU, CBNU and ONU and conversely: if $\overline{C} : 2^A \to 2^{2^A}$ satisfies the condition CNU, CBNU and ONU then for any NF if $\overline{C}(\{x,y\} \neq \emptyset$ and $C(\{x,y\})$ is NF for all $x, y \in X$, then $C(X) = V_R(X)$, where $R : x R y \longleftrightarrow \{x\} \in \overline{C}(\{x,y\})$.

The theorem 22 allows us to define the fuzzy NMS (FNMS) as $V(\mu) \in \overline{V}(\mu)$, where $\overline{V} : [0,1]^A \to 2^{[0,1]^A}$ is the choice function satisfied the conditions CFNU, CBFNU and OFNU. We will name $\overline{V}$ the extended NMS.

Theorem 22:

$V(\mu)$ is FNMS iff V is a solution of the system of equations

(15) $$V(\lambda V(\mu)) = \lambda V(\mu), \quad \lambda \in [0,1]^A$$

(16) $$V(\lambda V(\mu + (1-\lambda)\mu) = V(\mu), \ (1-\lambda(x) = 1-\lambda(x))$$

with inequality

(17) $$V(\mu)(x) \geq \inf V(\mu\chi_{\{x,y\}})(x)$$

where "inf" for all $y \in A$ and all $V(\mu\chi_{\{x,y\}}) \in \mathbb{V}(\mu\chi_{\{x,y\}})$.

It is clear, that condition (17) is CFNU. Condition (15) is CBFNU because any $\mu' \subset V(\mu)$ is equal to $\lambda\, V(\mu)$. Condition (16) is OFNU because if $\mu' : V(\mu) \leq \mu' \leq \mu$, then $\mu' = \lambda\, V(\mu) + (1-\lambda)\, \mu$.

Note that $V(\mu)(x) = 0$, $x \in A$ for all $\mu \in [0,1]^A$ is a solution of (15), (16), (17), i.e., FNMS but it is not NMS for non fuzzy set because R is not defined. The $V(\mu) \equiv \mu$ is also FNMS and for non-fuzzy set it is also NMS with $R = \emptyset$. We will name this two solutions trivial and below investigate only non-trivial solutions.

Let $V(\mu) = \mu \cap \mu_0$. It is easy to prove that this is also FNMS. If $\mu_0 = \chi_{X_0}$, then $V(X) = X \cap X_0$ is NMS for $R : x\, R\, y$ iff $x \in X_0$, $y \notin X_0$, but for $x, y \notin X_0$ the relation R is undefined.

Let us give an example of non-unique FNMS. Put $A = \bigcup_{i=1}^{n} A_1$ and $V_i(\mu) = \mu \cap \chi_{A_i}$ iff $\mu \cap X_{A_i} \neq \emptyset$. Then $\mathbb{V}(\mu) = \{V_i(\mu)\}$ is extended NMS. The relation

$$R = \bigcup_{i=1}^{n} (A_i \times (A - A_i)).$$

Now investigate the condition for fuzzy core to be FNMS.

Theorem 23:

The fuzzy choice function $V : 2^A \to [0,1]^A$ the core of fuzzy relation

$$R : V(X)(x) = C_R(X)(x) = 1 - \sup_{x \in X} \mu_R(y,x)$$

is FNMS iff for any $X \subset A$ any $x \in X$, $y \in X - N(V(X))$ and any $\epsilon > 0$ there exists

(18) $$y(\epsilon) \in N(V(X)),\ \mu_R(y(\epsilon), x) > \mu_R(y,x) - \epsilon\,.$$

Corollary 1:

If R is unfuzzy then condition (18) transforms into condition of the external stability.

Proof:

Let $x \notin V(X)$, the $y \in X$ exists: $y\ R\ x$, i.e., $\mu_R(y,x) = 1$, from ((18) $y(\epsilon) \in V(X)$ exists: $\mu_R(y(\epsilon),x) > 1-\epsilon$, i.e., $\mu_R(y(\epsilon),x) = 1$ and $y(\epsilon)\ R\ x$, q.e.d.

Corollary 2:

If for any $x, y \in A : \mu_R(x,y) < 1$ then $V = C_R$ is also FNMS.

