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An Alternative Proof for the Linear Utility Representation Theorem

by

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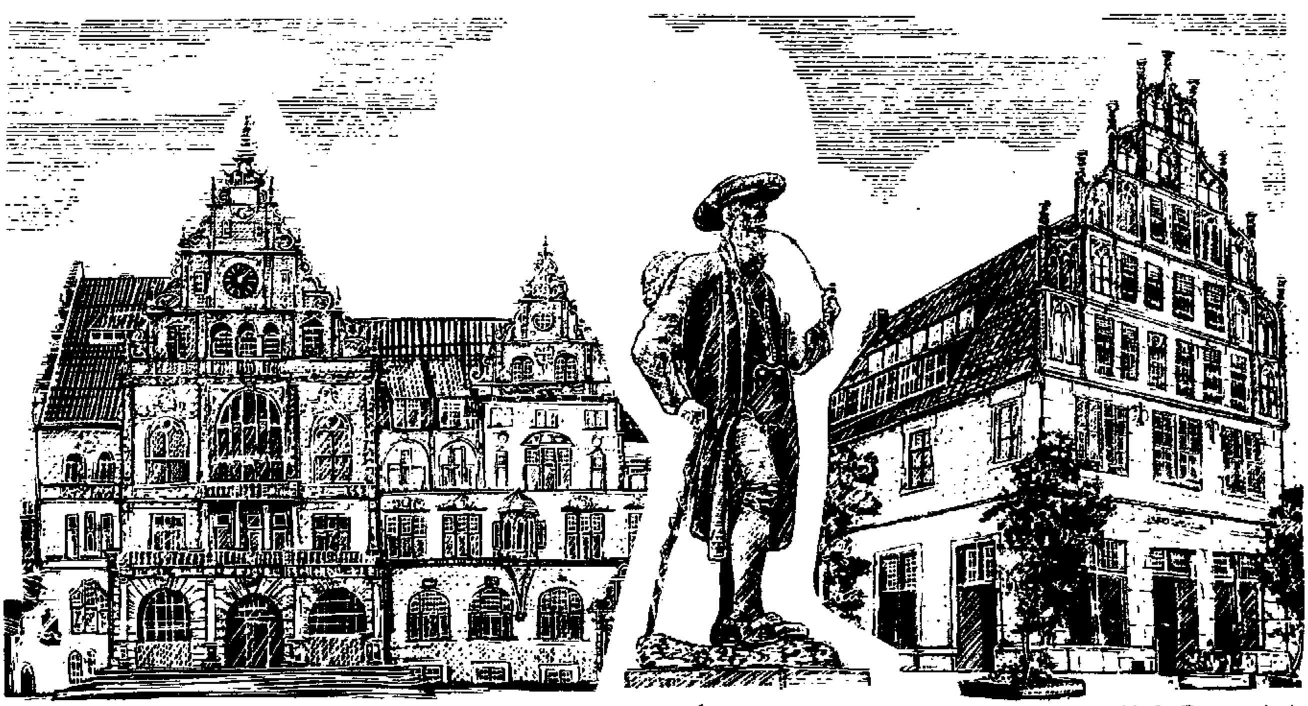

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Summary

The paper presents an alternative shorter proof for the linear utility representation theorem. In particular a special case of the theorem of Herstein and Milnor (1953) and a generalization of the theorem of Blackwell and Girshick (1954) are proved by exploiting the topological group structure of finite–dimensional Euclidean vector space.

1. Introduction

Looking at formulations of axioms from which linear utility functions may be derived one easily recognizes the presence of invariance postulates. The theorems of Herstein and Milnor (1953) and of Blackwell and Girshick (1954) are respectively, based on invariance postulates for the indifference relation and the preference relation.

These facts suggest to base a proof of this theorem on a systematic use of group theory. As continuity has to play its part one is forced to use topological groups. It turns out that standard arguments lead to a considerable abbreviation of the proof. In our proof of the Herstein and Milnor result in our present Theorem 1 we have to pay for the immense abridgement by restricting to $\mathbb{R}^\ell$ rather than working in general mixture sets. Regarding Blackwell and Girshick's result our Theorem 2 does not bring about very much of abbreviation, it provides however quite a different approach yielding a linear utility without imposing the assumption of monotonicity of preferences.

