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Linear Representability without Completeness and Transitivity

by

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1. Introduction

The linear utility representation theorem for preferences on a finite dimensional linear space is quite old. Different versions have been given, for instance, by Blackwell & Girshick (1954) and by Herstein & Milnor (1953), the latter one in the more general context of mixture sets. Both theorems rely on invariance and continuity assumptions. There is obviously a tradeoff between stronger (algebraic) invariance (Blackwell & Girshick) and a stronger (topological) continuity assumption (Herstein & Milnor). This is in a certain sense an analog to the result by Schmeidler (1971) which derives stronger ordering properties (completeness) from sufficient continuity of a binary relation. An alternative proof of the linear utility representation theorem has been given recently (Trockel (1991)).

It turns out that the techniques employed there can be used to show that given the translation–invariance imposed by Blackwell & Girshick, any non–trivial, reflexive binary relation, which is upper continuous at at least one point, is a continuous, complete preordering which moreover, can be represented by a linear utility function.

So in the Blackwell and Girshick result which plays a crucial role in welfaristic social choice theory the assumptions of transitivity, completeness and monotonicity cann altogether be dispensed with.

2. Notation and basic definitions

Let R be a binary relation on $\mathbb{R}^n$, $n \in \mathbb{N}$, which is reflexive, i.e. $\forall\, x \in \mathbb{R}^n : x\, R\, x$. For R we denote by I and P, respectively the induced relations defined as follows:

$$\forall\ x,y \in \mathbb{R}^n : \quad x\, I\, y \Leftrightarrow x\, R\, y \text{ and } y\, R\, x$$
$$x\, P\, y \Leftrightarrow x\, R\, y \text{ and not } y\, R\, x\,.$$

We shall employ the following notation:

$$R(x) := \{y \in \mathbb{R}^n \mid x\, R\, y\} \qquad R^{-1}(x) := \{y \in \mathbb{R}^n \mid y\, R\, x\}$$
$$P(x) := \{y \in \mathbb{R}^n \mid x\, P\, y\} \qquad P^{-1}(x) := \{y \in \mathbb{R}^n \mid y\, P\, x\}$$
$$I(x) := \{y \in \mathbb{R}^n \mid x\, I\, y\} =: I^{-1}(x)\,.$$

For convenience we shall denote for any $x \in \mathbb{R}^n$ the set $I(x)$ by I_x.

Definition 1: A binary relation R on $\mathbb{R}^n$ is called (translation–) invariant iff $\forall\ x,y,z \in \mathbb{R}^n : x\, R\, y \Leftrightarrow x + z\, R\, y + z$.

Definition 2: A binary relation R on $\mathbb{R}^n$ is called non–trivial iff $I \neq R$.

Definition 3: A binary relation R on $\mathbb{R}^n$ is upper (resp. lower) closed at $x \in \mathbb{R}^n$ iff $R^{-1}(x)$ (resp. $R(x)$) is closed. It is upper (resp. lower) open at $x \in \mathbb{R}^n$ iff $P^{-1}(x)$ (resp. $P(x)$) is open. It is upper (resp. lower) closed (resp. open) iff it is so at every $x \in \mathbb{R}^n$. It is upper (resp. lower) continuous [at $x \in \mathbb{R}^n$] iff it is upper (resp. lower) closed and upper (resp. lower) open [at x]. R is continuous [at x] iff it is upper and lower continuous [at x].

A continuous complete preordering is representable by a continuous utility function (cf. Debreu (1959)).

3. Result

<u>Proposition:</u> Let R be a non-trivial reflexive binary relation on $\mathbb{R}^n$ which is translation-invariant and upper continuous at some point $x \in \mathbb{R}^n$. Then R is a continuous complete preordering representable <u>by a linear utility function</u>.