The condition (18) we are able to interpret as the condition of external stability of FNMS, but it is not separate from condition of core belonging. What is the fuzzy external stability in general case is not clear.

REFERENCES:

Arakeljan, A.A. (1973) — Solutions with discrimination of two players in five–person constant–sum games, in: Theory of Games, Erevan, 60–63 (Russian)

Bondareva, O.N. (1963) — Some applications of linear programming methods to the theory of cooperative games, Problemy Kibernetiki 10, 119–139 (1963); German transl. in: Probleme der Kypernetik 10, 1963. English transl. in: Selected Russion papers on game theory 1959–1965, Princeton Univ., 1968

Bondareva, O.N. (1963) (1) — On uniqueness of the solution of (n–1)–games, Kibernetic 4, 118–122, English transl. in: Cybernetics, v.5, N4, 492–497

Bondareva, O.N. (1969) (2) — Solution for a class of games with empty core, Dokl.Akad.Nauk SSSR, tom 185, N2, 247–249; English transl. in: Soviet Math. Dokl., v.10, N2, 318–320

Bondareva, O.N. (1970) — A theorem on externally stable sets, Dokl.Akad.Nauk SSSR, tom 192, N2, 259–261; English transl. in: Soviet Math.Dokl., vol.II, N3, 602–604

Bondareva, O.N. (1972) — On some transformation preserving the solution, in: Operations Research and Statistic Modeling, Issue I, Leningrad Univ., 40–42 (Russian)

Bondareva, O.N. (1975) — Acyclic games, Vestnik LGU, N7, 16–22 (1975); English transl. in: Vestnik Leningrad Univ., Math.8, 183–190, 1980

Bondareva, O.N. (1976) Finite approximations for cores and solutions of cooperative games, Journ.Vychisl. Math. i Math. Fiz. tom 16, N3, 624–633; English transl. in: USSSR Computational Math. and Math. Physics 16, N3, 78–88

Bondareva, O.N. (1978) A convergence of spaces with relations and game–theoretical sequences, Journ. Vychisl. Math. i Math.Fiz., tom 18, N1, 84–92; English transl. in: USSR Computational Math. and Math.Physics 18, N1, 80–98

Bondareva, O.N. (1979) The solution of a classical cooperative 4–person game with a nonempty core (general case), Vestnik LGU 19, 14–19 (1979); English transl. in: Vestnik Leningrad Univ.Math. 12 (1980), 247–253

Bondareva, O.N. (1983) Extensive coverings and some necessary conditions for the existence of solutions of a cooperative game, Vestnik LGU 19, 5–12 (1983); English transl. in: Vestnik LLLGU 19, 5–12; English transl. in: Vestnik Leningrad Univ. Math. 16 (1984), 201–209

Bondareva, O.N. (1987) A finite approach to a choice on an infinite set, Izv.Akad.Nauk SSSR, Techn.Kibern.I, 18–22, English transl. in: Inginering Kybern.Soviet Journ. on Comp. and Syst.Science

Bondareva, O.N. (1988) Axiomatization of the core and von Neumann–Morgenstern solution as functions of unfuzzy and fuzzy choice, Vestnik LGU 8, 3–7; English transl. in: Vestnik Leningrad Univ.Math. 21, N2, 1–7

Bondareva, O.N. Vilkov, V.B., Kulakovskaia, T.E. Naumova, N.I., Sokolina N.A. (1977) A survey of soviet papers in the theory of cooperative games, in: Operations Research and Statistical Modeling, Issue 4, Leningrad Univ., 81–126 (Russian)

Bondareva, O.N. Kulakovskaia, T.E. Naumova, N.I. (1979) The solution of an arbitrary four–person game, Vestnik LLGU 7, 104–105 (Russian)

Bondareva, O.N. Naumova, N.I. (1971) On slight domination in cooperative games, Second All–Union Conf. Game Theory, Abstracts of Reports, Inst.Fiz.Math.Nauk Litovsk. SSR, Vilnius, 124–125 (Russian)

Djubin, G.N. (1973) Sufficient conditions for the coincidence of core and solution in a cooperative game, in: Game Theory, Erevan, 152–155 (Russian)