We shall collect the necessary notation and definitions in section 2.

Section 3 contains the results and proofs. The paper ends with concluding remarks in section 4.

2. The framework

Let R be a preference relation on $\mathbb{R}^\ell$, $\ell \in \mathbb{N}$, i.e. a complete transitive binary relation on $\mathbb{R}^\ell$. For given R we denote by I and P, respectively, the induced indifference and strict preference relation. Formally:

$$\forall\ x,y \in \mathbb{R}^\ell: \quad \begin{aligned} &x\ I\ y \Leftrightarrow x\ R\ y \text{ and } y\ R\ x \\ &x\ P\ y \Leftrightarrow x\ R\ y \text{ and not } y\ R\ x \end{aligned}$$

We shall use the following notation:

$$R(x) := \{y \in \mathbb{R}^\ell \mid x\ R\ y\} \qquad R^{-1}(x) := \{y \in \mathbb{R}^\ell \mid y\ R\ x\}$$

$$P(x) := \{y \in \mathbb{R}^\ell \mid x\ P\ y\} \qquad P^{-1}(x) := \{y \in \mathbb{R}^\ell \mid y\ P\ x\}$$

$$I(x) := \{y \in \mathbb{R}^\ell \mid x\ I\ y\} =: I^{-1}(x).$$

For convenience we shall denote the R–indifference set I(x) of x by I_x for any $x \in \mathbb{R}^\ell$.

Definition 1: The preference relation R on $\mathbb{R}^\ell$ is upper (resp. lower) semi–continuous at $x \in \mathbb{R}^\ell$ iff $R^{-1}(x)$ (resp. R(x)) is closed. R is upper (resp. lower) semi– continuous if it is so at every $x \in \mathbb{R}^\ell$. The preference relation R is continuous (at $x \in \mathbb{R}^\ell$) iff it is upper and lower semi–continuous (at $x \in \mathbb{R}^\ell$).

Recall that R is upper resp. lower semi–continuous resp. continuous iff it allows an upper resp. lower semi–continuous resp. continuous utility representation.

Definition 2: A binary relation T on $\mathbb{R}^\ell$ is called (translation–) invariant iff

$$\forall\ x,y,z \in \mathbb{R}^\ell : x\ T\ y \Leftrightarrow x + z\ T\ y + z$$

Definition 3: A preference relation R on $\mathbb{R}^\ell$ is trivial iff R = I.

3. Results

<u>Lemma:</u> Assume that R is non–trivial and continuous and I is translation–invariant. Then there is a hyperplane H of $\mathbb{R}^\ell$ through 0 such that

i) $\forall\ x \in \mathbb{R}^\ell : I_x = x + H$

ii) H separates $\mathbb{R}^\ell$ into two open half spaces which coincide with P(0) and $P^{-1}(0)$.

<u>Proof:</u> Let $x,y \in I_0$. As invariance yields $x - y, -y \in I_0$ the indifference class I_0 must be a subgroup of $(\mathbb{R}^\ell,+)$. Since $I_0 = R^{-1}(0) \cap R(0)$ is closed by continuity of R there are integers $p,r \leq \ell$ such that I_0 as a topological group is isomorphic to $\mathbb{R}^p \times \mathbb{Z}^{r-p}$, while the quotient group $\mathbb{R}^\ell/I_0$ is isomorphic to $T^{r-p} \times \mathbb{R}^{\ell-r}$ (cf. Theorem 6 in Morris (1977)). Here $T \cong \mathbb{R}/\mathbb{Z}$ denotes the circle group or torus group. Now for any $x,y \in I_0$ and $n,m \in \mathbb{Z}$ we have $nx + my \in I_0$. Then by invariance the same is true for any $n,m \in \mathbb{Q}$. Therefore the discrete factor $\mathbb{Z}^{r-p}$ of I_0 degenerates to 0, whence $r = p$ and $I_0 \cong \mathbb{R}^p$, $\mathbb{R}^\ell/I_0 \cong \mathbb{R}^{\ell-p}$.

