Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 193

The Least–Core, Nucleolus, and Kernel of Homogeneous Weighted Majority Games

by

Joachim Rosenmüller and Bezalel Peleg

August/September 1990

H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

ABSTRACT

Homogeneous weighted majority games were already introduced by von NEUMANN and MORGENSTERN [15]; they discussed uniqueness of the representation and the "main simple solution" for constant–sum games. For the same class of games PELEG [7,8] studied the kernel and the nucleolus. OSTMANN [6] and ROSENMÜLLER [9] described the nature of representations of general homogeneous weighted majority games; see also SUDHÖLTER [14]. The present paper starts out to close the gap: for the general homogeneous weighted majority game "without steps", we discuss the least core, the nucleolus, and the kernel and show their close relationship (coincidence) with the unique minimal representation.

SECTION 1 Introduction

Let $N = \{1,...,n\}$ denote the set of players and let $\underline{\underline{P}} = \{S \mid S \subseteq N\}$ denote the system of coalitions. If $v : \underline{\underline{P}} \to \{0,1\}$ is a mapping such that $v(\emptyset) = 0$, $v(N) = 1$, then the pair (N,v) is called a *simple game* ("in characteristic function form") or a simple coalitional game.

We shall fix N and, somewhat sloppily, speak of a mapping v as of a game as well.

A *representation* of v is a pair (M,λ) such that $M \geq 0$ is an additive function on $\underline{\underline{P}}$ and $0 < \lambda < M(N)$, satisfying

$$(1) \qquad v(S) = \begin{cases} 0 & M(S) < \lambda \\ 1 & M(S) \geq \lambda \end{cases} \qquad (S \in \underline{\underline{P}})$$

We shall also write $v = v^M_\lambda$ in order to indicate that (M,λ) is a representation of v.

An additive function $M \geq 0$ defined on $\underline{\underline{P}}$ is said to be homogeneous w.r.t. $\lambda \in (0,M(N))$ (written M hom λ) if the following holds true:

(2) For every $T \in \underline{\underline{P}}$ with $M(T) > \lambda$
there is $S \subseteq T$ such that $M(S) = \lambda$.

A game v is said to be homogeneous if there exists a homogeneous representation, i.e., a pair (M,λ) such that $v = v^M_\lambda$ and M hom λ.

The first to consider this type of characteristic function were von NEUMANN–MORGENSTERN [15]; they use the term "homogeneous weighted majority game". In [15] it is proved that the representation of a homogeneous game is unique up to multiplication by a constant, provided v is, in addition, a constant–sum game (and dummies get zero weight); in this case it is not hard to see that (M,λ) can be chosen to be integer.

OSTMANN [6] was the first to prove that the constant–sum version is a rather special case of a more general statement: a homogeneous game (no matter whether constant–sum or not, superadditive or not) has a *unique integer representation with minimal total number of votes* M(N). This representation is a homogeneous one and it respects types (i.e. attaches the same weight to players of equal type). In addition, OSTMANN introduces the property of players to be either *step, sum,* or *dummy.*

An alternative (recursive) definition of the minimal representation of a homogeneous game was given by ROSENMÜLLER [9], see also [10], [11]. The approach provides also an alternative proof to the existence of the minimal representation and in addition defines *characters* (called *steps, sums,* and *dummies*) which are, however, attached to *types* of players. Essentially a sum is a type the members of which in a certain minimal winning coalition can be replaced by smaller players and the weight of which is hence a sum of weights of smaller players. By contrast, a step is a type the members of which cannot be replaced in minimal winning coalitions. And it turns out that "*steps rule their followers*" (OSTMANN) in the sense that they must be members of any minimal winning coalition in which any of the smaller followers appear.

Note that the minimal representation is the von NEUMANN–MORGENSTERN representation in the zero–sum superadditive case (it attaches zero to dummies) and, in addition, it is non–degenerate (in the sense of ROSENMÜLLER – WEIDNER [12], see also [11]), that is, the minimal representation is uniquely defined by the system of minimal winning coalitions (via the appropriate system of linear equations).

Von NEUMANN and MORGENSTERN introduced the homogeneity condition not only as a structural tool but also in order to discuss a certain solution concept: the *vN–M–solution* or *stable set.* Indeed if, in the superadditive and constant sum case we normalize the unique representation, say, by

(3) $$(m,\alpha) = \frac{1}{M(N)}\ (M,\lambda)\ ,$$

then m is an imputation of the game. The *main simple solution* as defined in [15] is obtained by restricting M to all minimal winning coalitions and normalizing; this way one obtains a set of *imputations* (i.e. individually rational and Pareto efficient payoff vectors) which are then verified to constitute a stable set.

The first one to link the normalized representation (m,α) in the constant sum super additive case to a different solution concept was PELEG [7,8]. He proves that in this context m is indeed the nucleolus and discusses the shape of the kernel given additional requirements.

In view of the results that insure the existence of a unique representation it is natural to ask for the connection between the nucleolus and the unique representation in more general cases involving general homogeneous games. As OSTMANN has proved in a zero–sum game there is only one step present: the smallest nondummy. Also, it is easily seen that, whenever the smallest nondummy is the only step (he is always a step) then the representation is unique up to a multiple constant. A game in which the smallest nondummy is the only step will conveniently be called a *game "without steps"*. It would seem that the class of homogeneous games without steps is a first candidate for further tackling the question of relations between the unique representation and a solution concept like the nucleolus. Our results are as follows.

