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On Existence of Stable and Efficient Outcomes in Games with Public and Private Objectives

by

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ABSTRACT

It is well known that game-theoretic models of economies with externalities tend to exemplify the "contradiction between efficiency and stability". In formalizations and proofs of this rather loose statement some smoothness conditions are usually required. Here a situation of this kind is studied under continuity assumptions only. The most interesting thing is that there exist such (non-smooth) preferences which guarantee that this contradiction occurs under no circumstances. Assuming some sort of homogeneity of preferences over the set of the players, the necessary and sufficient conditions for such "persistent" existence of efficient and stable outcomes are derived.

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1. General Formulation

Consider a finite society N members of which will be called players. Each player $i \in N$ has his(her) set of strategies X_i and preferences over the set of outcomes $X = \times_{i \in N} X_i$. The peculiarity of the models to be dealt with here lies in the structure of the preferences. Suppose that there is a function $\psi : X \to \mathbb{R}$ and functions $\varphi_i : X_i \to \mathbb{R}$ ($i \in N$), and that the preferences of player i are expressed by his(her) utility function

$$u_i(x) = F(\psi(x), \varphi_i(x_i)), \qquad (1)$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ is a given increasing (at least, non-decreasing) function. So we have a normal form game with arbitrary strategy sets and specific utilities.

Models of this kind are rather usual in studying voluntary provision of a public good (or a public bad), see, e.g. Bergstrom, Blume, Varian (1986). We may suppose that each player has some amount of money which he(she) is to divide between his(her) personal and some public needs. The strategy sets then consist of all permissible allocations; the function φ_i expresses the level of player i's personal consumption; the function ψ the level of public facilities; the function u_i describes the "integral welfare" of player i. Instead of money we might consider vector resource allocation. Furthermore, the strategies may correspond to technological decisions, in which case φ_i describes net output or profit of player i, and ψ is some environmental characteristic.

In any case, what is essential is that each player has his(her) private welfare characteristic and there is a public welfare characteristic and every characteristic is scalar. In accordance with the previous experience of studying this kind of models (see, e.g., Feldman (1980), Moulin (1986)) it would be quite natural to expect the contradiction between stability and efficiency here.

To be more precise, denote by PO, NE, SE, respectively, the sets of all Pareto optima, all Nash equilibria, and all strong (coalitional) equilibria of a given normal form game (for the exact definitions see, e.g., Moulin (1986)). We shall never consider different games simultaneously, so shall need no special notation for the game itself. The said contradiction may be expressed as the equality $NE \cap PO = \emptyset$, while its absence is characterized by the condition

$$NE \cap PO \neq \emptyset, \qquad (2)$$

or, more strongly, by

$$SE \neq \emptyset. \qquad (3)$$

We shall call a function F weakly stable (respectively, stable), if for any finite N, any compacts X_i $(i \in N)$, any continuous ψ, φ_i $(i \in N)$ condition (2) (respectively, (3)) is satisfied. This means that (2) or (3) holds for the normal form game defined by the sets N, X_i $(i \in N)$ and the functions u_i $(i \in N)$ satisfying (1).

We have a special regard for the function F here, because it constitutes the most "subjective" part of the whole construction: the functions φ_i, ψ may easily be imagined as measurable in the quantitative sense, while F represents a purely personal assessment. Independence F of i means that the society is homogeneous in this respect. (It is worth noting that the presence of the same function F in Formula (1) for every player implies "ordinal level comparability" (Sen (1977)), i.e. a mo-

notonic transformation can be applied to all utilities simultaneously without changing anything, while independent transformations of utilities are not allowed.)

So the main problem of this paper can be formulated as follows: people of what kind could constitute a society which may live in harmony without any compulsory mechanism for implementation of public decisions (at least, decisions on provision of a public good or a public bad)? Or, interpreting the result in negative, which exactly features of preferences make such compulsory mechanisms indispensable in the real-world societies?

2. Main Result

There are some trivial stable functions: constant functions and two projections $F(\psi,\varphi)=\varphi$, $F(\psi,\varphi)=\psi$. The first non-trivial example of a stable function was discovered by Germeier and Vatel (1974); it is the minimum function: $F(\psi,\varphi)=\min(\psi,\varphi)$. In fact, they considered a less general model: strategies of players meant allocations of resources among public and private needs, and the functions ψ, φ_i were strictly increasing. Under these superfluous, as we see them now, assumptions the existence of a Pareto optimal Nash equilibrium was shown. Further investigations summed up in Kukushkin et al. (1985) showed stability in the above sense of the minimum function. Another example of a stable function is the maximum function, see Kukushkin (1989).

