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1. Introduction

The concept of an evolutionarily stable strategy introduced by J. Maynard Smith and G.R. Price has become the cornerstone of evolutionary game theory, a new branch of game theory concerned with applications to sociobiology (Maynard Smith and Price 1973). The famous book by von Neumann and Morgenstern has laid the foundation of both non-cooperative and cooperative game theory (von Neumann and Morgenstern 1944). John F. Nash proposed the notion of an equilibrium point and proved existence for finite games (Nash 1951). Modern non-cooperative game theory is almost exclusively devoted to the study of various kinds of equilibrium points and their properties. Evolutionarily stable strategies can be described as strategies in special equilibrium points. From the mathematical point of view, evolutionary game theory is a part of non-cooperative game theory.

Game theory has been developed as a theory of rational behavior in interpersonal conflict situations. Economics and other social sciences were the intended fields of application. It is surprising that a mathematical tool which has been tailored for the needs of the social sciences now finds its way into the natural sciences. Since game theory has been based on an idealized picture of human rationality, applicability to animal behavior is by no means obvious.

Evolutionary game theory does not suppose that animals are rational. Mindless forces of natural selection are assumed to exert pressure towards optimization. This creates the appearance of rationality.

Up to now, evolutionary game theory has been developed in the framework of normal form games. However, an adequate description of sequential features of animal conflicts requires extensive game models. It is the main purpose of this paper to generalize the concept of an evolutionarily stable strategy to extensive 2-person games. Unfortunately, this is not a straightforward task.

The remainder of the introduction presupposes some familiarity with game theoretical concepts. However, later everything will be explained in detail. In principle no prior knowledge of game theory is necessary in order to be able to read the paper, even if it will be of great help to be at least superficially acquainted with the field. The condensed and somewhat imprecise preview in the remainder of the introduction cannot completely avoid the use of unexplained terms.

For a long time it was a commonly held view among game theorists that an extensive game is adequately represented by its normal form. However, it turned out that important distinctions between different kinds of equilibrium points for extensive games cannot be based on the normal form. This has lead to the notion of a perfect equilibrium point (Selten 1965, 1973, 1975). The perfectness requirement excludes unreasonable decisions at unreached parts of the extensive game. Related but different difficulties are faced by the generalization of evolutionary game theory to the framework of extensive games.

Section 2 will introduce the definition of an evolutionarily stable strategy for a symmetric bimatrix game and its biological interpretation. The customary abbreviation ESS will be used for "evolutionarily stable strategy". The notion of an extensive 2-person game with perfect recall will be explained in section 3; these are the extensive games to be investigated here. It will be argued that the assumption of perfect recall is justified in the biological context. The relationship between mixed and behavior strategies will be discussed in section 4.

Biological game models are endowed with natural symmetries. Therefore, evolutionary game theory always deals with symmetric games. It is not immediately clear what should be understood under a symmetric extensive 2-person game. This problem is discussed in section 5. The notion of a symmetry will be introduced as a mapping from choices to choices. Since an extensive game may have several symmetries, one of them must be specified as the natural one. This leads to the definition of a symmetric extensive 2-person game.

A symmetric extensive 2-person game has a symmetric normal form. The symmetric normal from is an ordinary symmetric bimatrix game. The usual definition of an ESS can be applied to the symmetric normal form. However, it will be shown in section 6 why this way of generalizing evolutionary game theory is unsatisfactory. It turns out that a definition in terms of behavior strategies is preferable. This will yield the concept of a "direct ESS". The word "direct" indicates that the notion is the result of a direct translation of the usual definition to the space of behavior strategies.

Unfortunately, the direct ESS notion cannot be proposed as the final answer to the problem of defining a satisfactory ESS concept for symmetric extensive 2-person games. Many biological game models with intuitively plausible solutions do not have a direct ESS. Therefore, a more liberal ESS concept will be proposed in section 7.

The restrictive nature of the direct ESS concept is due to the fact that essentially all parts of the game must be reached with positive probability by a direct ESS. Unreached parts of the game cause instabilities since no selective presure is exerted there. It will be argued that this difficulty is due to an inherent overprecision of biological extensive game models. Actually, a non-optimal choice may be taken occasionally by mistake. Thereby, a part of the game can be reached which would remain unreached otherwise. This "trembling hand" approach leads to the definition of a perturbed game where some choices have to be taken with small minimum probabilities due to the possibility of mistakes. A "limit ESS" is then defined as a limit of direct ESS's for perturbed games whose minimum probabilities vanish in the limit. The limit ESS is the proposed generalization of the ESS concept to symmetric extensive 2-person games.

The trembling hand approach is essentially the same as in the refined perfectness definition (Selten 1975). However, there is one important difference: zero minimum probabilities in perturbed games are not excluded. This has the consequence that a direct ESS of an unperturbed game is always a limit ESS.

Curiously enough, the same tools which have been used to exclude unreasonable equilibrium points by the perfectness requirement also yield a less restrictive ESS definition.

In order to obtain insight into the nature of a limit ESS it is necessary to explore the properties of a direct ESS for a perturbed game. This is done in section 8. The property that, roughly speaking, essentially all parts of the game must be reached is called "pervasiveness". The pervasiveness of a direct ESS permits the derivation of useful local optimality properties.

A distinction between image confronted and image separated information sets has important consequences for a direct ESS of a perturbed game and for a limit ESS of an unperturbed game. This will be the subject matter of section 9. If at least one play intersects an information set and its symmetric image under the natural symmetry, then this information set is called "image confronted"; otherwise it is called "image detached". It will be shown that a direct ESS of a perturbed game prescribes strong local best replies at image detached information sets and that, therefore, a limit ESS prescribes pure local strategies at image detached information sets.

The necessary conditions obtained in sections 8 and 9 for a direct ESS of a perturbed game are summarized by the notion of a "locally stable strategy". A counterexample shows that these necessary conditions are not sufficient. This seems to exclude a convenient decentralized characterization of a direct ESS for a perturbed game.

Complex extensive games often permit a decomposition into subgames and truncations. In a truncation some subgames are replaced by payoff vectors derived from subgame solutions. Subgames and truncations may themselves be further decomposed in this way until finally some indecomposable "elementary games" are obtained. An elementary game is called symmetric if it is its own symmetric image under the natural symmetry; otherwise it is called asymmetric. Decomposition will be the subject matter of section 10. It will be shown that a direct ESS of a perturbed game induces direct ESSs on symmetric elementary games and gives rise to strong equilibrium points of asymmetric elementary games. A set of necessary conditions for a limit ESS concerning strategies induced on subgames and truncations will be summarized by theorem 10.

The necessary conditions of theorem 10 are a powerful instrument of analysis for an important special class of games, called "simultaneity games". In these games imperfect information may result from the fact that at some points in time both players have to make simultaneous decisions but they are always fully informed about each other's past choices. One can expect that many potentially interesting models of animal conflicts are of this type. It will be shown in section 11 that the necessary conditions for a direct ESS of a perturbed game which characterize a locally stable strategy are actually sufficient in the case of a simultaneity game. In order to derive sufficient local conditions for a limit ESS the notion of a regular ESS for symmetric bimatrix games will be introduced. Regularity excludes the possibility of alternative pure best replies which are not used with positive probability by the ESS.Theorem 12 contains the sufficient conditions for a limit ESS of a simultaneity game. A regular ESS must be induced on a symmetric elementary game and a strong equilibrium point must be induced by the limit ESS and its symmetric image on an asymmetric elementary game.

In section 12 the results on necessary and on sufficient conditions for a limit ESS will be applied to a many period model of animal conflicts with ritual fights and escalated conflicts. The model has the form of a simultaneity game. Apart from degenerate border cases the gap between necessary and sufficient conditions is insignificant. Probably this is typical for models of this kind. For non-degenerate parameter combinations the analysis achieves a complete overview over the important features of behavioral patterns corresponding to a limit ESS. It will be shown that a limit ESS is of one of two types. If the risk faced in one round of serious fight is not too high, one of both types completely avoids serious fights, whereas the other type has a positive probability of escalation. The role of ritual fights can be seen in the creation of asymmetries which lead to peaceful settlement.

Further introductary remarks can be found at the beginning of sections. Conceptual arguments behind definitions and the significance of results will often be discussed in special subsections. A reader who concentrates attention on the conceptual parts of the paper may gain considerable insight into the subject matter without bothering to look too closely at formal definitions and results.

Unfortunately, a precise exposition requires much more technical detail than one might think. Without such detail one may easily be mislead to wrong conclusions.

In the biological literature game theory is not only applied to abstract models of animal conflicts, but also in the context of empirical investigations of social interactions of specific animals like dung flies (Parker 1974) speckled wood butterflies (Davies 1978) and digger wasps (Dawkins and Brockmann 1980). Many examples of game theoretical interpretations of natural phenomena can be found in Dawkins' illuminating "Selfish Gene" and in the fascinating new book by Maynard Smith on "Evolution and the Theory of Games" (Dawkins 1976, Maynard Smith 1982). Applications to specific animal species will not be discussed in this paper.

2. Evolutionary stability in bimatrix games

Evolutionarily stable strategies have been defined by Maynard Smith and Price in the framework of bimatrix games (Maynard Smith and Price 1973). It will be convenient to desribe bimatrix games with the help of a notation which permits an easy extension to symmetric extensive 2-person games.

2.1 Bimatrix games: A bimatrix game $G = (\Pi,\Pi'; E,E')$ consists of two finite non-empty pure strategy sets Π and Π' for players 1 and 2, respectively and two payoff functions E and E' for players 1 and 2. Both E and E' are real functions defined on the set of all pure strategy pairs (π,π') with $\pi\in\Pi$ and $\pi'\in\Pi'$. The numbers $E(\pi,\pi')$ and $E'(\pi,\pi')$ are the payoffs for (π,π') of players 1 and 2, respectively.