In Section 2 we shortly discuss the graphical method which leads to characterizing games with two types only (and no steps) such that the nucleolus and the unique representation (normalized) coincide. In Section 3 we discuss the least core that was introduced by MASCHLER, PELEG, AND SHAPLEY [5] and we show that for homogeneous games (without dummies) the system of minimal winning coalitions is weakly balanced if some homogenous representation is an element of the least core. On the other hand, if the system of minimal winning coalitions is weakly balanced then any normalized homogeneous representation of a homogeneous game (without dummies) is an element of the least core. In Section 4 we discuss the nucleolus. For games without steps it turns out that the unique representation (normalized) is equal to the nucleolus if and only if the system of minimal winning coalitions is balanced. In Section 5 we prove that for games without steps the unique representation (normalized) is always an element of the kernel. Finally, Section 6 connects the results of Section 2 and further Sections and produces some additional insight w.r.t. games that may have steps. Thus, this paper essentially clears the connection between balancedness of the system of minimal winning coalitions and the coincidence of the nucleolus and the unique representation for games "without steps".

We shall use the notation

(4) $\mathbb{H} = \{v \mid v : \underline{\underline{P}} \to \mathbb{R}_+, v(\emptyset) = 0$, v is homogeneous, v has no dummies$\}$

and

(5) $\mathbb{H}_\Sigma = \{v \mid v \in \mathbb{H}$, v has "no steps"$\}$

$\underline{\underline{W}}^m$ denotes the system of minimal winning coalitions of a given function v. An additive function defined on $\underline{\underline{P}}$, say $x : \underline{\underline{P}} \to \mathbb{R}_+$, can, of course, be identified with a vector $x = (x_1,\ldots,x_n)$ via the usual convention

(6) $$x(S) = \sum_{i \in S} x_i \qquad (S \in \underline{\underline{P}}).$$

In this context for $S \in \underline{\underline{P}}$ and any imputation (i.e., $x \in \mathbb{R}^n_+$, $x(N) = 1$) x the *excess* of S w.r.t. x is given by

(7) $$e(S,x) = v(S) - x(S) \qquad (S \in \underline{\underline{P}}) .$$

In particular

(8) $$\Theta_1(x) = \max_{S \in \underline{\underline{P}}} (v(S) - x(S))$$

denotes the maximal excess of an imputation x (it is sufficient to consider minimal winning coalitions!).

We shall always assume that large players (and types) have small indices, that is, if $v = v^M_\lambda$ has a representation (M,λ) then we assume that

(9) $$M_1 \geq M_2 \geq \ldots \geq M_n .$$

We shall also assume that "M respects types". That is

(10) $$M_i = M_j \quad \text{if i and j belong to the same type.}$$

The decomposition of the set of players N into sets of players of equal type opens the road to a more condensed representation of games by turning to types instead of players and "profiles" instead of coalitions.

Indeed, given some game v let $K_1,\dots,K_r$ denote the sets of different types of players. That is, we have a decomposition

$$(11) \qquad N = \bigcup_{\rho=1}^{r} K_\rho \qquad (K_\rho \cap K_\sigma = \emptyset, \rho \neq \sigma)$$

of N, and if $k_1 = |K_1|,\dots,k_r = |K_r|$ denotes the numbers of players of each type then $k_1 + \dots + k_r = n$.

If (M,λ) is a representation of v respecting the types then M has equal weight for all players of a type, say $M_i = g_\rho$ $(i \in K_\rho, \rho = 1,\dots,r)$. The weights can be collected in a vector $g = (g_1,\dots,g_r)$ and the pair (g,k) essentially describes the additive function M via

$$(12) \qquad M(S) = \sum_{\rho=1}^{r} | S \cap K_\rho | \, g_\rho .$$

We shall use this notation in particular if M is integer. Thus, type respecting additive functions M and pairs of integer vectors (g,k) can be identified and if no confusion can arise we shall indeed use the letter M also to denote the pair (g,k).

In this context, a *profile* is a vector $s = (s_1,\dots,s_r) \in \mathbb{N}_0^r$, the coordinates of which are natural numbers or zero, such that

$$(13) \qquad s_\rho \leq k_\rho \quad (\rho = 1,\dots,r),$$

or, for short $s \leq k$. Obviously, profiles identify coalitions up to permutations of players of equal type and, if an integer additive function M is identified with (g,k) then it is convenient to use the notation

$$(14) \qquad M(s) = \sum_{i=1}^{r} s_i g_i ,$$

such that M gives rise to a linear function defined on profiles. Let $\underline{\underline{w}}^m$ denote the system of minimal winning profiles, given v. The lexicographic ordering on profiles is denoted by $\succ_L$.

The reader may now want to familiarize himself with the precise definition of steps, sums, and dummies in the context of the representation theory of homogeneous games. As we shall use exactly the notation offered in [10] we do not want to repeat the definitions explicitly, thus we refer to [10] Section 1 for the BASIC LEMMA, the recursive definition of the satellite measures and for the definition of characters. Of course the discussion in [10] starts out with an arbitrary homogeneous representation of some v. In our context we shall always assume that we have representations respecting types and, frequently we shall assume that we are dealing with the unique minimal representation. In case of $v \in \mathbb{H}_{\Sigma}$, this is actually no severe restriction as, up to a multiple constant, there is only one homogeneous representation.

