Universität Bielefeld/IMW

Working Papers
Institute of Mathematical Economics

Arbeiten aus dem
Institut für Mathematische Wirtschaftsforschung

Nr. 129

Reinhard Selten

Are Cartel Laws Bad for Business ?

January 1984

H. G. Bergenthal

Institut für Mathematische Wirtschaftsforschung
an der
Universität Bielefeld
Adresse/Address:
Universitätsstraße
4800 Bielefeld 1
Bundesrepublik Deutschland
Federal Republic of Germany

Are Cartel Laws Bad for Business ?

by

Reinhard Selten, Bielefeld

1. Introduction. Consider a cartel law which forbids any kind of collusion in oligopolistic markets and is actually enforced. In this kind of legal environment prices can be expected to be lower than in a situation where binding agreements are permissible. Therefore, cartel laws are in the interest of the consumer.

At first glance it may seem to be obvious that cartel laws are not in the interest of the producer. Oligopolists lose the opportunity to increase profits by collusion. However, this argument does not show that cartel laws are bad for business. Generally, joint profit maximization permits a greater number of competitors in a market than non-collusive behavior. It has been pointed out in the industrial organization literature that joint profit maximization may lead to overcrowded markets where each supplier has very low profits (Scherer 1970). Cartel laws exclude this possibility of excessive entry. It is quite possible that under cartel laws a market has fewer competitors with higher profits.

The term "entry effect" will be used for the decreasing influence of cartel laws on the number of competitors. The entry effect counteracts the obvious advantages of collusion. Is this counterinfluence sufficiently strong to make cartel laws desirable for business from the point of view of profit maximization? This question will be examined in the framework of a theoretical model.

The results of the analysis do not support the idea that cartel laws are bad for business. On the contrary, it will be shown that under reasonable assumptions on the distribution of market parameters the expected sum of all profits is increased by cartel laws.

In order to exhibit the intuitive reasons for the results it may be useful to present a preliminary and necessarily incomplete sketch of the theory.

The model will consider a great number of firms waiting for the opportunity to enter a new market. New markets are Cournot oligopolies with linear costs and linear demand. Costs are zero for those who do not enter but those who enter have fixed costs. The cost function is the same for each entrant.

The strategic situation resulting from the emergence of a new market depends on whether cartel laws are in force or not. Both cases will be modelled as non-cooperative extensive games. The game to be played under cartel laws will be called the <u>non-collusive</u> game and the game to be played in the absence of cartel laws will be referred to as the <u>collusive</u> game.

The non-collusive game has two stages: an <u>entry stage</u> and a <u>supply stage</u>. The collusive game has an additional stage, the <u>bargaining stage</u>, between the entry stage and the supply stage.

The entry stage is common to both games. A random ranking determines in which order the firms decide to enter or not. All rankings are equally probable. The firms know the cost and demand parameters. After the set of suppliers has been fixed by the entry stage, supply decisions have to be made in the non-collusive game. In the collusive game the suppliers first have the opportunity to reach a cartel agreement specifying supply quantities. The bargaining stage is modelled as a unanimity game. Every supplier independently and simultaneously proposes a cartel agreement. If all choose the same proposal it becomes a binding agreement. Otherwise supplies are determined in the supply stage where each supplier independently and simultaneously selects a supply quantity. Those who have entered receive their profits as payoffs; those who did not enter have zero payoffs.

The solution concept to be applied will determine a unique subgame perfect equilibrium point for a limited class of extensive games. In order to do this two selection principles of local symmetry and local efficiency will be employed.

In the solution of the non-collusive game, Cournot-Nash equilibrium is reached for the maximal number of competitors such that equilibrium profits are positive. In the collusive game the solution results in a symmetric cartel agreement for the maximal number of competitors such that cartel profits are positive.

The framework of our theory offers an obvious way to give a precise meaning to the question whether cartel laws are good or bad for business. One has to look at profits in both games. The sum of all solution profits obtained at the end is the relevant measure. It will be referred to as joint profits. Before the beginning of the game each of the many potential entrants has the same profit expectation, namely joint profits divided by the number of players. The profit sums for the collusive and non-collusive games will be called joint cartel profits and joint Cournot profits.

One could also compare individual cartel profits and individual Cournot profits for one supplier. This comparison is more favorable to the non-collusive case than the comparison of joint profits. This is due to the entry effect. However, this comparison neglects the fact that not only the profits of a firm in the market but also the chance to be among them is important for the evaluation of profit opportunities. Therefore, joint profits are compared in this paper.

For a single set of market parameters joint Cournot profits may be greater or smaller than joint cartel profits. A meaningful comparison must consider averages with respect to a distribution of market parameters. As far as the comparison is concerned, a market can be described by two parameters, monopoly gross profits M (sales minus variable costs) and size s; the size is the quotient $s = M/C$ of monopoly gross profits M over fixed costs C.

The size is an important characteristic of a market. The joint profits obtainable by a cartel are $M - nC$ where n is the number of competitors. In the collusive game solution n is the maximal integer below s. Joint cartel profits are $M(s-n)/s$. It will be shown that joint Cournot profits also can be written as a product of M and a factor which depends only on s.

It will be convenient to introduce a function $W(s)$, the weight of s. For a given probability density $f(M,s)$ the weight $W(s)$ is the marginal frequency density of s multiplied by the conditional expectation of M. One may think of $W(s)$ as a measure of the economic importance of markets of size s, since the gross profit opportunities offered by such markets are expressed by $W(s)$.

The answer to the question whether cartel laws are good or bad for business depends on the shape of $W(s)$. Casual empiricism suggests that the number of suppliers varies over a great range from product to product. The number of suppliers is related to the size of the market. Therefore, it seems to be plausible to expect that W(s) is quite flat over a wide range of sizes.

For the results to be obtained it will be sufficient to assume that the elasticity of W(s) with respect to s is smaller than 1. This is equivalent to the assumption that W(s)/s is decreasing. Moreover, it will be assumed that W(s) vanishes for market sizes above an upper bound $\bar{s}$ which is smaller than the number N of all firms. This means that the number of firms must be sufficiently great to exclude restriction of entrance by a lack of potential entrants. It will also be assumed that the range of W(s) extends beyond s = 2.

It will be shown that under the conditions on the shape of W(s) described above, cartel laws are good for business in the sense that joint Cournot profits are higher than joint Cartel profits.

The conclusion of the analysis depends on the assumptions on cooperative opportunities embodied in the collusive game. Agreements which restrict entry are excluded from consideration. It is reasonable to model the collusive case in this way, since entry restricting agreements would have to include almost all potential competitors in order to achieve their aim. Such agreements are not practicable if new firms can be formed just for the purpose to enter a new market.

Under plausible assumptions cartel laws are good for business in the sense of increased profits. Economic interests do not always lie in the naively expected direction.

2. Model description.

For the limited purpose of this paper it seems to be adequate to avoid a formal definition of an extensive game. The extensive games considered here will have infinitely many choices at some information sets, but otherwise they will not be different from finite extensive games with perfect recall. In particular, the length of a play is bounded from above. The usual game theoretical definitions of choices, information sets, strategies, etc. can be transferred without difficulty to such games [see Kuhn (1953) and Selten (1973) and (1975)]. Even if the game theoretical analysis of the game models considered here will be straightforward, a certain familiarity with basic definitions relating to extensive forms will have to be presupposed. In particular, the notion of a subgame will be used without explanation. Precise definitions can be found in the references given above.

It would be quite tedious to describe the game models introduced here in the terminology of extensive games. Instead of this, for each of both models a set of rules will be formulated which contains all the information necessary for the construction of the extensive game. Apart from inessential details like the order in which simultaneous decisions are represented in the game tree, the extensive game is fully determined by this description in an obvious way. Therefore, it will be sufficient to relate only some of the features of the model to the formal structure of the extensive game. This will be done after the description of the rules of both game models.

2.1 Cost and demand.

Both game models to be introduced have N players 1,...,N to be interpreted as firms waiting for the opportunity to enter a market. The play of the game determines a set Z of suppliers, a subset of the set of all players. In the following notations and assumptions concerning cost and demand will be introduced.