2.2 Interpretation: A bimatrix game $G = (\Pi,\Pi'; E,E')$ is played as follows: Each of both players selects one of his pure strategies. The strategy choices are simultaneous and independent of each other. Let (π,π') be the pair of pure strategies selected by the players.Then player 1 receives $E(\pi,\pi')$ as his payoff and player 2 receives $E'(\pi,\pi')$ as his payoff.

One can think of the payoff vectors $(E(\pi,\pi'), E'(\pi,\pi'))$ as arranged in a bimatrix whose rows correspond to the pure strategies of player 1 and whose columns correspond to the pure strategies of player 2. This explains the name "bimatrix game".

In biological applications the players are thought of as participants in an animal conflict between two members of the same species. Payoffs are in terms of incremental Darwinian fitness (up to positive linear transformations). Darwinian fitness may be thought of as the expected number of offsprings in the next generation, even if this is not always the correct interpretation. The application of game theoretical concepts is based on the idea that evolutionary processes have the tendency to optimize fitness and, therefore, produce results which look like rational behavior.

2.3 Mixed strategies: A mixed strategy q for player 1 in $G = (\Pi,\Pi'; E,E')$ is a probability distribution over Π; analogously a mixed strategy q' of player 2 is a probability distribution over Π'. The probability assigned to $\pi \in \Pi$ or $\pi' \in \Pi'$ is denoted by $q(\pi)$ or $q'(\pi')$, respectively. The symbol Q is used for the set of all mixed strategies of player 1. Analogously Q' is the set of all mixed strategies of player 2. Wherever this can be done without any danger of confusion, no distinction will be made between a pure strategy and that mixed strategy which assigns 1 to this pure strategy and zero to all others.

2.4 Payoffs for mixed strategies: The payoff functions E and E' are extended to pairs of mixed strategies (q,q') with $q \in Q$ and $q' \in Q'$ in the usual way:

$$(1) \qquad E(q,q') = \sum_{\pi \in \Pi} \sum_{\pi' \in \Pi'} q(\pi)q'(\pi')\, E(\pi,\pi')$$

$$(2) \qquad E'(q,q') = \sum_{\pi \in \Pi} \sum_{\pi' \in \Pi'} q(\pi)q'(\pi')\, E'(\pi,\pi')$$

The payoffs $E(q,q')$ and $E'(q,q')$ are the expected values of $E(\pi,\pi')$ and $E'(\pi,\pi')$, respectively,if the mixed strategies q and q' are played.

2.5 Best reply: $r \in Q$ is a best reply to $q' \in Q'$ in $G=(\Pi,\Pi';E,E')$ if the following is true:

$$(3) \qquad E(r,q') = \max_{q \in Q} E(q,q')$$

Analogously $r' \in Q'$ is a best reply to $q \in Q$ if we have

$$(4) \qquad E'(q,r') = \max_{q' \in Q'} E'(q,q')$$

2.6 Equilibrium point: The mixed strategy pair (r,r') with $r \in Q$ and $r' \in Q'$ is an equilibrium point of $G=(\Pi,\Pi'; E,E')$ if r and r' are best replies to each other.

2.7 Remark: John F. Nash has proved a fundamental existence theorem which implies that every bimatrix game has at least one equilibrium point (Nash 1951).

2.8 Symmetry: A bimatrix game $G = (\Pi,\Pi', E,E')$ is called symmetric if we have $\Pi=\Pi'$ and

$$(5) \qquad E(\pi,\pi') = E'(\pi',\pi)$$

for every pair of pure strategies π and π'. In view of (5) it is sufficient to specify the common pure strategy set Π and player 1's payoff function E in order to characterize a symmetric bimatrix game. Accordingly, a symmetric bimatrix game can be described as a pair (Π,E) where Π is the common pure strategy set of both players and E is player 1's payoff function.

2.9 Comment: Models of animal conflicts in evolutionary game theory usually take the form of symmetric games. Note that the definition of symmetry is not invariant with respect to a renaming of one of the player's pure strategies. In biological applications the pure strategies have meanings like "attack" or "flee" and it is important that the same strategy π describes the same behavior for both players. Payoffs must be symmetric in the sense of (5) with respect to this natural mapping between both players' strategic possibilities.

Animal conflicts may involve asymmetries like differences of weight and strength. Such asymmetries are not excluded by symmetric game models. The positions in which a player may find himself can be assigned randomly to both players in a symmetric fashion. Pure strategies describe behavior conditional on the position.

2.10 Example: Figure 1 represents a version of the famous hawk-dove-game (Maynard Smith and Price 1973). The terms hawk and dove refer to the character of the two strategies rather than to the animal contestants and have political rather than biological connotations. "e" for "escalate" is the hawkish strategy which means serious attack whereas "d" stands for

	e	d
e	$\frac{1}{2}(V-W)$; $\frac{1}{2}(V-W)$	V ; 0
d	0 ; V	$\frac{1}{2}V$; $\frac{1}{2}V$

e : escalate
d : display
$W>V>0$

V: value of victory
W: damage of wound

Figure 1: The hawk-dove-game. Rows correspond to player 1's pure strategies and columns to player 2's pure strategies. Player 1's payoffs are shown in the upper left corner and player 2's payoffs are shown in the lower right corner.

a threatening "display" of weapons like horns and teeth. If both contestants escalate the fight will go on until one of them is seriously wounded; the wounded animal flees and the other animal gains a valuable resource (e.g. a territory). Both contestants have the same probability to win the fight. The damage W caused by a serious wound is assumed to be higher than the value V of the resource. If only one of the animals escalates then the other one will flee and the resource will be won by the escalating player. If both choose to display, then some kind of unspecified random mechanism (e.g. ritual fight) will decide who gains the resource. Again, both players have the same chance to win.

The game has two equilibrium points in pure strategies, namely (d,e) and (e,d) and one equilibrium point in mixed strategies, namely (r,r') with

$$(6) \quad r(e) = r'(e) = \frac{V}{W}$$

and

$$(7) \quad r(d) = r'(d) = 1 - \frac{V}{W}$$

<u>2.11 Evolutionarily stable strategies</u>: Let $G = (\Pi, E)$ be a symmetric bimatrix game. The common set of mixed strategies is denoted by Q. A mixed strategy q^* is called an <u>evolutionarily stable strategy</u> or shortly an ESS for G, if the following two conditions (a) and (b) are satisfied:

(a) <u>Equilibrium condition</u>: (q^*,q^*) is an equilibrium point of G.

(b) <u>Stability condition</u>: If r is an alternative best reply to q^*, i.e. a best reply with $r \neq q^*$, then

$$(8) \quad E(q^*,r) > E(r,r)$$

<u>2.12 Interpretation</u>: The equilibrium condition (a) requires that q^* should be the equilibrium strategy of a symmetric equilibrium point where "symmetric" is understood in the sense that both players use the same strategy. Since the numbering of the animal contestants is arbitrary, evolution cannot produce a behavioral pattern which depends on this numbering.

An evolutionarily stable strategy is meant to describe an equilibrium of a population which is monomorphic in the sense that all animals use the same strategy q^*. Imagine that in a very large population of this kind suddenly a mutant appears who uses r instead of q^*; this mutant has a very small relative frequency $\varepsilon>0$. Thereby, a slightly perturbed situation arises where an opponent in a conflict will play q^* with probability $1-\varepsilon$ and the mutant strategy r with probability ε.

The mutant will be selected against if the following inequality is satisfied:

$$(9) \qquad E(q^*,(1-\varepsilon)q^* + \varepsilon r) > E(r,(1-\varepsilon)q^* + \varepsilon r)$$

The left hand side and the right hand side show the fitness payoffs obtained by playing q^* and r, respectively in the perturbed situation. Due to the bilinearity of the payoff function (9) can be rewritten as follows:

$$(10) \qquad (1-\varepsilon)E(q^*,q^*) + \varepsilon E(q^*,r) > (1-\varepsilon)E(r,q^*) + \varepsilon E(r,r)$$

Inequality (10) cannot be satisfied for all r and for sufficiently small ε unless (q^*,q^*) is an equilibrium point; otherwise we would have $E(q^*,q^*) < E(r,q^*)$ for a best reply r and the first terms on both sides would reverse the inequality for sufficiently small ε. Now suppose that (q^*,q^*) is an equilibrium point; if r is not a best reply to q^* then (10) will be satisfied for sufficiently small ε in view of $E(q^*,q^*) > E(r,q^*)$. If r is an alternative best reply to q^*, then the first terms on both sides of (10) are equal and (10) is satisfied if and only if (8) holds.

The hawk-dove-game example has only one symmetric equilibrium point, namely the equilibrium point in mixed strategies described by (6) and (7). It can be shown that the equilibrium strategy $r = r'$ is evolutionarily stable.

Not every symmetric bimatrix game has an evolutionarily stable strategy. It is known that the number of evolutionarily stable strategies is always finite (Haigh 1975). The notion of evolutionary stability is much more restrictive than that of an equilibrium point.

3. Extensive 2-person games

In this section the notion of an extensive 2-person game will be introduced. It will be useful to prepare the formal definition by an explanation of the graphical conventions used for the representation of extensive games. This will be done with the help of a simple example. With this example in mind the notational complications of the formal definition will be easier to understand.

3.1 A hawk-dove-game with incomplete information: The game situation to be explained in the following is similar to that of figure 1. Both players have the choice either to "escalate" or to "display" with the same interpretation as in figure 1. However, we now assume that one of the players is the "possessor" of a territory and the other one is an "intruder" who wants to conquer the territory; the territory can be either "good" or "bad". The payoffs are similar to those of figure 1 except that the value of a territory depends on whether it is good or bad.

	e	d
e	-4, -4	8, 0
d	0, 8	4, 4

V = 8
W = 16
good territory
probability .4

	e	d
e	-6, -6	4, 0
d	0, 4	2, 2

V = 4
W = 16
bad territory
probability .6

Figure 2: A hawk-dove-game with incomplete information. V is known to the possessor but not to the intruder. For the conventions of graphical representation and the meaning of symbols see figure 1.