SECTION 2: Graphical solution of games with two types

Consider $v = v_\lambda^M \in \mathbb{H}_\Sigma$ such that there are exactly two types of players. As the "larger" type is necessarily a sum, M has the shape

(1) $$M = (\underbrace{t,\dots,t}_{k_1}, \underbrace{1,\dots,1}_{k_2}) \qquad t \in \mathbb{N},\ t > 1,\ k_2 \geq t$$

while the majority level λ admits of two cases

(2) CASE A: $\lambda = ct$ $\qquad 1 \leq c \leq k_1$

(3) CASE B: $\lambda = k_1 t + c$ $\qquad 1 \leq c \leq k_2 - t$

The normalized representation is

$$v = v_\alpha^m$$

with

(4) $$m = \frac{M}{M(N)}, \qquad \alpha = \frac{\lambda}{M(N)}$$

where $M(N) = tk_1 + k_2$ denotes the total mass of M.

Note that the min win profiles

$$\{s = (s_1, s_2) \in \mathbb{N}_0^2 \mid s \leq k\}$$

for games described by (1), (2), (3) are listed at once:

CASE A: $(c,0)\ (c-1,t),\dots,(c-l,\ lt)$ where $l = \min\left[\left[\frac{k_2}{t}\right], c\right]$

CASE B: $(k_1,c)\ (k_1-1,t+c),\dots,(k_1-l,\ lt+c)$ where $l = \min\left[\left[\frac{k_2-c}{t}\right], k_1\right]$

($[\rho]$ is the largest integer not exceeding $\rho \in \mathbb{R}$). In the above list, profiles appear in lexicographic order.

As we want to study the relationship between the representation and the nucleolus, it is sufficient to consider symmetric imputations of the form

$$x = (\underbrace{\xi_1,\dots,\xi_1}_{k_1}, \underbrace{\xi_2,\dots,\xi_2}_{k_2})$$

In particular, $\mu = \frac{1}{M(N)}\,(t,\dots,t,1,\dots,1)$ denotes the normalized weights and $\nu = (\nu_1,\dots,\nu_1, \nu_2,\dots,\nu_2)$ denotes the nucleolus.

Next, introduce

$$(5) \qquad \Xi = \{\underline{X} = (\xi_1, \xi_2) \in \mathbb{R}^2_+ \mid k_1\,\xi_1 + k_2\,\xi_2 = 1\},$$

this is a one–dimensional simplex with extreme points

$$\xi^L = (\frac{1}{k_1}, 0)\ ,\ \xi^R = (0, \frac{1}{k_2})$$

Note that

$$\tilde{\mu} := \frac{1}{M(N)}\ (t,1)$$

and

$$\tilde{\nu} := (\nu_1,\nu_2)$$

are elements of Ξ.

For any profile $s = (s_1,s_2) \in \mathbb{N}_0^2$, $s \leq k$, consider the ("symmetrized") *excess function*

$$(6) \qquad \begin{aligned} &e(s,\cdot\,) : \Xi \longrightarrow \mathbb{R} \\ &e(s,\xi) = 1-s_1\,\xi_1 - s_2\,\xi_2 = 1-s\xi \end{aligned}$$

e is a linear function on Ξ, depicted in Fig. 1.

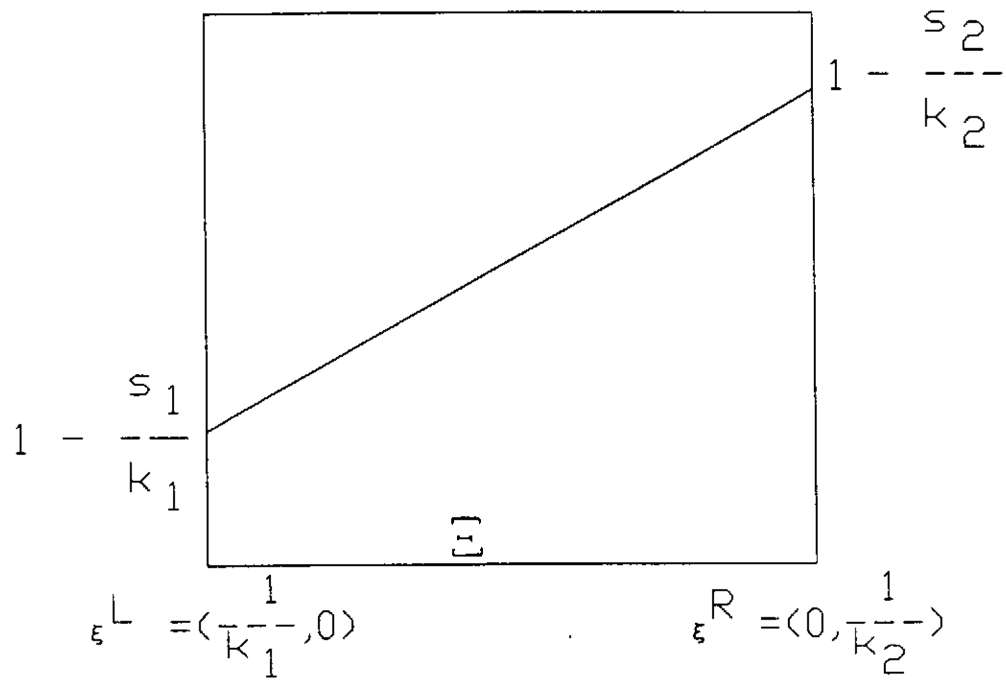

Fig.1

Lemma 2.1 1. For every $s \in \underline{\underline{w}}^m$, $e(s,\tilde{\mu}) = 1-\alpha$

2. Let $\hat{s}$ denote the lexicographically last min–win profile. Then $\nu = \mu$ if

(7) $$e(\hat{s}, \xi^L) > e(\hat{s}, \xi^R) .$$

Proof: Since v is homogeneous, we have for any $s \in \underline{\underline{w}}^m$

$$e(s,\tilde{\mu}) = 1 - \tilde{\mu}s = 1 - \mu(S) = 1 - \alpha$$

(where $S \in \underline{\underline{W}}^m$ has profile s). This proves our first statement.