Gillies, D.B. (1959) Solutions to general non–zero–sum games, in: Annals of Math. Stud.40, Princeton, 47–85

Kulakovskaia, T.E. (1969) Sufficient conditions for the coincidence of core and solution in a cooperative game, Litovsk.Mat.Sbornik 9, N2, 424–425 (Russian)

Kulakovskaia, T.E. (1971) Necessary and sufficient conditions for the coincidence of core and solution in a classical cooperative game, Dokl.Akad.Nauk SSSR, tom 199, N5, 1015–1017; English transl. in: Soviet Math. Dokl. vol.12, N4, 1231–1234

Kulakovskaia, T.E. (1976) Classical principles of optimality in infinite cooperative games, in: Recent Directions in Game Theory, Vilnius, 94–109 (Russian)

Kulakovskaia, T.E. (1979) The solution of a class of cooperative four–person games with non–empty core, Vestnik LGU 19, 42–47 (1979); English transl. in: Vestnik Leningrad Univ.Math 12 (1980), 286–292

Menshikova, O.R. (1976) Extreme points of C–core of symmetric games, Vestnik LGU, Mat. i Mechan., 5, 63–72; English transl. in: Vestnik Moskow Univ. Math.

Morosov, V.V. (1971) On some approach to cooperative games, Journ.Vichisl.Mat. i Mat.Fin., tom 13, N3, 781–787; English transl. in USSR Computational Math. and Math.Physics 13, N3, 320–330

Morosov, V.V. (1983) On convex solutions of cooperative games, Vestnik MGU Vichisl.Mat.Kibern., N4, 55–57, English transl. in: Vestnik Moskow Univ. Comp.Math.Kybern.

Naumova, N.I. (1971) On the core in a game with a countable set of players, Dokl. Akad.Nauk SSSR, tom 197, N1, 40–42; English transl. in: Soviet Math.Dokl., Vol.12, N2, 409–411

Naumova, N.I. (1972) A solution of infinite simple games, in: Operations Research and Statistical Modeling, Issue 1, Leningrad Univ., 126–135 (Russian)

Naumova, N.I. (1979) N–M–solutions of a class of cooperative four–person games with an empty core, Vestnik LGU 19, 52–60 (1979); English transl. in: Vestnik Leningrad Univ.Math.12 (1980), 301–314

von Neumann, J. Morgenstern, O. Theory of games and economic behavior, Princeton (1944)

Sokolina, N.A. (1974) On the structure of a solution that discriminates a coalition of two players, Vestnik LGU 13, 151–153 (1974); English transl. in: Vestnik Leningrad Univ.Math 7

Sokolina, N.A. (1985) Existence of solutions that discriminates a two–person coalition, Vestnik LGU 22, 30–37; English transl. in: Vestnik Leningrad Univ.Math. 18, N4, 37–44

Sokolina, N.A. (1986) Some solutions of a special form, discriminating the coalition of two players, Vestnik LGU, Issue 1, N2, 33–41; English transl. in: Vestnik Leningrad Univ.Math. 19, N2, 40–48

Vilkov, V.B. (1972) Theorems on core–solution for games without side payments, Vestnik LGU 19, 5–8 (1972); English transl. in: Vestnik Leningrad Univ.Math. 5 (1978), 300–304

Vilkov, V.B. (1973) Necessary and sufficient conditions for the existence of a core in cooperative games without side payments on polyhedral sets, Vestnik LGU 19, 21–27 (1973); English transl. in: Vestnik Leningrad Univ.Math. 6 (1979), 323–330

Vilkov, V.B. (1976) Σ-domination in the classical cooperative games, Vestnik LGU 19, 16–29 (1976); English transl. in: Vestnik Leningrad Univ.Math. 9 (1981)

Vilkov, V.B. (1977) Convex games without side payments, Vestnik LGU 7, 21–26 (1977); English transl. in: Vestnik Leningrad Univ.Math. 10 (1982), 115–119

Vilkov, V.B. Kulakovskaia, T.E. (1975) On core-solution in cooperative games without side payments, Vestnik LLGU 13, 14–18 (1975); English transl. in: Vestnik Leningrad Univ.Math. 10 (1982), 115–120