The possessor knows the value of the territory but this value is unknown to the intruder. The numerical assumptions are shown in figure 2. A good territory has a value of 8 and a bad territory has a value of 4. If these values together with W = 16 are inserted in figure 1, one obtains the payoffs shown in figure 2. It is assumed that the territory is good with probability .4 and bad with probability .6. The intruder does not know whether the territory is good or bad but he "knows" these probabilities or, in less antropomorphic terms, evolution has adapted his behavior to these relative frequences of good and bad territories.

An initial random choice assigns the roles of possessor and intruder to the two players. Both players have the same chance to become the possessor.

<u>3.2 Graphical representation</u>: The hawk-dove-game with incomplete information described above will serve to illustrate the notion of an extensive game and the conventions of graphical representation used for such games. The structure of an extensive game specifies which decisions have to be made by whom in what order and under which information on the past history of the game; it also specifies the probabilities of random choices and the final payoffs at the end of the game.

Figure 3 represents the hawk-dove-game with incomplete information as an extensive game. The drawing shows a tree structure which grows from below to above. The vertices of the tree correspond to situations which may arise in the game. The edges show the possibilities of continuation.

The "origin" o at the bottom of the drawing represents the beginning of the game. The number 0 left of vertex o indicates that a random decision has to be taken there. The random decision determines which player will be the possessor of the territory; the other is the intruder. The two possible random choices correspond to the edges ox_1 and

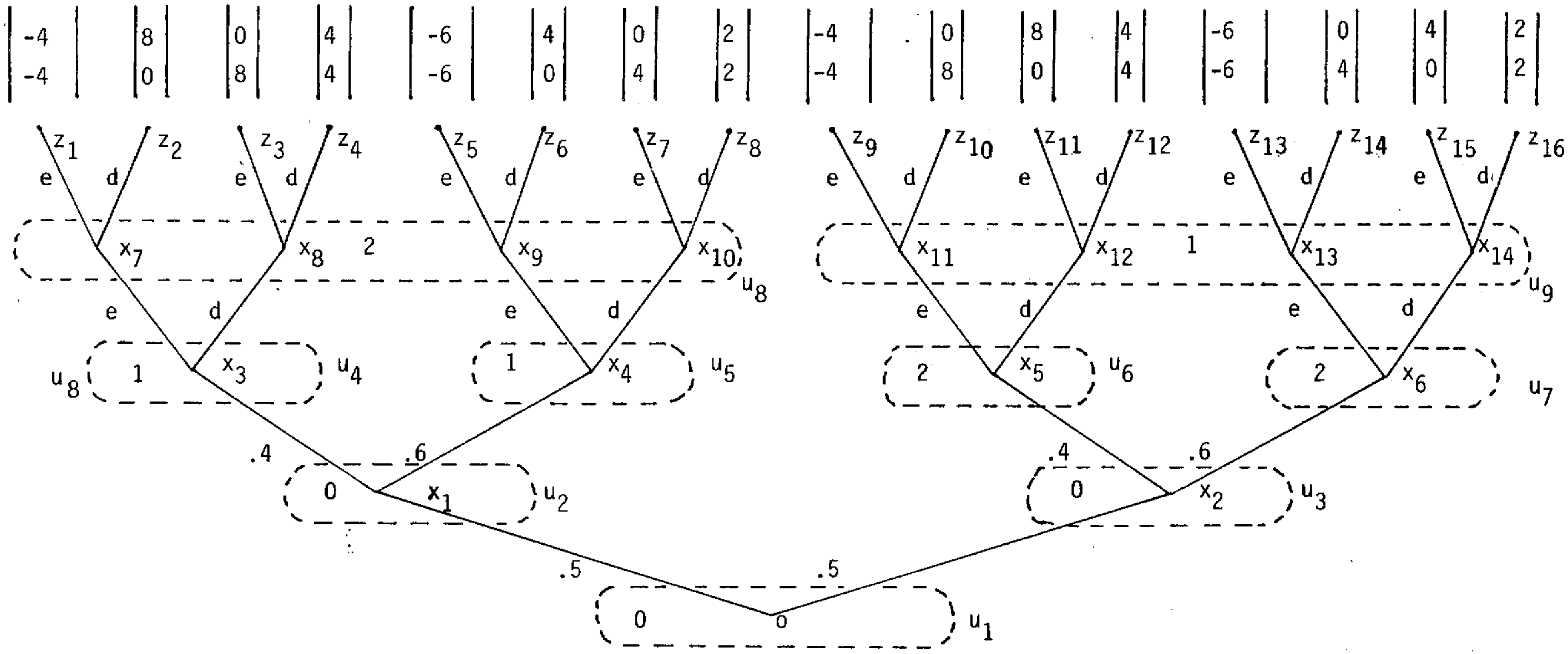


Figure 3: The extensive form of the hawk-dove-game with incomplete information described in 3.1 - Information sets are represented by dashed lines. Choices are indicated by the letters e and d. The numbers 0, 1, and 2 show where a random decision or a decision of player 1 or player 2 has to be made. Payoff vectors are indicated by column vectors above the corresponding endpoints. Probabilities of random choices are shown left of the edge representing the alternative.

ox_2. The probability .5 of these choices is indicated left of ox_1 and ox_2.

At x_1 the role of possessor has been assigned to player 1. Now a second random decision decides whether the territory is good (edge x_1x_3) or bad (edge x_1x_4). At x_1 and x_2 player 1 has to decide whether he wants to escalate or to display. This is indicated by the number 1 left of x_3 and x_4 and by the letters e and d left of the edges from x_3 and x_4 to x_7, x_8, x_9 and x_{10}.

At x_7, x_8, x_9 and x_{10} player 2 has to decide whether he wants to escalate or to display. He has to do this without knowing which of the vertices x_7, x_8, x_9 and x_{10} has been reached or, in other words, without knowing whether the territory is good or bad and whether player 1 escalates or displays. This state of his information is expressed by saying that x_7, x_8, x_9 and x_{10} belong to the same "information set". This information set $u_8 = \{x_7, x_8, x_9, x_{10}\}$ is graphically indicated by a dashed line encircling x_7, x_8, x_9 and x_{10}. Player 1 knows the result of the previous random decisions when he has to act at x_3 and x_4. Therefore $u_4 = \{x_3\}$ and $u_5 = \{x_4\}$ are separate information sets. This is shown by the dashed lines around x_3 and x_4.

Vertices where a random choice is made are called random decision points. Formally, also random decision points are assigned to information sets. Each random decision point always forms a separate information set. Therefore o, x_1 and x_2 are enclosed by separate dashed lines.

The right hand side of the drawing is analogous to the left hand side; the roles of players 1 and 2 are reversed. Here player 2 is the possessor who knows the value of the territory and player 1 is the intruder who does not know this value.

The endpoints $z_1,\ldots,z_{12}$ correspond to the possible outcomes of the game. The payoffs associated with each of the 12 possible outcomes are shown as column vectors between vertical bars above the corresponding endpoints. The upper entry is player 1's payoff and the lower entry is player 2's payoff.

At $u_8 = \{x_7, x_8, x_9, x_{10}\}$ player 2 has two choices, e and d. If he selects e he does not know which of the edges x_7z_1, x_8z_3, x_9z_5 and $x_{10}z_7$ will be the continuation of the play. Therefore, his choice e is formally defined as the set of these four edges. In the mathematical description of extensive forms choices will always be sets of edges.

Actually the decisions of players 1 and 2 for e and d are simultaneous, even if figure 3 suggests a sequential structure where the possessor acts first. However, the fact that player 1 and 2 act simultaneously rather than one after the other does not have any strategic significance in itself. The only aspect of this simultaneity which does have strategic importance is the informational one. Neither player 1 nor player 2 knows the decision of the other player when he has to make his decision. This is correctly expressed by the drawing of the extensive form.

In cases where simultaneous decisions are made an extensive game model must impose an arbitrary sequential order which is without any significance. The information structure is the feature which counts.

<u>3.3 Perfect recall</u>: Perfect recall is a property of extensive games which has been introduced by H.W. Kuhn as the formalization of the idea that a player does not forget his own previous decisions (Kuhn 1953). The extensive games considered in this paper will always be games with perfect recall.

Obviously, it is reasonable to assume perfect recall, if a player is thought of as a fully rational decision maker. In the context of evolutionary game theory perfect recall has a different interpretation. The assumption of perfect recall does not impute any intellectual capabilities to animals, but simply asserts that naturally evolved behavior may depend on individual past experience in any conceivable way which proves to be advantageous in terms of Darwinian fitness. Obviously, this is nothing more than a simple consequence of the general principle of fitness optimization. Therefore, it is reasonable to construct extensive game models of animal conflicts as games with perfect recall. Any deviation from perfect recall would need special justification as a restriction of the general principle of fitness optimization.

3.4 Formal definition of an extensive 2-person game: The notational conventions adopted here are essentially the same as in an earlier paper by the author (Selten 1975). The words extensive game will always refer to a finite 2-person game in extensive form with perfect recall. A game of this kind is described by a septuple

(11) $\Gamma = (K, P, U, C, p, h, h')$

where the constituents K, P, U, C, p, h and h' of Γ are as follows:

(a) The game tree: The game tree K is a finite tree with a distinguished vertex o, the origin of K. The sequence of vertices and edges connecting o to a vertex x is called the path to x. We say that y comes before x or that x comes after y if y is different from x and on the path to x. An endpoint is a vertex z such that no vertex comes after z. A decision point is a vertex which is not an endpoint. The set of all endpoints is denoted by Z. The set of all decision points is denoted by X. A path to an endpoint is called a play. The edges are also called alternatives. An alternative at x is an edge which connects x with a vertex after x.

(b) The player partition: The player partition $P = (P_0, P_1, P_2)$ partitions X into a random decision set P_0, a decision set P_1 of player 1 and a decision set P_2 of player 2.