Since R is non–trivial there exists $z \in P^{-1}(0)$. We will show that this implies $-z \in P(0)$. Now, $-z \in I_0$ is impossible since by invariance it would imply $z \in I_0$, a contradiction. Assume $-z \in P^{-1}(0)$. Then $z\ I\ -z$ is impossible because by invariance it would imply $2z \in I_0$, hence $z \in I_0$, again a contradiction.

So assume without loss of generality $z\ P\ -z\ P\ 0$.
Then $[0,z] = ([0,z] \cap P(-z)) \cup ([0,z] \cap P^{-1}(-z))$ is impossible, since both sets in the union are non–empty, relatively open in $[0,z]$ and disjoint. Therefore there exists $z' \in [0,z] \cap I(-z)$. As $0 \in [-z, z']$ this implies $-z \in I_0$, a contradiction. So, P(0) and $P^{-1}(0)$ are disjoint,

non–empty open sets with $P^{-1}(0) \cup P(0) = \mathbb{R}^{\ell}\backslash I_0$. Hence $\mathbb{R}^{\ell}\backslash I_0$ cannot be connected which implies dim $I_0 \geq \ell - 1$. Non–triviality then yields dim $I_0 = \ell - 1$. Define $H := I_0$. Obviously $I_x = x + H$ is a hyperplane for any $x \in \mathbb{R}^{\ell}$. So i) is established.

Next let z P 0 and p be a linear functional on $\mathbb{R}^{\ell}$ with kernel H and $p\,z > 0$. Let $H_+ := \{y \in \mathbb{R}^{\ell} | p \cdot y > 0\}$ and $H_- := -H_+$. Since H_+, H_-, $P(0), P^{-1}(0)$ are non–empty and open and H_+, H_- are connected and since z P 0, the equality $H_+ = (H_+ \cap P(0)) \cup (H_+ \cap P^{-1}(0))$ implies that $H_+ \cap P(0) = \emptyset$. This implies $H_+ \subset P^{-1}(0)$ and $P(0) \subset H_-$. Since both $\{P^{-1}(0), P(0)\}$ and $\{H_+, H_-\}$ cover $\mathbb{R}^{\ell}\backslash H$ equality of $P^{-1}(0)$ and H_+ is equivalent to equality of P(0) and H. Assume $P^{-1}(0) \cap H_- \neq \emptyset$ and $z' \in P^{-1}(0) \cap H_-$. As was shown above this implies $-z' \in P(0) \cap H_+$, a contradiction. Hence, $P^{-1}(0) = H_+$ and thus, $P(0) = H$.

Q.E.D.

Next we shall use this lemma to prove the linear utility representation theorem in the version of Herstein and Milnor (1953). It should be noted however that the much longer original proof holds true for any mixture set while the present proof exploits the specific framework of a finite–dimensional group.

<u>Theorem 1:</u> Let R be continuous with translation–invariant I. Then R is representable by a linear utility function.

<u>Proof:</u> A trivial R is represented by the linear map $x \mapsto 0$. Now let R be non–trivial. By the lemma I_0 is a linear subspace, in particular the kernel of the linear functional p. Moreover,

$$H_+ = \{y \in \mathbb{R}^{\ell} | p \cdot y > 0\} = P^{-1}(0).$$

To show that p represents R means to establish

$$\forall\ x,y \in \mathbb{R}^{\ell} : p\,x \geq p\,y \Leftrightarrow x\,R\,y.$$

As x I y if and only if p x = p y it suffices to show that

$$\forall\ x,y \in \mathbb{R}^{\ell} : p\,x > p\,y \Rightarrow x\ P\ y.$$

If p x > 0 > p y the lemma yields x P 0 P y.
Now let p x > p y > 0. Since I_y is a hyperplane separating I_x from

$$y + H_{-} \text{ and as } [0,y] \subset y + H_{-} \text{ we get } [0,y] \cap I_x = \emptyset.$$

Hence $[0,y] = ([0,y] \cap P(x)) \cup ([0,y] \cap P^{-1}(x))$. As $[0,y]$ is closed and connected and $0 \in P(x)$ the set $[0,y] \cap P^{-1}(x)$ must be empty.
Hence x P y.
Finally, let 0 > p x > p y.
Since $[0,x] = ([0,x] \cap P(y)) \cup ([0,x] \cap P^{-1}(y))$ and $0 \in P^{-1}(y)$ we have $[0,x] \cap P(y) = \emptyset$, thus x P y.

