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ON THE MINIMAL REPRESENTATION
OF HOMOGENEOUS GAMES

Axel Ostmann

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H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

ABSTRACT

It is known that the lattice-minimal representation of a weighted majority game maybe not unique and may lack of equal-treatment. The same is true for total-weight-minimal representation. Maybe both concepts coincide. We focus on the total-weight-version for homogeneous games.

A Lemma tells us that all representations of a game induce the same order on the set of equivalence classes of the players. So we can compare players by their weights and can speak of greater or smaller ones.

To calculate adequate weights for a condidate for a minimal representation, we look at the incidence matrix of the lexicographically ordered minimal winning coalitions and we distinguish two sorts of players: sums and steps. A sum is defined by the existence of a substituting coalition of some "smaller" players; and a step is a non-sum.

Now we can calculate a candidate inductively going from smaller players to the greater ones. This candidate is a representation of the game we are looking at, and it is homogeneous. The smallest non-dummy gets one and the representation assigns the same weight to players that are equivalent. Furthermore we can show that the candidate is the unique minimal representation.

§ 1 BASIC NOTATIONS

A simple game is a pair (N,v) with $N = \{1,2,\dots,n\}$, a finite subset of $\mathbb{N}$, called players, and $v : \underline{\underline{P}}(N) \to \{0,1\}$, $v(\emptyset) = 0$. The elements of $\underline{\underline{P}}(N) := \{S; S \subset N\}$ are called coalitions. Sometimes we identify $\underline{\underline{P}}(N)$ with 2^N by $S \to 1_S$.

Let $W = W(N,v)$ be the set of winning coalitions, i.e. $W := v^{\leftarrow}(1)$. Correspondingly $L := v^{\leftarrow}(0)$ is the set of loosing coalitions.

(N,v) is called superadditive iff $v(S) + v(T) \leq v(S \cup T)$ for all $S,T \in 2^N$, $S \cap T = \emptyset$.

(N,v) has zero-sum iff $v(S) + v(N \setminus S) = v(N)$ for all $S \in 2^N$.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We identify $\mathbb{N}_0^n$ with the integral measures on N and write mS or $1_S m$ for $\sum_{i \in S} m_i$.
We speak of a weighted majority game (w.m.g.) iff the simple game (N,v) is "representable by integral measures", i.e. iff there exist $m \in \mathbb{N}_0^n$, $mN \neq 0$ and $\mu \in \mathbb{N}$, $\mu > mN/2$ with

$$(*) \qquad v = 1_{[\mu,\, m(N)]} \circ m$$

the pairs $(\mu,m) \in \mathbb{N} \times \mathbb{N}_0^n$, $m(N) \neq 0$, $\mu > m(N)/2$ with $(*)$ are called representations of (N,v).

We denote the set of representations by $R = R(N,v)$.

A w.m.g. is superadditive and simple by definition, and so we can study its structure by the following two subsets of 2^N:

First the set W_* of minimal winning coalitions,
$W_* := \{S \in W \; ; \bigwedge_{i \in S} S \setminus \{i\} \in L\}$, and next
the set L^* of maximal loosing coalitions,
$L^* := \{S \in L; \bigwedge_{i \notin S} S \cup \{i\} \in W\}$.

If a w.m.g. is zero-sum, then the application $W_* \to L^* : S \to N \setminus S$ is well-defined and bijective.

Total mass $m(N)$ induces a preference on R by $(\mu, m) \overset{N}{\to} (\mu', m')$ iff $mN \geq m'N$. The optimal elements of $(R, \overset{N}{\to})$ are called minimal representations.

For some purposes it is instructive to look at the real version $\hat{R}$ of R by substituting $\mathbb{R}^+ \times (\mathbb{R}_0^+)^n$ to $\mathbb{N}^+ \times \mathbb{N}_0^n$. $\hat{R}$ is a convex cone. R and $\hat{R}$ are the solutions of the following system of inequalities:

$$\begin{cases} mS \geq \mu & , S \in W_* \\ mS < \mu & , S \in L^* . \end{cases}$$

Let us conclude by a short look at the symmetry group $\Gamma = \Gamma(N,v)$ of a simple game. Permutations of N induce motions in coalitions, games, and - for w.m.g. - on $\mathbb{N}^+ \times \mathbb{N}_0^n$ resp. $\mathbb{R}^+ \times (\mathbb{R}_0^+)^n$. The symmetry group Γ is defined by $\Gamma := \{\pi$ is permutation of $N; \; v = \pi \, v\}$.