(c) The information partition: For i = 1, 2 a subset u of P_i is called eligible (as an information set), if u is not empty, if every play intersects u at most once and if the number of alternatives at x is the same for every $x \in u$. A subset $u \in P_0$ is eligible, if it contains exactly one vertex.
The information partition U is a refinement of the player partition into eligible subsets of P_0, P_1 and P_2. These subsets u are called information sets. For i = 0, 1, 2 the set of all

information sets $u \subseteq P_i$ is denoted by U_i. The elements of U_1 and U_2 are called information sets of player 1 and player 2, respectively.

(d) The choice partition: For $u \in U$ the set of all alternatives at vertices $x \in u$ is denoted by A_u. A subset c of A_u is called eligible (as a choice), if it contains exactly one alternative at every $x \in u$. The choice partition C partitions the set of all edges of K into eligible subsets c of the A_u. These sets are called choices. The choices $c \subseteq A_u$ are called choices at u. The set of all choices at u is denoted by C_u. The union of all C_u with $u \subseteq P_i$ is denoted by C_i. The choices in C_1, C_2 and C_o are called choices of player 1, of player 2 and random choices, respectively. We say that the vertex x comes after the choice c if one of the edges in c is on the path to x. In this case we also say that c is on the path to x. The choice c precedes an information set u if c is on the path to at least one $x \in u$. The information set v precedes an information set u if a choice at v precedes u. The choice c necessarily precedes an information set u if c is on every path to a vertex $x \in u$. The information set v necessarily precedes u if a choice at v necessarily precedes u.

(e) The probability assignment: A probability distribution p_u over C_u is completely mixed, if it assigns a positive probability $p_u(c)$ to every $c \in C_u$. The probability assignment p is a function which assigns a completely mixed probability distribution p_u ober C_u to every $u \in U_o$.

(f) The payoff functions: The payoff functions h and h' of players 1 and 2, respectively assign real numbers h(z) and h'(z),respectively to every $z \in Z$. The numbers h(z) and h'(z) are called payoffs of player 1 and player 2 at z, respectively.

(g) Perfect recall: The extensive games considered here are games with perfect recall which means that the following condition must be satisfied for every information set u of player 1 or 2: If u is player i's information set then every choice c of player i which precedes u, necessarily preceds u.

3.5 Subgames: Let $\Gamma = (K,P,U,C,p,h,h')$ be an extensive game and let y be a decision point in Γ which is different from the origin o. The subtree K_y at y consists of y and all vertices of K after y together with all edges of K connecting such vertices. We say that y is a decomposition point of Γ, if y is a decision point different from the origin and in addition to this the subtree K_y at y has the following property: If an information set u of Γ contains vertices in K_y, then all vertices in u belong to K_y. Let P_y, U_y, C_y, p_y, h_y and h'_y be the restictions of P, U, C, p, h and h' to K_y. If y is a decomposition point, then the game $\Gamma_y = (K_y, P_y, U_y, C_y, p_y, h_y, h'_y)$ is a subgame of Γ, referred to as the subgame at y.

The notational conventions introduced in 3.4 are transferred to subgames in the following way: the lower indices 1, 2 and 0 pointing to players 1 and 2 and to random events, respectively, are used as second lower indices, e.g. $P_y = (P_{y0}, P_{y1}, P_{y2})$.

It is clear that a subgame Γ_y of Γ has all the properties required in (a) to (g) including perfect recall. The origin o of Γ is not a decomposition point and Γ itself is not a subgame of Γ.

3.6 Comment: A subgame is a part of an extensive game which can be looked upon as a game in itself. Not every subtree belongs to a subgame. It is important that each player must know for sure that he is in the subgame when he has to make a decision there. For this reason the definition of a decomposition point requires non-overlapping information sets in the subtree at the decomposition point. The game of figure 3 has two subgames at x_1 and x_2. There is no subgame at x_3 since u_8 contains the vertices x_9 and x_{10} which do not belong to the subtree at x_3. At u_8 player 2 does not know whether he is in that subtree or not.

Subgames are important substructures of extensive games. Once a subgame has been reached everything outside the subgame becomes irrelevant as far as the strategic situation is concerned.

4. Strategies

The definitions introduced in the following refer to a fixed extensive 2-person game $\Gamma = (K, P, U, C, p,h,h')$.

4.1 Pure strategies: A pure strategy π of player 1 is a function which assigns a choice c at u to every $u \in U_1$; analogously a pure strategy π' of player 2 assigns a choice c at u to every $u \in U_2$. The symbols Π and Π' are used for the set of all pure strategies of player 1 and player 2, respectively.

4.2 Interpretation: A pure strategy is a complete plan of behavior for a player. In the game of figure 3 each pure strategy of player 1 can be characterized by a string of three letters like "ede" where the letters stand for the decisions at u_4, u_5 and u_9, respectively. Since there are two choices e and d at each of the three information sets, there are altogether 8 such strings. Each of both players has 8 pure strategies.

It may happen that a choice at an earlier information set excludes the possibility that a later information set of the same player is reached. It is important that also in this case a choice has to be specified at the later information set. The completeness requirement for the behavior plan expressed by a pure strategy covers all situations which may arise in the game, regardless of whether they can occur or not if the strategy is actually used.

Mixed strategies have been defined already in subsection 2.3. The definitions and notations introduced there are also applied to extensive games.

In extensive games a player may also randomize locally over the choices at his information sets instead of randomizing globally over his pure strategies. The possibility of local randomization leads to the notion of behavior strategies which will be defined below. In this paper, behavior strategies will be much more important than mixed strategies.

<u>4.3 Local strategies</u>: A local strategy b_u at the information set $u \in U_i$ is a probability distribution over the set C_u of choices at u; the probability assigned to a choice c at u is denoted by $b_u(c)$. A local strategy is called <u>pure</u> if it assigns 1 to one choice c and 0 to all other choices. Wherever this can be done without danger of confusion no distinction will be made between a choice c and the pure local strategy with probability 1 for c. The set of all local strategies at u is denoted by B_u.

<u>4.4 Behavior strategies</u>: A behavior strategy b for player 1 is a function which assigns a local strategy $b_u \in B_u$ to every $u \in U_1$. Analogously a behavior strategy b' for player 2 assigns a local strategy $b_u \in B_u$ to every $u \in U_2$. The set of all behavior strategies of player 1 is denoted by B and the set of all behavior strategies of player 2 is denoted by B'. Since choices are special local strategies, pure strategies are special behavior strategies.

<u>4.5 Expected payoffs</u>: For every endpoint z the set of all choices which contain alternatives on the play to z is denoted by C(z). Consider a <u>behavior strategy pair</u> (b,b'), i.e. a pair with $b \in B$ and $b' \in B'$. The probability that a vertex x is reached, if (b,b') is played, is denoted by $\gamma(x,b,b')$. This probability, called the <u>realization probability of</u> x <u>under</u> (b,b'), is the product of all $p_u(c)$, all $b_u(c)$ and all $b'_u(c)$ assigned by p, b and b' to choices c on the path to x. Player 1's <u>payoff</u> is computed as follows:

$$(12) \qquad E(b,b') = \sum_{z \in Z} \gamma(z,b,b')h(z)$$

Player 2's payoff E'(b,b') is defined analogously:

$$(13) \qquad E'(b,b') = \sum_{z \in Z} \gamma(z,b,b')h'(z)$$

Since pure strategies are special behavior strategies, these formulas apply to pure strategy pairs (π,π') as well.

<u>4.6 Normal form</u>: The pure strategy sets Π and Π' of Γ together with the payoff functions E and E' restricted to pure strate-

gy pairs (π,π') with $\pi \in \Pi$ and $\pi' \in \Pi'$ form a bimatrix game $G = (\Pi,\Pi';E,E')$. This bimatrix game is called the normal form of Γ.

4.7 Comment: It has been mentioned already in the introduction that the normal form cannot replace the extensive game. It will be necessary to make a sharp distinction between evolutionarily stable strategies of the normal form and of the extensive game. The latter will be defined in terms of behavior strategies rather than mixed strategies. It will turn out that the difference between mixed strategies and behavior strategies becomes very important as soon as one deals with evolutionarily stable strategies.

Equilibrium points of the extensive game will be defined in terms of behavior strategies, too. In order to avoid any risk of confusion we shall always sharply distinguish between equilibrium points of the extensive game and equilibrium points of the normal form.

4.8 Best replies: A behavior strategy r of player 1 is called a best reply to a behavior strategy b' of player 2 if we have:

$$E(r,b') = \max_{b \in B} E(b,b') \tag{14}$$

Analogously r' is a best reply to b if we have:

$$E'(b,r') = \max_{b' \in B'} E'(b,b') \tag{15}$$

4.9 Equilibrium point: A pair of behavior strategies (r,r') with $r \in B$ and $r' \in B'$ is an equilibrium point of the extensive game Γ if r and r' are best replies to each other.

4.10 Relationship between mixed and behavior strategies: There is a natural way to assign a unique mixed strategy to every behavior strategy. Consider a behavior strategy b for player 1; for every $\pi \in \Pi$ let $q(\pi)$ be the product of all probabilities $b_u(c)$ assigned by b to choices c with

$c = \pi(u)$. In this fashion a mixed strategy q is defined which is called the mixed representative of b. The mixed representative of a behavior strategy $b' \in B'$ is defined analogously.

A behavior strategy and its mixed representative are realization equivalent in the sense that for any fixed behavior of the other player the probability that a vertex x is reached is the same for both strategies. A more precise definition of realization equivalence can be found elsewhere (Selten 1975). For the purposes of this paper it is sufficient to rely on the informal explanation given above.

H.W. Kuhn has proved a theorem which shows that in extensive games with perfect recall not only a realization equivalent mixed strategy can be found for every behavior strategy, but also a realization equivalent behavior strategy can be found for every mixed strategy (Kuhn 1953). This shows that a player's strategic possibilities in an extensive game with perfect recall are fully represented by his behavior strategies.