Next, observe that for two profiles s, s' $\in \underline{\underline{w}}^m$ such that $s \succ_L s'$ it follows that

$$e(s, \xi^L) < e(s', \xi^L), e(s, \xi^R) > e(s', \xi^R) .$$

From this it follows that the family of lines representing the graphs of functions $e(s,\cdot)$ ($s \in \underline{\underline{w}}^m$) is ordered by $\succ_L$ in such a way that all lines pass through $(\tilde{\mu}, 1-\alpha) \in \Xi \times (0,1)$ and lexicographically smaller s induce smaller slopes of $e(s,\cdot)$; see Fig.2.

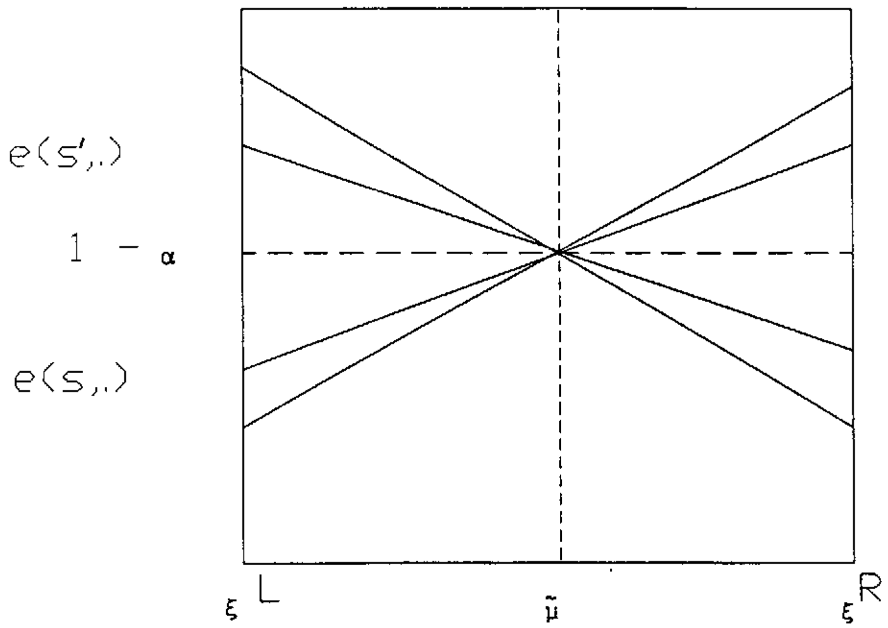

Fig.2

The maximal excess (in terms of profiles of $\underline{\underline{w}}^m$)

$$\Theta_1 : \Xi \longrightarrow \mathbb{R}$$

$$\Theta_1(\xi) = \max_{s \in \underline{\underline{w}}^m} e(s,\xi)$$

is geometrically represented by the upper envelope of the graphs $e(s,\cdot)$, $s \in \underline{\underline{w}}^m$. Clearly, if the slope of the lexicographically last graph, i.e., of $e(\hat{s},\cdot)$ is negative, then

$$\min_{\xi} \Theta_1(\xi) = \Theta_1(\tilde{\mu}) = 1 - \alpha$$

and $\tilde{\mu}$ is the unique minimizer, thus μ is the unique symmetric imputation minimizing maximal excess. I.e., $\mu = \nu$ if $e(\hat{s},\cdot)$ is represented by a line with negative slope – which is equivalent to (7). q.e.d.

Corollary 2.2 Let $v = v_{\lambda}^{M} \in \mathbb{H}_{\Sigma}$ and suppose M and λ are specified as in (1), (2), (3). Let $[\rho]$ be the largest integer not exceeding $\rho \in \mathbb{R}$. Then $\nu = \mu$ if

(8) in CASE A: $\frac{lt}{k_2} > \frac{c-l}{k_1}$ $\quad l = \min\left[\left[\frac{k_2}{t}\right], c\right]$

(9) in CASE B: $\frac{lt+c}{k_2} > \frac{k_1-l}{k_1}$ $\quad l = \min\left[\left[\frac{k_2-c}{t}\right], k_1\right]$

Proof: In CASE A we have $\hat{s} = (c-l, lt)$, hence condition (7) reads

$$1 - \frac{c-l}{k_1} > 1 - \frac{lt}{k_2}.$$

Accordingly, in CASE B where

$$\hat{s} = (k_1-l, lt+c)$$

$$1 - \frac{k_1-l}{k_1} > 1 - \frac{lt+c}{k_2}.$$

Note: l is the number of "blocks" of small players of exactly weight t that may replace a big player. If we fix k_1 and t (the characteristics of big players) then, for $k_2 \geq ct$ the right hand side in (8) equals zero. Hence, as $c \leq k_1$, (8) will be satisfied for "sufficiently many small players".

SECTION 3: The least–core

The graphical method discussed in SEC.2 provides a nice tool in order to compute the min–max excess for coalitions (profiles) in $\underline{\underline{W}}^m$ ($\underline{\underline{w}}^m$). In some cases, this yields the opportunity to compute the nucleolus.

