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Abstract

In this paper the class of homogeneous n-person games "without dummies and steps" is characterized by two algebraic axioms. Each of these games induces a natural vector of length n, called incidence vector of the game, and vice versa. A geometrical interpretation of incidence vectors allows to construct all of these games and to enumerate them recursively with respect to the number of persons.

In addition an algorithm is defined, which maps each directed game to a minimal representation of a homogeneous game. Moreover both games coincide, if the initial game is homogeneous.

§1 Basic Notations and Definitions

A <u>simple</u> <u>n-person</u> <u>game</u> is a pair (Ω, v). Here $\Omega = \{1, \ldots, n\}$ is called the set of <u>players</u>, and $v : \mathcal{P}(\Omega) \longrightarrow \{0,1\}$, $v(\emptyset) = 0$, is the "characteristic function" in the sense of Game Theory. An element of $\mathcal{P}(\Omega)$, i.e. a subset of Ω, is a <u>coalition</u>.

A coalition S is <u>winning</u> if $v(S) = 1$ and <u>losing</u> otherwise. If each proper sub-coalition of a winning coalition is losing, this winning coalition is called <u>minimal</u>.

The set of minimal winning coalitions is denoted by $W_\star$ or $W_\star(\Omega, v)$, if the dependence of the game is to be stressed.

Sometimes the expression "n-person" is deleted.

Let $m = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ and let $\lambda \in \mathbb{N}$. In the context to be discussed we shall call m_i player i's weight and $m(S) := \sum_{i \in S} m_i$ the <u>weight</u> of coalition S. Finally λ is called the level. This terminology is justified by the following construction: Assume that $0 < \lambda \leq m(\Omega)$ and define a simple game (Ω, v) by

$$v(S) = \begin{cases} 1 \text{ , if } m(S) \geq \lambda \\ 0 \text{ , if } m(S) < \lambda, \end{cases}$$

v is also written

$$v = 1_{[\lambda, m(\Omega)]} \circ m = v_\lambda^m$$

where 1_T is the indicator function of T. Intuitively we interpret v ("the characteristic function") as to represent the power structure of a parliament or committee where each member (or rather each party) is commanding a certain number of votes according to its weight. Obviously, various pairs (λ, m) may result in the same v.

Thus a pair (λ, m) resulting in a function v as specified above is called a re-presentation of (Ω, v); we shall use notations like $(\lambda, (m_1, \ldots, m_n))$ and $(\lambda; m_1, \ldots, m_n)$ simultaneously.

A simple game having a representation is called <u>weighted</u> <u>majority</u> <u>game</u>. It should be noted that such a game is completely determined by its minimal winning coalitions.

If a weighted majority game has a representation (λ,m) such that all minimal winning coalitions are exactly of weight λ, then both the simple game and the representation are called <u>homogeneous</u>. For a special case the terms "simple", "weighted majority", and "homogeneous" were introduced by von NEUMANN and MORGENSTERN [13]. SHAPLEY [12] considered homogeneous games in general. ISBELL [1,2,3], OSTMANN [5], PELEG [7], and ROSENMÜLLER [8,9,10,11] should also be mentioned in this context.

Following OSTMANN [5] each player i of Ω belongs to an equivalence class $\tilde{i}$, called <u>type of</u> i: i ~ j, if there is a permutation Π of Ω such that $v = v \circ \Pi$ and $\Pi(i) = j$. Let $\tilde{i}$ be the set $\{j\in\Omega|i{\sim}j\}$. In the paper just mentioned it is shown that all representations of a given weighted majority game (Ω,v) are inducing the same order of the types of Ω, i.e. for all types $t_1 \neq t_2$ and all players $i_1\in t_1$, $i_2\in t_2$ either $m_{i_1} > m_{i_2}$ or $m_{i_1} < m_{i_2}$ for all representations (λ,m).

Let (Ω,v) be a simple game. The relation $\preccurlyeq \subseteq \Omega^2$, defined by $i \preccurlyeq j$, if $v(\{i\} \cup S) \leq (\{j\} \cup S)$ for all S with $\{i,j\} \cap S = \emptyset$, is called <u>desirability</u> relation of (Ω,v). For this definition we refer to [4]. The simple game (Ω,v) is called <u>ordered</u>, if its desirability relation is complete, and is called <u>directed</u>, if additionally $1 \succcurlyeq 2 \succcurlyeq 3 \succcurlyeq \ldots \succcurlyeq n$ is valid.

Two simple games (Ω,v) and (Ω,v') are <u>equivalent</u>, if there is a permutation Π of Ω such that $v \circ \Pi = v'$. Consequently the equivalence class of an ordered game can be identified with its unique directed representative. As our interest is restricted to these equivalence classes of ordered games only, it is assumed w.l.o.g. from now on that all considered games are directed.

Each weighted majority game is ordered and thus directed by the assumption, which implies that it has a <u>directed</u> representation, i.e. a representation (λ,m) with the property $m_1 \geq \ldots \geq m_n$.

For these definitions and assertions we refer to [6].

The representations (λ,m) of a weighted majority game are ordered by <u>total mass</u> $m(\Omega)$. OSTMANN [5] has shown that there is a unique minimal representation of a homogeneous game, which is automatically homogeneous. Two further proofs are contained in [10]. Therefore a homogeneous game is often identified with its minimal representation, which automatically is directed, since in [5] it is shown that players of the same type have the same weights in the minimal representation.

Let $D = D(\Omega,v) = \{i \in \Omega | v(S\cup\{i\}) = 0$ for all losing coalition $S\}$ be the set of <u>dummies</u>.

The fact that two coalitions S and T do not differ earlier than at i, formally meaning $S \cap [1,i-1] = T \cap [1,i-1]$, is abbreviated by $S \sim_i T$. If i and j are nonnegative integers, we put $[i,j]_n := [i,j] \cap \mathbb{N}_0$. Sometimes, if a misunderstanding is excluded, the lower index "n" is deleted.

A player $i \in \Omega \setminus D$ is a <u>sum</u>, iff there are two coalitions $S, T \in W_*$ such that $i \in S$, $i \notin T$ and $S \sim_i T$, and is a <u>step</u> otherwise. If $D(\Omega,v)$ is nonvoid, the decreasing order of the weigths m_i cause the existence of a first dummy player j, i.e. $D(\Omega,v) = \{j,\ldots,n\}$.

If (λ,m) is the minimal representation of a homogeneous n-person game, the following assertion is obviously true:

$$m_i = \begin{cases} 0, & \text{iff } i \text{ is a dummy} \\ \lambda, & \text{iff } \{i\} \text{ is a one-person winning coalition} \end{cases}$$

<u>Lemma 1.1:</u>   There are canonical bijections

$$\left\{ (\lambda,m) \,\middle|\, \begin{array}{l} (\lambda,m) \text{ is the minimal representation of} \\ \text{a homogeneous n-person game} \end{array} \right\}$$

$$\longrightarrow$$

$$\bigcup_{t=1}^{n} \left\{ (\lambda,m) \,\middle|\, \begin{array}{l} (\lambda,m) \text{ is the minimal representation of a} \\ \text{homogeneous t-person game without dummies} \end{array} \right\}$$

and

$$\left\{ (\lambda,m) \,\middle|\, \begin{array}{l} (\lambda,m) \text{ is the minimal representation of a} \\ \text{homogeneous n-person game without dummies} \end{array} \right\}$$

$$\longrightarrow$$

$$\{(1;\underbrace{1,\ldots,1)}_{\text{n-times}}\} \cup \bigcup_{t=2}^{n} \left\{ (\lambda,m) \,\middle|\, \begin{array}{l} (\lambda,m) \text{ is the minimal representa-} \\ \text{tion of a homogeneous t-person} \\ \text{game without dummies and with-} \\ \text{out one-person winning coalitions} \end{array} \right\}$$

<u>Proof:</u>   Definition of the first mapping:

If (λ,m) is a minimal representation of a homogeneous game, then there is a player $1 \leq i_0 \leq n$ such that $D(\Omega,v) = \{i_0+1,\ldots,n\}$. The vector $(\lambda;m_1,\ldots,m_{i_0})$ clearly is the minimal representation of a homogeneous game without dummies and

the mapping $(\lambda,m) \longmapsto (\lambda;m_1,\ldots,m_{i_0})$ is bijective.

Definition of the second mapping:

Distinguish two cases:

1. $(\lambda,m) = (1;\underbrace{1,\ldots,1}_{\text{n-times}}) \longmapsto (\lambda,m)$

2. If (λ,m) does not coincide with some $(1;1,\ldots,1)$, then there is an i_0, $1 \leq i_0 \leq n-1$, such that $m_1 = \ldots = m_{i_0-1} = \lambda > m_{i_0}$. The vector $(\lambda;m_{i_0},\ldots,m_n)$ is the minimal representation of a homogeneous game without one-person winning coalitions.

The inverse function is constructed as follows:

If (λ,m) is the minimal representation of a homogeneous t-person game without one-person winning coalitions, define the image of this function as $(\lambda; \underbrace{\lambda,\ldots,\lambda}_{\text{(n-t)-times}}, m_1,\ldots,m_t)$, otherwise $(1; \underbrace{1,\ldots,1}_{\text{n-times}})$, i.e. if

$(\lambda,m) = (1; \underbrace{1,\ldots,1}_{\text{n-times}}).$ \hfill q.e.d.

In view of the last lemma only homogeneous games without dummies and without one-person winning coalitions are being considered in the following presentation. Furthermore, the pathological case, that Ω is the only winning coalition, is excluded without being always explicitely mentioned.

Let (Ω,v) be a directed n-person game. The matrix

$$X := X(\Omega,v) := \begin{bmatrix} \vdots \\ 1_S \\ \vdots \end{bmatrix}_{S \in W_*}$$

with lexicographically ordered rows, this order being induced by $O(S) := \sum_{i \in S} 2^{n-i}$, is called <u>incidence matrix of</u> (Ω,v).

Let (Ω,v) be a homogeneous n-person game without dummies and one-person winning coalitions. From OSTMANN [5] we know the following algorithm, which generates the minimal representation of this game:

If i is a sum, let $S(i)$, $T(i)$ be the lexicographically first pair in W_* such

that $S(i) \sim_i T(i)$, $i \in S(i)$, $i \notin T(i)$.

If i is a step, define $h_m(i) := \max \{m(H) \mid H \in \{[i,n] \setminus S \mid i \in S, S \in W_*\}\}$, $m \in \mathbb{N}^n$.

Define

$$m_i := \begin{cases} m(T(i) \setminus S(i)), & \text{if } i \text{ is a sum} \\ 1 + h_m(i) & \text{, if } i \text{ is a step,} \end{cases}$$

$\lambda = m(S^1)$, where $S^{(1)}$ is the lexicographically first minimal winning coalition.

It follows

Lemma 1.2: (λ,m) is the minimal representation of (Ω,v).

Remark 1.3:

1. (λ,m) is well defined, since it can be calculated successively by starting at n:

$$m_n = 1 + h_m(n) = 1 + m(\emptyset) = 1.$$

2. It is not necessary to use the lexicographically first pair $S(i)$, $T(i)$, in case i is a sum, to construct the minimal weights, only the fact $S(i) \sim_i T(i)$, $i \in S(i)$, $i \notin T(i)$ is needed.

3. In the following all games considered are assumed to have no one-person winning coalitions, unless otherwise specified.

In order to classify homogeneous games without dummies we can restrict our attention to those "without" steps, i.e. with just one step, since player n always is a step. Indeed, if H_n denotes the set of minimal representations of homogeneous n-person games without dummies, steps and one-person winning coalition, the following assertion is true.