The supply of firm $i \in Z$ is denoted by x_i. The quantity x_i is a non-negative number. (There is no capacity limit). The cost function is the same for each firm $i \in Z$:

(1) $$K_i = C + \gamma x_i$$

C and γ are positive parameters. Total supply

(2) $$x = \sum_{i \in Z} x_i$$

determines price p:

(3) $$p = \begin{cases} \beta - \alpha x & \text{for } 0 \leq x \leq \frac{\beta}{\alpha} \\ 0 & \text{for } x > \frac{\beta}{\alpha} \end{cases}$$

α and β are positive parameters. We assume $\beta > \gamma$. (There is no loss of generality entailed by the exclusion of the case $\beta \leq \gamma$ since such markets do not offer any incentive to enter.) The profit margin g is defined as follows:

(4) $$g = p - \gamma$$

In view of (3) we have:

(5) $$g = \begin{cases} \beta-\gamma-\alpha x & \text{for } 0 \leq x \leq \frac{\beta}{\alpha} \\ -\gamma & \text{for } x \geq \frac{\beta}{\alpha} \end{cases}$$

Gross profits G_i and net profits P_i of a firm $i \in Z$ are as follows:

(6) $$G_i = x_i g$$

(7) $$P_i = G_i - C$$

Total gross profits

(8) $$G = xg$$

is the sum of individual gross profits of the firms $i \in Z$. With the help of (5) it is easy to compute the maximal value M of G, referred to as monopoly gross profits:

(9) $$M = \max_{x \geq 0} x(\beta-\gamma-\alpha x) = \frac{(\beta-\gamma)^2}{4\alpha}$$

We call

(10) $$s = \frac{M}{C}$$

the <u>size</u> of the market. Since both M and C are positive the size s is positive. It is assumed that the number N of all firms is greater than the size of the market. We have:

(11) $$N > s > 0$$

As has been explained already in the introduction the assumption $N > s$ prevents restriction of entry by a lack of potential competitors.

2.2 The entry stage.

The entry stage is common to both game models to be introduced. It begins with the random choice of a ranking $r = (i_1, \ldots, i_N)$ of all players. Mathematically r is a permutation of the numbers $1, \ldots, N$. Let R be the set of all these permutations. It is assumed that each of these rankings is chosen with the same probability $1/N!$

The random choice of r is immediately made known to all players $1, \ldots, N$. All players $1, \ldots, N$ know the market parameters α, β, γ and C.

The random choice of r is followed by N substages of the entry stage.

In the k-th substage the k-th player i_k in the ranking $r = (i_1,\ldots,i_N)$ has to make his entry decision modelled as the selection of a zero-one variable $z(i_k)$. He either selects $z(i_k) = 0$ which means that he does not enter or $z(i_k) = 1$ which indicates entry. His entry decision is immediately made known to all players. Therefore a player is fully informed about all previous entry decisions when he has to make his entry decision.

The entry stage ends after all players $1,\ldots,N$ have made their entry decisions. The set of all players i with $z(i) = 1$ is denoted by Z.

In both game models the game ends after the entry stage in the special case that Z is empty. If this happens all players receive zero payoffs.

2.3 Interpretation of the entry stage rules.

The rules of the entry stage are based on the idea that due to random factors different firms are more or less well prepared to enter a newly emerging market. Therefore each of them has a different delay time before the entry decision can be made. Therefore entry is modelled as sequential rather than simultaneous. Entry is described as irreversible. In order to justify this feature of the rules one may think of fixed costs as sunk costs. A decision not to enter cannot be reconsidered once it has been made. This is a simplifying assumption which can be justified as follows: The analysis of the game models shows that it is advantageous to be as early in the ranking as possible. Nothing can be gained by delaying the entry decision beyond the necessary delay time.

Profits obtained during the entry stage are neglected by the game models presented here. The entry stage is assumed to be short relative to the life time of the market. Only the long run profits obtained after the entry stage matter.

It seems to be possible to construct much more realistic but also much more complicated models which yield the same conclusion. One could introduce the option of exit and model fixed costs as partially sunk. The analysis can be expected to be much more complicated without substantially different results.

2.4 The supply stage.

Our theory compares two game models, the non-collusive and the collusive game. If the entry stage has determined a non-empty set Z of suppliers the supply stage follows in the non-collusive game.

The collusive game may also reach a supply stage. This happens in two cases. The first case arises if there is only one player in Z. Then the supply stage follows the entry stage. (Cartel bargaining makes no sense in this case.) If there are at least 2 players in Z then the bargaining stage follows the entry stage in the collusive game. The second case where a supply stage is reached in the collusive game occurs if no cartel agreement is reached in the bargaining stage.

In the supply stage each player $i \in Z$ selects his supply x_i, a non-negative real number. These decisions are made simultaneously and independently of each other.

In both game models the game ends after the supply stage if it is reached.

2.5 The bargaining stage.

The bargaining stage follows the entry stage in the collusive game if there are at least 2 players in the set Z of those who have entered. In the bargaining stage each player $i \in Z$ proposes a supply vector

(12) $$Y_i = (y_{ij})_{j \in Z}$$

which contains a proposed supply quantity $y_{ij} \geq 0$ for every $j \in Z$ as indicated by the subscript $j \in Z$. The supply vector Y_i is called the proposal of i. A supply vector

(13) $$Y = (y_j)_{j \in Z}$$

becomes a binding agreement,if and only if the following is true:

(14) $$Y_i = Y \qquad \text{for every } i \in Z.$$

The players $i \in Z$ make their proposals Y_i simultaneously and independently of each other.

After the bargaining stage the proposals Y_i of all players $i \in Z$ are made known to all players.

2.6 The non-collusive game.

In the non-collusive game the entry stage is immediately followed by the supply stage. The game ends after the supply stage. The payoffs for the players $i \in Z$ are the net profits P_i computed according to (5), (6) and (7). The players $i \notin Z$ receive zero payoffs.

2.7 The collusive game.

In the collusive game the supply stage follows the entry stage only if exactly one player has entered in the entry stage. Otherwise the bargaining stage follows the entry stage. If no binding agreement is reached in the bargaining stage then the supply stage follows the bargaining stage. Whenever the supply stage is reached, payoffs are as in the non-collusive game. If a binding agreement

(15) $Y = (y_j)_{j \in Z}$

is reached then the game ends after the bargaining stage. The supplies x_i are fixed by the agreement:

(16) $x_i = y_i$ for every $i \in Z$.

The players $i \in Z$ receive their net profits computed on the basis of (16) as their payoffs. The players $i \notin Z$ receive zero payoffs.

2.8 Some features of the extensive game representations.

In spite of the fact that detailed formal descriptions of both game models are not needed, it may be useful to point out some of their features. Let us denote the extensive form representation of the non-collusive game by Γ^1 and the extensive form representation of the collusive game by Γ^2. (The symbol Γ will be used for extensive games). The representation of decisions follows the order of stages and within the entry stage the order of substages. Simultaneous decisions are represented in the order given by the numbering of the players, the lower numbers coming first. The arbitrary convention about simultaneous decisions is needed since the tree structure of the extensive form requires a successive representation of simultaneous choices.

The entry stage begins with a random choice among N! alternatives. After each of these branches follow information sets for all players where the entry decisions have to be made. Since the players are informed on previous entry decision, when they have to make their entry decision, each of these information sets continues only one node of the tree. Up to the beginning of the stage after the entry stage both games are like games with perfect information.

The entry stage can end in $2^{N!}$ different ways. The set Z of suppliers contains a given player in $2^{N!-1}$ of these cases. Therefore in Γ^1 each player i has $2^{N!-1}$ information sets corresponding to possible decision situations in the supply stage.

In Γ^2 a bargaining stage arises if the number of players in Z is at least 2. This happens in $2^{N!-1}-N!$ cases. Therefore in Γ^2 each player i has $2^{N!-1}-N!$ information sets corresponding to decision situations in the bargaining stage.