Γ splits the player set N into transitive classes $\tilde{i}$ called types:
$i \sim j$ iff player i is element of the orbit of j, i.e. $i \in \Gamma j$.
Let $\tilde{i} = \{j \in N, \; i \sim j\}$ and $\tilde{N} = N/_\Gamma = \{\tilde{i}; \; i \in N\}$.
Let us define two special types $D = D(N,v)$ and $E = E(N,v)$:
$D := N \setminus \bigcup_{W_*} S$ is the dummy type and
$E := \bigcap_{W_*} S$ is the type of unavoidable players (or "veto players"; iff $E \neq 0$ the game is called weak).

§ 2 THE NATURAL ORDER

For further considerations we need the following

LEMMA (2.1): Let (N,v) be a w.m.g.
All representations are inducing the same order on $\tilde{N}$.

Proof. We assume that there are different orders induced. Then there are $i, j \in N$, $i \neq j$ and two representations (μ,m), $(\mu',m') \in R$ with $m_i > m_j$ and $m'_j > m'_i$.

We shall prove $i \sim j$; this contradiction will complete the proof of the lemma.

It is enough to prove that the permutation

$$\pi(\nu) = \begin{cases} \nu & \nu \neq i,j \\ i & \nu = j \\ j & \nu = i \end{cases}$$

is element of the symmetry group Γ of the game. To this end, consider W_*.
Let $S \in W_*$. If $i, j \in S$ or $i, j \notin S$ then $\pi S \in W_*$. Hence, w.l.o.g. let $i \in S$, $j \notin S$. Then $\pi S \in W$ by $\mu' \leq m'(S) < m'S - m'_i + m'_j = m'\pi S$.

Moreover for $k \in S$ we shall get $S \setminus k \in L$:

(1) If $k \neq i$ then $\pi(S \setminus k) = \pi S \setminus k$ and
$\mu > m(S) - m_k > mS - m_i + m_j - m_k = m\,\pi\,S - m_k$

(2) If $k = i$ then $m(\pi(S \setminus i)) = m(S \setminus i)$.
So we got $\pi S \in W_*$.

§ 3 SUMS AND STEPS AND ZERO-SUM

Lemma 2.1 ensures that we can define the smallest non-dummy type $F := \tilde{i}$ if $m_i = \min_{N \setminus D} m_\nu$ for some $(\mu,m) \in R$.

Since Γ is transitive on the types $\tilde{i}$ we can restrict our attention to one fixed order on N and can assume $m_i \geq m_j$ for all $i \leq j$.

A representation (μ,m) is said to preserve types if $i \sim j \Rightarrow m_i = m_j$. If a minimal representation does not preserve types then it is not unique-

DEFINITION 3.1 $$X := X(N,v) = \begin{pmatrix} \vdots \\ 1_S \\ \vdots \end{pmatrix}_{S \in W_*}$$

with the lexicographic order on W_* is called incidence matrix (of the game (N,v), or of W_*). The lexicographic order $\underline{L}$ on W_* is induced by $\varphi_S := \sum_N 1_S(i) \cdot 2^{n-i}$ and $S \,\underline{L}\, T$ iff $\varphi_S > \varphi_T$.

(φ_S is the corresponding dual number).

Example 3.2 Let (N,v) be the game defined by the representation $(9;5,4,3,2,2)$, so

$$X = \begin{pmatrix} 1&1&0&0&0 \\ 1&0&1&1&0 \\ 1&0&1&0&1 \\ 1&0&0&1&1 \\ 0&1&1&1&0 \\ 0&1&1&0&1 \end{pmatrix} \qquad \tilde{N} = \{\{1\},\{2\},\{3\},\{4,5\}\}$$

the given representation preserves types.

$D = \emptyset$, $E = \emptyset$, $F = \{4,5\}$.

Example 3.3 For (5;3,2,2,1) we get

$$X = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

DEFINITION 3.4 Let $\sim j$, $j \in N$ be the relation on 2^N defined by

$$S \sim j\ T \quad \text{iff} \quad \bigwedge_{i<j} 1_S(i) = 1_T(i)$$

If $S \sim j\ T$ the both coalitions differ not earlier then at j. The induced relations on W_* can be visualized by submatrices of X of length $n-j$ from behind.