The main theorem of this paper shows that maximum and minimum do not exhaust the scope of the stable functions, but every such function is in a sense constructed of these two.

To get necessary and sufficient conditions for the stability of a function F, we shall restrict the scope of the functions considered. Only continuous and monotonic functions will be allowed. Continuity needs no comment, and monotonicity will be understood in the following sense: if both arguments have increased, the value of the function must also increase. By this restriction we exclude constant functions.

Theorem. For any continuous monotonic function $F : \mathbb{R}^2 \to \mathbb{R}$ the following three statements are equivalent:

(a) F is weakly stable;

(b) F is stable;

(c) F can be described by the formula:

$$F(\psi,\varphi) = \min \{ \max (\lambda_1(\psi), \lambda_2(\varphi)), \lambda_3(\varphi)\}, \qquad (4)$$

where $\lambda_1, \lambda_2, \lambda_3$ are strictly increasing continuous functions, $\inf \lambda_3 \geq \sup \lambda_2$, one of the functions λ_1, λ_2 may have $-\infty$ as a value, one of the functions λ_1, λ_3 may have $+\infty$ as a value.

There is a kind of geometric interpretation for Formula (4). First of all, a function satisfying (4) may only have lines of constant value of the following four types:

Second, if the function has both maximum-like and minimum-like lines, then whether the line of constant value drawn through a given point (ψ,φ) of the plane is maximum-like or minimum-like depends on ψ only.

It is easy to show that the only smooth functions satisfying (4) are two projections. Furthermore, there is no strictly

increasing stable function.

The quantification on N and X_i is, in fact, unessential. As may be easily seen from the proof below, Formula (4) is sufficient for stability w.r.t. any N, X_i and necessary for stability w.r.t. $|N|=2$, $|X_i|=2$.

Without pretending on serious applications, consider a "fairy" example just to clarify the meaning of Formula (4) in the general case when both maximum-like and minimum-like lines of constant value are present. The players are dwellers of a street by which they walk every morning to a railway station. The utility function of each player evaluates the state of his feet on arriving there. There is some lexicography in this evaluation: first of all, everyone wants to have his feet dry. If it has proved impossible, the utility function measures wetness of the feet regardless of any other characteristic. If the feet are dry then their outlook becomes essential, so the utility function measures somehow the quality of this outlook. Strategies describe allocation of money, time, effort, etc. by each player among enhancing the quality of street pavement and of his personal boots or shoes. Now we may suppose with some plausibility that wetness of your feet depends on qualities of street pavement and of your boots in a maximum-like fashion (on a well-paved street you have no need for heavy boots, and with good boots on you are indifferent, to an extent, to the quality of street pavement), while the outlook depends on them in a minimum-like fashion (rough pavement spoils your fine shoes beyond recognition).

So far our assumptions sound plausible, though not necessarily quite convincing (e.g. even after a rough walk good shoes may still have a better outlook than initially bad ones). But to obtain Formula (4) we need further assumptions. First: there exists, in principle, such street pavement that guarantees you dry feet even if you walk with bare feet, and the cheapest version of such a pavement is infinitely rough, i.e. you would have an infinitely ugly outlook after walking on it ($\lambda_1(\psi) = \sup\lambda_2$). Second: achieving the "completely dry" state of your feet relying solely on your boots is only possible by use of infinitely expensive and infinitely ugly boots. These additional assumptions may seem rather far-fetched.

Perhaps, the most natural interpretation of the theorem is negative: there exists no stable function except such exotic ones as described by Formula (4). In this case I can claim the most general formulation of this quite anticipated negative result (there was no word about smoothness in the theorem). On the other hand, though, more simple maximum or minimum functions may be quite relevant in some appropriate circumstances, so the positive side of the theorem may eventually find its applications.

3. Proof

It is obviously sufficient to prove two implications: (c) ⇒ (b), (a) ⇒ (c).

1. Given a function F satisfying (4), a finite set N, $|N|=n$, compacts X_i ($i\in N$), continuous functions $\psi : X \to \mathbb{R}$ and $\varphi_i : X_i \to \mathbb{R}$ ($i\in N$), we have to prove that SE $\neq \emptyset$ for the normal form game defined by the sets X_i and the utility functions u_i satisfying (1).

In a sense, we may treat the function F as if it were either just minimum (case 1) or just maximum (case 2). Denote

$\psi^{max} = \max\{\psi(x) \mid x\in X\}$ and consider two possibilities.