However, the appropriate concept based on the min–max excess is the *least–core* as defined by MASCHLER, PELEG and SHAPLEY [5]. Hence, we shall first deal with this notion of a solution concept.

Moreover, the (number theoretical) conditions as obtained in (8) and (9) of SEC.2 (Corollary 2.2), although admitting of a nice interpretation (a "limit theorem": "convergence" of ν and μ for a large set of players) at this stage slightly blur the view on some more game theoretical criteria related to the coincidence of μ and ν.

After all, what is the connection between our results in SEC.2 and PELEG's [7] result that $\mu = \nu$ in the zero–sum case? As we know by OSTMANN's [6] work, the homogeneous zero–sum game shows "no steps" in its unique minimal representation, hence zero–sum games form a subclass of $\mathbb{H}_\Sigma$.

It turns out that balancedness of the system of min–win coalitions provides the clue for linking the various aspects. This concept, introduced by SHAPLEY [13] and used (among other applications) by KOHLBERG [4] is decisive: it is related to the coincidence of μ and ν and may be expressed in number–theoretical terms.

We shall first exhibit this fact by discussing the least–core concept.

Let us repeat the definition as provided in [5].

Definition 3.1 Let $v : \underline{\underline{P}} \rightarrow \mathbb{R}_+$ present a game and denote

(1) $$\mathscr{X} = \mathscr{X}(v) = \{x \in \mathbb{R}^n_+ \mid x(N) \leq v(N)\}.$$

Also let

(2) $$\epsilon_0 = \epsilon_0(v) := \min_{x \in \mathscr{X}} \max_{\substack{S \in \underline{\underline{P}} \\ S \neq \emptyset, N}} (v(S) - x(S))$$

The *least–core* of v is defined by

(3) $$\mathscr{L}(v) = \{x \in \mathscr{X} \mid x(S) \geq v(S) - \epsilon_0 \quad (S \in \underline{\underline{P}};\ S \neq \emptyset, N)\}\ .$$

Now we have

Theorem 3.2 Let $v \in \mathbb{H}$ and let (m,α) be *any homogeneous* representation which is normalized (i.e. $m(N) = 1$).

If $m \in \mathscr{L}$, then $\underline{\underline{W}}^m$ is weakly balanced.

Proof: We may assume $\alpha < 1$, for otherwise $\underline{\underline{W}}^m = \{N\}$ is even balanced.

Consider the linear programming problem designed in variables $(x,\beta) = (x_1,\dots,x_n,\beta) \in \mathbb{R}^n \times \mathbb{R}$ and indicated by

(P) $$\min \left\{ \beta \mid \begin{array}{lll} (x,\beta) \in \mathbb{R}^n \times \mathbb{R} & & \\ x(S) + \beta \geq 1 & & (S \in \underline{\underline{W}}^m) \\ x(N) = 1 & & \\ x_i \geq 0 & & (i=1,\dots,n) \end{array} \right\}$$

(the "primal program").

It is seen that $\epsilon_0 = \min \{\beta \mid \dots\}$ is the value of this program and $\mathscr{L}$ is the projection on $\mathbb{R}^n$ of the set of optimal solutions of (P).

By assumption the representing measure m satisfies $m \in \mathscr{L}$. Because of homogeneity, we have $m(S) = \alpha\ (S \in \underline{\underline{W}}^m)$, thus it follows at once that

(4) $$\epsilon_0 = 1 - m(S) = 1 - \alpha > 0 \quad (S \in \underline{\underline{W}}^m)\ .$$

Next we consider the dual program corresponding to (P). This is an LP in variables

$$(c_S)_{S \in \underline{\underline{W}}^m}\ ,\ c_N$$

indicated by

$$\text{(D)} \quad \max\left\{1 + c_N \;\middle|\; \begin{array}{ll} (c_S)_{S\in\underline{\underline{W}}^m \cup \{N\}} \in \mathbb{R}^{|\underline{\underline{W}}^m|+1} & \\ \sum_{S \in \underline{\underline{W}}^m} c_S 1_S + c_N 1_N & \leq 0 \\ \sum_{S \in \underline{\underline{W}}^m} c_S & = 1 \\ c_S & \geq 0 \quad (S \in \underline{\underline{W}}^m) \end{array}\right\}$$

(1_S is the indicator function of $S \subsetneq N$, defined on N).

Now, let $(\bar{c}_S)_{S\in\underline{\underline{W}}^m \cup \{N\}}$ be an optimal solution of (D). By the duality theorem of LP–theory we know that

$$1 + \bar{c}_N = \epsilon_0 = 1 - \alpha$$

i.e.

$$\text{(5)} \qquad \bar{c}_N = -\alpha < 0 \,.$$

Moreover, since m > 0 and (m, 1–α) is a solution of (P) we conclude that

$$\text{(6)} \qquad \sum_{S \in \underline{\underline{W}}^m} \bar{c}_S 1_S = -\bar{c}_N 1_N = \alpha 1_N \,.$$

Thus, $\underline{\underline{W}}^m$ is weakly balanced. q.e.d.

Theorem 3.3 Let $v \in \mathbb{H}$ and let (m,α) be a homogeneous representation which is normalized.

If $\underline{\underline{W}}^m$ is weakly balanced, then $m \in \mathscr{L}$.