Lemma 1.4: There is a canonical bijection from

$$H_{n+1} \setminus \{n; 1,\ldots,1)\}$$

to

$$\left\{ (\lambda,m) \mid \begin{array}{l} (\lambda,m) \text{ is the minimal representation of a} \\ \text{homogeneous n-person game without dummies} \end{array} \right\}$$

<u>Proof:</u>

$(\lambda; m_1, \ldots, m_{n+1}) \longmapsto (\lambda; m_1, \ldots, m_n)$ has the desired properties:

$(\lambda; m_1, \ldots, m_n)$ is a homogeneous representation of a simple game and the above algorithm shows the minimality - note: $m_n = 1$ if $(\lambda; m_1, \ldots, m_{n+1})$ has no steps. The element $(n; \underbrace{1, \ldots, 1}_{(n+1)\text{-times}})$ must be excluded since the image would have only one

winning coalition. On the other hand the algorithm also shows that

$$(\lambda; m_1, \ldots, m_n) \longmapsto (\lambda, m_1, \ldots, m_n, 1)$$

maps homogeneous games to those without steps. This map is obviously inverse to the first. q.e.d.

By using the identification of homogeneous games with their minimal representations, we also identify H_n with

$$\{(\Omega, v) \mid (\Omega, v) \text{ has a representation } (\lambda, m) \in H_n\}$$

and denote this set again by H_n.

§2 The Incidence Vector of a Homogeneous Game

The incidence matrix of a simple game is frequently of a respectable size. It is desirable to select an appropriate submatrix which allows for a unique identification of the game. Given the incidence matrix of a homogeneous game without dummies and steps we are going to show that there exists an n×n submatrix which completely determines the game. The n rows defining the submatrix are chosen in such a way that for each player i ≠ n there exist at least two rows S,T with i ∈ S, i ∉ T and $S \sim_i T$.

At first some notation is needed. Let (Ω, v) be a directed game, not necessarily homogeneous and without dummies and steps.

Definition 2.1:

For a nonempty coalition let $l(S)$ be the <u>length</u> of S, meaning the one player of S who has the highest index, i.e.

$$l(S) := \max \{j \mid j \in S\}.$$

Given $S \in W_*$, let j be such that $[j, l(S)] \subseteq S$.

If $S \setminus \{j\} \cup [l(S)+1, n]$, is winning, define

$$\rho_j(S) := S \setminus \{j\} \cup [l(S)+1, t],$$

where t is minimal such that $S \setminus \{j\} \cup [l(S)+1, t]$ is winning.

For a minimal winning coalition T, which is not the lexicographically maximal one, define

$$\varphi(T) := T \cup \{r\} \setminus [t, l(T)],$$

where $r = \max \{j \notin T \mid j < l(T)\}$ and

$$t = \min \{t' \mid T \cup \{r\} \setminus [t', l(T)] \text{ is winning}\}.$$

Player r exists, since T is not lexicographically maximal, and player t exists, since (Ω, v) is directed.

Remark 2.2:

(i) $\rho_j(S)$ is the lexicographically next minimal winning coalition to S, in which j is substituted by players of smaller or equal type.

(ii)   With the above notations the following holds true:

$$\rho_r(\varphi(T)) = T, \quad \varphi(\rho_j(S)) = S.$$

(iii)   From the "Basic Lemma" of ROSENMÜLLER ([8,9]) we know the follow-
          ing: If (λ,m) is homogeneous, then

$$m_j = m([1(S)+1,t]),$$

$$m_r = m([t, 1(T)]).$$

The existence of n rows of the incidence matrix with the desired properties is a
direct consequence of the following

Theorem 2.3:

Let (Ω,v) be a game of H_n and S_1 be the lexicographically maximal minimal
winning coalition.

If $S_2,\ldots,S_k$ are minimal winning coalitions and $k < n$, such that for all j
$\in [2,k]$ there is an $i < j$ with

$$\rho_{j-1}(S_i) = S_j,$$

then there is $i_0 \in [1,k]$ such that

$$S_{k+1} := \rho_k(S_{i_0})$$

is defined.

Note that the property $\rho_{j-1}(S_i) = S_j$ can be replaced by $\varphi(S_j) = S_i$ (see
Remark 2.2 (ii)).

Proof:

Assume the contrary.
For each coalition S define

$$r(S) = \max (\{j \mid j \notin S, j < 1(S)\} \cup \{0\}).$$

Let $\bar{S}$ be a coalition in $\left\{ S \,\middle|\, \begin{array}{l} S \text{ is a minimal winning coalition and} \\ \rho_k(S) \text{ is defined} \end{array} \right\}$

– which is indeed nonempty, since (Ω,v) has no steps except n –, such that
$r := r(\bar{S})$ is maximal, thus $r \geq 1$, since $\bar{S}$ cannot be the lexicographically maxi-
mal minimal winning coalition S_1. We distinguish two cases.

1. $1(S_{r+1}) < 1(\bar{S})$. Then $1(S_{r+1}) < k$, otherwise $\rho_k(S_{r+1})$ would be defined.
       Therefore

$$\rho_k(\bar{S} \cup \{r\} \setminus [1(\varphi(S_{r+1}))+1, 1(S_{r+1})])$$

is defined, contradicting the maximality of r.

2. $1(S_{r+1}) \geq 1(\bar{S})$. Then it is obvious that $1(S_{r+1}) > 1(\bar{S})$ and $1(\varphi(S_{r+1})) < k$, otherwise $\rho_k(S_{r+1})$ resp. $\rho_k(\varphi(S_{r+1}))$ would be defined. As a direct consequence
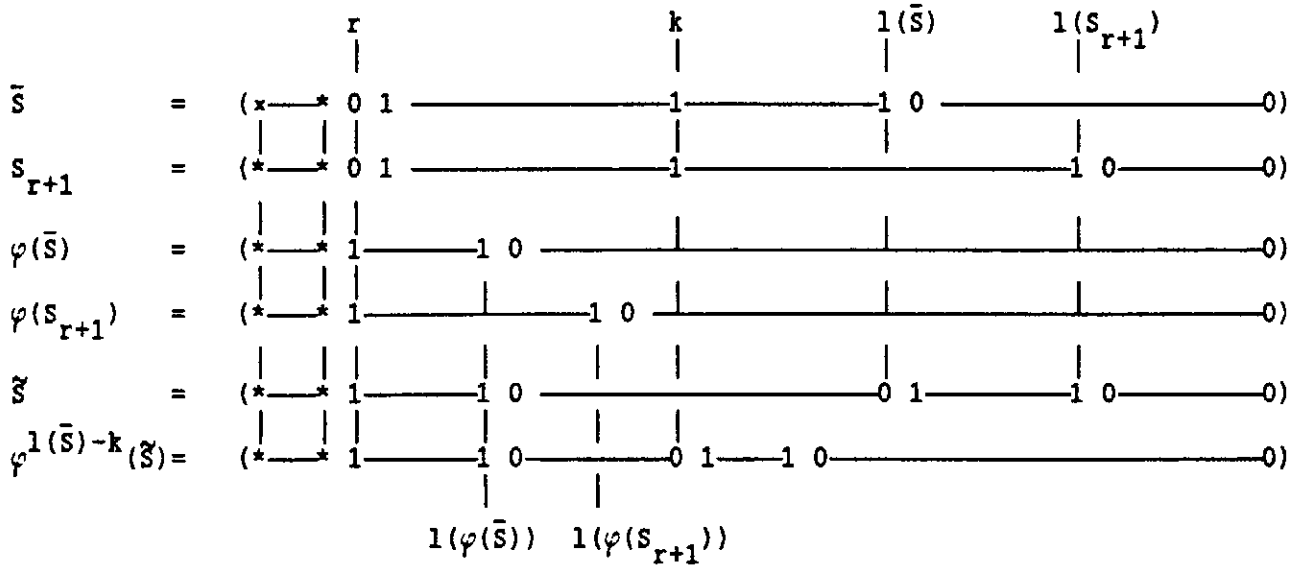
$$\tilde{S} := S_{r+1} \cup \{r\} \setminus [1(\varphi(\bar{S}))+1, \ 1(\bar{S})]$$

is a minimal winning coalition with $[k-1, \ 1(\bar{S})] \cap \tilde{S} = \emptyset$.

A simple computation shows that

$$\rho_k(\varphi^{1(\bar{S})+1-k} (\tilde{S})) = \varphi^{1(\bar{S})-k}(\tilde{S}),$$

but $r < k-1$, a contradiction to the maximality of r. Let us illustrate this situation by an example:

q.e.d.

With the help of the last important theorem we will define a unique sequence of minimal winning coalitions recursively.

Definition 2.4:

Let (Ω, v) be a game in H_n and S_1^v be the lexicographically maximal minimal winning coalition.

If $S_2^v, \ldots, S_k^v$ $(k < n)$ are already defined, then

$$S_{k+1}^v := \rho_k (S_{i_o}^v),$$

where
$$i_0 = \min \{t \leq k \mid 1(S_t) = \min \{1(S_r) \mid r \leq k, \ 1(S_r) \geq k\}\}.$$

In view of Theorem 2.3 the sequence $S_1^V, \ldots, S_n^V$ is well defined and hence unique. Besides note that it may be useful to compare this procedure with the context of section 5, pp.324-327, in [10].

Corollary 2.5:

The function
$$H_n \longrightarrow (\wp(\Omega))^n,$$

defined by
$$(\Omega, v) \longmapsto (S_1^V, \ldots, S_n^V),$$

is injective.

Proof:

Let (Ω, v) be a game with the desired properties.

Define successively
$$m_n = 1, \quad m_i = m \ (S_{i+1}^V \setminus S_j^V),$$

where $\quad \varphi(S_i^V) = S_j^V,$

and $\quad \lambda = m(S_1^V).$

With this notation it can be shown analogously to Lemma 1.2 that $(\lambda; m_1, \ldots, m_n)$ is the minimal representation of (Ω, v). \hfill q.e.d.

Corollary 2.6:

The mapping
$$H_n \longrightarrow \{(1_1, \ldots, 1_n) \in \mathbb{N}^n\},$$
$$(\Omega, v) \longmapsto (1(S_1^V), \ldots, 1(S_n^V))$$

is injective.

Proof:

From $(1(S_1^V), \ldots, 1(S_n^V))$ the sequence $(S_1^V, \ldots, S_n^V)$ can be reconstructed successively:

$$S_1^v = \{i \in \Omega \mid 1 \leq i \leq 1(S_1^v)\},$$

if $S_1^v, \ldots, S_k^v$ are already constructed, then

$$S_{k+1}^v = S_j^v \setminus \{k\} \cup \{1(S_j^v)+1, \ldots, 1(S_{k+1}^v)\},$$

where

$$j = \min \{t \leq k \mid 1(S_t) = \min \{1(S_r) \mid r \leq k, 1(S_r) \geq k\}\}.$$

<div align="right">q.e.d.</div>

In the following it will be shown that the image of the mapping given in Corollary 2.5, i.e. the vectors $(1(S_1^v), \ldots, 1(S_n^v))$, can be characterized by algebraic means.

Lemma 2.7:

If (Ω, v) is a homogeneous n-person game without steps and dummies, then

(i) $\quad 1(S_i^v) > \min \{1(S_j^v) \mid 1 \leq j \leq i-1, 1(S_j^v) \geq i-1\}$ for all $2 \leq i \leq n$

(ii) $\quad 1(S_{i+1}^v) \leq 1(S_i^v)$, if $\min \{1(S_j^v) \mid j \leq i-1, 1(S_j^v) \geq i-1\} \geq i$.

Proof:

Assertion (i) is a direct consequence of the successive construction of the sequence $S_1^v, \ldots, S_n^v$.

The order of the minimal weights, i.e. $m_1 \geq \ldots \geq m_n$ for $(\lambda; m_1, \ldots, m_n)$ being the minimal representation of the game, directly implies assertion (ii).

<div align="right">q.e.d.</div>

We show now, roughly speaking, that the converse is also true, i.e. every vector which fulfills (i) and (ii) of the last lemma is of the form $(1(S_1^v), \ldots, 1(S_n^v))$ for some homogeneous game (Ω, v). Thus, a new characterization of this class of homogeneous games is obtained as soon as a proof of the theorem, containing the above mentioned assertion, is provided.

The following notation simplifies the formulation of this important result. For technical reasons $n \geq 3$ is presumed.