For every player i = 1,...,N the collusive game Γ^2 has infinitely many information sets corresponding to decision situations in the supply stage. Cartel bargaining can break down in infinitely many ways.

It will be important for the game theoretical analysis of Γ^1 and Γ^2 that these games have subgames. These subgames will be named according to the decisions which have to be made at the beginning of the subgame. An entry subgame is a subgame which begins with an entry decision, a bargaining subgame begins with supply vector proposals and a supply subgame represents supply decisions.

Each node, where an entry decision has to be made, is the origin of an entry subgame. The supply subgames of Γ^1 have their origins at $2^{N!}-N!$ nodes representing possible endings of the entry stage. N! of these nodes are endpoints since they represent situations where nobody has entered.

The collusive game Γ^2 has two kinds of supply subgames. If in the entry stage only one player enters, then the node representing the end of the entry stage is the origin of a supply subgame. Other supply subgames represent situations where cartel bargaining has broken down; there are infinitely many such supply subgames in Γ^2. The bargaining subgames of Γ^2 have their origins at nodes representing possible ends of the entry stage, where at least 2 players have entered.

A subgame which contains at least one information set and which is not the whole game itself is called a proper subgame. An extensive game is called indecomposable, if it does not have any proper subgames; otherwise it is called decomposable. Obviously, the supply decision subgames of Γ^1 and Γ^2 are indecomposable. The bargaining subgames of Γ^2 and the entry subgames are decomposable. These subgames have supply subgames as proper subgames.

3. The solution concept.

A definite normative answer to the question how players should behave in a non-cooperative game must take the form of an equilibrium point in the sense of Nash (1951). Theories which prescribe non-equilibrium behavior are self-destroying prophecies since they create incentives for deviations from their own prescriptions.

Equilibrium properties should hold not only in the game as a whole but also in its subgames. This leads to the notion of a subgame perfect equilibrium point. Originally, the term "perfect" was used for such equilibrium points (Selten 1965 and 1973) but later a refined notion of perfectness has been introduced which makes it necessary to distinguish perfectness and subgame perfectness (Selten 1975).

The solution concept applied here will single out a unique subgame perfect equilibrium point in pure strategies wherever it can be successfully applied. It is defined by two selection principles applied to indecomposable games and by a recursive decomposition procedure which works its way backwards from the end of the game. The procedure serves the purpose to reduce the task of solving a decomposable game to the task of solving indecomposable games.

Indecomposable games are solved with the help of a symmetry principle and an efficiency principle. The symmetry principle requires that the solution should reflect symmetries of the game. The efficiency principle requires that there is no other equilibrium point where all players except those without strategic influence are better off.

The indecomposable games which have to be solved in the process of finding the solution of decomposable games capture local features of such games. Therefore, these indecomposable games will be called the local games. The symmetry and efficiency principles are applied to these local games. In this sense we speak of local symmetry and local efficiency as selection principles employed by the solution concept applied here.

A similar but much more elaborate approach to the problem of defining a solution for a limited class of extensive games has been presented elsewhere (Selten 1973).There it was necessary to define the solution in a much more complicated way even if the basic ideas underlying the construction are essentially the same as in the present approach.

The general equilibrium selection theory proposed by John C. Harsanyi and the author (Harsanyi-Selten 1980, 1982) also embodies principles of perfectness, local symmetry and local efficiency. An application to the problem at hand would require an approximation of our game models by finite games. It can be expected that the results would not be essentially different from those obtained here but the derivation would be burdened with a lot of technical detail. Therefore, a more direct approach has been chosen here.

3.1 Subgame perfect equilibrium points.

In this paper we shall only consider pure strategies. For every information set u in an extensive game Γ let Ψ_u be the set of choices at u. A <u>pure strategy</u> φ_i of player i in Γ is a function which assigns a choice $\varphi_i(u) \in \Psi_u$ to every information set u of player i. The symbol Φ_i will be used for the set of all pure strategies of player i in Γ.

Assume that Γ has N players 1,...,N. A pure strategy combination φ is an N-tuple

(17) $$\varphi = (\varphi_1,\dots,\varphi_N)$$

with $\varphi_i \in \Phi_i$. The set of all strategy combinations φ is denoted by Φ. For every $\varphi \in \Phi$ a vector

(18) $$H(\varphi) = (H_1(\varphi),\dots,H_N(\varphi))$$

of expected payoffs $H_i(\varphi)$ for the players 1,...,N is computed in the usual way. The pure strategy sets Φ_i together with the <u>payoff function</u> H constitute the <u>normal form</u> $(\Phi_1,\dots,\Phi_N;H)$ of Γ.

We shall always refer to pure strategies where we speak of strategies since no other strategies are considered here. Similarily, a strategy combination will always be a pure strategy combination.

It is convenient to introduce the following notation. If in a strategy combination $\varphi = (\varphi_1,\dots,\varphi_N)$ the i-th component is replaced by ψ_i, then a new strategy combination results which is denoted by φ/ψ_i. A strategy $\pi_i \in \Phi_i$ with

(19) $$H_i(\varphi/\pi_i) = \max_{\psi_i \in \Phi_i} H_i(\varphi/\psi_i)$$

is called a <u>best reply</u> to $\varphi \in \Phi$. An <u>equilibrium point</u> (in pure strategies) is a strategy combination $\pi = (\pi_1,\dots,\pi_N)$ with the property that for every i = 1,...,n the strategy π_i is a best reply to π.

Let Γ' be a subgame of Γ. We say that a strategy φ_i' of player i for Γ' is <u>induced</u> by a strategy φ_i of player i in Γ if φ_i and φ_i' assign the same choices to information sets of player i in the subgame Γ'. Analogously, the strategy combination $\varphi' = (\varphi_1',\dots,\varphi_N')$ <u>induced</u> on a subgame Γ' of Γ by a strategy combination $\varphi = (\varphi_1,\dots,\varphi_N)$ for Γ contains the strategies φ_i' induced by the corresponding strategies φ_i.

An equilibrium point π of Γ is called <u>subgame perfect</u> if an equilibrium point π' of Γ' is induced by π on every subgame Γ' of Γ.

<u>3.2 The symmetry principle</u>.

Let Γ be an indecomposable extensive game and let $(\Phi_1,\dots,\Phi_N,H)$ be the normal form of Γ. Intuitively, a symmetry is a renaming of players and strategies. Formally, a symmetry is a pair (σ,τ) where σ is a one-to-one mapping of the player set onto itself; $\tau=(\tau_1,\ldots,\tau_N)$ is a system of mappings such that τ_i is a one-to-one mapping of Φ_i onto $\Phi_{\sigma(i)}$. Moreover, the definition of a symmetry requires that (σ,τ) is payoff preserving in the following sense. For every $\varphi = (\varphi_1,\ldots,\varphi_N) \in \Phi$ let $\tau(\varphi)$ be that strategy combination $\psi \in \Phi$ which contains the strategies $\psi_{\sigma(i)} = \tau_i(\varphi_i)$. The pair (σ,τ) is payoff preserving if the following is true:

(20) $$H_i(\varphi) = H_{\sigma(i)}(\tau(\varphi))$$

for every $\varphi \in \Phi$ and for $i = 1,\ldots,N$. An equilibrium point $\pi \in \Phi$ is called symmetry invariant if we have

(21) $$\tau(\pi) = \pi$$

for every symmetry (σ,τ) of G.

The symmetry principle requires that the solution of an indecomposable game is a symmetry invariant equilibrium point.

3.3 The efficiency principle.

Let Γ be an indecomposable game and let $(\Phi_1,\ldots,\Phi_N,H)$ be the normal form of Γ. A player i is called inessential if Φ_i contains only one pure strategy. Otherwise, he is called essential. An equilibrium point $\pi \in \Phi$ is called efficient if there is no other equilibrium point $\psi \in \Phi$ with $H_i(\psi) > H_i(\varphi)$ for every essential player i.

The efficiency principle requires that the solution of an indecomposable game is efficient.