Proof: We may assume $\alpha < 1$, for otherwise the least core equals the set of imputations.
If $m \notin \mathscr{L}$, then for some $S_0 \in \underline{\underline{W}}^m$

$$\alpha = m(S_0) < v(S) - \epsilon_0 = 1 - \epsilon_0 \,.$$

Hence $x \in \mathscr{L}$ satisfies

(7) $$x(S) \geq 1 - \epsilon_0 > \alpha = m(S) \quad (S \in \underline{\underline{W}}^m) .$$

As $\underline{\underline{W}}^m$ is weakly balanced, we conclude that, for a suitable set of weights $(c_S)_{S \in \underline{\underline{W}}^m}$,

$$\begin{aligned} 1 = x(N) &= \sum_{S \in \underline{\underline{W}}^m} c_S \, x(S) \\ &> \sum_{S \in \underline{\underline{W}}^m} c_S \, m(S) = m(N) = 1; \end{aligned}$$

the inequality following from (7). This contradiction verifies $m \in \mathscr{L}$. q.e.d.

Remark 3.4 Consider the situation discussed in SEC.2, i.e., $v = v_\lambda^M \in \mathbb{H}_\Sigma$ with two types only.

Using our notation as previously, assume that the lexicographically last min–win profile $\hat{s}$ satisfies

(8) $$e(\hat{s}, \xi^L) = e(\hat{s}, \xi^R) .$$

This means that the slope of $e(\hat{s},\cdot)$ is zero – and by inspecting Fig.3 it is seen that in this case the "intervall" $[\xi^L, \tilde{\mu}]$ constitutes the symmetric imputations of the least core, say

(9) $$\mathscr{L}_{symm} = [\xi^L, \tilde{\mu}]$$

(with a suitable interpretation of the interval).

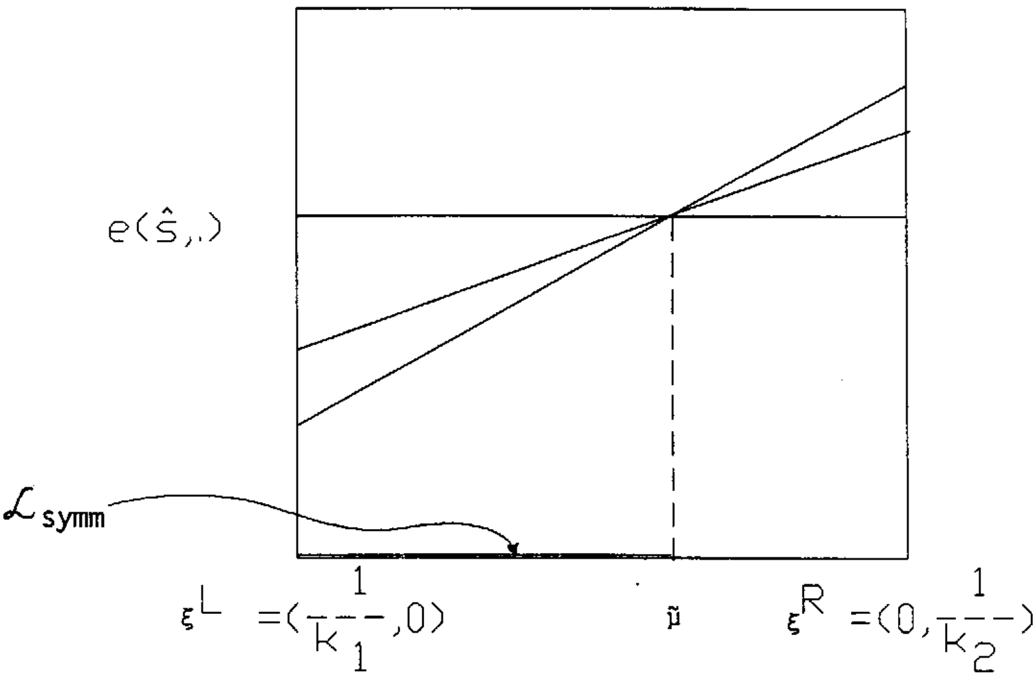

Fig.3

In this case the nucleolus $\tilde{\nu}$ does not necessarily equal $\tilde{\mu}$.

Remark 2.3 The graphical method may also be used to provide counter examples. Consider the case g = (4,1), k = (5,7), $\lambda = 16$ which is conventionally written

$$[\,16;\ 4,4,4,4,4,1,1,1,1,1,1,1,]$$

Hence $t = g_1 = 4$, $l = 1$ ("one block") and

$$\frac{l\,t}{k_2} = \frac{4}{7} < \frac{3}{5} = \frac{c-l}{k_1}.$$

Thus, the slope of the line corresponding to $e(\hat{s},\cdot)$ is positive, from which it follows that $\nu = (\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},0,0,0,0,0,0,0) \neq m$.

SECTION 4: The nucleolus

Theorem 4.1 Let $v \in \mathbb{H}$. Suppose there is $\alpha \in (0,1)$ such that (ν,α) is a homogeneous representation of v. Then $\underline{\underline{W}}^m$ is balanced.

Proof: Let

(1)
$$\epsilon = \max_{\substack{S \in \underline{\underline{P}} \\ S \neq \emptyset, N}} e(S,\nu)$$

be the maximal excess at ν. As ν is a homogeneous representation, the excess for any $S \in \underline{\underline{W}}^m$ at ν is

$$e(S,\nu) = 1 - \nu(S) = 1 - \alpha$$

and as v has no dummies the excess for any other coalition at ν is smaller. Thus $\epsilon = 1 - \alpha$ and the system of coalitions of maximal excess at ν is $\underline{\underline{W}}^m$.

By a theorem of KOHLBERG [4], $\underline{\underline{W}}^m$ is balanced. q.e.d.