Remark 3.5 If X has a row $1_S(j) = 0$, $1_S(i) = 1$ with $j < i$, then there exists $T \in W_*$ with $S \sim j\ T$ and $1_T(j) = 1$.

This is a direct consequence of Lemma 2.1.

DEFINITION 3.6 A player $i \in N \setminus D$ is called sum iff there exist $S,T \in W_*$ with $S \sim i\ T$ and $i \in S$, $i \notin T$. Otherwise $i \in N \setminus D$ is called step. A step i is called final iff there exists $S \in W_*$, $i \in S$ with $j \notin S$ for all $j > i$.

The members of E are steps. The last player in $N \setminus D$ is a final step. In 3.2 there is only one step; in 3.3 the last two players are final steps.

DEFINITION 3.7 A w.m.g. (N,v) is called homogeneous iff there exists a representation (μ,m) with

(∗) $$Xm = \mu 1_N$$

A representation fullfilling (∗) is called homogeneous.

LEMMA 3.8 For a homogeneous representation (μ,m)

1. the weight of a sum is a sum of some smaller players weights;
2. the weight of a final step exceeds the sum of the smaller players weights.

Proof: 1. Let $S(i)$, $T(i)$ be the lexicographically first pair in W_* with $S(i) \sim_i T(i)$ and $i \in S(i)$, $i \in T(i)$. Look at the corresponding rows of (*) - they are in lexicographic order: at i is the first difference, after i both $S(i)$ and $T(i)$ are of type (1...1 0...0). $T(i)$ has to have a longer period of ones by the fixed order on N. So we get:

$$m_i = \sum_{\gamma=1}^{\gamma_i} m_{i+\gamma} .$$

Similarly we get 2.

LEMMA 3.9 Let j be a step and $i \in N$ with $i > j$. If $i \in S \in W_*$ then $j \in S$.

I.e. Steps rule its followers.

Proof: Let us assume there would exist $T \in W_*$ with $i \in T$, $j \notin T$. Then Remark 3.5 tells us that j is not a step.

LEMMA 3.10 Zero-sum w.m.games have only one step.

Proof: W.l.o.g. let $D = \emptyset$.
Let us assume there would be another step $j \neq n$. There is $S \in W_*$ with $n \in S$. By Lemma 3.9 we know $j \in S$. Zero sum yields $N \setminus S \in L^*$, but $(N \setminus S) \cup \{n\}$ is loosing too.

Lemma 3.10 is a sharper version of the known fact that homogeneous zero-sum games without dummies are nondegenerate, i.e. rank $(X) = n$.

But the step-version is also true for game that are not homogeneous. For inhomogeneous games nondegeneracy and rank $(X) = n$ is not the same.

A zero-sum game (N,v) is called majority game iff there exists $(\mu,m) \in R(N,v)$ with $mN = 2\mu - 1$ (cf. ISBELL 1956).

<u>LEMMA 3.11</u> If (μ,m) is a minimal representation of a zero-sum w.m. game then $mN = 2\mu - 1$. I.e. it is no difference between zero-sum w.m. and majority games.

Proof: $mN < 2$ by definition. There is a coalition S with $mS = \mu - 1$, else we can lower μ to μ' and for (μ',m) we can lower some weight, since a minimal representation (μ',m) guarantees for all $i \in N \setminus D$ the existence of a coalition T , $i \in T$ with $mT = \mu'$. Now we have $mS = \mu - 1$, $m(N \setminus S) \geq \mu$ and $mN \geq 2\mu - 1$. q.e.d.

$mN = 2\mu - 1$ is the condition for the nucleolus to generate the minimal representation (PELEG 1968).

Now let (μ,m) a homogeneous representation of a zero-sum game. By Lemma 3.9 we can solve the equation

$$X m = \mu 1$$

from behind, analoguesly to Lemma 3.8.

Let X_C be the submatrix of X of the rows used; then by elementary operations on rows we get

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ 0 & & \ddots \\ \cdots & \cdots & 1 \end{pmatrix} m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & & 0 & a_1 \\ & \ddots & & \vdots \\ 0 & & 1 & \vdots \\ & & & a_n \end{pmatrix} m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu \end{pmatrix}$$

or

$$\begin{cases} m_i = - a_i \, m_n & (i \neq n) \\ a_n \, m_n = \mu \end{cases}$$

For a minimal element of the homogeneous representations we can set $m_n = 1$. This is the only minimal one.