<u>Proof:</u>

<u>1. $\exists\, p \leq n : I_0 = \mathbb{R}^p$</u>

As R is upper continuous at some $x \in \mathbb{R}^n$, the sets $R^{-1}(x)$ and $P^{-1}(x)$ are, respectively, closed and open. Hence I_x is closed and so is $I_0 = -x + I_x$. For any $x, y \in I_0$ invariance yields $x - y, -x \in I_0$. Thus I_0 is a closed subgroup of $\mathbb{R}^n$. Invariance again yields $n\,x + m\,y \in I_0$ for any $n,m \in \mathbb{Z}$ and therefore for any $n,m \in \mathbb{Q}$. So I_0 is even a linear subspace of $\mathbb{R}^n$.

2. <u>$p = \ell - 1$</u>

As $\mathbb{R}^n \neq I$ there exists $z \in P^{-1}(0) \cup P(0)$. Assume w.l.o.g. $z\ P\ 0$. From invariance we get $0\ P\ -z$ from invariance. By definition we have $P(0) \subset (R^{-1}(0))^C$. (The superscript C denotes the complement set). We have $P^{-1}(0) \cap (R^{-1}(0))^C = \emptyset$, $z \in P^{-1}(0)$, $-z \in P(0) \subset (R^{-1}(0))^C$. As both sets in this intersection are non-empty and open the set $\mathbb{R}^n \setminus I_0$ covered by them cannot possibly be connected. Therefore $\dim I_0 \geq \ell - 1$. Non-triviality then implies $\dim I_0 = \ell - 1$.

3. <u>R is complete and linearly representable</u>

Define $H := I_0$. Then $I_x = x + H$. Next let $z\ P\ 0$ and $p : \mathbb{R}^n \to \mathbb{R}$ be linear with kernel H and $p \cdot z > 0$. Let $H_+ := \{y \in \mathbb{R}^n \mid p \cdot y > 0\}$ and $H_- := -H_+$. As H_-, H_+, $P^{-1}(0)$, $(R^{-1}(0))^C$ are non-empty and open, and H_-, H_+ are connected, we get from

$H_+ = (H_+ \cap P^{-1}(0)) \cup (H_+ \cap (R^{-1}(0))^C)$ and $z\ P_0$ that
$H_+ \cap (R^{-1}(0))^C = \emptyset$.

This implies $H_+ \subset P^{-1}(0)$ and $P(0) \subset (R^{-1}(0))^C \subset H_-$. Now assume there exists $z' \in P^{-1}(0) \cap H_-$. Then $-z' \in H_+$ and, as above, $-z' \in P(0)$, hence $-z' \in P(0) \cap H_+ \subset H_+ \cap (R^{-1}(0))^C$, a contradiction. Therefore, $P^{-1}(0) = H_+$ and $P(0) = (R^{-1}(0))^C = H_-$. So R is complete and represented by p, hence transitive.

□

4. Concluding remarks

It is not hard to show that none of the employed assumptions can be dropped. Without any continuity for instance, our framework would admit the lexicographic ordering. Without reflexivity the meaning of indifference would become most dubious. Technically I_0 would fail to be a group.

One might be tempted also to require invariance only for the indifference relation. Even in the context of complete preorderings this would require full continuity as a substitute. In the present general framework even continuity would not compensate the lacking invariance of P since no transitivity or completeness are available.

Compared with the Blackwell & Girshick result which uses the same invariance, and besides transitivity and completeness even monotonicity to get a linear representation, the present result appears to be built on a weakest possible set of assumption for binary relations which suffices to yield linear representability.

Given the key role the Blackwell and Girshick result has played in the social choice literature to get utilitarianism results (cf. d'Aspremont and Gevers (1977), Maskin (1978), Gevers (1979), Roberts (1980)) it is clear that a much more general utilitarianism result can be derived from our present proposition. Clearly, an aggregate relation whose invariance property takes regard of individuals' possibility to choose between different cardinal utility representations but which is not based on undue rationality assumptions like transitivity and completeness is much more appealing. A proper application to this social choice problem will be given elsewhere.

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