Case 1. Let $\lambda_1(\psi^{max}) \geq \sup\lambda_2$. To obtain an outcome $x\in$SE we shall lexicographically maximize the utilities in the increasing order. For any $x\in X$ define $\vartheta_1(x), \ldots, \vartheta_n(x)$ as the result of ordering of the list of utilities $(u_i(x))_{i\in N}$, i.e. $\vartheta_1(x)\leq\ldots\leq\vartheta_n(x)$ and $u_i(x)=\vartheta_{\sigma(i)}(x)$ for all $i\in N$, where σ is a one-to-one mapping of N onto $\{1,\ldots,n\}$. We shall say that an outcome $y\in X$ lexicographically dominates another outcome $x\in X$ if there exists $m\in\{1,\ldots,n\}$ such that $\vartheta_m(y)>\vartheta_m(x)$ while $\vartheta_l(y)=\vartheta_l(x)$ for $l=1,\ldots,m-1$. Following d'Aspremont and Gevers (1977) denote Leximin the set of all such outcomes that are lexicographically dominated by no outcome $y\in X$. It is easy to show that every function ϑ_l is continuous and $\emptyset \neq$ Leximin $\subseteq$ PO.

Pick an outcome $x\in$Leximin and show that $x\in$SE. Suppose to the contrary that there exists a coalition $I\subseteq N$, a player $k\in I$ and an outcome $y\in X$ such that $u_k(y)>u_k(x)$, $u_i(y)\geq u_i(x)$ for every $i\in I$, $x_j=y_j$ for every $j\in N\setminus I$.

Lemma 1.1. $u_j(x)\geq\sup\lambda_2>\lambda_2(\varphi_j(x_j))$ for every $j\in N$.

Otherwise any outcome z with $\psi(z)=\psi^{max}$ would lexicographically (in fact, even Pareto) dominate the outcome x.

Lemma 1.2. $u_j(y)>\lambda_2(\varphi_j(y_j))$ for every $j\in N$.

The inequality $u_j(y)\leq\lambda_2(\varphi_j(y_j))$ for some $j\in N$ would imply $\lambda_1(\psi(y))<\sup\lambda_2$, hence $u_j(y)<u_j(x)$ for every $j\in N$, contrary to our presumption on y.

The two lemmas show, in fact, that considering outcomes x and y we may treat the function F as if it were just $\min\{\lambda_1(\psi),\lambda_3(\varphi)\}$.

Now obtain the required contradiction. Denote $J=\{j\in N \mid u_j(y)<u_j(x)\}$, $w^*=\min\{u_j(y) \mid j\in J\}$ ($J\neq\emptyset$ because $x\in$PO and

$u_k(y) > u_k(x)$). Let $j \in J$ be such that $u_j(y) = w^*$, then $j \notin I$, hence $x_j = y_j$, hence $\varphi_j(x_j) = \varphi_j(y_j)$, therefore, $u_j(y) = \lambda_1(\psi(y)) < \lambda_3(\varphi_j(y_j)) \geq u_j(x)$. Now we have $u_k(x) < u_k(y) \leq \lambda_1(\psi(y)) = w^*$. By the definition of w^* we have $u_i(x) \leq u_i(y)$ for every $i \in N$ satisfying $u_i(y) < w^*$. It follows immediately that y lexicographically dominates x, contrary to our choice of $x \in$ Leximin.

Case 2. Let $\lambda_1(\psi^{max}) \leq \inf \lambda_3$. It means that for any $x \in X$, $i \in N$ the equality $u_i(x) = \max\{\lambda_1(\psi), \lambda_2(\varphi)\}$ holds. Now we may argue quite similarly to the previous case, picking $x \in$ Leximax (see d'Aspremont and Gevers 1977) instead of $x \in$ Leximin and changing appropriately signs of some inequalities.

More precisely, if $x \in$ Leximax, $y \in X$ is such that $u_k(y) > u_k(x)$ for some $k \in I$, $u_i(y) \geq u_i(x)$ for every $i \in I$, $x_j = y_j$ for every $j \in N \setminus I$ then we denote $J = \{j \in N \mid u_j(y) < u_j(x)\}$ (it is non-empty for just the same reason as above), $w^* = \max\{u_j(x) \mid j \in J\}$. Pick $j \in J$ for which $u_j(x) = w^*$, then $j \notin I$, hence $x_j = y_j$, hence $\varphi_j(x_j) = \varphi_j(y_j)$, therefore, $u_j(x) = \lambda_1(\psi(x)) > \lambda_3(\varphi_j(x_j)) \leq u_j(y)$. Now we have $u_k(y) > u_k(x) \geq \lambda_1(\psi(x)) = w^*$. By the definition of w^* we have $u_i(x) \leq u_i(y)$ for every $i \in N$ satisfying $u_i(x) > w^*$. It follows immediately that y lexicographically (in the Leximax sense) dominates x, contrary to our choice of $x \in$ Leximax.