Since different behavior strategies have different mixed representatives the set of all behavior strategies of a player can be identified with a subset of his mixed strategies. In many cases, the number of dimensions of the set of all behavior strategies is much lower than that of the set of all mixed strategies. For example, in the game of figure 3 a behavior strategy of player 1 can be characterized by 3 parameters, namely the probabilities to choose e at the information sets u_4, u_5 and u_9. In order to characterize a mixed strategy of player 1 one needs 7 parameters, since there are 8 pure strategies (see 5.2).

Already here it can be seen why the description of a player's strategic possibilities by his mixed strategies involves much spurious duplication which is avoided by a conceptualization in terms of behavior strategies. Later it will become clear why this spurious duplication has desastrous consequences for a definition of evolutionarily stable strategies in terms of mixed strategies if it is applied to extensive games.

4.11 Existence of equilibrium points: The existence of at least one equilibrium point for every extensive 2-person game with perfect recall is an immediate consequence of Kuhn's theorem mentioned above and Nash's existence theorem for finite games in normal form (Nash 1951, Kuhn 1953).

5. The symmetry problem

The definition of an evolutionarily stable strategy in section 2 refers to a symmetric bimatrix game G = (Π,E). In a game of this kind the pure strategy sets of both players coincide. It is clear what it means that both players use the same strategy.

Consider an extensive 2-person game Γ; here the pure strategy sets of both players never coincide even if the game is obviously symmetric in any reasonable sense. The game of figure 3 may serve as an example. A pure strategy of player 1 assigns choices to the information sets of player 1 and a pure strategy of player 2 assigns choices to information sets of player 2. Both must be different, simply because both players have different information sets.

Nevertheless, it is clear that in figure 3 the information sets u_4, u_5 and u_9 of player 1 correspond to the information sets u_5, u_6 and u_8 of player 2, respectively and that both players "use the same strategy" in an intuitive informal sense if the probabilities for e and d agree at corresponding information sets of both players.

The example of figure 3 suggests that a formal definition of symmetry in an extensive 2-person game should involve mappings of one player's information sets onto those of the other and of one player's choices onto those of the other. The definition of a symmetry given below will take the form of a mapping f from the choice set C onto itself with certain special properties.

An extensive 2-person game may be symmetric in several different ways or,in other words, it may have several symmetries in the sense to be made precise below. In such cases one must specify which of the symmetries is the natural one which connects biologically equal behaviors of both players. Therefore, a symmetric extensive game will not be defined as an extensive game in the usual sense, but as an extensive game together with one of its symmetries.

The questions raised by the meaning of symmetry must be clarified by formal definitions. This is a necessary first step in any attempt to transfer the concept of an evolutionarily stable strategy to the framework of the extensive game.

5.1 Definition of a symmetry: Consider an extensive 2-person game $\Gamma = (K,P,U,C,p,h,h')$. A symmetry f of Γ is a mapping from the choice set C onto itself with the following properties (a) to (f)

(a) If $c \in C_0$ then $f(c) \in C_0$ and $p(f(c)) = p(c)$

(b) If $c \in C_1$ then $f(c) \in C_2$

(c) $f(f(c)) = c$ for every $c \in C$

(d) For every $u \in U$ there is a $u' \in U$ such that for every choice c at u, the image f(c) is a choice at u'. - The notation f(u) is used for this information set u'.

(e) For every endpoint $z \in Z$ there is a $z' \in Z$ with $f(C(z)) = C(z')$, where $f(C(z))$ is the set of all images of choices in C(z), - The notation f(z) is used for this endpoint z'

(f) $h(f(z)) = h'(z)$ and $h'(f(z)) = h(z)$

5.2 The example of figure 4: This game has the following symmetry f:

(16) $f(e_1) = e_2$

(17) $f(d_1) = d_2$

(18) $f(e_2) = e_1$

(19) $f(d_2) = d_1$

It can be seen easily that f satisfies the properties (a) to (c). The choices at u_1 have images at u_2 and vice versa. (d) is satisfied, too. The basic mapping f is defined for choices.

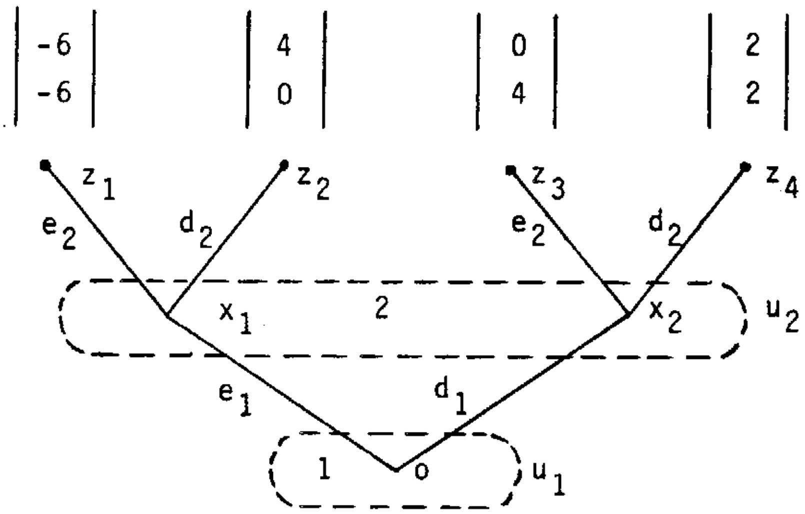


Figure 4: The hawk-dove-game of figure 1 with V = 4 and W = 16 as an extensive game

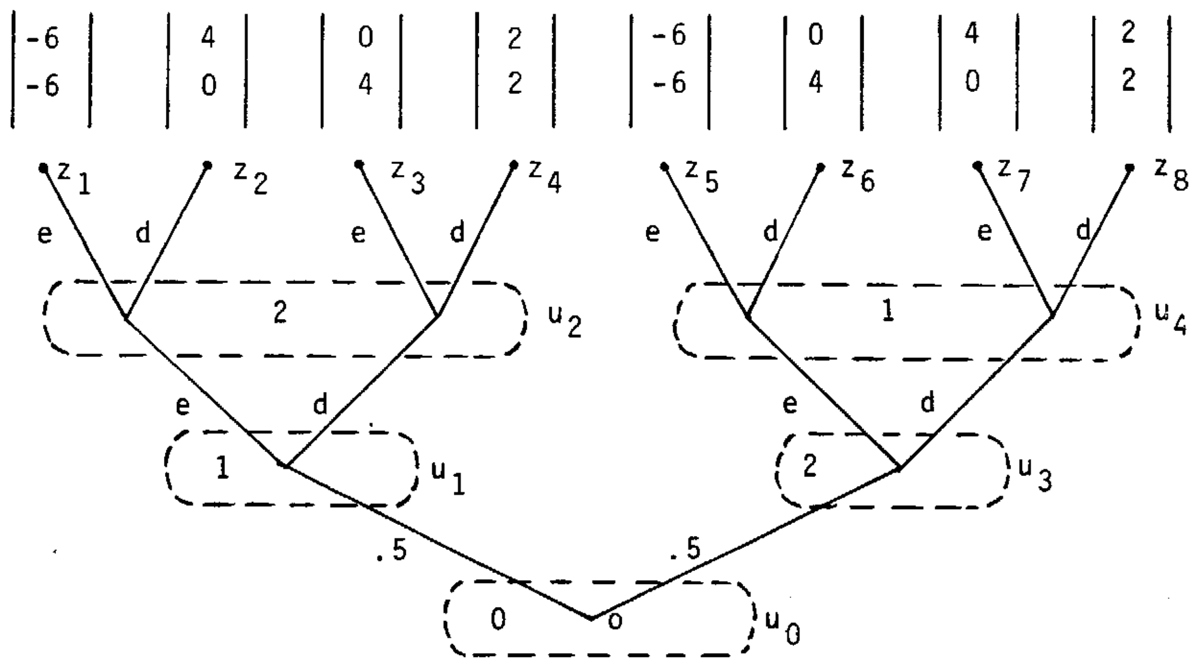


Figure 5: An example with two symmetries

Mappings for information sets and endpoints are induced by the basic mapping. For the sake of simplicity the symbol f is also used for the induced mappings.

Note that no one to one mapping is induced on the set X of all decision points. The image u_2 of the one element set u_1 has two elements.

The endpoints z_1, z_2, z_3 and z_4 have the images z_1, z_3, z_2 and z_4, respectively. It can be seen immediately that (e) is satisfied. For example, z_2 is mapped to z_3 and we have $C(z_2) = \{e_1,d_2\}$ and $C(z_3) = \{e_2,d_1\}$. Property (f) requires that both payoffs at an endpoint are exchanged by the mapping. This is the case in figure 4.

5.3 The example of figure 3: The natural symmetry f of this game maps the choice e and d at u_4, u_5, u_6, u_7, u_8 and u_9 to e and d, respectively at u_6, u_7, u_4, u_5, u_9 and u_8, respectively. The left random choice at u_1 is mapped to the right one and vice versa and the left and right choices at u_2 are mapped to the left and right choices at u_3 and vice versa. It can be seen here, too, that (a) to (f) are satisfied.

In the case of figure 3 one could describe the symmetry by a mapping from vertices to vertices, but as we have seen with the help of figure 4, this is not always possible.

5.4 An example with two symmetries: Consider the game of figure 5. This game has two symmetries f_1 and f_2. Symmetry f_1 maps each of both initial random choices to itself; moreover, $f_1(u_1) = u_2$ and $f_1(u_3) = u_4$. The images of choices, information sets and endpoints on the left hand side of the figure remain on the left and those on the right remain on the right.

The second symmetry f_2 maps the left initial random choice to the right one and vice versa, u_1 and u_2 are mapped to u_3 and u_4, respectively. The choices, information sets and endpoints on the left hand side are moved to the right and vice versa.