3.4 The solution of indecomposable games.

An indecomposable extensive game is called solvable if it has exactly one efficient and symmetry invariant equilibrium point in pure strategies. If Γ is a solvable indecomposable game, then its efficient and symmetry invariant equilibrium point $\pi \in \Phi$ is called the solution of Γ. The solution of Γ is denoted by $L(\Gamma)$. The class of all solvable indecomposable games is denoted by $\mathfrak{K}_1$. The function L which assigns $L(\Gamma)$ to every $\Gamma \in \mathfrak{K}_1$ is called the solution function for indecomposable games.

3.5 Comment.

Features of indecomposable games which are not captured by the normal form are neglected by our approach. However, this limitation is not a serious one as long as the indecomposable games to be solved have the strategic structure of normal forms. This is the case in the application to our game models. The indecomposable games arising there represent situations where each of the essential players makes just one decision; all of them act simultaneously.

3.6 Recursive decomposition.

In the following the recursive decomposition procedure will be introduced which serves the purpose to reduce the task of solving decomposable games to the task of solving indecomposable games.

An extensive game Γ is called truncatable if it is decomposable and if in addition to this all indecomposable subgames of Γ are solvable. For every truncatable extensive game we shall define the truncation $T(\Gamma)$. The truncation is a new game which results from Γ in the following way. The indecomposable proper subgames are cut off in the sense that the origins of these subgames become endpoints. The payoff vectors attached to these endpoints in $T(\Gamma)$ are the payoff vectors connected to the solutions of the respective indecomposable subgames of Γ. Outside of the indecomposable proper subgames of Γ the truncation $T(\Gamma)$ agrees with Γ.

The recursive decomposition procedure consists in the repeated application of the operation of forming a truncation. Let Γ_1 be the game $T(\Gamma)$. If Γ_1 is truncatable, then $\Gamma_2 = T(\Gamma_1)$ is formed, etc. In this way, one obtains a sequence of games $\Gamma, \Gamma_1, \ldots, \Gamma_k$ where each of the games $\Gamma_1, \ldots, \Gamma_k$ is the truncation of the preceding one in the sequence. The sequence is continued until it terminates in a non-truncatable game Γ_k. This may either be a decomposable game with at least one non-solvable indecomposable proper subgame or it may be an indecomposable game. If Γ_k is indecomposable and solvable, then Γ is called solvable. The local games of Γ are the indecomposable proper subgames of the games $\Gamma, \Gamma_1, \ldots, \Gamma_k$ and the game Γ_k, if it is indecomposable. Obviously, Γ is solvable, if and only if all its local games are solvable. The class of all solvable extensive games is denoted by $\mathfrak{S}$.

3.7 Comment.

In the following sense the recursive decomposition procedure achieves a decomposition of solvable decomposable games Γ into local games: Every information set of Γ belongs to a uniquely determined local game Γ.

The solution of a decomposable solvable game will be composed of the solutions of its local games. This idea is expressed by the extension principle formulated below.

3.8 Extension principle.

The following extension principle extends the solution function L from the class $\mathfrak{K}_1$ of solvable indecomposable games to the class $\mathfrak{K}$ of all solvable extensive games: If $\Gamma \in \mathfrak{K}$ is decomposable then at every information set of one of the players $i = 1,\ldots,N$ the choice prescribed by $L(\Gamma)$ is the choice prescribed by the solution of that local game of Γ to which this information set belongs.

3.9 Remarks.

The extension principle completes the definition of the solution concept applied here. The definition automatically yields a subgame perfect equilibrium point. This can be proved easily by induction on the number of truncations to be formed in the recursive decomposition procedure. A proof shall not be given here. A theorem which yields the assertion as a conclusion has been proved elsewhere (theorem 1, p.152 in Selten 1973).

3.10 Further comments.

The local games capture the local interests of the players. A player who has to make a decision at an information set should be motivated by the features of the relevant local game if he expects that in later local games the players will behave as prescribed by the solution concept. Therefore, the efficiency principle should be applied locally rather than globally. This is important since there may be a conflict between local and global efficiency. A simple numerical example which illustrates the point has been presented elsewhere (Selten 1973,p.166).

4. Solution of the game models.

The discussion of the extensive game representations Γ^1 and Γ^2 of the non-collusive and the collusive game in section 2.8 has exhibited the subgames of these games. To each of these subgames corresponds a local game. The local game represents the decisions of the concerning stage or substage (in the case of entry decisions). We shall speak of local entry games, local bargaining games and local supply games when we refer to the local games corresponding to entry subgames, bargaining subgames and supply subgames, respectively.

The determination of the solution of Γ^1 and Γ^2 will begin with the analysis of the local supply games which, of course, are nothing else than the supply subgames. Then we shall consider the local bargaining games of Γ^2 and finally the local entry games of both game models.

4.1 Solution of the supply subgames.

The structure of a supply subgame depends only on the set Z of players who have entered the market. It does not matter after which of the N! initial random choices Z has resulted from the entry decisions and it does not matter whether one looks at Γ^1 or Γ^2 or in which way cartel bargaining has broken down in the case of the collusive game.

Consider a supply subgame Γ'. Let $Z = \{i_1,...,i_z\}$ the set of players who have entered the market in Γ'. Formally, the players not in Z are also players of Γ' but since they have no decisions to make each of them has only one strategy in the normal form. Moreover, they receive payoffs zero, no matter what the essential players in Z do.

It is well known that the symmetric Cournot oligopoly with linear costs and demand as defined in 2.1 has exactly one equilibrium point in pure strategies, namely the Cournot solution where each of the players $i \in Z$ supplies the same quantity:

$$(22) \qquad x_i = \frac{\beta - \gamma}{\alpha(z+1)}$$

where z is the number of suppliers in Z. Equilibrium net profits are as follows:

$$(23) \qquad P_i = \frac{(\beta-\gamma)^2}{\alpha(z+1)^2} - C$$

for every $i \in Z$. In view of (9) and (10) this yields:

$$(24) \qquad P_i = \left(\frac{4}{(z+1)^2} - \frac{1}{s}\right) M$$

for $i \in Z$. These are the payoffs obtained by the players $i \in Z$ in the payoff vector connected to the solution of Γ'. The other players receive zero payoffs.

4.2 Solution of the local bargaining games.

Let Γ' be a local bargaining game of Γ^2. For similar reasons as in the case of supply subgames the structure of Γ' depends only on the set $Z = \{i_1,...,i_z\}$ of the z players who have entered. Moreover, z is at least 2. Formally, the players not in Z are also players of Γ' but they are inessential and receive zero payoffs no matter what the players in Z do.

If no binding agreement is reached then the players in Z receive the solution payoffs (24) of the subsequent supply subgame. Obviously Γ' has infinitely many equilibrium points. Those who result in no binding agreement are not efficient. Other equilibrium points are connected to common supply vector proposals which yield net profits at least as high as those in (24) for every $i \in Z$. Among these equilibrium points only those are efficient which yield net profits summing up to 1.

Obviously Γ' has many symmetries (σ,τ). Every permutation σ of $\{1,\dots,N\}$ which maps Z onto Z combined with identical mappings τ_i yields a symmetry of Γ'. The only pure strategy equilibrium point π of Γ' which satisfies $\pi = \tau(\pi)$ for such symmetries,specifies the following system of proposals:

(25) $\quad Y_i = Y = (y_j)_{j \in z} \quad$ for every $i \in Z$

with

(26) $\quad y_j = \frac{\beta-\gamma}{2\alpha z} \quad$ for every $j \in Z$

The net profits connected to this common proposal Y are the same for every $i \in Z$:

(27) $\quad P_i = \frac{(\beta-\gamma)^2}{4\alpha z} - C$

Since 4z is smaller than $(z+1)^2$ for $z \geq 2$ these net profits are higher than those in (23) which are obtained in the case of a breakdown of cartel bargaining.

We have shown that the system of proposals (25) is an efficient equilibrium point π of Γ' and that no other efficient equilibrium point in pure strategies can be symmetry invariant. However, we did not yet show that π is symmetry invariant.