Definition 2.8:

(1) A vector $1 = (1_1, \ldots, 1_n) \in \{2, \ldots, n\}^n$ is called an n-person _inci-_

dence vector, iff - for all $i \in [2,n]$ -

(i) $\qquad l_i > \min \{l_j | 1 \leq j \leq i-1, \; l_j \geq i-1\},$

(ii) $\qquad l_{i+1} \leq l_i,$ if $\min \{l_j | 1 \leq j \leq i-1, \; l_j \geq i-1\} \geq i.$

(2) $\qquad$ The sequence of coalitions $(S_1^l, \ldots, S_n^l)$ is __associated__ to l, iff

$$S_1^l = \{1, \ldots, l_1\}, \; S_{k+1}^l = S_i^l \setminus \{k\} \cup \{l_i+1, \ldots, l_{k+1}\},$$

where

$$i = \min \{j \leq k \mid l_j = \min \{l_t \mid t \leq k, \; l_t \geq k\}\}.$$

(3) $\qquad$ l __generates__ $(\lambda; m_1, \ldots, m_n)$ via

$$m_n = 1, \; m_i = m_{j+1} + \ldots + m_{l_{i+1}}$$
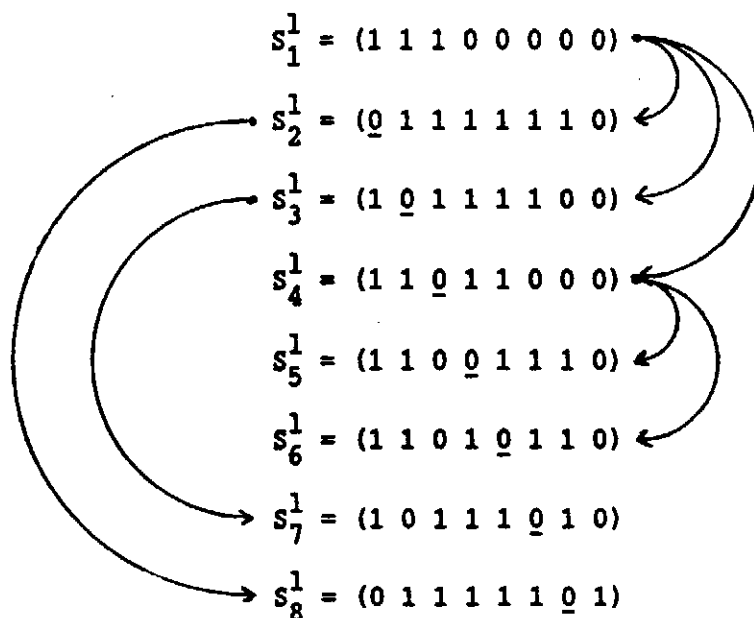
where

$$j = \min \{l_k \mid k \leq i, \; l_k \geq i\}, \; \lambda = m(S_1^v).$$

(4) $\qquad$ Let I_n denote the set of n-person incidence vectors.

In order to illustrate the last definition we give an explicit example:

The vector l = (3,7,6,5,7,7,7,8) is an 8-person incidence vector.
The associated sequence of coalitions is

$$S_1^l = (1\;1\;1\;0\;0\;0\;0\;0)$$
$$S_2^l = (\underline{0}\;1\;1\;1\;1\;1\;1\;0)$$
$$S_3^l = (1\;\underline{0}\;1\;1\;1\;1\;0\;0)$$
$$S_4^l = (1\;1\;\underline{0}\;1\;1\;0\;0\;0)$$
$$S_5^l = (1\;1\;0\;\underline{0}\;1\;1\;1\;0)$$
$$S_6^l = (1\;1\;0\;1\;\underline{0}\;1\;1\;0)$$
$$S_7^l = (1\;0\;1\;1\;1\;\underline{0}\;1\;0)$$
$$S_8^l = (0\;1\;1\;1\;1\;1\;\underline{0}\;1)$$

The coalition at the origin of each arrow is needed to construct the coalition at the top of the arrow.

The generated representation turns out to be

$$(15; \ 6,5,4,2,2,1,1,1).$$

Remark 2.9:

(1)   A direct consequence of the definition of an incidence vector 1 is that both the sequence $S_1^1,\ldots,S_n^1$ and the generated vector $(\lambda;m_1,\ldots.m_n)$ are well defined.

(2)   If (Ω,v) is a homogeneous game in some H_n, then by Lemma 2.7 the vector $1 := (1(S_1^v),\ldots,1(S_n^v))$ is an incidence vector, $(S_1^v,\ldots,S_n^v)$ is associated to 1 and 1 generates a tupel (λ,m), which is - this can be shown analogously to Lemma 1.2 by the way - the minimal representation of (Ω,v).

The main result of this chapter is stated in form of the following

Theorem 2.10:

Each homogeneous n-person game without dummies and steps can be identified with some n-person incidence vector and vice versa, formally:

$$L_n \ : \ H_n \longrightarrow I_n$$

$$(\Omega,v) \longmapsto (1(S_1^v),\ldots,1(S_n^v))$$

is bijective.

Proof:

The injectivity is already shown. So it is enough to prove that the vector $(\lambda;m_1,\ldots,m_n)$, generated by an incidence vector 1, is the minimal representation of some homogeneous game without dummies and steps. The associated sequence $S_1^1,\ldots,S_n^1$, the members of which must then be minimal winning coalitions, guarantees that there cannot be dummies or steps and that (λ,m) is a minimal representation, if it is a representation of a homogeneous game at all.

First of all, the order of the weights, i.e. $m_1 \geq \ldots \geq m_n$, is shown by induction on n-i:

$$m_n = 1 \leq m_i \ \text{for all} \ 1 \leq i \leq n.$$

Assume $m_i \geq \ldots \geq m_n$, then

$$m_{i-1} = m_{j+1} + \ldots + m_{1_i},$$

where

$$j = \min \{1_t \mid t \leq i-1, 1_t \geq i-1\}.$$

Two cases are distinguished:

1. $1_i > j \geq i$: Then $m_i = m_{j+1} + \ldots + m_{1_{i+1}}$ and $1_{i+1} \leq 1_i$

                                   (see Definition 2.8, (ii)), thus $m_{i-1} \geq m_i$.

2. $j = i-1$: Then $m_{i-1} = m_i + \ldots + m_{1_i} \geq m_i$.

Referring to Definition 2.8 (i) the case $j \geq 1_i$ cannot occur, thus the induction is finished.

The first part of this proof implies that $(\lambda;m)$ is a representation of some simple game (Ω,v). It remains to show the homogeneity of (Ω,v):

Let $|\{1_1,\ldots,1_n\}| =: r$ for some $2 \leq r \leq n-1$ and write

$$\{1_1,\ldots,1_n\} = \{1^1,\ldots,1^r\}$$

such that

$$1_1 = 1^1 < \ldots < 1^r = 1_n = n, \quad 1^0 := 0.$$

It is enough to show per induction on $0 \leq i \leq r-1$:

if $S \in W_*(\Omega,v)$ with $1^i < 1(S) \leq 1^{i+1}$, then $1(S) = 1^{i+1}$ and $m(S) = \lambda$.

For $i = 0$ the assertion is immediately implied, since S_1^1 is the lexicographically first minimal winning coalition in (Ω,v).

Let S_0 be a coalition in $\{S \in W_* \mid 1(S) = \min \{1(T) \mid T \in W_* \text{ and } 1(T) > 1^i\}\}$, such that $r := r(S_0)$ is maximal, where $r(S)$ is defined as in the proof of Theorem 2.3. Define $1_0 := 1(S_0)$. Observe that $1_0 \leq 1^{i+1}$.

It can be remarked that $r > 0$ since S_0 cannot be the lexicographically first coalition in W_*.

The inductive hypothesis implies

$$1(\varphi(S_0)) \leq 1^i, \quad m(\varphi(S_0)) = \lambda.$$

Let l_j be minimal with $l_j \geq r$, thus $r \in s_j^1$ - otherwise $l_j > 1(\varphi(s_j^1)) \geq r$. The minimality of l_j shows that

$$l_j \leq 1(\varphi(S_0)),$$

thus

$$\max \{1(S) \mid S \in W_*, \; 1(S) < 1(s_j^1)\} < r.$$

If $l_j = 1(\varphi(S_0))$ nothing remains to be shown, because of the definition of s_{r+1}^1.

Therefore, assume $l_j < 1(\varphi(S_0))$ and define

$$\tilde{S}_0 = s_j^1 \setminus \{r\} \cup \{1(\varphi(S_0)) +1,\ldots,1(S_0)\}.$$

With this notation

$$1(\tilde{S}_0) = 1(S_0) \text{ and } m(\tilde{S}_0) = m(S_0).$$

Since additionally

$$r < 1(\varphi(S_0)) \notin \tilde{S}_0,$$

this assumption contradicts the maximality of r.   q.e.d.

The proof of this theorem also implies the following

Corollary 2.11:

Let $(\Omega,v) \in H_n$ and $S \in W_*(\Omega,v)$.

Then there is a $j \in [1,n]$, not necessarily unique, such that

$$1(s_j^v) = 1(S).$$

Remark 2.12:

The identification L_n of homogeneous games and incidence vectors permits us to provide an upper bound for the number of these n-person games:

If $1 \in I_n$, then l_1 is determined by the other components of the vector, since Definition 2.8 (i), (ii) guarantees that

$$l_2 \geq \ldots \geq l_{l_1+1} < l_{l_1+2},$$

showing that

$$l_1 = \max \{t \mid 2 \leq t \leq n, \; l_2 \geq \ldots \geq l_t\} -1.$$

Additionally the just mentioned definition implies

$$3 \leq l_2 \leq n, \quad k \leq l_k \leq n \text{ for all } k \in [3,n].$$

Therefore l_2 can run through at most n-2 values and l_k can run through at most n-k+1 values. This implies

$$| \mathbf{H}_n | \leq (n-2)! \ (n-2) < (n-1)!$$

So far the number (n-1)! was the smallest known upper bound for the cardinality of the set of homogeneous n-person zero-sum games (see [3]). Clearly this set is by comparison a very small subset of the considered class of simple games $\mathbf{H}_n$;

hence it would seem that the preceding result is certainly an improvement. However, in the next chapter it will turn out, that we can achieve much more: We will construct an explicit recursive formula for the number of incidence vectors.

§3 Geometrical Description of Incidence Vectors as Anti Step Functions, Providing a Recursive Formula for the Number of Homogeneous Games

It is the aim of this chapter to enumerate the homogeneous games without dummies and steps recursively w.r.t. the number of players. This will be done by partitioning the corresponding class of n-person incidence vectors into certain subsets which will be defined later on.

We assume $n \geq 3$, unless otherwise specified.

Definition 3.1:

A function $f : [0, n-1] \longrightarrow [0, n-1]$ is a <u>quadratic</u> n-<u>person step function</u>, iff

(i) $f(0) = 0$,

(ii) there is a natural number r such that there are $k_i \in \mathbb{N} \cup \{0\}$ for all integers $i \in [0,r]$ with $0 = k_0 < k_1 < ... < k_r = n-1$, $k_1 \geq 2$ and

$f(x) = k_i$ for all $x \in (k_{i-1}, k_i]$, if $i \in [1,r]_n$.

The denotation "quadratic" reflects the fact that each step of a quadratic step function is as high as long.

Since a vector $(k_1,...,k_r)$, $k_i \in \mathbb{N}$ for all i, $1 \leq i \leq r$ and $2 \leq k_1 < ... < k_r = n-1$, uniquely determines a quadratic n-person step function and vice versa (see Definition 3.1 (ii)), this function is often identified with its vector of values, the zero being sometimes deleted.