It is important to know which of the two symmetries is the natural one. The answer cannot be deduced from the structure of figure 5. It depends on the interpretation of the game.

The initial random decision can be <u>distinguishing</u> or <u>neutral</u> in the following sense: Suppose that the left random choice determines player 1 as the possessor of the territory and the right random choice determines player 2 as the possessor. Afterwards both players know who is the possessor and who is the intruder. In this case, the random decision is <u>distinguishing</u>. The roles of possessor and intruder can be distinguished. A player cannot make his behavior dependent on whether he is player 1 or player 2, but he can make his behavior dependent on his role. Information set u_1 where player 1 is the possessor must be mapped to u_2 where player 2 is the possessor. Clearly, this interpretation of figure 2 leads to f_2 as the natural symmetry.

Now consider another interpretation of the random choices in Figure 5. Suppose that the left choice means rain and the right one means sunshine. Since rain and sun are impartial to both animals, player 1's information set u_1 after rain must now be mapped to player 2's information set u_2 after rain and similarly the information sets u_3 and u_4 are mapped onto each other. In this case, the random decision is <u>neutral</u> and f_1 is the natural symmetry of the game.

In figure 5 payoffs are determined by the players' decisions for e and d. The random choice at the beginning has no influence on the payoff. The role distinction in the first interpretation is <u>payoff irrelevant</u> in this sense (Hammerstein 1981). As we shall see later a payoff irrelevant role distinction may have important strategic consequences. Maynard Smith and Parker were the first to observe this fact (Maynard Smith and Parker 1976).

5.5 Definition of a symmetric game: A symmetric extensive 2-person game, also shortly called a symmetric game, is a pair (Γ,f) where $\Gamma = (K,P,U,C,p,h,h')$ is an extensive 2-person game with at least one symmetry and f is one of these symmetries. f is called the natural symmetry of (Γ,f).

5.6 Symmetric images of strategies: Let (Γ,f) be a symmetric game as defined in 5.5 and let b_u be a local strategy at an information set u. Let v be the symmetric image f(u) of u. The local strategy b_v at v with

(20) $$b_v(f(c)) = b_u(c) \text{ for all } c \in C_u$$

is called the symmetric image of b_u. The notation $f(b_u)$ is used for the symmetric image of b_u. In view of properties (c) and (d) in the definition of a symmetry the following is true for every information set u:

(21) $$f(f(u)) = u$$

Therefore for every local strategy b_u we have

(22) $$f(f(b_u)) = b_u$$

Consider a pair of behavior strategies b and b' for player 1 and 2, respectively. Let b_u and b'_u be the local strategies assigned by b and b' to information sets u of players 1 and 2, respectively. We say that b' is the symmetric image of b and b' is the symmetric image of b and we write $b' = f(b)$ and $b = f(b')$ if for every information set u of player 1 the local strategy b'_v at $v = f(u)$ is the symmetric image $f(b_u)$ of b_u. In view of (21) and (22) we have:

(23) $$f(f(b)) = b$$

for every $b \in B$ and

(24) $$f(f(b')) = b'$$

for every $b' \in B'$.

5.7 Symmetry and expected payoff: Consider a behavior strategy pair (b,b'). The realization probability $\gamma(z,b,b')$ of an endpoint z (see 4.5) does not change if z, b and b' are replaced by their images under f:

$$(25) \qquad \gamma(f(z),\ f(b'),\ f(b)) = \gamma(z,b,b')$$

This can be seen immediately with the help of property (e) of a symmetry. In view of property (f) this yields the following conclusions:

$$(26) \qquad E(f(b'),\ f(b)) = E'(b,b')$$

$$(27) \qquad E'(f(b'),\ f(b)) = E(b,b')$$

for every $b \in B$ and every $b' \in B'$. Equations (26) and (27) show that the payoffs of both players are interchanged if the behavior strategy of each player is replaced by the symmetric image of the other player's strategy. This property is analogous to the symmetry property (5) of symmetric bimatrix games (see 2.8).

5.8 The symmetric normal form: Consider a symmetric extensive 2-person game (Γ,f). The normal form $G = (\Pi,\Pi',E,E')$ of Γ is not a symmetric bimatrix in the sense of 2.8 since Π and Π' are different from each other. However, it is easy to construct a symmetric bimatrix game which has essentially the same relationship to (Γ,f) as the normal form to the extensive form. In this bimatrix game both players have the same pure strategy set Π. Player 1's payoff $E(\pi,\varphi)$ if player 1 selects $\pi \in \Pi$ and player 2 selects $\varphi \in \Pi$ is defined as follows:

$$(28) \qquad E(\pi,\varphi) = E(\pi,f(\varphi))$$

For the sake of notational parsimony no new symbol is introduced for the new payoff function on the left hand side. No confusion can arise from this convention since the new payoff function is defined on a region which has no intersection with that of the old one. The common pure strategy set Π for both players together with the payoff function E for pairs (π,φ) with $\pi,\varphi\in\Pi$ form a symmetric bimatrix game (Π,E). This symmetric bimatrix game is called the symmetric normal form of (Γ,f).

Definition 2.11 of an evolutionarily stable strategy for a symmetric bimatrix game can be applied to the symmetric normal form of (Γ,f). This is maybe the most obvious way in which one may try to generalize the notion of an evolutionarily stable strategy to extensive games. However, as we shall see in section 7, it is better to take a different approach based on behavior strategies rather than mixed strategies.

5.9 Subgame preservation: A symmetry f for Γ is called subgame preserving, if in addition to the properties (a) to (f) the following condition (g) is satisfied:

(g) For every subgame Γ_x of Γ there is a subgame Γ_y of Γ such that every information set u of Γ_x has its image f(u) in Γ_y and every information set u of Γ_y has its image f(u) in Γ_x . (The subgames Γ_x and Γ_y may or may not be different from each other.)

We write $\Gamma_y = f(\Gamma_x)$ if Γ_y and Γ_x are related as in condition (g). Obviously, in this case we also have $\Gamma_x=f(\Gamma_y)$.

Figure 6 shows an example of a symmetry which fails to be subgame preserving. The subgame at x is not mapped into a subgame. v' = f(v) belongs to a subgame but this subgame contains u' as well. However, figure 6 has the peculiar feature that there is only one choice at some information sets. As we shall see this special property is crucial for the example.

5.10 Theorem 1 (subgame preservation): Let f be a symmetry of an extensive 2-person game Γ. If in Γ there are at least two choices at every information set, then f is subgame preserving.

Proof: Let y be a decomposition point of Γ and let Γ_y be the subgame at y. For the purposes of this proof we shall introduce the following notations: U_y is the set of all information sets in Γ_y. The set of all information sets which intersect the path to y but do not belong to U_y is denoted by V_y. The set of all information sets of Γ not in U_y or V_y is denoted by V_{-y}. The sets of all choices at information sets in U_y, V_y and V_{-y}, respectively, are denoted by C_y, D_y and D_{-y}, respectively.

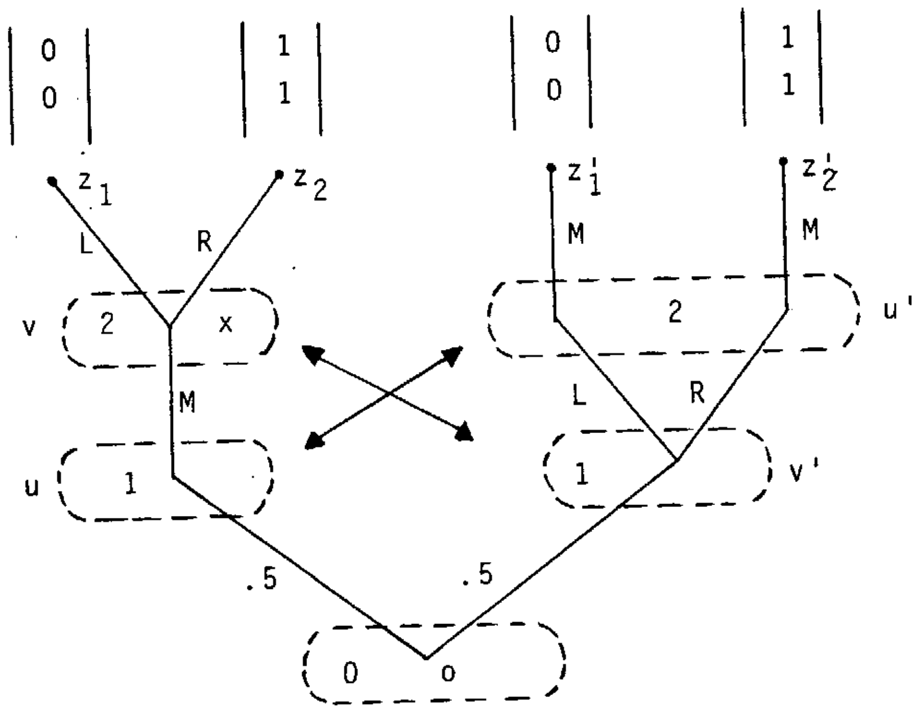


Figure 6: A symmetry which fails to be subgame preserving. Double arrows show which information set is mapped to which. Choices L, R and M are mapped to L, R and M, respectively.

Since Γ_y is a subgame all choice in D_y necessarily precede all information sets in U_y. We are going to show that all choices in the image set $f(D_y)$ of D_y necessarily precede all information sets in the image set $f(U_y)$ of U_y.

Consider a choice $c \in D_y$ and an information set $u \in U_y$. Assume that $f(c)$ does not necessarily precede $f(u)$. Then there must be a vertex x in $f(u)$ such that the path to x does not contain an edge of $f(c)$. Nevertheless, it follows by (e) that an edge of $f(c)$ must be on every play through x. Therefore, $f(c)$ is a choice at an information set v which contains at least one vertex x' after x. Since there are at least two choices at v, there must be a play through x which does not contain an edge on $f(c)$. This cannot be true, since all plays through x contain an edge of $f(c)$. Consequently, the choices in $f(D_y)$ necessarily precede the information sets in $f(U_y)$.