Theorem 4.2 Let $v \in \mathbb{H}_\Sigma$, i.e., assume that v has "no steps".

If $\underline{\underline{W}}^m$ is balanced then there is $\alpha \in (0,1]$ such that (ν,α) is a representation of v (the unique representation of v!).

Proof: The case $\underline{\underline{W}}^m = \{N\}$ (and hence $\alpha = 1$) again is trivial. Let us assume that $\underline{\underline{W}}^m \neq \{N\}$ holds true. In this case, as v has no steps, the minimal representation (M,λ) has the property that M is *nondegenerate* with respect to λ (see ROSENMÜLLER – WEIDNER [12] for this notion), and the normalized version $\frac{1}{M(N)}(M,\lambda) = (m,\alpha)$ has the same property. Nondegeneracy means that the system

$$\underline{\underline{Q}}_\alpha = \{S \in \underline{\underline{P}} \mid m(S) = \alpha\}$$

determines m *uniquely* in the sense that the linear system of equations in variables $x_1,\dots,x_n$

$$\sum_{i\in S} x_i = \alpha \qquad\qquad (S \in \underline{\underline{Q}}_\alpha)$$

has the *unique* solution m.

Hence, nondegeneracy implies the *uniqueness* of the representation (normalized); for the fact that m n.d. α see OSTMANN [6] (Remark 5.5). Compare also ROSENMÜLLER [9], Theorem 4.2, [10], Remark 1.2, and the discussion in [11], Section 2 and Section 3.

Of course, if $v \in \mathbb{H}$, then

(3) $$\underline{\underline{Q}}_\alpha = \underline{\underline{W}}^m.$$

Next, consider the set

(4) $$Y = \{x \in \mathbb{R}^n_+ \mid x(N) = 1,\ x(S) \geq m(S) \qquad (S \in \underline{\underline{W}}^m)\}\ .$$

By definition of ν it is seen at once that $\nu \in Y$. As $\underline{\underline{W}}^m$ is balanced we have, with suitable coefficients $(c_S)_{S\in\underline{\underline{W}}^m}$,

(5) $$\begin{aligned} 1 = \nu(N) &= \sum_{S\in\underline{\underline{W}}^m} c_S\, \nu(S) \\ &\geq \sum_{S\in\underline{\underline{W}}^m} c_S\, m(S) \\ &= m(N) = 1 \end{aligned}$$

Hence, all inequalities used in (5) must be equations. This means

$$\nu(S) = m(S) \qquad\qquad S \in \underline{\underline{W}}^m = \underline{\underline{Q}}_\alpha$$

By nondegeneracy of m w.r.t. α it follows that $\nu = m$, q.e.d.

Remark 5.4 Let v ∈ $\mathbb{H}$ be homogeneous *with* steps (other than the smallest non-dummy, that is). There are profiles in which steps appear without their satellites (Remark 5.5 in ROSENMÜLLER [9], see also Lemma 4.1). On the other hand "steps rule their followers" (Lemma 4.8 in OSTMANN [6]), hence smaller players cannot enter min–win coalitions without all the preceding steps. From this it can be inferred that $\underline{\underline{W}}^m$ cannot be balanced.

By KOHLBERG [4] we know that the system of coalitions of maximal excess "at ν" must be balanced. Thus, ν cannot be equal to *any* normalized representation of v.

SECTION 5: The kernel

Lemma 5.1 Let $v \in \mathbb{H}_{\Sigma}$. If $\underline{\underline{W}}^m \neq \{N\}$, then $\underline{\underline{W}}^m$ is completely separating, i.e., for any two players $i,j \in N$ there exist coalitions $S, T \in \underline{\underline{W}}^m$ such that $i \in S$, $j \notin S$ and $j \in T$, $i \notin T$.

Proof: Let (m,α) be the unique normalized representation, then $m_1 \geq m_2 \geq \ldots \geq m_n$. Consequently, it is sufficient to prove that if $1 \leq i \leq n-1$ then there exists $S \in \underline{\underline{W}}^m$ such that $n \in S$ and $i \notin S$. Thus let $i < n$. By assumption i is a sum. Hence, by Definition 4.6 of OSTMANN [6], there exist coalitions T and S_1 such that

$$\{1,\ldots,i-1\} \cap S_1 = \{1,\ldots,i-1\} \cap T, \quad S_1, T \in \underline{\underline{W}}^m$$

and $i \in T-S_1$. Let $j = \max \{k \mid k \in S_1\}$. If $j = n$ the proof is complete. Otherwise, $j > i$ and j is a sum. Therefore, we may replace j by "smaller" players, that is, by a set $A \subset \{j+1,\ldots,n\}$. If $n \in S_2$, where $S_2 = (S_1 - \{j\}) \cup A \in \underline{\underline{W}}^m$, then the proof is complete. Otherwise, we continue the replacement process until we obtain the desired coalition S. q.e.d.

Theorem 5.2 Let $v \in \mathbb{H}_{\Sigma}$ and let (m,α) be its unique homogeneous representation. Then $m \in \mathcal{K}(v)$, i.e., the unique representation is an element of the kernel of v.

Proof: This is a direct consequence of Lemma 5.1.