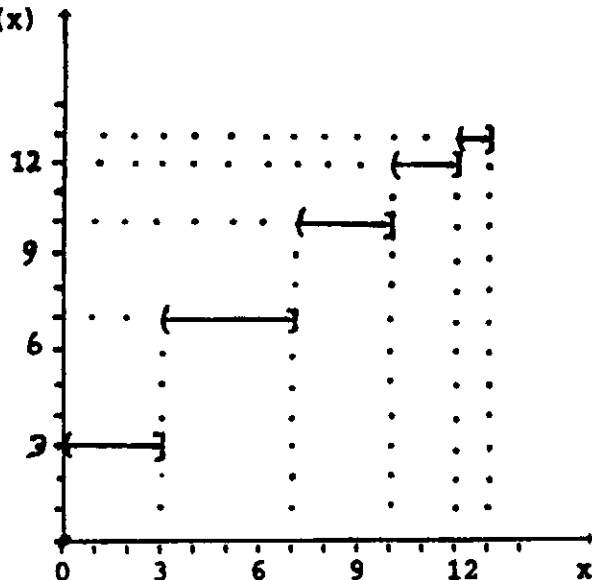The following sketch provides an illustration of the quadratic 14-person step function $(3,7,10,12,13)$:

Figure 1

Definition 3.2:

A function h : $[0, n-1] \longrightarrow (0,n]$ is called <u>n-person anti step function</u>, iff there is a quadratic n-person step function f with the vector of values $(0, k_1, \ldots, k_r)$ such that

(i) $h([0, n-1]) = \{k_i \mid 1 \leq i \leq r\} \cup \{n\}$,

(ii)   h, restricted to the left open and right closed interval $(j, j+1]$, is constant, formally $h_{|(j,j+1]} = $ const for all $0 \leq j \leq n-2$,

(iii)   h, restricted to sections where f is constant, is - not necessarily strictly - decreasing, i.e.

$h_{|(k_i, k_{i+1}]}$ is not increasing for all i, $0 \leq i \leq r - 1$,

(iv) $h(x) > f(x)$ for all $x \in [0, n-1]$.

Let us introduce the following notations:

$$Q_n := \{f \mid f \text{ is a quadratic n-person step function}\}$$

$$A_n := \{h \mid h \text{ is an n-person anti step function}\}.$$

It is obvious from the last definition, part (i), that the anti step function h uniquely determines a quadratic step function f, so we define for each quadratic step function f:

$$A(f) := \begin{bmatrix} h \mid h \text{ is an anti step function which determines } f \text{ by} \\ \text{Definition 3.2} \end{bmatrix}.$$

An n-person anti step function h is often identified with its vector of values $(h(0), \ldots, h(n-1))$.

Note that A(f) is nonvoid since

$$(k_1, k_2, \ldots, k_2, k_3, \ldots, k_3, \ldots, k_r, \ldots, k_r, n, \ldots, n)$$
$$\underbrace{\hphantom{k_1 \text{ times}}}_{k_1 \text{ times}} \underbrace{\hphantom{(k_2-k_1)}}_{(k_2-k_1) \text{ times}} \underbrace{\hphantom{(k_{r-1})}}_{\substack{(k_{r-1}-k_{r-2}) \\ \text{times}}} \underbrace{\hphantom{(k_r)}}_{(k_r-k_{r-1}) \text{ times}}$$

is the vector of values of an anti step function, which is a member of A(f), where $(0, k_1, \ldots, k_r)$ is the vector of values of f.

Example 3.3:

The vector $(3,14,14,7,13,12,10,10,14,14,13,13,13,14) = (h(0),\ldots,h(13))$ is an anti step function in $A(f)$, where f is the above defined quadratic 14-person step function $(3,7,10,12,13)$.

Figure 2 illustrates the graphs of h and f.
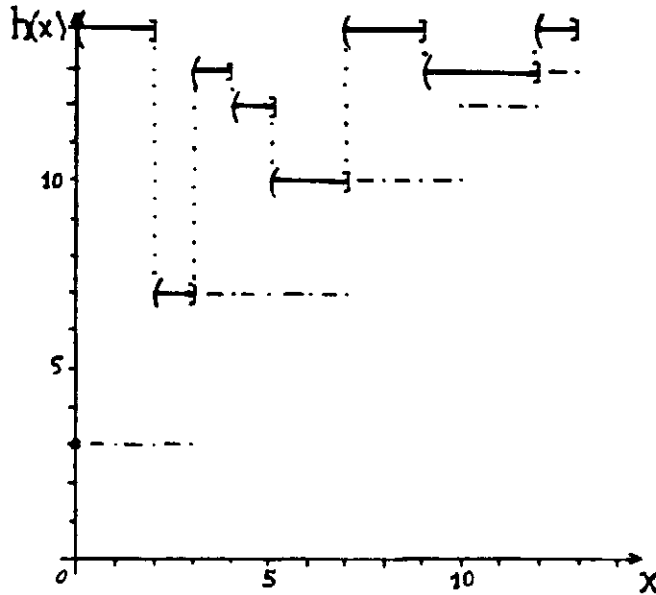
The lines "·—·—·—·" represent the graph of f.

Figure 2

Remark 3.4:

$$| Q_n | = 2^{n-3}.$$

This assertion is verified by induction on n:

For $n = 3$ there is only one quadratic step function with the vector of values $(0,2)$. This is a direct consequence of Definition 3.1.

From a quadratic n-person step function f resp. $k = (k_1,\ldots,k_r)$ two quadratic (n+1)-person step functions are constructed, namely

$$f^1 \text{ defined by } k^1 = (k_1,\ldots,k_r,\ n)$$

and

$$f^2 \text{ defined by } k^2 = (k_1,\ldots,k_{r-1},\ n).$$

The maps $k \longmapsto k^1$ and $k \longmapsto k^2$ reps. $f \longmapsto f^1$ and $f \longmapsto f^2$ are injective and have disjoint images. It is obvious that the union of these images contains all quadratic (n+1)-person step functions, thus the proof is finished.

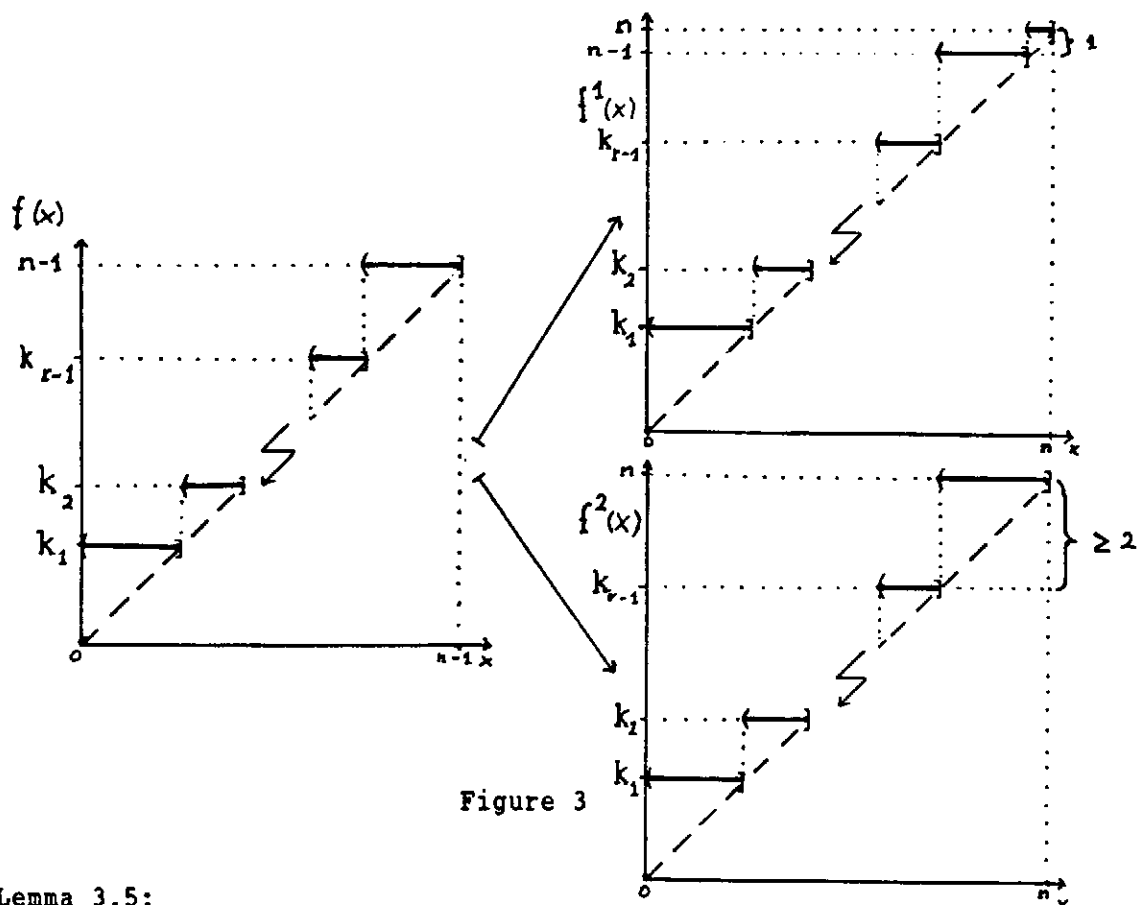A sketch of the preceding mappings is given in the following illustration:

Figure 3

Lemma 3.5:

(i)   If $h \in A_n$, then $(h(0),\ldots,h(n-1)) \in I_n$.

(ii)   The mapping

$$A_n \longrightarrow I_n$$

$$h \longmapsto (h(0),\ldots,h(n-1))$$

identifies the n-person anti step functions and the n-person incidence vectors, i.e. is bijective.

Proof:

Let h be a member of A_n, say $h \in A(f)$ for some $f \in Q_n$ with values $0 < k_1 < \ldots < k_r = n-1$. Then

(i) $(h(0),\ldots,h(n-1)) \in \{2,\ldots,n\}^n$,

(ii) $h(i) > \min \{h(j) \mid j \in [0,i-1]_n, \, h(j) \geq i-1\}$ and

(iii) $h(i+1) \leq h(i)$, if $\min \{h(j) \mid j \in [0,i-1]_n, \, h(j) \geq i-1\} \geq i$.

Assertion (i), (ii) resp. (iii) is a consequence of Definition 3.2 (i) and $k_1 \geq 2$, Definition 3.2 (iv) resp. Definition 3.2 (iii).

Thus $(h(0),\ldots,h(n-1))$ is an incidence vector.

If $1 = (1_1,\ldots,1_n)$ is an incidence vector, then the inverse arguments to (i) - (iii) show that this vector is the vector of values of some anti step function.

q.e.d.

In view of the last lemma incidence vectors will often be identified by anti step functions. If $h \in A_n$, we define $\bar{1}(h) := (h(0),\ldots,h(n-1))$. In order to enumerate the homogeneous games the set of anti step functions will be decomposed into subsets, the members of which having a common property called type. To begin with some notation will be needed.

Definition 3.6:

Let $h \in A_n$. Then $h \in A(f)$ for some $f \in Q_n$ with values $0 < k,<\ldots<k_r= n-1$. Define

$$\alpha_i := |\{j \in \mathbb{N} \mid j \in (k_{i-1},k_i] \text{ and } h(j) = n\}|.$$

Then there is a chain

$$i_1 <\ldots< i_t$$

for some $t \in \mathbb{N}$ such that

$$\{i_1,\ldots,i_t\} = \{i \in \mathbb{N} \mid \alpha_i \neq 0 \text{ and } i \in [1,r]\}.$$

Since $\alpha_r \neq 0$, this last set must be nonvoid. The vector $(\alpha_{i_1},\ldots,\alpha_{i_t})$ is called ceiling of h.

With this notation $(2,2,1)$ is the ceiling of the anti step function h, given in Example 3.3, since $(2,0,2,0,1) = (\alpha_1, \alpha_2,\ldots,\alpha_5)$.