Plays through information sets in D_{-y} do not intersect information sets in U_y. Therefore, plays through information sets in the image set $f(V_{-y})$ of V_{-y} do not intersect information sets in the image set $f(U_y)$ of U_y.

Let z be an endpoint of Γ_y. From what has been said before it follows that the choices on the play to $f(z)$ belong to $f(D_y)$ and $f(C_y)$; moreover, all choices in $f(D_y)$ are on the play to $f(z)$ and these choices precede the information sets in $f(U_y)$. Therefore, the play to $f(z)$ contains a vertex $\bar{y}$ such that the choices in $f(D_y)$ are on the path to $\bar{y}$ and the remaining ones are on the connection from $\bar{y}$ to $f(z)$. This vertex $\bar{y}$ is independent of the choice of the endpoint z of Γ_y since the same choices in $f(D_y)$ are made in all cases before the first information set in $f(U_y)$ is reached. This shows that $\bar{y}$ is the origin of a subgame $\Gamma_{\bar{y}}$ whose information sets are those in $f(U_y)$. It is now clear that f is subgame preserving.

6. Towards an ESS definition for extensive games

The treatment of the symmetry problem in the previous section has prepared the ground for a generalization of the concept of an evolutionarily stable strategy to the framwork of the extensive game. A first step in this direction will be the introduction of the notion of a "direct ESS". Even if the direct ESS is not yet a final concept it will be useful to discuss the difference between a direct ESS which is defined in terms of behavior strategies and a mixed strategy ESS for the symmetric normal form. It will become clear why a definition in terms of behavior strategies is preferable to a definition in terms of mixed strategies. Later it will be shown that the direct ESS definition is unduly restrictive and the more liberal concept of a limit ESS will be prepared by the discussion of an instructive example.

6.1 Definition of a direct ESS: Let (Γ,f) be a symmetric extensive 2-person game. A direct evolutionarily stable strategy or shortly a direct ESS for (Γ,f) is a behavior strategy b for player 1 in Γ with the following two properties (a) and (b).

(a) Equilibrium condition: $(b,f(b))$ is an equilibrium point of Γ.

(b) Stability condition: If r is an alternative best reply to $f(b)$, i.e. a best reply with $r \neq b$, then

$$E(b,f(r)) > E(r,f(r)) \tag{29}$$

A behavior strategy b which satisfies (a) but not necessarily (b) is called a symmetric equilibrium strategy. An equilibrium point of the form $(b,f(b))$ is called a symmetric equilibrium point.

6.2 Comment: It can be seen immediately that the conditions (a) and (b) in 6.1 are direct translations of conditions (a) and (b) in the definition 2.11 of an evolutionarily stable strategy for a bimatrix game. The biological interpretation remains the same as before.

6.3 Direct ESS versus ESS of the symmetric normal form:
As has been pointed out in 5.8, at least at first glance it may seem to be a natural idea to define an ESS of a symmetric extensive game (Γ,f) as an ESS of its symmetric normal form (Π,E). It is important to point out that this definition in terms of mixed strategies is different from the definition in terms of behavioral strategies given in 6.1. This will be shown with the help of the example of of figure 7. The discussion will also illustrate the advantages of a definition in terms of behavior strategies.

The game of figure 7 is similar to the subgame after the left initial random choice in figure 3. Here, too, one may think of a fight over a territory which may be good or bad, but now with equal probabilities, contrary to figure 3; moreover, now both players know whether the territory is good or bad.

The subgame at x_1 and x_4 in figure 6 are hawk-dove-games with $W = 16$ and with $V = 8$ and $V = 4$, respectively (see figure 2). With the help of (6) and (7) evolutionarily stable strategies can be computed for these hawk-dove-games. If this is done we actually obtain a direct ESS for the game of figure 7, namely the following behavior strategy b^* for player 1:

(30) $b^*_u(e) = .5$

(31) $b^*_v(e) = .25$

Since the probabilities for e and d sum up to 1, a behavior strategy for player 1 is fully described by the probabilities assigned to e at u and v.

It can be shown easily that the game of figure 7 has exactly one direct ESS as defined in 6.1, namely the behavior strategy b^* with (30) and (31). For the sake of brevity no proof will be given here.

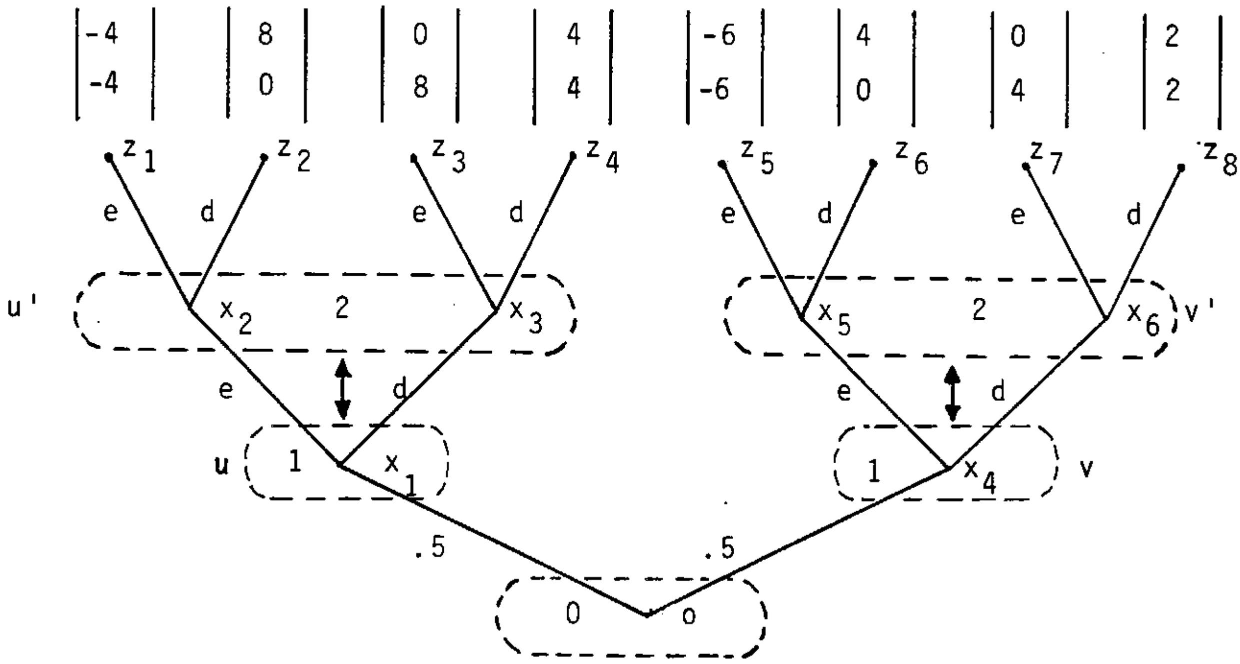


Figure 7: A hawk-dove-game with complete information on a resource value with random variation. The natural symmetry is indicated by double arrows.

Figure 8 contains the symmetric normal form of the game of figure 7. It can be shown without much difficulty that this bimatrix game has no evolutionarily stable strategy. It will be illuminating to discuss the main reasons for this phenomenon without giving a detailed proof.

It can be shown without difficulty that all symmetric equilibrium points of the game of figure 8, i.e. all equilibrium points of the form (q^*,q^*) are characterized by the following equations:

(31) $q^*(ee) + q^*(ed) = .5$

(32) $q^*(ee) + q^*(de) = .25$

	ee	ed	de	dd
ee	-5, -5	0, -2	1, -3	6, 0
ed	-2, 0	-1, -1	4, 2	5, 1
de	-3, 1	-2, 4	-1, -1	4, 2
dd	0, 6	1, 5	2, 4	3, 3

Figure 8: The symmetric normal form of the symmetric extensive game of figure 7. The first and second letter in the two letter symbols for pure strategies stand for player 1's choice at u and v, respectively.

Equations (32) and (33) permit the interpretation that q^* must result in the same probabilities for e and d at u and v as the direct ESS in behavior strategies described by (30) and (31).

It can be seen easily that there are infinitely many strategies q^* which satisfy (32) and (33). In the range $0 \leq q^*(ee) \leq .5$ the probability for ee can be selected arbitrarily; then the other probabilities are determined by (32) and (33).

It can be shown that all mixed strategies q^* which satisfy (32) and (33) are <u>payoff</u> <u>equivalent</u> in the sense that regardless of which r is used by player 2 the payoffs $E(q^*,r)$ and $E(r,q^*)$ do not change if one mixed strategy q^* is replaced by another.

Suppose that q^* satisfies the equilibrium condition (a) in the ESS definition of 2.11. Then (32) and (33) are satisfied. Let r^* be a different strategy such that (32) and (33) are satisfied with r^* instead of q^*, too. Since q^* and r^* are payoff equivalent we must have

(34) $$E(q^*,r^*) = E(r^*,r^*)$$

Consequently, the stability condition (b) in the ESS definition of 2.11 cannot be satisfied by q^*. Therefore, the symmetric bimatrix game of figure 8 has no evolutionarily stable strategy.

The fact that there are infinitely many strategies q^* which satisfy (32) and (33) is due to the spurious duplication in the description of strategic possibilities by mixed strategies (see 4.10). This spurious duplication results in a multitude of payoff equivalent alternative best replies to any symmetric equilibrium strategy and, thereby, destroys the chance to satisfy the stability condition (b) in the ESS definition of 2.11.

The example shows that the direct ESS defined in 6.1 is different from an ESS of the symmetric normal form. It also shows why an ESS definition in terms of behavior strategies is preferable. The payoff equivalent alternative best replies to a symmetric equilibrium strategy of the bimatrix game in figure 8 do not really describe different forms of behavior. Since differences between two mixed strategies satisfying (32) and (33) are unobservable, it seems to be inadequate to think of them as destabilizing sources of genetic drift.