SECTION 6: Balancedness, profiles and number theory

Let $N = \bigcup_{\rho=1} K_\rho$ denote a decomposition of N into *disjoint* subsets $K_\rho \subsetneq N$ ($\rho = 1,\ldots,n$) representing "types of players" and let $P : \mathbb{R}^n \to \mathbb{R}^r$ be defined by

(1) $$(Px)_\rho = \sum_{i \in K_\rho} x_i \, .$$

P is a linear mapping and maps indicators 1_S ($S \in \underline{\underline{P}}$) into profiles s, that is

$$P(1_S) = s\ ,\ s_\rho = |S \cap K_\rho| \qquad (\rho = 1,\ldots,r).$$

Therefore any *symmetric* decomposition of the unit

$$\sum_{S \in \underline{\underline{W}}^m} c_S 1_S = 1_N$$

provides a decomposition of $k = (|K_1|,\ldots,|K_r|)$ via

(2) $$k = P(1_N) = \sum_{S \in \underline{\underline{W}}^m} c_S P(1_S) = \sum_{s \in \underline{\underline{w}}^m} \sum_{\substack{S \in \underline{\underline{W}}^m \\ P(1_S) = s}} c_S P(1_S)$$

$$= \sum_{s \in \underline{\underline{w}}^m} c_s s \sum_{\substack{S \in \underline{\underline{W}}^m \\ P(1_S) = s}} 1 = \sum_{s \in \underline{\underline{w}}^m} n_s c_s s$$

with $n_s = |\{S \mid P(1_S) = s\}|$, $c_s = c_S$ $(P(1_S) = s)$.

Thus, we call a system of profiles $\underline{\underline{s}}$ (weakly) *balanced* if there are (nonnegative) positive coefficients d_s ($s \in \underline{\underline{s}}$) such that $\sum_{s \in \underline{\underline{s}}} d_s s = k$.

Now, let us return to our class of examples in SEC.2.

Remark 6.1 In SEC.2, CASE A yields the min–win profiles

(3) $$(c, 0), (c-1, t), \ldots, (c-l, lt).$$

The system $\underline{\underline{w}}$ as described by (3) is balanced if there exist positive numbers d_0, $d_1, \ldots, d_l$ such that

$$d_0(c, 0) + d_1(c-1, t) + \ldots + d_l(c-l, lt) = k,$$

i.e.

$$c\left(\sum_{i=0}^{l} d_i\right) - \sum_{i=1}^{l} id_i = k_1, \quad t \sum_{i=1}^{l} id_i = k_2$$

or

(4) $$d_0 + \sum_{i=1}^{l} d_i = \frac{k_1 + \frac{k_2}{t}}{c}, \quad \sum_{i=1}^{l} id_i = \frac{k_2}{t}.$$

Now, the system (4) has solutions $d \in \mathbb{R}^{l+1}$, $d > 0$ if and only if

(5) $$\min\left\{\sum_{i=1}^{l} d_i \mid d \in \mathbb{R}^l, d \geq 0, \sum_{i=1}^{l} id_i = \frac{k_2}{t}\right\} < \frac{k_1 + \frac{k_2}{t}}{c}$$

holds true. The minimum on the left side in (5), however, is attained at $\bar{d} := (0, \ldots, 0, \frac{k_2}{lt})$ and equals $\frac{k_2}{lt}$. Hence, $\underline{\underline{w}}$ is balanced if and only if

(6) $$\frac{k_2}{lt} < \frac{k_1 + \frac{k_2}{t}}{c}, \text{ i.e., } \frac{lt}{k_2} > \frac{(c-l)}{k_1},$$

which is formula (8) in Corollary 2.2.

LITERATURE

[1] Isbell, J.R.:
A class of majority games.
Quarterly Journal Math.7 (1956), pp.183–187

[2] Isbell, J.R.:
A class of simple games.
Duke Math.Journal 25 (1958), pp.423–439

[3] Isbell, J.R.:
On the enumeration of majority games.
Math.Tables Aids Comput, 13 (1959), pp.21–28

[4] Kohlberg, E.:
On the nucleolus of a characteristic function game.
SIAM Journal Appl.Math. 20 (1971), pp.62–66

[5] Maschler, M., Peleg, B., and Shapley, L.S.:
Geometric properties of the kernel, nucleolus, and related solution concepts.
Math. of Operations Research 4 (1979), pp.303–338

[6] Ostmann, A.:
On the minimal representation of homogeneous games.
Int.Journal of Game Theory 16 (1987), pp.69–81

[7] Peleg, B.:
On the kernel of constant–sum simple games with homogeneous weights.
Ill.Journal Math. 10 (1966), pp.39–48

[8] Peleg, B.:
On weights of constant–sum majority games.
SIAM Journal Appl.Math. 16 (1968), pp.527–532

[9] Rosenmüller, J.:
Homogeneous games: recursive structure and computation.
Math.of Operations Research 12 (1987), pp.309–330

[10] Rosenmüller, J.:
Homogeneous games with countably many players.
Math.Social Sciences 17 (1988), pp.131–159

[11] Rosenmüller, J.:
The role of non–degeneracy and homogeneity in n–person game theory: an equivalence theorem.
Journal of Economics 47 (1987), pp.367–389

[12] Rosenmüller, J. and Weidner, H.G.:
Extreme convex set functions with finite carrier: general theory.
Discrete Math. 10 (1974), pp.348–382

[13] Shapley, L.S.:
On balanced sets and cores.
Naval Res.Logist.Quarterly 14 (1967), pp.453–460

[14] Sudhölter, P.:
Homogeneous games as anti step functions.
Int. Journal of Game Theory 18 (1989), pp.433–469

[15] von Neumann, J. and Morgenstern, O.:
Theory of games and economic behavior.
Princeton Univ.Press, NJ., (1944)