Now

$$\begin{cases} m_i = - a_i & (i \neq n) \\ m_n = 1 \\ \det X_C = \mu \end{cases}$$

These arguments proved the following

LEMMA 3.12: There is a unique minimal element (μ,m) of the homogeneous representations of a zero-sum game; m_i equals one for $i \in F$ and (μ,m) preserves types.

§ 4 THE HOMOGENEOUS ZERO-SUME CASE

THEOREM: If (N,v) is a homogeneous zero-sum game, then its minimal representation (μ,m)

(a) is unique
(b) is homogeneous
(c) preserves types
(d) has unit weight on the smallest type
(e) has a level μ equal to the determinant of a submatrix of the incidence matrix
(f) is an absolute majority game with an odd number of votes, namely $2\mu - 1$.

Proof. In the light of §3 it is enough to prove that a minimal representation has to be homogeneous.

Now let (μ,m) be the homogeneous candidate for the minimal representation and (μ',m') be a minimal representation. Let P be the matrix of the elementary operations on X_C with

$$PX_C = \begin{pmatrix} 1 & \ddots & 0 & -m_1 \\ & & & \vdots \\ 0 & & 1 & -m_{n-1} \\ 0 & \cdots & 0 & \mu \end{pmatrix}$$

Let $X_C\, m' =: b$ and $PX_C\, m' = Pb =: b'$.
With $\hat{b} := (b'_1 \dots b'_{n-1}, 0)$ we get the vector equation

$$(*) \qquad m' = \hat{b} + m'_n\, m$$

i.e. we can see a representation as affine transformation of the minimal homogeneous (μ, m).

Now, for the first i with $\hat{b}_i < 0$ from behind (consequently $i \neq n$) take the lexicographicly first $S \in W_*$, $T \in L^*$ with $S \sim_i T$, $i \in S$, $i \notin T$. Such pairs must exist.

Then $0 < m'S - m'T = m'_n\, (mS - mT) + \hat{b}S - \hat{b}T$.

Since homogeneous representations of zero-sum games are homogeneous from below, we get

$$m'S - m'T = m'_n\, (\mu - (\mu - 1)) + \hat{b}S - \hat{b}T\ .$$

Taking the lexicographicly first S, T the difference $1_S - 1_T$ is not positive up to $(1_S - 1_T)\,(i)$, that is

$$m'S - m'T = m'_n + \hat{b}_i - \sum_{j>i} \beta_j\, \hat{b}_j\ , \quad \beta_j \in \{0,1\}$$

We got

$$(**) \qquad 0 \leq \sum_{j>i} \beta_j\, \hat{b}_j < m'_n + \hat{b}_i\ .$$

If $m'_n = 1$ this inquality implies the wanted contradiction: $\hat{b}_i \geq 0$.

So we assume $m'_n > 1$,
from (*) and (**) we get for all $i \in N$

$$m'_i = \hat{b}_i + m'_n\, m_i > -\, m'_n + m'_n\, m_i$$

and

$$m'_i \geq m'_n\, (m_i - 1) + 1\ .$$

Our assumption $m'_n > 1$ ensures

$$m'_i \geq 2(m_i - 1) + 1 = 2m_i - 1 \geq m_i$$

or as vector-inequality

$$m' \geq m \quad \text{and} \quad m' \neq m$$

in contradiction with $m'(N) \leq m(N)$.

The contradiction shows $\hat{b} \geq 0,\ m'_n = 1$.

Consequently $m' = m$.

§ 5 THE GENERAL HOMOGENEOUS CASE

In this paragraph we deal with homogeneous w.m.g. and we assume w.l.o.g. that there are no dummies.

DEFINITION 5.1: If i is a sum, let $S(i)$, $T(i)$ be the lexicographically first pair in W_* with $S(i) \sim_i T(i)$ and $i \in S(i)$, $i \notin T(i)$.

If i is a step, then $\mathbf{H}(i) := \{\{i,\ldots,n\} \setminus S,\ i \in S,\ S \in W_*\}$.

Let $h_m(i) := \max \{mH;\ H \in \mathbf{H}\}$.

For sums $T(i) \setminus S(i)$ is a substitute for i (cf. Lemma 3.8). For steps the elements of $\mathbf{H}(i)$ are too small to be substitute, and there is no substitute at all.