Lemma 3.7:

If $\beta = (\beta_1,\ldots,\beta_t)$ is the ceiling of some n-person anti step function and $\bar{\beta} = (\bar{\beta}_1,\ldots,\bar{\beta}_t) \in \mathbb{N}^t$ is a vector such that $\sum_{i=1}^{t} \beta_i = \sum_{i=1}^{t} \bar{\beta}_i$, then there is a canonical bijection from

$$\{h \in A_n \mid \text{the ceiling of h is } \beta\}$$

to
$$\{h \in A_n \mid \text{the ceiling of } h \text{ is } \tilde{\beta}\}.$$

Proof:

Let h be an anti step function of ceiling β, let us say $h \in A(f)$ for some f with values $0 < k_1 < \ldots < k_r = n-1$ and let $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $(\alpha_{i_1}, \ldots, \alpha_{i_t})$ be defined according to Definition 3.6, implying

$$\beta = (\alpha_{i_1}, \ldots, \alpha_{i_t}).$$

We put

$$\tilde{\alpha}_k := \begin{cases} \tilde{\beta}_j, & \text{if } k = i_j \text{ for some } 1 \le j \le t \\ 0, & \text{otherwise} \end{cases} \quad ; \quad 1 \le k \le r$$

and

$$\tilde{k}_j := k_j - \alpha_j - \alpha_{j-1} - \ldots - \alpha_1 + \tilde{\alpha}_j + \ldots + \tilde{\alpha}_1, \quad 1 \le j \le r.$$

Conclude that

$$\tilde{k}_i - \tilde{k}_{i-1} = k_i - k_{i-1} + \tilde{\alpha}_i - \alpha_i \ge \tilde{\alpha}_i$$

and thus

$$\tilde{k}_i - \tilde{k}_{i-1} - \tilde{\alpha}_i = k_i - k_{i-1} - \alpha_i.$$

Observe that $h(i) = n$ is equivalent to $i \in [k_j+1, \ k_j+\alpha_{j+1}]_n$ for some $j \in [0, r-1]_n$ and define analogously

$$\tilde{h}(i) := \begin{cases} n, & \text{if } i \in [\tilde{k}_j+1, \ \tilde{k}_j+\tilde{\alpha}_{j+1}]_n \text{ for some } 0 \le j \le r-1 \\ \tilde{k}_l, & \text{if } i = \tilde{k}_j+\tilde{\alpha}_{j+1}+s \le \tilde{k}_{j+1} \text{ for some } j \text{ and } s, \text{ such that } h(k_j+\alpha_{j+1}+s) = k_l. \end{cases}$$

A simple computation shows that $\tilde{h}$ is an anti step function of ceiling $\tilde{\beta}$ and $\tilde{h} \in A(\tilde{f})$, where $\tilde{f}$ is the quadratic step function with values $0 < \tilde{k}_1 < \ldots < \tilde{k}_r = n-1$.

The inverse mapping can be defined analogously by interchanging the rôles of $\tilde{\beta}$ and β. $\hspace{4cm}$ q.e.d.

The following example graphically represents the preceding canonical bijection:
Let
$$\beta = (2,2,1), \quad \tilde{\beta} = (1,1,3)$$
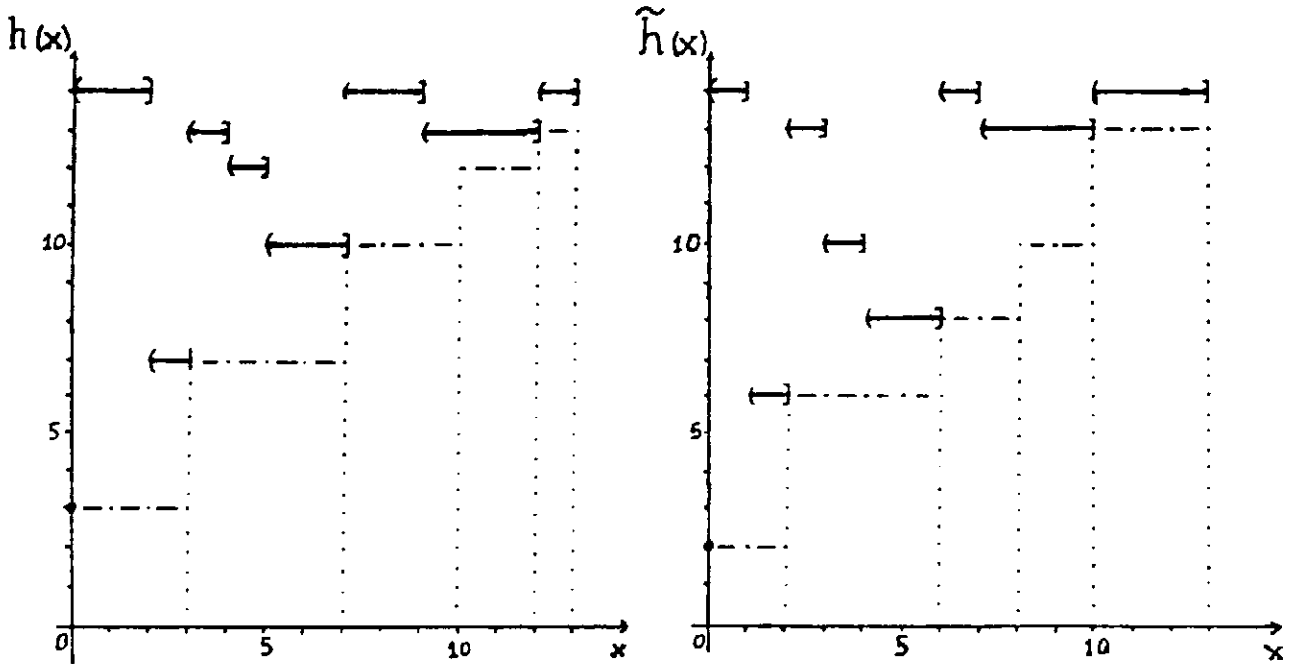
and h be the anti step function given in Example 3.3.

With the notation used in the above proof we get

$$(\alpha_1, \ldots, \alpha_r) = (2,0,2,0,1), \quad (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r) = (1,0,1,0,3)$$

and

$$(\check{k}_1, \ldots, \check{k}_r) = (2,6,8,10,13).$$

Figure 4 illustrates the anti step function h and $\check{h}$.

The lines "·—·—·" represent the underlying quadratic step functions.

Figure 4

Definition 3.8:

Let h be a member of A_n with ceiling $(\beta_1, \ldots, \beta_t)$.

Then (t,p) is called type of h, if

$$p = \sum_{i=1}^{t} \beta_i - t.$$

The subset of A_n, whose elements are of type (k,p), is denoted by $A_n^{k,p}$ and, in addition, the cardinality of this set is abbreviated by $a_n^{k,p}$, formally:

$$A_n^{k,p} = \{h \in A_n \mid h \text{ is of type } (k,p)\}, \quad a_n^{k,p} = |A_n^{k,p}|.$$

Note that if $a_n^{k,p} \neq 0$, then $k \in \mathbb{N}$, $p \in \mathbb{N}_0$.

Additionally we introduce the notation

$$I_n^{k,p} := \{\bar{1}(h) \mid h \in A_n^{k,p}\}$$

and observe that $|I_n^{k,p}| = a_n^{k,p}$.

The rest of this chapter will be used to give a recursive description of the cardinalities $a_n^{k,p}$ and starts with the following important

Theorem 3.9:

Let $n \geq 3$. Then the following assertions are valid.

(i) $a_{n+1}^{1,0} = \sum\limits_{k \geq 1} \sum\limits_{p \geq 0} a_n^{k,p}$;

(ii) $a_{n+1}^{k,0} = \sum\limits_{\tilde{k} \geq k-1} \binom{\tilde{k}}{k-1} \sum\limits_{p \geq 0} a_n^{\tilde{k},p} - a_n^{k-1,0}$, if $k \geq 2$;

(iii) $a_{n+1}^{k,p} = a_n^{k,p-1} \cdot \dfrac{p+k-1}{p}$, if $p \geq 1$.

Proof:

The canonical identification of anti step functions with incidence vectors (see Lemma 3.5) simplifies the arguments.

ad (i): If h is a member of $\bigcup\limits_{k \geq 1} \bigcup\limits_{p \geq 0} A_n^{k,p}$, then

$(h(0),\ldots,h(n-1),\ n+1) \in I_{n+1}^{1,0}.$

Conversely, if $\tilde{h} \in A_{n+1}^{1,0}$, then

$(h(0),\ldots,h(n-1)) \in \bigcup\limits_{k \geq 1} \bigcup\limits_{p \geq 0} I_n^{k,p}.$

These considerations induce a function and its inverse and thus verify assertion (i).

ad (ii): If $h \in A_{n+1}^{k,0}$, then h has the ceiling $(\underbrace{1,\ldots,1}_{k \text{ times}})$.

Put $1 := \bar{1}^{-1}(h)$. If $k_1 < \ldots < k_r = n$ are the values of this anti step function (exept n+1), then $k_{r-1} = n-1$ and the vector $(0,k_1,\ldots,k_r)$ defines a quadratic

step function f, such that $h \in A(f)$. Define

$$\tilde{h}(x) := \min \{h(x), n\} \text{ for all } x \in [0,n-1].$$

Let $\tilde{f}$ be the quadratic step function defined by the vector of values $(0,k_1,\ldots,k_{r-1})$ and observe that

$$\tilde{h} \in A(\tilde{f}).$$

If $(\beta_1,\ldots,\beta_t)$ is the ceiling of $\tilde{h}$, then it is clear that

$$\sum_{i=1}^{t} \beta_i \geq k-1 \text{ and } t \geq k-1,$$

since $\quad \{i \leq n \mid h(i) = n+1\} \subseteq \{i \leq n \mid \tilde{h}(i) = n\}.$

As there is at least one player i such that $h(i) = n = \tilde{h}(i)$, the case $\sum_{i=1}^{t} \beta_i = k-1 = t$ cannot occur. Put $\tilde{l} := \bar{l}^{-1}(\tilde{h})$.

For the converse let $l = (l_1,\ldots,l_n)$ define a member of $A_n^{\tilde{k},p}$, i.e. $\bar{l}(h)=l$, of some ceiling $(\beta_1,\ldots,\beta_{\tilde{k}})$ such that

$$p + \tilde{k} \geq k, \quad \tilde{k} \geq k-1.$$

Let $0 < k_1 < \ldots < k_r = n-1$ define the quadratic step function f such that $h \in A(f)$ and put

$$\alpha_i = |\{j \in \mathbb{N} \mid l_j = n \text{ and } k_{i-1} < j-1 \leq k_i\}|$$

(the vector $(\alpha_1,\ldots,\alpha_r)$ has already been constructed in Definition 3.6).

Following this definition there is an increasing subsequence $(i_1,\ldots,i_{\tilde{k}})$ of $(1,\ldots,r)$ such that

$$(\beta_1,\ldots,\beta_{\tilde{k}}) = (\alpha_{i_1},\ldots,\alpha_{i_{\tilde{k}}}).$$

Let T be an arbitrary subset of $\{1,\ldots,\tilde{k}\}$ of cardinality k-1.
Define
$$l_i^T := \begin{cases} n+1, & \text{if } i=k_{i_s-1}+1 \text{ for some } s \in T \text{ or } i=n+1 \\ l_i, & \text{otherwise} \end{cases}$$

To verify that $l^T = (l_1^T,\ldots,l_{n+1}^T)$ is an incidence vector in $I_{n+1}^{k,0}$ is straightforward and therefore skipped here.

Note that the case $\tilde{k} = k-1$ and $p = 0$ must be excluded, since then T automatically coincides with $\{1,\ldots,k-1\}$ and 1^T cannot be an incidence vector ($1^T_{n+1} = n+1 = \min \{1^T_j \mid 1 \leq j \leq n, 1^T_j \geq n\}$, thus condition (i) of Definition 2.8 is violated).

With the above notation the following assertions are valid:

(i) $1^T = 1$,

(ii) For each 1 with $\bar{1}^{-1}(1) \in A^{k,0}_{n+1}$ there is a unique $T \subseteq \{1,\ldots,\tilde{k}\}$ of cardinality $k-1$ such that $\hat{1}^T = 1$ and $\tilde{1} \in I^{\tilde{k},p}_n$.