As $\inf \lambda_3 \geq \sup \lambda_2$, the two cases cover all possibilities (even with some overlapping).

2. Suppose that $F : \mathbb{R}^2 \to \mathbb{R}$ is a weakly stable continuous monotonic function.

Lemma 2.1. If $\psi' < \psi''$, $\varphi' < \varphi''$, $F(\psi', \varphi'') = F(\psi'', \varphi') = u$ then there exists $\bar{\psi} \in [\psi', \psi'']$ such that $F(\bar{\psi}, \varphi') = F(\bar{\psi}, \varphi'') = u$.

Denote $\psi^1 = \sup\{\psi \mid F(\psi, \varphi'') = u\}$, $\psi^2 = \inf\{\psi \mid F(\psi, \varphi') = u\}$; clearly, $\psi^1 \geq \psi'$, $\psi^2 \leq \psi''$. If $\psi^1 \geq \psi^2$ then we may pick $\bar{\psi} = \psi^1$ (in fact, in this

case the equality $\psi^1 = \psi^2$ must hold). Otherwise, for any $\tilde{\psi} \in]\psi^1, \psi^2[$ we have $F(\tilde{\psi}, \varphi'') > u$, $F(\tilde{\psi}, \varphi') < u$; for the rest of the proof of the lemma fix one such $\tilde{\psi}$.

Denote $u^+ = F(\tilde{\psi}, \varphi'') > u$, and let δ_1 be the radius of a neighbourhood of the point (ψ'', φ') where F is less than u^+, δ_2 be the radius of a neighbourhood of the point $(\tilde{\psi}, \varphi')$ where F is less than u . Define $\Delta = \min\{\delta_1, \delta_2\}/2$, $u^- = F(\tilde{\psi}, \varphi' + \Delta)$, $u^0 = F(\psi'' + \Delta, \varphi' + \Delta)$; it is easy to see that $u^- < u < u^0 < u^+$.

Now we may choose $X_1 = X_2 = \{1, 2\}$, and define ψ, φ_i as follows:

$$\varphi_i(1) = \varphi' + \Delta, \quad \varphi_i(2) = \varphi'' \quad (i = 1, 2),$$
$$\psi(1,1) = \psi'' + \Delta, \quad \psi(2,2) = \psi',$$
$$\psi(1,2) = \psi(2,1) = \tilde{\psi}.$$

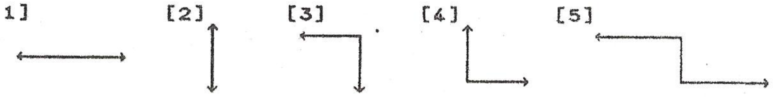
Applying Formula (1) we obtain the following bimatrix game:

(u^0, u^0)	(u^-, u^+)
(u^+, u^-)	(u, u)

The strategies $x_1 = 2$, $x_2 = 2$ are dominant for respective players, so NE={(2,2)} and this unique Nash equilibrium is Pareto dominated by the outcome (1,1). (In fact, this game is a version of the prisoner's dilemma). This contradiction with the presumed weak stability of F proves Lemma 2.1.

Lemma 2.2. For every $u \in \mathbb{R}$ the line of constant value $F^{-1}(u)$ (if not empty) follows one of the five patterns:

[1] [2] [3] [4] [5]



where abscissae correspond to ψ, ordinates to φ, open ends of the lines should be continued to infinity.