Now we define recursively a suitable candidate (μ,m) for the minimal homogeneous representation of the given game:

DEFINITION 5.2:

$$m_i := \begin{cases} m(T(i) \setminus S(i)) & i \text{ is sum} \\ 1 + h_m(i) & i \text{ is step} \end{cases}$$

$\mu := mS^{(1)}$ with $S^{(1)}$ the first element of $(W_*, \underline{L})$

(μ,m) is well defined, since we can calculate it "from behind" $(m_n = 1 + h_m(i) = 1 + m\emptyset = 1)$.

First we shall prove that (μ,m) is a homogeneous representation of the given game. We proceed by two steps:

LEMMA 5.3: If $S \in W_*$ then $mS = \mu$

LEMMA 5.4: If $S \in L_*$ then $mT < \mu$

Proof of the first Lemma: Let $S^{(k)}$ be the k-th coalition in the list $(W_*, \underline{L})$. $mS^{(1)} = \mu$ by definition. We prove: If $mS^{(1)} = \ldots = mS^{(k)} = \mu$ then $mS^{(k+1)} = \mu$. Let j be the first player not common to $S^{(k)}$ and

$S^{(k+1)}$. So we have $S^{(k)} \sim_{j \notin} S^{(k+1)}$ and by $\underline{L}$: $j \in S^{(k)}$, $j \in S^{(k+1)}$.
This implies that j is a sum. Let $S^{(1)}$ be the lexicographically first coalition with $S^{(1)} \sim_j S^{(k)}$; similarly to Lemma 3.8 we get $S^{(k+1)} \setminus S^{(1)} = T(j) \setminus S(j)$. By $1 \leq k$ we get $mS^{(1)} = mS^{(k)}$.

Now we can calculate $mS^{(k+1)}$:

$$mS^{(k+1)} = mS^{(1)} - m_j + m(S^{(k+1)} \setminus S^{(1)})$$

$$= mS^{(k)} - m_j + m(T(j) \setminus S(j)) = mS^{(k)}$$

<u>REMARK 5.5:</u> From the proof above we also lern rank $X = 1 + n - |\text{steps}|$.

Proof of the second Lemma: If $S \in L^*$ let $j(S)$ be the maximal player for whom exist some $T \in W_*$ with $S \sim_{j(S)} T$.

Assume $j(S) \in S$. So $j(S) \notin T$, but $T \cup \{j(S)\} \in W$. By omitting smaller players (there are such ones)we get a $T' \in W_*$ with $T' \sim_{j(S)+1} S$. (Otherwise $S \supset T$). This is a contradiction against the maximality of $j(S)$. Hence $j(S) \notin S$.

We got that $j(S)$ is "still available", and $S \cup j(S) \in W$. By Lemma 5.3 $m(S \cup j(S)) \geq \mu$. Let T be the first coalition in W_* with $S \sim_{j(S)} T$ and $T \subset S \cup j(S)$. It is constructed by omitting some smaller players. Let $j = j(S)$. To complete the proof it is enough to prove $m(S \setminus T) < m_j$.

Since T is lexicographically first $S \subset (T \setminus j) \cup (S \setminus T)$ holds true. If $m(S \setminus T) < m_j$, then $mS \leq m((T \setminus j) \cup (S \setminus T)) = mT - m_j + m(S \setminus T) < mT$, and $mT = \mu$ by the first Lemma.

If j is a step, so $S \setminus T \subset \{j, \ldots, n\} \setminus T \in \mathbb{H}(i)$ and $m(S \setminus T) \leq h(j) < m_j$. Remember, n is a step.

Now let us assume $mS < \mu$ is proven for all $S \in L^*$ with $j(S) > j_o$. We prove $mS < \mu$ holds for all $S \in L$ with $j(S) = j_o$.

If j_0 is a step then there is nothing to add. So let j_0 be a sum, and $S \in L^*$ with $j(S) = j_0$. Maximality of j_0 implies that there is no coalition $T' \in W_*$ with $S \sim_{j_0} T'$ and $j \notin T'$. So $S \cup \{j_0 + 1,\dots,n\}$ is loosing too, and by $S \in L^*$ follows $\{j_0 + 1,\dots,n\} \subset S$. But j_0 is a sum, and $T(j_0) \setminus S(j_0)$ can be used to construct a coalition S' with $mS = mS'$ and $j_0 \in S'$. Since the game is homogeneous S' is loosing too and we can add some smaller players to reach L^*. Say the constructed coalition is S''. $j(S'') > j_0$ and so we have $mS = mS' \leq mS'' < \mu$.