Q.E.D.

The next result is a generalization of the linear utility representation theorem due to Blackwell and Girshick (1954) as formulated in their Theorem 4.3.1 and the associated problem.

Theorem 2: Let R be translation–invariant and semi–continuous at some point $x \in \mathbb{R}^{\ell}$. Then R is representable by a linear utility function.

Proof: W.l.o.g. assume R to be non–trivial. Translation–invariance implies $R(x) = x + R(0)$, $R^{-1}(x) = x + R^{-1}(0)$ for any $x \in \mathbb{R}^{\ell}$ and $R^{-1}(0) = -R(0)$. Therefore semi–continuity of R at some x implies continuity of R. Now the assumptions of Theorem 1 are fulfilled.

Q.E.D.

It is immediate that any preference relation represented by a linear utility function must have the properties stated in the assumptions of the two theorems. Therefore these respective sets of assumptions are sufficient and necessary.

4. Concluding Remarks

The proofs for the two theorems are based on the structure of $\mathbb{R}^\ell$ as a finite dimensional topological group and are in fact quite different from the original ones.

As to Theorem 1 it is in fact weaker than the Herstein–Milnor–Theorem. The latter is formulated for general mixture sets. Accordingly the continuity assumption is formulated there in a different way. However, if the mixture set is a finite–dimensional Euclidean space the continuity used by Herstein and Milnor (Axiom 2) coincides with the usual one. While full continuity is used in Theorem 1 (and in Herstein and Milnor (1953)) translation invariance is required only for the indifference relation. Translation–invariance for the preference–relation can be derived then.

Our Theorem 2 is a stronger result that that due to Blackwell and Girshick (1954). No monotonicity assumption is used in our Theorem 2. The proof of Blackwell and Girshick, however, crucially depends on that assumption. As the two versions of the linear utility representation result show there is a trade–off between invariance and continuity. Full continuity allows to derive translation–invariance of the preference relation from that of the associated indifference relation. On the other hand, translation invariance of the preference relation allows to derive full continuity from semi–continuity at some point.

One might be tempted to try to derive linear utility representability from a combination of the weaker continuity and the weaker invariance assumption. This is not possible. Even with semi–continuity at a point one may have invariance for I without having it for R. Also invariance for I alone does not suffice to derive continuity of R from semi–continuity at some point.

The continuity of R has been used in the proof of our theorems in a twofold way.

First, the closedness of indifference sets I_x, $x \in \mathbb{R}^\ell$ was caused by continuity.

Secondly, the ordering of the closed indifference manifolds in such a way that "above" each indifference set are only better elements and "below" are only worse ones is also due to continuity.

Without continuity of R we are left with co–sets I_x, $x \in \mathbb{R}^\ell$ of a subgroup I_0 of $\mathbb{R}^\ell$ which, however, need not be closed, and the indifference manifolds could be ordered in a quite arbitrary way. This latter one would be the case even if the indifference sets would assumed to be closed.

This is possible since the continuity of R which is equivalent to the continuity of its representation $u : \mathbb{R}^\ell \to \mathbb{R}$ amounts to the simultaneous continuity of both of the two maps proj and ũ into which u can be decomposed as follows:

$$\mathbb{R}^\ell \xrightarrow{\text{proj}} \mathbb{R}^\ell/I \xrightarrow{\tilde{u}} \mathbb{R} : x \xmapsto{\text{proj}} I_x \xmapsto{\tilde{u}} u(x)$$

Here $\mathbb{R}^\ell/I$ is endowed with the identification topology.

Exploiting the isomorphism between the topological groups $(\mathbb{R}^\ell,+)$ and $(\mathbb{R}^\ell_{++},\cdot)$ described by the coordinatewise exponential map, one can restate the above theorems as statements about preferences which are invariant under stretching of the axes of the space.

This invariance called budget–invariance was used in Trockel (1989) to characterized Cobb–Douglas representable preferences by continuity, monotonicity and price–invariance. The present Theorem 2 allows to extend the result in Trockel (1989) to non– monotonic preferences.

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