Suppose that there exist $\psi^1<\psi^2<\psi^3$, $\varphi^1>\varphi^2>\varphi^3$ such that $F(\psi^1,\varphi^1)=F(\psi^2,\varphi^2)=F(\psi^3,\varphi^3)=u$. Then $\psi^1,\psi^3,\varphi^3,\varphi^1$ satisfy the conditions of Lemma 2.1, therefore, there exists an appropriate $\bar{\psi}\in[\psi^1,\psi^3]$. If $\bar{\psi}<\psi^2$ then $F(\bar{\psi},\varphi^3)=u=F(\psi^2,\varphi^2)$ contradicts monotonicity, as $\varphi^3<\varphi^2$; if $\bar{\psi}>\psi^2$ then $F(\bar{\psi},\varphi^1)=u=F(\psi^2,\varphi^2)$ contradicts monotonicity, as $\varphi^1>\varphi^2$; so $\bar{\psi}=\psi^2$. If there exist $\varphi>\varphi^1$, $\psi\le\psi^1$ such that $F(\psi,\varphi)=u$ then we may apply Lemma 2.1 to $\psi,\psi^3,\varphi^3,\varphi$, obtaining an appropriate $\bar{\psi}$. As above, there must hold $\bar{\psi}=\psi^2$, but then $F(\psi^2,\varphi)=u=F(\psi^1,\varphi^1)$ contradicts monotonicity. So $F(\psi,\varphi^1)=u$ for every $\psi\le\psi^1$. Quite similar reasoning shows that $F(\psi,\varphi^3)=u$ for every $\psi\ge\psi^3$. We see that the line $F^{-1}(u)$ follows the pattern [5].

It is easy to show that any line which does not contain three points situated as it was supposed above must follow one of the other four patterns (regardless even of the stability of F). So Lemma 2.2 is proved.

The image $F(\mathbb{R}^2)$ must be an open interval; denote it $]u_{-\infty},u_{+\infty}[$. For $k=1,\dots,5$ denote $R_k=\{\,u\in\mathbb{R}\mid F^{-1}(u)$ follows the pattern $[k]\}$. So we have

$$]u_{-\infty},u_{+\infty}[\;=\bigcup_{k=1}^{5} R_k\,.$$

Lemma 2.3. The set R_5 is empty.

It is easy to see that R_5 is open. So, if not empty, it would consist of a finite or infinite number of open intervals. Let $]u^-,u^+[$ be one of them. For every $u\in]u^-,u^+[$ the line $F^{-1}(u)$ is defined by the three parameters: $\varphi^+(u)>\varphi^-(u)$ and $\psi^0(u)$ ($(\psi^0(u),\varphi^+(u))$ are the coordinates of the upper corner, $(\psi^0(u),\varphi^-(u))$ the lower corner); it is easy to see that the functions $\varphi^+(\cdot)$, $\varphi^-(\cdot)$, $\psi^0(\cdot)$ are continuous and increasing.

Pick arbitrarily $u^3\in]u^-,u^+[$; pick $u^1<u^3$, $u^5>u^3$ close enough to u^3 to fulfill the inequalities $\varphi^+(u^1)>\varphi^-(u^3)$ $\varphi^-(u^5)<\varphi^+(u^3)$, and pick arbitrarily $u^2\in]u^1,u^3[$, $u^4\in]u^3,u^5[$.

Now we are able to construct our example violating the supposed stability of F. Let $X_1=X_2=\{1,2\}$, $\varphi_1(1)=\varphi^-(u^2)$ $\varphi_1(2)=\varphi^-(u^3)$, $\varphi_2(1)=\varphi^+(u^3)$, $\varphi_2(2)=\varphi^+(u^4)$, $\psi(1,1)=\psi^0(u^5)$, $\psi(1,2)=\psi^0(u^4)$, $\psi(2,1)=\psi^0(u^3)$, $\psi(2,2)=\psi^0(u^1)$. So we have the following bimatrix game:

(u^2,u^5)	(u^2,u^4)
(u^3,u^3)	(u^1,u^4)

It is easy to see that NE=∅ for the game (remember, that $u^1<u^2<u^3<u^4<u^5$). Lemma 2.2 is proved.

Now list some properties of the sets R_k ($k=1,\dots,4$) omitting their quite straightforward proofs.

(i) The sets R_1 and R_2 can not be non-empty simultaneously.

(ii) The sets R_1 and R_2 are closed in $]u_{-\infty},u_{+\infty}[$.

(iii) The sets R_3 and R_4 are open.

(iv) If $u\in R_3$ and $u'<u$ then $u'\in R_3$.

(v) If $u\in R_4$ and $u'>u$ then $u'\in R_4$.