Assume that (σ,τ) is a symmetry of Γ' with $\pi \neq \tau(\pi)$. Obviously, σ must map Z onto Z since the players not in Z have only one strategy in the normal form of Γ' and the τ_i are one-to-one mappings. Since the symmetry is payoff preserving we must have

(28) $\quad H'_i(\pi) = H'_{\sigma(i)}(\tau(\pi))$

for every $i \in Z$ where H' is the payoff function of the normal form of Γ'. We know that $H_i'(\pi)$ is equal to the right hand side of (27) for every $i \in Z$. Therefore, the same is true for $H_i'(\tau(\pi))$. This means that not only π but also $\tau(\pi)$ maximizes joint profits and divides them equally among the players in Z. However, the common proposal in π is the only supply quantity vector with this property. Therefore we cannot have $\pi \neq \tau(\pi)$.

It is now possible to draw the conclusion that the system of proposals specified by (25) and (26) is the uniquely determined efficient and symmetry invariant equilibrium point in pure strategies of Γ' or, in other words, the solution $L(\Gamma')$ of Γ'.

4.3 Solution of the local entry games of the non-collusive game.

The payoffs at the endpoints of the truncation $T(\Gamma^1)$ of the non-collusive game are zero for those who did not enter and are given by (24) for those who have entered. Consider a local entry game Γ' of Γ^1. Assume that player i_j, the j-th in the ranking fixed by the initial random choice, is the player who has to make his entry decision in Γ'.

It will be important to compare the number of entrants up to the beginning of Γ' with the maximal number of entrants compatible with non-negative Cournot net profits. In order to be able to describe this maximal number in a convenient way we shall use the notation int μ for the greatest integer not greater than μ where μ is a real number. Define

$$(29) \qquad m = \begin{cases} \text{int } 2\sqrt{s} & \text{for } s \geq 1 \\ 1 & \text{for } 0 < s < 1 \end{cases}$$

It follows by (24) that the maximal number of entrants compatible with non-negative Cournot net profits is $m-1$.

The non-collusive game Γ^1 is not solvable in the sense of the solution concept applied here if $2\sqrt{s}$ happens to be integer. In this border case some local entry games arise in the recursive decomposition procedure, where entry and non-entry both yield zero payoffs. For our purposes it is not necessary to define solutions for such border cases since the exceptional cases do not influence the integral which evaluates average joint Cournot profits.

In the following we shall assume $m \neq 2\sqrt{s}$. Let k be the number of players who have entered before player i_j has to make his entry decision in Γ'. It will be shown that the solution of Γ' is as follows:

$$(30) \qquad z(i_j) = \begin{cases} 0 & \text{for } k \geq m - 1 \\ 1 & \text{for } k < m - 1 \end{cases}$$

This means that player i_j enters, if and only if after his entry the number of entrants is at most the maximal number $m - 1$ of entrants compatible with positive Cournot net profits. It will be shown by induction on $N - j$ that (30) correctly describes the entry decision of player i_j specified by the solution $L(\Gamma')$ of Γ'.

It is clear that for $N - j = 0$ the only optimal entry decision is given by (30). Suppose that (30) correctly describes the solution of local entry games for $N - j = 0,\ldots,h$ and assume $N - j = h + 1$.

Consider the case $k \geq m - 1$. If player i_j enters nobody will enter after him. The number of suppliers at the end of the entry stage will be $z = k + 1 \geq m$. Therefore player i_j's payoff for $z(i_j) = 1$ in Γ' is negative. $z(i_j) = 0$ is optimal since it yields zero payoffs.

Now consider the case $k < m - 1$. For $h + k + 1 < m - 1$ all n players who have to make entry decisions after i_j will enter and finally there will be $z = h + k + 1 < m - 1$ suppliers in the market. For $h + k + 1 \geq m - 1$ the next $m - 2 - k$ players in the ranking will enter and later players will not enter. Finally, there will be $z = m - 1$ suppliers in the market. For $z \leq m - 1$ Cournot net profits in (24) are positive. Therefore $z(i_j) = 1$ is the only optimal choice of i_j in Γ' for $k < m - 1$.

The difficulty in the border case $m = 2\sqrt{s}$ arises for $k = m - 2$. The concerning local entry game is not solvable since both entry and non-entry yield zero payoffs. There are two pure strategy equilibrium points, namely $z(i_j) = 0$ and $z(i_j) = 1$. Both of them fail to be symmetry invariant since an obvious symmetry of the local game maps one to the other.

<u>4.4 Solution of the local entry games of the collusive game</u>.

In the case of the collusive game Γ^2 the local entry games can be solved in a similar way as in the case of the non-collusive game. Let Γ' be a local entry game of Γ^2 and let player i_j, the j-th in the ranking fixed by the initial random choice, be the player who has to make his entry decision in Γ'.

The cartel net profits obtained in the solution of a local bargaining game with $z \geq 2$ suppliers is given by (27). In view of (9) and (10) equation (27) can be rewritten as follows:

$$P_i = \left(\frac{1}{z} - \frac{1}{s} \right) M \quad \text{for } i \in Z \tag{31}$$

Note that (31) also describes the Cournot net profits obtained by a single supplier in the supply subgame reached for $z = 1$. Define

$$n = \text{int } s \tag{32}$$

Equation (31) shows that n is the maximal number of suppliers compatible with non-negative net profits.

The border case $n = s$ has to be excluded for the same reasons as $m = 2\sqrt{s}$ in section 4.3. In the border case $n = s$ the non-collusive game Γ^2 is not solvable since local entry games fail to be solvable if exactly $n - 1$ players have entered before the beginning of the local entry game.

In the following we shall assume $n \neq s$. Let k be the number of players who have entered before player i_j has to make his decision in Γ'. The solution of Γ' is as follows:

$$z(i_j) = \begin{cases} 0 & \text{for } k \geq n \\ 1 & \text{for } k < n \end{cases} \tag{33}$$

This means that player i_j enters,if and only if after his entry the number of entrants is at most the maximal number n of entrants compatible with positive cartel net profits. The proof of the assertion that (33) correctly describes the solution $L(\Gamma')$ of Γ' will not be given here since it is analogous to the proof of the assertion expressed by (30) in section 4.3.

4.5 Properties of the solution of the non-collusive game.

It has been shown that Γ^1 is solvable with the exception of the border case where $2\sqrt{s}$ is an integer. In the following we shall assume that $2\sqrt{s}$ is not an integer. By definition the solution $L(\Gamma^1)$ of Γ^1 agrees with the solutions of the local games of Γ^1. Therefore, the subgame perfect equilibrium point singled out by the solution concept applied here is fully described by sections 4.1 and 4.3. However, we did not yet look at the question which plays of Γ^1 result and which expected payoffs are obtained if $L(\Gamma^1)$ is played.

By definition we have $m = 1$ in the case $0 < s < 1$. It follows by (30) that in this case no player enters if $L(\Gamma^1)$ is played.

Now consider the case $s > 1$ (we have excluded $s = 1$ by $m \neq 2\sqrt{s}$). It has been assumed that N is greater than s (see (10) in section 2.1). For $s > 1$ we have:

(34) $$(s + 1)^2 - 4s = (s - 1)^2 > 0$$

and therefore:

(35) $$s + 1 > 2\sqrt{s} > m$$

This yields

(36) $$N > m - 1$$

For $s < 1$ inequality (36) holds, too. (36) has the consequence that the number of players is sufficiently great to permit entrance by the maximal number of suppliers compatible with non-negative Cournot net profits.

We shall now describe what happens if $L(\Gamma^1)$ is played. First a ranking $r = (i_1,\ldots,i_n)$ is fixed by the initial random choice. Then in view of (30) the players $i_1,\ldots,i_{m-1}$ enter the market. The players $i_m,\ldots,i_N$ do not enter the market. This yields $Z = \{i_1,\ldots,i_{m-1}\}$. In the supply stage every supplier chooses his Cournot supply:

(37) $$x_i = \frac{\beta-\gamma}{\alpha m} \qquad \text{for } i = i_1,\ldots,i_{m-1}$$

At the end of the game the players in Z receive their Cournot net profits.