<u>THEOREM 5.6:</u> Let (N,v) be a homogeneous w.m.g. The minimal homogeneous representation

(a) is unique

(b) has a unit player

(c) preserves types

Proof of the theorem: It is enough to prove that any $(\mu',m') \in R_h(N,v)$ with $m \neq m'$ and $mN \geq m'N$ is not homogeneous.

Let j be the first player from behind with $m_j > m'_j$.

We assume j is a step. Let S resp. $H = \{j,\dots,n\} \setminus S$ hold $mH = h_m(j)$. Then

$$(S \setminus j) \cup H \notin W$$

$$m'H \geq mH$$

$$m'_j < m_j = mH + 1$$

We get a contradiction by

$$m'((S \setminus j) \cup H) = m'S - m'_j + m'H > m'S - m_j + mH$$
$$= m'S - 1 \geq \mu' - 1 .$$

So j is a sum. The calculation

$$m'S(j) - m'T(j) = m'_j - m'(T(j) \setminus S(j))$$
$$< m_j - m(T(j) \setminus S(j)) = 0$$

shows (μ',m') is not homogeneous.

REMARK 5.7: The minimal homogeneous representation can be used to generate all homogeneous w.m.g. (by the procedure given in ROSENMOLLER 1982, theorem 1.6 and 2.1 using that the minimal weight of the players is one).

THEOREM 5.8: The minimal homogeneous representation is the unique minimal representation.

PROOF: Let us denote the minimal homogeneous representation by (μ, m) and some minimal representation by (μ', m'). From $m_n = 1$ follows $m'_n \geq m_n$.

We shall prove that for any $i \in N : m'_i \geq m_i$. Let $m'_i \geq m_i$ for $i > j$. If j is a step, $(j \neq n)$ we take a maximal non-substitute H w.r.t. m (i.e. $mH = h_m(i)$) and get:

$$m'_j \geq 1 + m'H \geq 1 + mH = m_j \ .$$

Now let j be a sum . $S(j)$ and $T(j)$ are defined above. Let k be the last player of $S(j)$. We look at the family of all $T \in W_*$ with $T \sim_{k+1} T(j)$. The last player appearing in this family is called l . Player l is a final step (maybe it is equal to $k+1$ or to n). So $R := T \setminus l \cup \{l+1 \ldots n\} \in L^*$ and the following four equations hold:

$$\begin{cases} mT = \mu \\ m_l = 1 + \sum\limits_{\nu > l} m_\nu \\ mR = \mu - 1 \\ m(R \setminus S(j)) = m_j - 1 \end{cases}$$

for m' we have $m'S(j) > m'R$.

By $S(j) \sim_j R$ that inequality implies $m'(S(j) \cap \{j \ldots n\}) > m'(R \cap \{j \ldots n\})$, and diminishing it by the weight of $\{j+1, \ldots, k\}$:

$m'_j > m'(R \cap \{k+1, \ldots, n\}) = m'(R \setminus S(j))$.

But $m'(R \setminus S(j)) \geq m(R \setminus S(j)) = m_j - 1$. So we have shown $m_j \leq m'_j$.

For being minimal m' has to be equal to m; $\mu' = \mu$ follows.

<u>REMARK 5.9:</u> Theorem 5.8 tells us, that if there are different minimal representations, then the game has to be inhomogeneous. ISBELL gave the example (99; <u>38</u>, 31, 31, 28, 23, 12,11, 8, 6, 5, 3, 1), (99; <u>37</u>, 31, 31, 28, 23, 12, 11, 8, 7, 5, 3, 1). The example has |2| minimal winning coalitions. My smallest example is (16; 6, 5, 4, 3, 2, 2), (16; 5, 5, 4, 4, 2, 2) with five minimal representations and $|W_*| = 7$, $|L^*| = 12$.

If the smallest non-dummy gets more than one, then the game has to be inhomogeneous too. For majority games ISBELL 1959 got $n \geq 7$ for that case. We get $n \geq 5$ for w.m.g. by (9; 5, 4, 3, 2, 2).

We are interested in the game (9; 5, 4, 3, 2, 2) by another reason too. Let us aks for the maximal minimal representation for n-person-games.

For majority games ISBELL found a close connection with the Fibonacci-Numbers. We conjecture that the maximal size of a house mN to represent all w.m.g. with $|N| = n$ is 2^{n-1} .

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