From combinatorics it is known that the binomial coefficient $\binom{\tilde{k}}{k-1}$ describes the cardinality of the set of subsets of $\{1,\ldots,\tilde{k}\}$ containing $k-1$ elements, thus assertion (ii) of the theorem is shown.

Figure 5 illustrates per example how the map $1 \longmapsto 1^T$ works.

ad (iii): The mapping
$$(\beta_1,\ldots,\beta_k) \longmapsto \{\beta_1, \beta_1+\beta_2, \ldots, \beta_1+\ldots+\beta_{k-1}\}$$
yields a bijection from
$$\{(\beta_1,\ldots,\beta_k) \mid \sum_{i=1}^{k} \beta_i - k = p, \beta_i \in \mathbb{N}\} = T^{k,p}$$
to
$$\{M \subseteq \{1,\ldots,k+p-1\} \mid |M| = k-1\}$$
for each $k \in \mathbb{N}$, $p \in \mathbb{N}$,
thus
$$|T^{k,p}| = \binom{k+p-1}{k-1}.$$
Analogously it can be shown:
$$|\{(\beta_1,\ldots,\beta_k) \in T^{k,p} \mid \beta_k \geq 2\}| = \binom{k+p-2}{k-1}$$

Let 1 define a member of $A^{k,p}_{n+1}$ with a ceiling $(\beta_1,\ldots,\beta_k)$ such that $\beta_k \geq 2$. Then define for each $1 \leq i \leq n$

$$\tilde{1}_i = \begin{cases} 1_i, & \text{if } 1_i \leq n-2 \\ n-1, & \text{if } 1_i = n \\ n, & \text{if } 1_i = n+1 \end{cases}$$

The case $l_i = n-1$ cannot occur since $\beta_k \geq 2$ directly implies $\{i \leq n+1 \mid l_i = n-1\} = \emptyset$. It is clear (see Definition 2.8) that $\tilde{l} = (\tilde{l}_1, \ldots, \tilde{l}_n)$ is an incidence vector, thus $\tilde{l} \in I_n^{k,p-1}$.

Furthermore the function

$$\{l \in I_{n+1}^{k,r} \mid l \text{ has the ceiling } (\beta_1, \ldots, \beta_k) \text{ with } \beta_k \geq 2\} \longrightarrow I_n^{k,p-1}$$
$$l \longmapsto \tilde{l}$$

is bijective, because it is obvious how to define the inverse mapping.

Combining the above observations and definitions, and using Lemma 3.7, we get:
Fix a ceiling $\beta = (\beta_1, \ldots, \beta_k)$ such that $\sum\limits_{i=1}^{k} \beta_i - k = p \geq 1$, then

$$\left|\{h \in A_{n+1}^{k,p} \mid h \text{ has the ceiling } \beta\}\right| \cdot \binom{k+p-1}{k-1} = a_{n+1}^{k,p}$$

$$\left|\{h \in A_{n+1}^{k,p} \mid h \text{ has the ceiling } \beta\}\right| \cdot \binom{k+p-2}{k-1} =$$

$$\left|\{h \in A_{n+1}^{k,p} \mid h \text{ has the ceiling } (\tilde{\beta}_1, \ldots, \tilde{\beta}_k) \text{ with } \tilde{\beta}_k \geq 2\}\right| = a_n^{k,p-1}.$$

Consequently

$$a_{n+1}^{k,p} = a_n^{k,p-1} \cdot \frac{\binom{k+p-1}{k-1}}{\binom{k+p-2}{k-1}} = a_n^{k,p-1} \cdot \frac{p+k-1}{p}. \qquad \text{q.e.d.}$$

The following example illustrates the maps $l \longmapsto l^T$ constructed in part (ii) of the preceding proof.

Example 3.10:

Let $n = 7$, $k = 3$, and $l = (3,7,7,5,7,6,7)$. Then $l \in I_7^{3,1}$, since $(2,1,1)$ is the ceiling of l. There are exactly 3 subsets of $\{1,2,3\}$ with two elements, namely $T_1 = \{1,2\}$, $T_2 = \{1,3\}$, $T_3 = \{2,3\}$. So we have to construct l^{T_i}, $i = 1,2,3$.

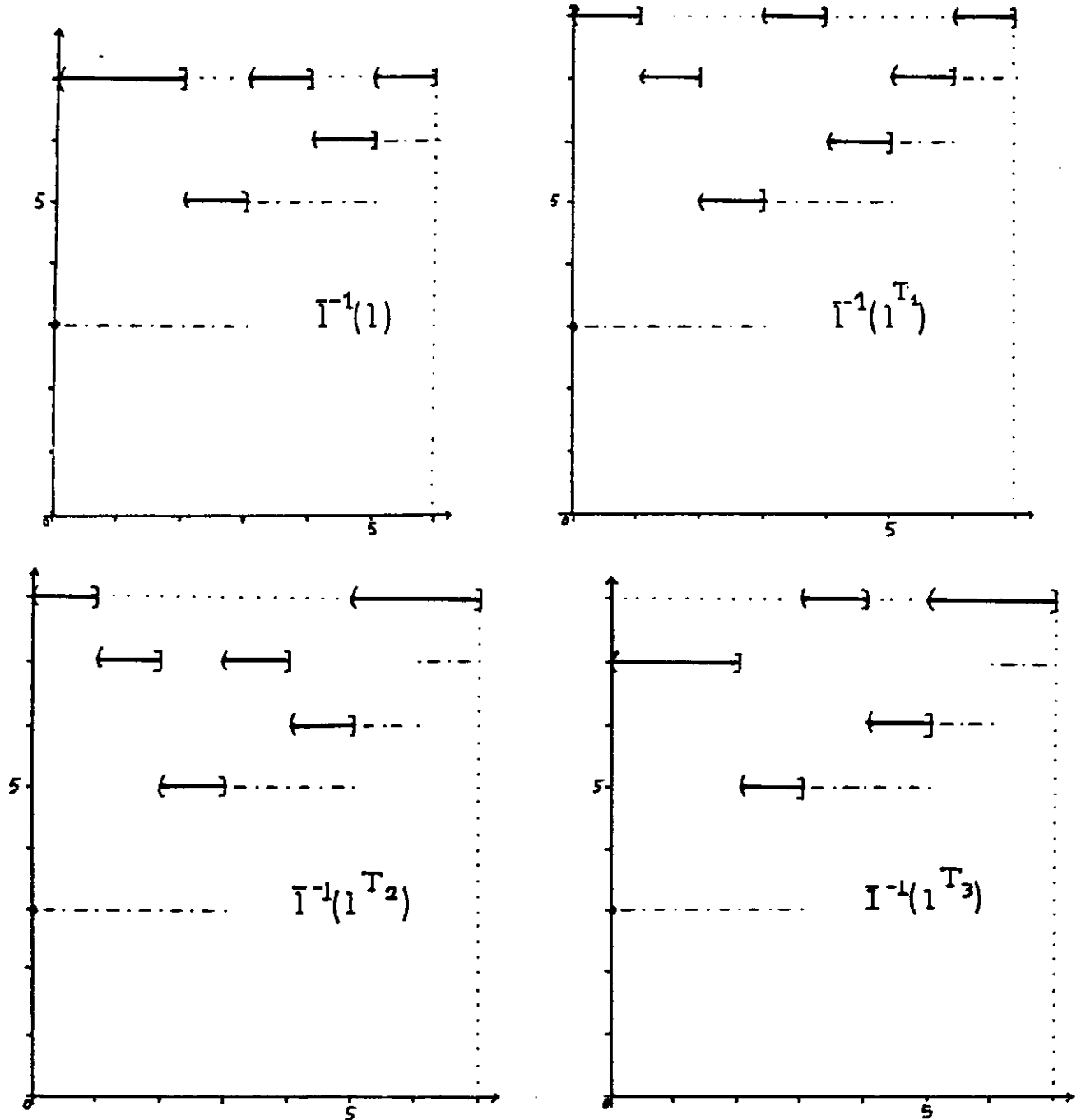Since $\tilde{l}^{-1}(l) \in A(f)$, where f is the quadratic step function, defined by the values $(0,3,5,6)$, we get:

$$1^{T_1} = (3,8,7,5,8,6,7,8),$$

$$1^{T_2} = (3,8,7,5,7,6,8,8)$$

$$1^{T_3} = (3,7,7,5,8,6,8,8).$$

These incidence vectors are sketched in the following diagramms (as graphs of the corresponding anti step functions).

The lines "·—·—·" represent the underlying quadratic step functions

Figure 5

<u>Corollary 3.11:</u>

(i) $a_n^{1,n-3} = 0$

(ii)   Let $(k,p) \neq (1,n-3)$. Then $a_n^{k,p} \neq 0$ if and only if the following

holds true:

(a) $k \leq [\frac{n}{2}]$

(b) $p + 2k \leq n$.

(c) $(k,p) \in \mathbb{N} \times \mathbb{N} \cup \{0\}$.

These assertions can be verified by induction on n:
      Since $(2,3,3)$ is the only 3-person incidence vector, the corollary is
valid in the case $n = 3$. The last theorem directly completes the proof.

Let us introduce the following notation: $D_n = \{(k,p) \mid a_n^{k,p} \neq 0\}$.

The next assertion is a direct consequence of the last corollary.

<u>Corollary 3.12:</u>

$$|D_n| = \begin{cases} \dfrac{n^2-5}{4}, & \text{if } n \text{ is odd} \\ \\ \dfrac{n^2-4}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Since the number of homogeneous n-person games without dummies and steps equals
$a_{n+1}^{1,0}$ by Theorem 3.9 (i), it is very useful to eliminate the $a_n^{k,p}$, $p > 0$. Define

$$a_n^k := a_n^{k,0}$$

and for technical reasons

$$a_2^1 := 1.$$

With these notations the following recursive formulae are valid.

<u>Theorem 3.13:</u>

(i) $$a_{n+1}^1 = \sum_{k=1}^{[\frac{n}{2}]} \sum_{p=0}^{n-2k} \binom{k+p-1}{p} a_{n-p}^k$$

$$= \sum_{p=0}^{n-2} \sum_{k=1}^{[\frac{n-p}{2}]} \binom{k+p-1}{p} a_{n-p}^k$$

(ii) $\quad a_{n+1}^k = \sum_{\tilde{k}=k}^{[\frac{n}{2}]} \binom{\tilde{k}}{k-1} a_n^{\tilde{k}} + \sum_{\tilde{k}=k-1}^{[\frac{n-1}{2}]} \binom{\tilde{k}}{k-1} \sum_{p=1}^{n-2\tilde{k}} a_{n-p}^{\tilde{k}} \binom{\tilde{k}+p-1}{p}$

$\qquad\qquad = \sum_{p=0}^{n-2k+2} \left[\sum_{\tilde{k}=k-1}^{[\frac{n-p}{2}]} \binom{\tilde{k}}{k-1} \binom{\tilde{k}+p-1}{p} a_{n-p}^{\tilde{k}} \right] - a_n^{k-1},$

$\qquad$ if $2 \leq k \leq [\frac{n+1}{2}]$.

The proof of this theorem will be postponed, as we shall have to apply the following formula which describes $a_n^{k,p}$ in terms of some $a_{\tilde{n}}^k$.