(38) $$P_i = (\frac{4}{m^2} - \frac{1}{s})M \qquad \text{for } i = i_1,\ldots,i_{m-1}$$

The other players receive payoffs zero. The sum of all payoffs obtained at the end is called <u>joint Cournot profits</u> and is denoted by P_C:

(39) $$P_C = (m-1)(\frac{4}{m^2} - \frac{1}{2})M$$

Let us now turn our attention to the expected payoffs for the solution $L(\Gamma^1)$ of Γ^1. Let $H^1 = (H_1^1,\ldots,H_N^1)$ be the payoff function of the normal form of Γ^1. All rankings r are equally probable. Therefore every player i has the same probability $(m-1)/N$ to be in the set Z of suppliers. This together with (38) yields:

(40) $$H_i(L(\Gamma^1)) = \frac{m-1}{N}(\frac{4}{m^2} - \frac{1}{s})M$$

for $i = 1,\ldots,N$. Equations (39) and (40) hold for the case $0 < s < 1$, too, since m is defined as 1 in this case.

4.6 Properties of the solution of the collusive game.

In the following it will be assumed that s is not an integer since otherwise Γ^2 is not solvable. The subgame perfect equilibrium point $L(\Gamma^2)$ singled out by the solution concept applied here is fully described by sections 4.1, 4.2 and 4.4. In the following we shall describe what happens if $L(\Gamma^2)$ is played.

First a ranking $r = (i_1,\ldots,i_N)$ is fixed by the initial random choice. Then, in view of (33) the players $i_1,\ldots,i_n$ enter the market. (It has been assumed that N is greater than s which is greater than n). The remaining players $i_{n+1},\ldots,i_N$ do not enter the market. In the case $n=1$ player i_1 offers the monopoly supply in the supply stage and receives monopoly net profits at the end of the game. In the case $n > 1$ the bargaining stage follows the entry stage and each player in $Z = \{i_1,\ldots,i_n\}$ makes the same proposal:

(41) $\quad Y_i = Y = (y_j)_{j\in Z} \qquad$ for $i = i_1,\ldots,i_n$

with

(42) $\quad y_j = \frac{\beta-\gamma}{2\alpha n} \qquad$ for $j = i_1,\ldots,i_n$

According to (31) the net profits connected to this common proposal are as follows:

(43) $\quad P_i = (\frac{1}{n} - \frac{1}{s})M \qquad$ for $i = i_1,\ldots,i_n$

At the end of the game players $i_1,\ldots,i_n$ receive these <u>cartel net profits</u> as payoffs. (43) also describes the payoffs obtained by player i_1 in the case $n = 1$, where the supply stage is reached after the entry stage. The players $i_{n+1},\ldots,i_N$ receive zero payoffs.

In the case $0 < s < 1$ nobody enters and the sum of all payoffs is 0. The sum of all payoffs obtained for $s > 1$ is called <u>joint cartel profits</u> and is denoted by P_M:

(44) $\quad P_M = (1 - \frac{n}{s})M \qquad$ for $s > 1$

Let $H^2 = (H_1^2,\ldots,H_N^2)$ be the payoff function of the normal form of Γ^2. Since all rankings are equally probable each player i has the same probability n/N to be in the set Z of supplies. This together with (43) yields:

$$(45) \qquad H_i(L(\Gamma^2)) = \begin{cases} \frac{1}{N}\left(1 - \frac{n}{s}\right)M & \text{for } s > 1 \\ 0 & \text{for } 0 < s < 1 \end{cases}$$

for $i = 1,\dots,N$.

5. Comparison of average joint profits.

The non-collusive and the collusive games are theories on the strategic structure of the situation arising with the emergence of a new market. The non-collusive game applies to an institutional environment with strictly enforced cartel laws and the collusive game describes a situation without such restrictions of collusion. The solutions of both games can be meaningfully compared with each other since the same Cournot oligopoly model with linear costs and demand underlies both games.

The way in which a precise meaning will be given to the question whether cartel laws are good or bad has already been indicated in the introduction. Within the theoretical framework presented here the question boils down to a comparison of expected joint Cournot profits and expected joint cartel profits under reasonable assumptions on the joint distribution of market parameters.

5.1 Average joint profits.

For $0 < s < 1$ no player enters in both game models and all players receive zero payoffs. Obviously, this interval is without significance for the comparison. Therefore, we shall restrict our attention to parameter combinations with $s \geq 1$.

For integer values of s the collusive game Γ^2 is not solvable. The non-collusive game is not solvable if $2\sqrt{s}$ is an integer. This does not matter as far as the comparison is concerned. Average joint profits will be computed as expectations under a continuous joint probability distribution of the market parameters. Therefore, we simply shall proceed as if the expressions derived for joint Cournot profits and joint cartel profits were valid for all $s \geq 1$.

In the following we shall repeat those few formulas of section 4 which are needed for the comparison of joint profits.

(46) $m = \text{int } 2\sqrt{s}$

(47) $n = \text{int } s$

(48) $P_C = (m-1)(\frac{4}{m^2} - \frac{1}{s})M$

(49) $P_M = (1 - \frac{n}{s})M$

where int μ denotes the greatest integer not greater than μ. In (46) we have made use of the fact that we assume:

(50) $s \geq 1$

A Cournot market with linear costs and linear demand as introduced in 2.1 has four parameters α, β, γ and C. The joint profits P_C and P_M depend only on the two parameters M and s defined by (9) and (10). Therefore, it is convenient to make assumptions directly on the joint probability distribution of the pair (s,M).

It is assumed that (s,M) is continously distributed. Let f(s,M) be the probability density of (s,M). This density is to be understood as conditional on $s \geq 1$ since we are only interested in markets which are profitable in the sense of (50). We assume that f has a closed bounded range and is continuously differentiable over this range. Since the range is bounded there are constants $\bar{s}$ and $\bar{M}$ such that f(s,M) is zero outside the following rectangle

(51) $1 \leq s \leq \bar{s}$

(52) $0 \leq M \leq \bar{M}$

Clearly, one has to assume an upper bound $\bar{s} < N$ if one wants to stay in the framework of the two game models since otherwise the case of entry restriction by a lack of potential competitors could arise. An upper bound $\bar{M}$ on gross profit opportunities hardly needs any justification.

We are interested in the expected values $E(P_C)$ and $E(P_M)$ of P_C and P_M :

(53) $$E(P_C) = \int_1^{\bar{s}} \int_0^{\bar{M}} P_C f(s,M)\,dM\,ds$$

(54) $$E(P_M) = \int_1^{\bar{s}} \int_0^{\bar{M}} P_M f(s,M)\,dM\,ds$$

In view of the assumptions on f these expectations are finite. Both P_C and P_M are products of M and a factor depending only on s. The notations $Q_C(s)$ and $Q_M(s)$ are introduced for these factors:

(55) $$Q_C(s) = (m-1)\left(\frac{4}{m^2} - \frac{1}{s}\right)$$

(56) $$Q_M(s) = 1 - \frac{n}{s}$$

We shall refer to $Q_C(s)$ as the Cournot profit factor and to $Q_M(s)$ as the cartel profit factor. The profit factors express joint net profits as a fraction of monopoly gross profits and therefore can be looked upon as conversion factors which measure how much of the maximal gross profits achievable is transformed into net profits.

In order to be able to rewrite (53) and (54) in a simpler way we introduce a function W(s) called the weight of s:

(57) $$W(s) = \int_0^{\bar{M}} Mf(s,M)\,dM$$

The weight W(s) can be interpreted as the conditional expectation of M given s multiplied with the marginal density of s. One can also think of W(s) as the contribution of markets of size s to the expectation of M. Therefore W(s) is a measure of the importance of the gross profit opportunities offered by markets of size s. The assumptions on f have the consequence that W(s) is defined for all s with $1 \leq s \leq \bar{s}$ and is bounded and continuous over this range.