Suppose that R_2,R_3 and R_4 are non-empty, so $R_3=]u_{-\infty},\bar{u}[$, $R_2=[\bar{u},\bar{\bar{u}}]$, $R_4=]\bar{\bar{u}},u_{+\infty}[$. The position of a line $F^{-1}(u)$ is described by two real values $\psi(u)$, $\bar{\varphi}(u)$ (the coordinates of the corner) for $u\in R_3$, by a real value $\psi(u)$ for $u\in R_2$, by two real values $\psi(u)$, $\bar{\bar{\varphi}}(u)$ (the coordinates of the corner) for $u\in R_4$. It is easy to see that the function $\psi(\cdot)$ is increasing and continuous on the whole interval $]u_{-\infty},u_{+\infty}[$, while the functions $\bar{\varphi}(u)$, $\bar{\bar{\varphi}}(u)$ are increasing and continuous on the intervals $]u_{-\infty},\bar{u}[$ and $]\bar{\bar{u}},u_{+\infty}[$, respectively, and $\bar{\varphi}(\bar{u})=+\infty$, $\bar{\bar{\varphi}}(\bar{\bar{u}})=-\infty$. Now we may define

$\lambda_1(\cdot)=\psi^{-1}(\cdot)$, $\lambda_2(\cdot)=\bar{\varphi}^{-1}(\cdot)$, $\lambda_3(\cdot)=\bar{\bar{\varphi}}^{-1}(\cdot)$. Verification of Formula (4) is straightforward; note that $\sup\lambda_2=\bar{u}\le\bar{\bar{u}}=\inf\lambda_3$.

If only one of the sets R_3 or R_4 is non-empty, the situation is even more simple (in fact, in these cases Formula (4) is reduced to pure maximum or minimum).

Suppose now that R_1, R_3, R_4 are non-empty, i.e. $R_3=]u_{-\infty},\bar{u}[$, $R_1=[\bar{u},\bar{\bar{u}}]$, $R_4=]\bar{\bar{u}},u_{+\infty}[$. By quite similar reasoning we can obtain the "dual" formula:

$$F(\psi,\varphi) = \min\{\max\{\lambda_1(\varphi),\lambda_2(\psi)\},\lambda_3(\psi)\}, \qquad (5)$$

where $\lambda_1,\lambda_2,\lambda_3$ are continuous increasing functions, $\sup\lambda_2\le\inf\lambda_3$, one of the functions λ_1,λ_2 may have $-\infty$ as a value, one of the functions λ_1,λ_3 may have $+\infty$ as a value. Moreover, as $R_3\ne\emptyset\ne R_4$, the inequalities $\inf\lambda_1<\sup\lambda_2$ and $\sup\lambda_1>\inf\lambda_3$ must hold.

Lemma 2.4. The function F can not satisfy (5) with all the conditions listed below it.

Supposing the contrary, let $X_1=X_2=\{1,2\}$. Pick $\psi_{12}<\psi_{21}$ so that $\lambda_1(\psi_{21})>-\infty$, $\lambda_3(\psi_{12})<+\infty$; pick $\psi_{22}<\psi_{12}$, $\psi_{11}>\psi_{21}$, $\varphi_2^-<\varphi_2^+<\bar{u}\le\bar{\bar{u}}<\varphi_1^-<\varphi_1^+$ so that the following inequalities be fulfilled:

$$\lambda_1(\varphi_2^-)<\lambda_2(\psi_{21})<\lambda_1(\varphi_2^+)<\lambda_2(\psi_{11}),$$
$$\lambda_3(\psi_{12})<\lambda_1(\varphi_1^-)<\lambda_1(\varphi_1^+)<\lambda_3(\psi_{21}).$$

Assuming now $\varphi_i(1)=\varphi_i^-$, $\varphi_i(2)=\varphi_i^+$, $\psi(i,j)=\psi_{ij}$ $(i,j=1,2)$ we obtain the following bimatrix game

$(\lambda_1(\varphi_1^-),\lambda_2(\psi_{11}))$	$(\lambda_3(\psi_{12}),\lambda_1(\varphi_2^+))$
$(\lambda_1(\varphi_1^-),\lambda_2(\psi_{21}))$	$(\lambda_3(\psi_{12}),\lambda_1(\varphi_2^+))$

It is easy to see that NE=$\emptyset$ in the game. Lemma 2.4 is proved, and so is the theorem.

Remark. It is easy to show that demanding the functions

$\lambda_1,\lambda_2,\lambda_3$ in Formula (4) to be only non-decreasing, we would obtain a stable function satisfying a weaker monotonicity condition: if neither of the arguments has decreased, the value of the function must not decrease. It remains an open question sc far, whether every weakly stable continuous function satisfying this monotonicity condition can be described by Formula (4) with appropriate non-decreasing lambdas.

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