Lemma 3.14:

$$a_n^{k,p} = a_{n-p}^k \binom{k+p-1}{p} \text{ for all } k \leq [\frac{n}{2}], \ p \leq n-2.$$

Proof (by induction on n):

In the case $p = 0$ nothing remains to be shown. Since $(2,3,3)$ is the only 3-person incidence vector, we have

$$a_3^{1,1} = 1 = a_2^1.$$

If $p \geq 1$, then by Theorem 3.9 (iii):

$$a_{n+1}^{k,p} = a_n^{k,p-1} (\frac{p+k-1}{p})$$

$$\qquad = a_{n-p+1}^{k,0} \binom{k+p-2}{p-1} (\frac{p+k-1}{p}) \text{ (by inductive hypothesis)}$$

$$\qquad = a_{n+1-p}^{k,0} \binom{k+p-1}{p}. \qquad\qquad\qquad\qquad \text{q.e.d.}$$

We now proceed by proving the theorem:
It is straightforward - by interchanging the summation indices - that the second equality of (i) resp. (ii) holds.
The first equalities are shown by induction on n:
by Theorem 3.9 (i) resp. (ii) we get

$$a_4^1 = 1 \text{ resp. } a_4^2 = 1$$

which coincides with

$$\sum_{k=1}^{1} \sum_{p=0}^{1} \binom{k+p-1}{p} a_{3-p}^{k} \quad \text{resp.} \quad \sum_{k=2}^{1} \binom{k}{1} a_{3}^{k} + \sum_{k=1}^{1} \binom{k}{1} \sum_{p=1}^{1} \binom{k+p-1}{p} a_{3-p}^{k}.$$

Assume the validity of the assertions for some n.

Then this assumption, Lemma 3.14 and Theorem 3.9 (i) resp. (ii) imply

$$a_{n+2}^{1} = \sum_{k=1}^{[\frac{n+1}{2}]} \sum_{p=0}^{n+1-2k} a_{n+1}^{k,p} = \sum_{k=1}^{[\frac{n+1}{2}]} \sum_{p=0}^{n-2k} a_{n+1-p}^{k} \binom{k+p-1}{p}$$

resp.

$$a_{n+2}^{1} = \sum_{\breve{k}=k-1}^{[\frac{n+1}{2}]} \binom{\breve{k}}{k-1} \sum_{p=0}^{n+1-2\breve{k}} a_{n+1}^{\breve{k},p} - a_{n+1}^{k-1}$$

$$= \sum_{\breve{k}=k}^{[\frac{n+1}{2}]} \binom{\breve{k}}{k-1} a_{n+1}^{\breve{k}} + \sum_{\breve{k}=k-1}^{[\frac{n}{2}]} \binom{\breve{k}}{k-1} \sum_{p=1}^{n+1-2\breve{k}} a_{n+1-p}^{\breve{k}} \binom{\breve{k}+p-1}{p},$$

if $k \geq 2$.   q.e.d.

A sketch of the recursive development will be given in the following figures. We restrict our attention to the case that n is even and k is at least two. The other three cases can be treated analogously. We presume that all a_{t}^{j}, $t \leq n-1$, are already known.

The element in the j-th row and 1-th column of Figure 6, which is often deleted for clearness reasons, shall represent a_{j}^{1}.

Figure 7 is Pascal's triangle, rotated to the left by $1\frac{1}{2}$ right angles. Thus the element in the j-th row and 1-th column shall represent $\binom{j+1}{j}$.

The marked areas of Figures 7 and 6 cover each other and the distances of the vertical axes are equal. The binomial coefficients of the first column of the marked area of Fig.7 have to be multiplied with the first element in the weak marked (k-1)-st row, the second column with the second element and so on.

The number a_{n}^{k} will be obtained by summing up all the products of the coefficients in the marked areas of Fig.6 and the modified coefficients of the marked area of Fig.7 (elementwise).
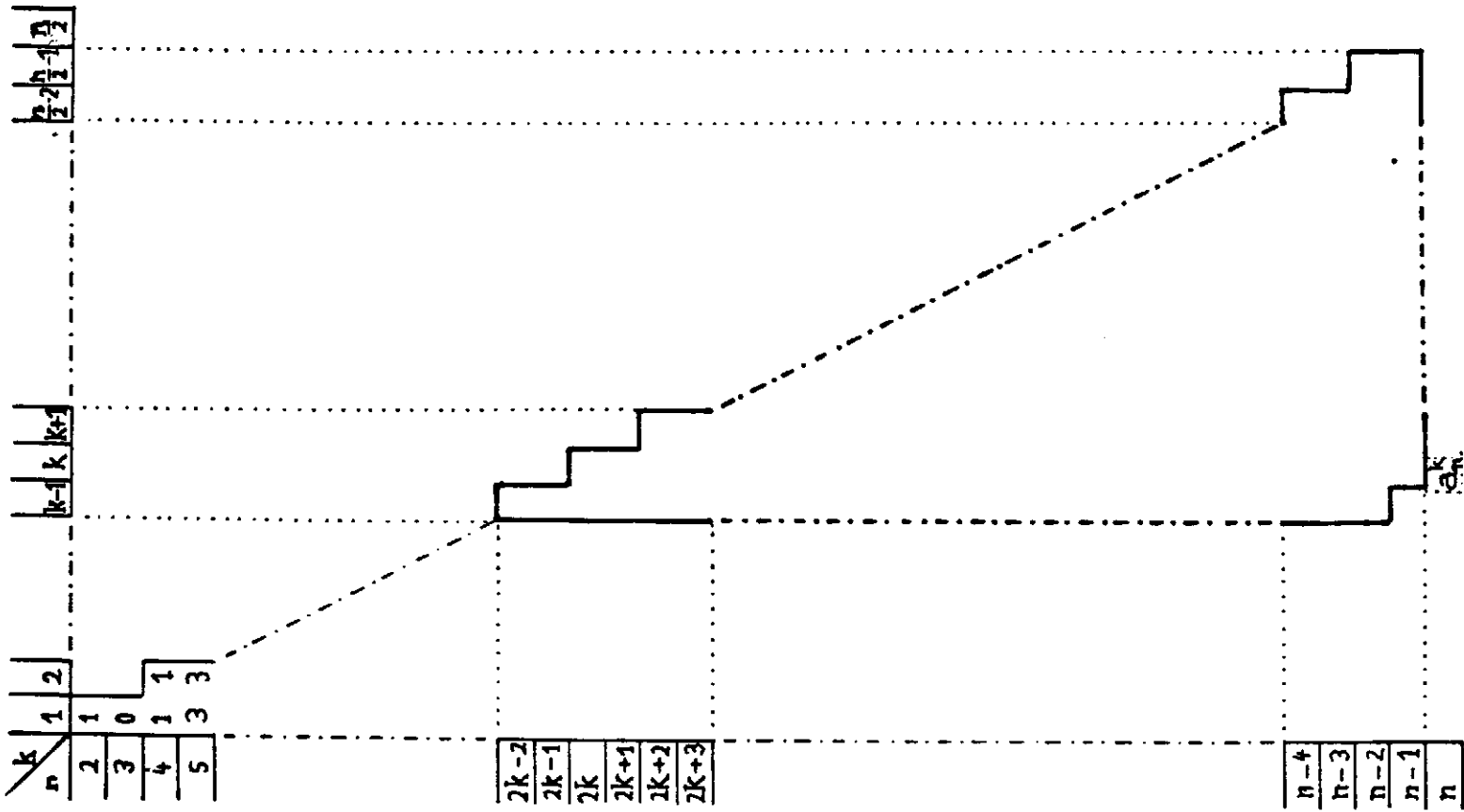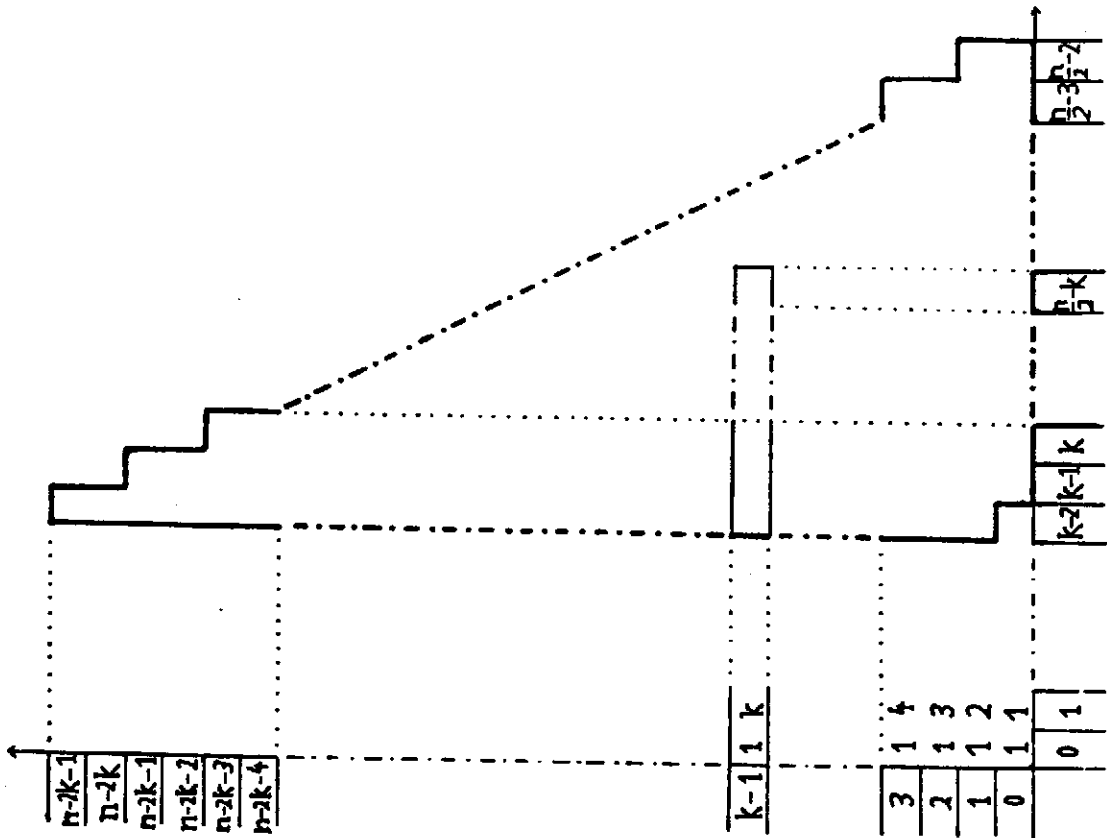
Figure 6

Figure 7

§4 Test of Homogeneity

In this chapter an algorithm is constructed which enables us to decide whether a directed game is homogeneous or not.

In the following the homogeneity of a directed game (Ω,v) is tested.

Definition 4.1:

Let (Ω,v) be a directed n-person game without one-person winning coalitions and let S_1 be the lexicographically maximal minimal winning coalition. If $S_1,\ldots,S_k$ are already constructed, define

$$S_{k+1} = \begin{cases} \rho_k(S_i), & \text{if } i = \min\{1\leq j\leq k \mid l(S_j)=\min\{l(S_t)\mid 1\leq t\leq k, l(S_t)\geq k\}\}, \\ & \text{the preceding set is nonvoid and } \rho_k(S_i) \text{ exists} \\ \emptyset, & \text{otherwise} \end{cases}$$

$\Omega \setminus \bigcup\limits_{i=1}^{n} S_i$ is called set of pseudo-dummies. If $S_{i+1} = \emptyset$ and i is not a pseudo-dummy then it is called pseudo-step. Let $\{k_0+1,\ldots,n\}$ be the set of pseudo-dummies. Then $(l_1,\ldots,l_{k_0+1})$ is an incidence vector, where

$$l_j = \begin{cases} l(S_j), & \text{if } S_j \neq \emptyset \\ k_0+1, & \text{otherwise} \end{cases}$$

If (λ,m) is the minimal representation, generated by $(l_1,\ldots,l_{k_0+1})$ (for the expression "generated representation" Definition 2.8 is referred to), define

$$\iota(\Omega,v) = (\lambda; m_1,\ldots,m_{k_0}, \underbrace{0,\ldots,0}_{(n-k_0)\text{-times}}).$$

A straightforward consequence of this definition, Remark 2.9 and Theorem 2.3 is the following

Theorem 4.2:

Let (Ω,v) be a directed n-person game without one-person winning coalitions.

Then $\iota(\Omega,v)$ is the minimal representation of a homogeneous game, where exactly the pseudo-steps resp. pseudo-dummies of (Ω,v) are the steps resp.

dummies of $\iota(\Omega,v)$.

Additionally, if (Ω,v) is homogeneous, then $\iota(\Omega,v)$ is the minimal representation of the same game.