With the help of (55), (56) and (57) the expectations of P_C and P_M can be expressed as an integral over a function of s alone:

$$(58)\qquad E(P_C) = \int_0^{\bar{s}} Q_C(s)W(s)ds$$

$$(59)\qquad E(P_M) = \int_0^{\bar{s}} Q_M(s)W(s)ds$$

$E(P_C)$ and $E(P_M)$ are the average joint profits to be compared. There are intervals for s where $Q_C(s)$ is greater than $Q_M(s)$ and there are other intervals where the opposite is true. Therefore, the shape of W(s) is important for the comparison.

5.2 Assumption on the shape of the weight function.

As has been pointed out before, W(s) can be looked upon as a measure of the gross profit opportunities offered by markets of size s. There is no reason to suppose that W(s) grows very fast in some parts of the interval $1 \leq s \leq \bar{s}$. In both game models the number of suppliers, namely m - 1 and n, respectively, is closely related to the market size s. Casual empiricism suggests that the number of suppliers varies over a great range. It does not seem to be the case that gross profit opportunities are concentrated in a small part of this range.It is more plausible to suppose that W(s) is quite flat over the interval $1 \leq s \leq \bar{s}$.

Consider the elasticity of W(s) with respect to s. Since f is assumed to be continuously differentiable this eleasticity exists. The idea that W(s) does not increase too fast with increasing s can be given a more precise form by the assumption that the elasticity of W(s) with respect to s is smaller than 1:

$$(60)\qquad \frac{dW(s)}{ds} \cdot \frac{s}{W(s)} < 1 \qquad \text{for } 1 \leq s < \bar{s}$$

The comparison of average joint profits will rely on this assumption on the shape of the weight function. Condition (60) can be restated in an equivalent and more convenient form. For this purpose we introduce the <u>average weight function</u> w(s)

$$(61)\qquad w(s) = \frac{W(s)}{s}$$

Differentiation of the right hand side shows that w(s) is decreasing over the range $1 \leq s \leq \bar{s}$, if and only if condition (60) is satisfied.

An additional assumption on W(s) concerns the upper bound $\bar{s}$. The range where W(s) is positive should be sufficiently wide. The following assumption will be made:

$$(62)\qquad \bar{s} > 2$$

In the interval $1 \leq s \leq 2$ the joint profits P_C and P_M are equal since only one player enters in both game models.

5.3 First intermediary result.

It is our final aim to prove that under assumption (60) on the shape of the weight function, average joint Cournot profits $E(P_C)$ are greater than average joint cartel profits $E(P_M)$ for $\bar{s} > 2$. In this section,we shall derive an intermediary result which shows that it is sufficient to examine the special case w(s) = 1.

With the help of (61) the difference between both average joint profits can be expressed as follows:

$$(63)\qquad E(P_C) - E(P_M) = \int_1^{\bar{s}} (Q_C(s) - Q_M(s))sw(s)ds$$

It is convenient to introduce the notation D(s) for the intergral in (63):

$$(64)\qquad D(s) = (Q_C(s) - Q_M(s))s$$

In view of (55) and (56) we have:

$$(65)\qquad D(s) = (m-1)\left(\frac{4s}{m^2} - 1\right) - (s-n)$$

The integral on the right hand side of (63) can be rewritten as a double integral:

$$\text{(66)} \qquad \int_1^{\bar{s}} D(s)w(s)ds = \int_1^{\bar{s}} \int_0^{w(s)} D(s)dtds$$

The right hand side of (66) can be interpreted as an area integral of D(s). In order to see this imagine a rectangular coordinate system which shows s horizontally and t vertically. Let F be the area below t = w(s), right of s = 1 and above t = O. The right hand side of (66) is the integral of D(s) over F. The same area integral will now be evaluated in a different way. Let v(u) be the inverse of w(s) over the interval $O \leq u \leq w(1)$. Since w(s) is monotonically decreasing this inverse v(u) exists. v(u) is monotonically decreasing, too. The area integral of D(s) over F permits the following alternative evaluation:

$$\text{(67)} \qquad \int_1^{\bar{s}} D(s)w(s)ds = \int_0^{w(1)} \int_1^{v(u)} D(s)dsdu$$

The right hand side of (67) is positive for $\bar{s} > 2$ if we have:

$$\text{(68)} \qquad \int_1^{S} D(s)ds > O \qquad \text{for every } S > 2$$

For $1 \leq S \leq 2$ the integral assumes the value zero since D(s) is zero in the interval $1 \leq s \leq 2$. This is due to the fact that in this interval P_C and P_M are both equal to monopoly net profits. In view of (63) inequality (68) is equivalent to the assertion that $E(P_C)$ is greater than $E(P_M)$ for $\bar{s} > 2$ in the special case $w(s) = 1$ for $1 \leq s \leq \bar{s}$.

Result: Under assumption (60) on the shape of the weight function we have:

$$\text{(69)} \qquad E(P_C) > E(P_M)$$

for $\bar{s} > 2$ if (68) is satisfied.

5.4 Second intermediary result.

In this section, it will be shown that inequality (68) holds for every $S > 2$ if it holds for integer values of S with $S > 2$.

The right hand side of (65) can be transformed as follows:

$$(70) \qquad D(s) = -\left(\frac{m-2}{m}\right)^2 s - m + n + 1$$

Let k be an integer with $k \geq 2$ and consider the interval $k \leq s < k+1$. In this interval we have $n = k$. The number m may not be constant in this interval since m changes its value from h to $h+1$ at points of the form

$$(71) \qquad s = \left(\frac{h+1}{2}\right)^2$$

where h is an integer. There may be a point of this kind at $k+.25$. This happens e.g. in the case $k = 2$. However, whether a point of this kind is in the interval or not, the function $D(s)$ is non-increasing in the whole interval since $(m-2)/m$ is increased if m is increased. In fact for $k = 3,4,\ldots$ the function $D(s)$ is decreasing in the whole interval. The case $k = 2$ merits special attention. We have:

$$(72) \qquad D(s) = 1 \qquad \text{for } 2 \leq s < 2.25$$

and

$$(73) \qquad D(s) = -\frac{1}{9}s \qquad \text{for } 2.25 \leq s < 3$$

It will be convenient to use the following notation:

$$(74) \qquad A(S) = \int_1^S D(s)ds$$

It does not matter whether one integrates from 1 to S or 2 to S since $D(s)$ is zero for $1 \leq s < 2$.

Suppose that $A(S)$ is not positive for some S with $k < S < k+1$ where k is an integer. Consider the case $D(S) \geq 0$. In view of (72) and (73) we must have $k > 2$ in this case. $D(s)$ is decreasing in the interval $k \leq s < k+1$. Therefore $D(s)$ is positive for $k \leq s < S$ and we must have $A(k) < A(S)$.

Now consider the case $D(S) < 0$. In this case $D(s)$ is decreasing for $S \leq s < k+1$ and we must have $A(k+1) < A(S)$. We have seen that either $A(k)$ or $A(k+1)$ must be negative if $A(S)$ is not positive for some S with $k < S < k+1$.

Result: Under the assumption (60) on the shape of the weight function we have:

(75) $$E(P_C) > E(P_M)$$

for $\bar{s} > 2$ if we have:

(76) $$A(S) > 0 \qquad \text{for } S = 3,4,\ldots$$

where A is defined by (74).

5.5 Derivation of the final result.

In order to derive the final result it is sufficient to show (76). In view of (65) the function $A(S)$ can be split into two parts:

(77) $$A(S) = \int_1^S (m-1)\left(\frac{4s}{m^2} - 1\right)ds - \int_1^S (s-n)\,ds$$

If S is integer then the second integral is nothing else than $(S-1)/2$. The first part will be evaluated in subintervals where m is constant. For this purpose we introduce the auxiliary variable Q_h

(78) $$Q_h = \int_{\frac{1}{4}h^2}^{\frac{1}{4}(h+1)^2} (m-1)\left(\frac{4s}{m^2} - 1\right)ds$$

for every $h = 2,3,\ldots$ we have $m = h$ in the interval of integration. The evaluation of the integral yields:

(79) $$Q_h = \frac{h-1}{2}\left(\frac{2h+1}{2h}\right)^2$$

For every $h = 3,4,\ldots$ we obtain:

(80) $$A\left(\frac{h^2}{4}\right) = -\frac{1}{2}\left(\frac{h^2}{4} - 1\right) + \sum_{j=2}^{h-1} \frac{j-1}{2}\left(\frac{2j+1}{2j}\right)^2$$

One receives a lower bound of $A(h^2/4)$ if one inserts 1 instead of $(2j+1)^2/4j^2$ in (80). This yields:

(81) $$A(\frac{h^2}{4}) > -\frac{1}{2}(\frac{h^2}{4} - 1) + \frac{1}{4}(h-1)(h-2)$$

Consider an integer S with

(82) $$\frac{h^2}{4} \leq S < (\frac{h+1}{4})^2$$

In view of (77) and (81) we have:

(83) $$A(S) > -\frac{1}{2}(S-1) + \frac{1}{4}(h-1)(h-2) + \frac{h-1}{h^2} \int_{\frac{h^2}{4}}^{S} (4s-h^2)\,ds$$

Let B(S) denote the right hand side of (84). We shall look at B(S) as continuously depending on S, even if we are interested in integer values of S only. Consider the derivative B'(S) of B(S) with respect to S:

(84) $$B'(S) = -\frac{1}{2} + \frac{h-1}{h^2}(4S-h^2)$$

Obviously, this derivative is increasing with S. The function B(S) assumes its minimum in the interval (82) where this derivative vanishes. Let $\tilde{S}$ be the value of S where this is the case:

(85) $$\tilde{S} = \frac{h^2}{4} + \frac{1}{2} \cdot \frac{h^2}{h-1}$$

Evaluation of $B(\tilde{S})$ yields:

(86) $$B(\tilde{S}) = -\frac{1}{2}(\frac{h^2}{4} + \frac{1}{2}\frac{h^2}{h-1} - 1) + \frac{1}{4}(h-1)(h-2)$$
$$+ \frac{h-1}{h^2} \cdot \frac{1}{2} \cdot \frac{h^2}{h-1} \cdot \frac{1}{4}\frac{h^2}{h-1}$$

(87) $$B(\tilde{S}) = \frac{1}{8}h^2 - \frac{1}{8}\frac{h^2}{h-1} - \frac{3}{4}h + 1$$

In view of $h/(h-1) \leq 2$ we have:

(88) $B(\hat{\tilde{S}}) > \frac{1}{8}h^2 - h + 1$

This shows that $B(\hat{\tilde{S}})$ is positive for $h \geq 8$. Consequently, $B(\hat{\tilde{S}})$ is positive for $\hat{\tilde{S}} \geq 16$. It follows that $A(S)$ is positive for all integers S with $S \geq 16$. Numerical computation shows that $A(S)$ is positive for $S = 3, \ldots, 15$, too. A table for the values of $A(S)$ up to $S = 30$ has been prepared in order to give an impression of the way in which the sequence develops with increasing S. The following theorem states the main result of this paper.

Theorem: Under assumption (60) on the shape of the weight function we have:

(89) $E(P_C) > E(P_M)$

for $\bar{s} > 2$.

S	A(S)	S	A(S)	S	A(S)
		11	2.57	21	6.90
2	.00	12	3.46	22	6.89
3	.03	13	3.53	23	7.28
4	.64	14	3.64	24	8.07
5	.51	15	4.25	25	9.25
6	1.14	16	5.34	26	8.93
7	1.22	17	5.06	27	8.97
8	1.52	18	5.21	28	9.37
9	2.46	19	5.81	29	10.13
10	2.24	20	6.84	30	11.25

Table: Values of $A(S)$ for $S = 2, \ldots, 30$.

5.6 Concluding remark

It has been shown that under plausible assumptions the theoretical framework presented here yields the conclusion that cartel laws are good for business in the sense of greater average joint profits. The assumptions are sufficient conditions for the result but they are far from necessary. However, it is not obvious what kind of weaker assumptions could lead to the same result.

References

Cournot, A. (1838) Recherches sur les principes mathématiques de la théorie des richesses, Paris 1838.

Harsanyi, John-C. and R.Selten (1980) A General Theory of Equilibrium Selection in Games, Chapter 2, Working Paper No. 105, Institute of Mathematical Economics, University of Bielefeld.

Harsanyi, John-C. and R.Selten (1982) A General Theory of Equilibrium Selection in Games, Chapter 3, Working Paper No. 114, Institute of Mathematical Economics,University of Bielefeld.

Kuhn, H.W. (1953) Extensive Games and the Problem of Information. In H.W. Kuhn and A.W. Tucker (eds.) Contributions to the Theory of Games, Vol.II, Annals of Mathematics Studies 28, p.193-216.

Nash, J.F. (1951) Non-Cooperative Games, Annals of Mathematics 54, p.155-162.

Scherer, F.M. (1970) Industrial Market Structure and Economic Performance, Chicago 1970.

Selten, R. (1965) Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit, Zeitschrift für die gesamte Staatswissenschaft 121, p.301-324, 667-689.

Selten, R. (1973) A Simple Model of Imperfect Competition, where 4 are Few and 6 are Many, International Journal of Game Theory, Vol.2, 3, p.141-201.

Selten, R. (1975) Reexamination of the Perfectness Concept for Equilibrium Points of Extensive Games, International Journal of Game Theory, p.25-55

"WIRTSCHAFTSTHEORETISCHE ENTSCHEIDUNGSFORSCHUNG"

A series of books published by the Institute of Mathematical Economics, University of Bielefeld.

Wolfgang Rohde
Ein spieltheoretisches Modell eines Terminmarktes (A Game Theoretical Model of a Futures Market).
The model takes the form of a multistage game with imperfect information and strategic price formation by a specialist. The analysis throws light on theoretically difficult empirical phenomena.

Vol. 1 176 pages price: DM 24,80

Klaus Binder
Oligopolistische Preisbildung und Markteintritte (Oligopolistic Pricing and Market Entry).
The book investigates special subgame perfect equilibrium points of a three-stage game model of oligopoly with decisions on entry, on expenditures for market potential and on prices.

Vol. 2 132 pages price: DM 22,80

Karin Wagner
Ein Modell der Preisbildung in der Zementindustrie (A Model of Pricing in the Cement Industry).
A location theory model is applied in order to explain observed prices and quantities in the cement industry of the Federal Republic of Germany.

Vol. 3 170 pages price: DM 24,80

Rolf Stoecker
Experimentelle Untersuchung des Entscheidungsverhaltens im Bertrand-Oligopol (Experimental Investigation of Decision-Behavior in Bertrand-Oligopoly Games).
The book contains laboratory experiments on repeated supergames with two, three and five bargainers. Special emphasis is put on the end-effect behavior or experimental subjects and the influence of altruism on cooperation.

Vol. 4 197 pages price: DM 28,80

Angela Klopstech
Eingeschränkt rationale Marktprozesse (Market processes with Bounded Rationality).
The book investigates two stochastic market models with bounded rationality, one model describes an evolutionary competitive market and the other an adaptive oligopoly market with Markovian interaction.

Vol. 5 104 pages price: DM 29,80

Hansjörg Haas
Optimale Steuerung unter Berücksichtigung mehrerer Entscheidungsträger (Optimal Control with Several Policy Makers).
The analysis of macrroeconomic systems with several policy makers as noncooperative and cooperative dynamic games is extensively discussed and illustrated empirically by econometric models of Pyndick for the US and Tintner for Austria.

Vol. 6 213 pages price: DM 42,--

Ulrike Leopold-Wildburger
Gleichgewichtsauswahl in einem Verhandlungsspiel mit Opportunitätskosten (Equilibrium Selection in a Bargaining Game with Opportunity Costs).
After a detailed introduction to the relevant parts of the Harsanyi-Selten equilibrium selection theory, this theory is applied to a noncooperative game model of a bargaining problem with opportunity costs of participating in negotiations.

Vol. 7 155 pages price: DM 38,80

Orders should be sent to:

Pfeffersche Buchhandlung, Alter Markt 7, 4800 Bielefeld 1, West Germany.