Proof:

The first part is obvious from Corollary 2.11 and Lemma 1.4. As the chain $S_1,\ldots,S_n$ constructed in the last definition, does not depend on the representation but only on the game it can be started with a minimal representation, which is itself automatically homogeneous, if the game is. Thus, the second part again is implied using the just mentioned assertions.   q.e.d.

Remark:

It is obvious how to generalize the preceding definitions and assertions to directed games containing one-person winning coalitions.

Corollary 4.3:

A directed game (Ω,v) is homogeneous, iff the incidence matrices of (Ω,v) and $\iota(\Omega,v)$ coincide.

A practicable, slightly modified test of homogeneity, which already has been implemented on a computer, is presented in what follows.

Let (Ω,v) be a directed game. A minimal winning coalition S is called shiftminimal, if $S \cup \{i+1\} \setminus \{i\}$ is losing for all players i such that $i \in S$, $i+1 \notin S$. (For this notation we refer to $[6]$). The matrix

$$X^* := X^*(\Omega,v) = \begin{bmatrix} \vdots \\ 1_S \\ \vdots \end{bmatrix}_{S \text{ shiftminimal}}$$

with lexicographically ordered rows is called shiftminimal matrix of (Ω,v). OSTMANN $[6]$ has shown that $X^*(\Omega,v)$ uniquely determines (Ω,v) and that neighboring players i, $i+1$ are of different type (for the definition of the term "type" we refer to section 1), iff there is a shiftminimal coalition S with $i \in S$, $i+1 \notin S$. The term "type" can easily be generalized to coalitions: S and S' have the same type - $S \sim S'$ -, iff there is a permutation Π of Ω such that $\Pi(S) = S'$ and $\Pi(i) \sim i$ for all $i \in \Omega$. With this definition it is obvious that all coalitions of one type are winning resp. minimal winning if only one does.

In the homogeneous case this notation trivially implies: each minimal winning coalition S corresponds to a unique shiftminimal coalition $\check{S} =: SH(S)$ satisfying $\check{S} \sim S$. In the general case $SH(S)$ is to be defined as the lexicographically last coalition such that $SH(S) \sim S$.

Lemma 4.4:

If (Ω, v) is a homogeneous n-person game, not necessarily without dummies and steps, and $S_1, \ldots, S_n$ are constructed according to Definition 4.1, then the following assertions are equivalent.

(i) $i \not\sim i+1$

(ii) There is a j, $1 \leq j \leq n$, such that $\{i, i+1\} \cap SH(S_j) = \{i\}$.

Proof:

We only have to show that (i) implies (ii).

Assume $i \not\sim i+1$. If $S_{i+1} = \rho_i(S_j)$ for some $j \leq i$, then $i \in SH(S_j)$ by definition. If $i+1 \notin SH(S_j)$, nothing remains to be shown. In the other case $i+1 \in S_j$, thus $S_{i+2} = \rho_{i+1}(S_j)$ (see Definition 2.8). Consequently $SH(S_{i+2}) \cap \{i, i+1\} = \{i\}$.

If $S_{i+1} = \emptyset$, two cases may occur:

1. None of the coalition $S_1, \ldots, S_i$ contains player i. Then i and thus i+1 are pseudo-dummies of (Ω, v). Since $\iota(\Omega, v)$ represents (Ω, v), both players are dummies and consequently of the same type, which contradicts assumption (i).

2. Player i is a pseudo-step of (Ω, v), thus a step. The fact that $\iota(\Omega, v) = (-\lambda; m_1, \ldots, m_{k_o}, 0, \ldots, 0)$ is the minimal representation of (Ω, v) implies $m_i > m_{i+1}$ (see section 1) and thus - by Definition 2.8 - the existence of t such that $i \in S_t$, $i+1 \notin S_t$. Then $SH(S_t)$ satisfies property (ii) by definition.   q.e.d.

Lemma 4.5:

Let X^* be the shiftminimal m × n matrix of a homogeneous game. Then the following assertions are valid:

(*)

(i) If $t < 1(X_1^*.)$, $t \notin X_1^*.$, then $t \sim 1(X_1^*.)$

(ii) If $t_0 := \min \{j \mid j \in X_i^*., j \notin X_{i+1}^*.\} < t < 1(X_{i+1}^*.)$,

$t \notin X_{i+1}^*.$, then $t \sim 1(X_{i+1}^*.)$, for all $1 \leq i < m$.

Proof:

ad (i): The fact that $X_1^*. = SH(S)$, where S is the lexicographically maximal minimal winning coalition, directly implies (i).

ad (ii): Again $X_{i+1}^*. = SH(\rho_{t_0}(X_i^*.))$ directly implies (ii).   q.e.d.

Definition 4.6:

Let X^* be a shiftminimal matrix of a directed n-person game (Ω, v), satisfying condition (*). Let $i_1 < \ldots < i_r = n$ be the last players of the different types of the game and $i_0 = 0$. Let S_1^{Tv} be the first row of X^*. If $S_1^{Tv}, \ldots, S_k^{Tv}$ are already constructed and $k < r$, define

$$S_{k+1}^{Tv} = \begin{cases} SH(\rho_{i_k}(S_j^{Tv})), & \text{if } j=\min\{j \leq i_k \mid j \in \{t \in \{t \leq k \mid 1(S_k^{Tv}) \geq i_k\} \mid |[i_k+1,n] \cap S_t^{Tv}| \text{ minimal}\}\} \\ & \text{the preceding set is nonvoid and } \rho_k(S_j^{Tv}) \text{ exists} \\ \emptyset, & \text{otherwise} \end{cases}$$

Analogously to Definition 2.8 (3) the matrix

$$\bar{S}^{Tv} = \begin{bmatrix} S_1^{Tv} \\ \vdots \\ S_r^{Tv} \end{bmatrix}$$

generates a vector (λ, m) via

$m_{k_0+1} = \ldots = m_n = 0$, where $k_0 = \max \{1(S_i^{Tv}) \mid 1 \leq i \leq r\}$, let us say

$k_0 = i_{t_0}$; $m_{i_{t_0}-1} = \ldots = m_{i_{t_0}} = 1$. If $m_{i_t+1}, \ldots, m_n$ are already constructed for some $t \leq t_0$ and $t \geq 1$, define

$$m_{i_{t-1}+1} = \ldots = m_{i_t} = \begin{cases} m([i_t+1,n] \cap S_{t+1}^{Tv} \setminus S_j^{Tv}), & \text{if } S_{t+1}^{Tv} \neq \emptyset, \text{ where } SH \, \rho_{i_t}(S_j^{Tv}) = S_{t+1}^{Tv} \\ 1+m([i_t+1,n] \setminus S_j^{Tv}), & \text{otherwise where } i_t \in S_j^{Tv} \text{ and } |S_j^{Tv} \cap [i_t+1,n]| \text{ is} \\ & \text{minimal with this property.} \end{cases}$$

$$\lambda := m(S_1^{Tv}).$$

Lemma 4.7:

Let $X^*(\Omega,v)$ satisfy (*). Then the representation $\iota(\Omega,v)$ coincides with the vector (λ,m) generated by $\bar{S}^{Tv}$.

Proof:

Let $S_1,\ldots,S_n$ be defined according to Definition 4.1. Then it is obvious that $S_1 \sim S_1^{Tv}$. Using an inductive argument we easily see that there is a sequence $0 = j_0 < j_1 < \ldots < j_r = n$ such that $S_{j_{t-1}+1} \sim \ldots \sim S_{j_t} \sim S_t^{Tv}$ for all t, $1 \leq t \leq r$. It is straightforward to finish the proof by comparing Definitions 2.8 (3) and 4.6. q.e.d.

Summarizing the preceding notation and assertions we get

Theorem 4.8:

Let (Ω,v) be a directed game, whose shiftminimal $(m \times n)$ matrix X^* satisfies condition (*).

(Ω,v) is homogeneous, iff the following conditions are valid:

(i) $i \not\sim i+1$, iff there is a t such that $i \in S_t^{Tv}$, $i+1 \notin S_t^{Tv}$.

(ii) $m(S) = \lambda$ for each row S of X^*, where (λ,m) is the vector generated by $\bar{S}^{Tv}$.

(iii) Let $j \in [1,m]_n$. If $i \in X_j^*$, $i \leq t_0 = \min \{t | t \in X_j^*, t \notin X_{j+1}^*\}$, $i+1 \not\sim i \not\sim t_0$, where X_{m+1}^* is the empty coalition, then $m([i+1,n] \setminus X_j^*) < m_i$.

Proof:

Assume (Ω,v) is homogeneous. Assertion (i) is a direct consequence of Lemma 4.4 and the proof of Lemma 4.7 and again Lemma 4.7 implies (ii). Condition (iii) follows from the fact that SH(S) is shiftminimal for each minimal winning coalition S.

Conversely if X^* satisfies (*), the matrix $\bar{S}^{Tv}$ is well defined and generates the minimal representation (λ,m) of a homogeneous game $(\Omega,\tilde{v})$ by Lemma 4.7. Each winning coalition w.r.t. (Ω,v) does win w.r.t. $(\Omega,\tilde{v})$, since for each shiftminimal coalition S the equality $m(S) = \lambda$ holds true. Assume there is a losing coalition S w.r.t. (Ω,v) which wins w.r.t. $(\Omega,\tilde{v})$. W.l.o.g. let S be shiftminimal w.r.t. $(\Omega,\tilde{v})$. Then there is a unique $j \in [1,m]$ such that $0(X^*_{j.}) > 0(S) > 0(X^*_{j+1.})$. Define $\tilde{\imath} = \min\{t \mid t \in X^*_{j.}, t \notin S\}$ and $i = \max\{t \mid t \sim \tilde{\imath}\}$. Then it is obvious that $i \preceq t_0$, thus $i \sim t_0$ by condition (iii). Consequently, $X^*_{j+1.}$ has a proper subcoalition $\tilde{S}$ with $m(\tilde{S}) = \lambda$, thus $1(X^*_{j+1.})$ is a dummy w.r.t. (λ,m). By (i) we have $i := 1(X^*_{j+1.}) \leq t_0$, $i \nsim t_0$, which contradicts (iii).   q.e.d.

The following 3 examples show that none of the conditions (i), (ii), (iii) can be deleted in Theorem 4.8. The shiftminimal matrices of the considered games, which all are weighted majority games, given in terms of representations, have property (*). Since it is straightforward to verify the relevant properties of these games, the concerning proofs are skipped.

The game (Ω,v) represented by $\begin{cases}(7;5,3,3,1,1,1)\\(10;5,5,2,2,2,2,2)\\(29;19,12,12,5,5,5,2,1,1)\end{cases}$

satisfies conditions $\begin{cases}(i),(ii)\\(i),(iii)\\(ii),(iii)\end{cases}$ but not $\begin{cases}(iii)\\(ii)\\(i)\end{cases}$ respectively

The following diagram illustrates some properties of these games.

representation of (Ω,v)	$X^*(\Omega,v)$	$\bar{S}^{Tv}$	representation, generated by $\bar{S}^{Tv}$	satisfied conditions
$(7;5,3,3,1,1,1)$	$\begin{pmatrix}101000\\100011\\011001\end{pmatrix}$	$\begin{pmatrix}101000\\011001\\100011\end{pmatrix}$	$(5;3,2,2,1,1,1)$	(i), (ii)
$(10;5,5,2,2,2,2,2)$	$\begin{pmatrix}1100000\\0100111\\0011111\end{pmatrix}$	$\begin{pmatrix}1100000\\0100111\end{pmatrix}$	$(6;3,3,1,1,1,1,1)$	(i), (iii)
$(29;19,12,12,5,5,5,2,1,1)$	$\begin{pmatrix}101000000\\100011000\\011001000\\001111100\\001111011\end{pmatrix}$	$\begin{pmatrix}101000000\\011001000\\100011000\\000000000\\000000000\end{pmatrix}$	$(5;3,2,2,1,1,1,0,0,0)$	(ii),(iii)

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