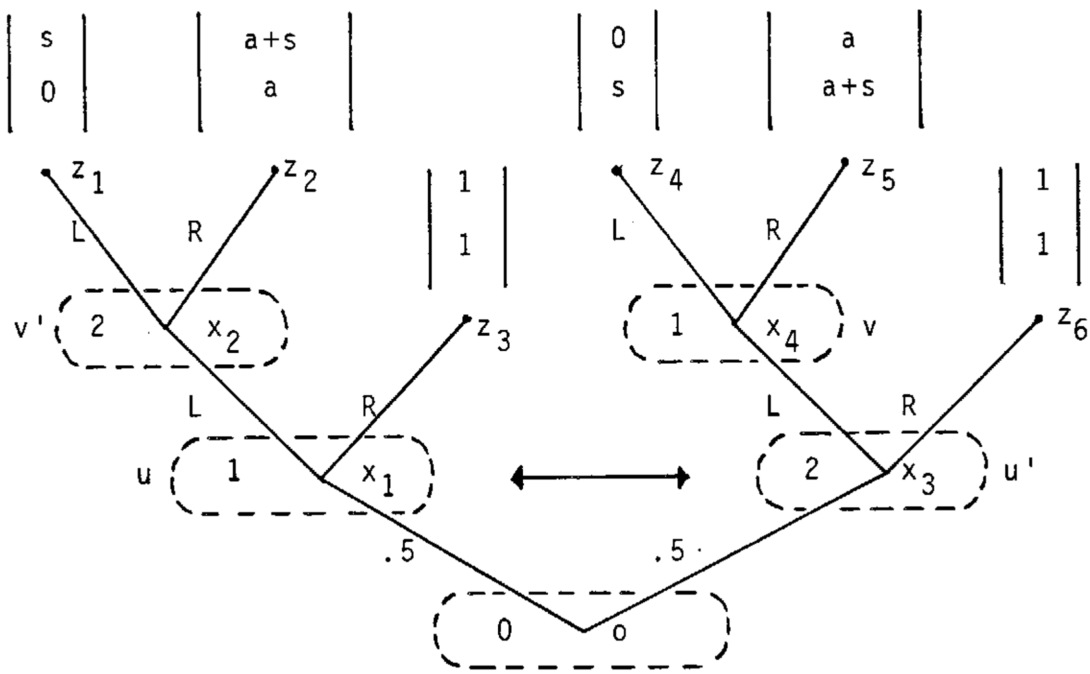
Since the stability condition in the ESS definition breaks down in the presence of payoff equivalent alternative best replies,it is important to avoid spurious duplication in the description of strategic possibilities.

6.4 Towards a less restrictive ESS definition: The concept of a direct ESS introduced in section 6.1 it not yet completely satisfactory. Many potentially interesting biological extensive game models must be expected to have no direct ESS.

A simple example called the "male desertion game" will serve to illustrate the source of the difficulty. It will be argued that the non-existence problem is due to the overexactness of the game model. The difficulty can be overcome if one permits the possibility that choices which are never taken intentionally are nevertheless taken with a very small probability by mistake.

The basic ideas underlying the less restrictive notion of an ESS to be proposed in this paper will be discussed informally with the help of the male desertion game example. Formal definitions will follow in section 7.

6.5 A male desertion game: An admittedly not very realistic model suggested by the literature (Dawkins 1976, chapter 9) may serve to illustrate the non-existence problem faced by the direct ESS definition.



$0 < a < 1$

$0 < s$

Figure 9: A male desertion game. - The meaning of symbols is explained in the text. The natural symmetry f maps u to u' and v to v'.

The game model is shown in figure 9. The initial random choice assigns the roles of "male bird" and "female bird" to players 1 and 2.If chance moves to the left, player 1 is the male bird and player 2 is the female bird. The roles are interchanged on the right hand side of the figure.

After the role assignment the male bird has to choose between L and R where L stands for "leave" and R stands for "raise". If he selects R, he decides to help to raise his young, if he chooses L he leaves the female and abandons his young in order to look for a second chance to find a female in the same season.

If the male bird has chosen L, a further decision has to be made by the female bird. R means that she decides to raise her young without the help of the male bird and L means that she abandons her young, too.

The incremental fitness obtained, if both cooperate in raising their young is normalized to 1. The incremental fitness obtained, if the female alone raises her young, is denoted by a. The fitness value of a second chance to find a male is expressed by s. Of course, we must have $s > 0$ and $0 < a < 1$.

<u>6.6 Analysis of the male desertion game</u>: It is important to distinguish two cases:

(35) $\quad a + s > 1$

and

(36) $\quad a + s < 1$

The limiting case $a+s=1$ will not be considered here. The first case (35) raises no difficulties. In this case, the game has a uniquely determined direct ESS which prescribes L at u and R at v. It is advantageous for the male bird to leave, and after the male bird has left, it pays for the female to raise her young alone.

From a theoretical point of view the case (36) is the more interesting one. No direct ESS exists in this case. There are infinitely many symmetric equilibrium strategies of the following form: At u player 1 chooses R and at v player 1 chooses an arbitrary local strategy b_v. If player 2 plays R at u' it does not matter what player 1 does at v.

Let b and r be two different symmetric equilibrium strategies, i.e. two different behavior strategies of player 1 which prescribe R at u. Clearly, b and r are best replies to each other. Obviously, we have:

(37) $E(b,r) = E(r,r) = 1$

This shows that b fails to be a direct ESS. No direct ESS exists.

6.7 Interpretation of the non-existence phenomenon: If one looks at the case $a+s<1$ of the male desertion game in an unprejudiced way, one immediately finds the intuitively obvious solution: R should be chosen everywhere. Clearly, R is better than L at u and u'. Once v or v' has been reached R is better than L there, too. Nevertheless, this obvious solution is not evolutionarily stable. It is instructive to examine this fact in the light of its biological interpretation.

Since the female never has to make a decision, if the male always decides to cooperate in child rearing, no selective pressure is exerted on the choice the female would have to make in case of male desertion. Therefore, at least at first glance nothing seems to prevent genetic drift with respect to the female decision.

Is it really justified to accept the idea that no selective pressure is exerted on the choice of the female, if the male always decides to choose R? Suppose for example that the male bird does not come back to the nest because it has become the victim of a bird of prey. From the point of view of the female an event of this kind is indistinguishable from male desertion. Even if such events may be rare they can exert sufficient selective pressure towards the intuitive solution where the female raises her children alone, should this become necessary.

6.8 Small mistake probabilities: Since physiological mechanisms are not absolutely precise and subject to malfunction caused by outside disturbances, it must be expected that on rare occasions an animal will "make a mistake" and take a choice which is not the one prescribed by its fitness optimizing genetically fixed behavioral program.

In the male desertion game it may for example happen that the male bird accidentally hits a hard object (e.g. a window pane) and, thereby, receives a shock which erases the memory of his mate and induces him to look for a new one.

Biological models are always slightly misspecified since they ignore all kinds of very rare events which may cause deviations from the genetically fixed behavioral program described by an evolutionarily stable strategy. It would be futile to try to construct very complicated games which explicitely model these events. Instead of this it shall be assumed that wherever a choice has to be made there may be a small probability of mistake. The nature of the mistakes and the exact size of the mistake probabilities need not be specified.

In the case of the male desertion game with $a+s<1$, it is sufficient to assume that the male cannot avoid a small minimum probability ε for his wrong choice L. Even if R is his optimal choice, at rare occasions he will select L by mistake. The unavoidable mistake probability of ε perturbs the game situation; the original game is slightly changed to a "perturbed game" where at u and u' local strategies must be played which assign a probability of at least ε to L.

Unlike the original game, the perturbed game does have a direct ESS; this ESS selects R with the maximum probability of $1-\varepsilon$ in the male role and with probability 1 in the female role. (One may also assume a minimum probability of L for the female, too, but it is not necessary to do so.) As ε goes to zero, this ESS approaches the intuitive solution where R is always chosen everywhere.

If one thinks of the unperturbed game as an approximation of an unknown perturbed game it makes sense to define an ESS as a limit of direct ESS's for perturbed games in a sequence approaching the unperturbed game. In this sense the strategy to choose R everywhere is an ESS of the male desertion game.

The approach outlined above will be used in order to define a "limit ESS". The limit ESS is the less restrictive ESS notion proposed in this paper.

7. The concept of a limit ESS

The idea to look at sequences of perturbed games with small mistake probabilities has been used as the basis of a perfectness definition for equilibrium points of extensive games (Selten 1975). Perfectness excludes unreasonable choices in unreached parts of the game. The set of equilibrium points is considerably reduced by the perfectness requirement. Following a suggestion by R.J. Aumann the use of perturbed game sequences is often referred to as the "trembling hand approach".

In the following the trembling hand approach is used in order to arrive at a less restrictive ESS definition. It is somewhat surprising that essentially the same approach on the one hand makes the equilibrium point notion more restrictive and on the other hand makes the ESS notion less restrictive.

7.1. Perturbed games: Let (Γ,f) be a symmetric extensive 2-person game. A perturbance of (Γ,f) is a function η which assigns a minimum probability η to each choice $c \in C_1 \cup C_2$ in Γ, such that the following conditions are satisfied.

(38) $\quad \eta_c \geq 0 \quad$ for every $c \in C_1 \cup C_2$

(39) $\quad \sum_{c \in C_u} \eta_c < 1 \quad$ for every $u \in U_1 \cup U_2$

(40) $\quad \eta_{c'} = \eta_c$ for $c' = f(c)$

A perturbed game of (Γ,f) is a triple $\bar{\Gamma} = (\Gamma,f,\eta)$ where η is a perturbance of (Γ,f).

A local strategy b_u at an information set u of player 1 or 2 is called a local strategy for $\bar{\Gamma} = (\Gamma,f,\eta)$ if it satisfies the condition:

(41) $\quad b_u(c) \geq \eta_c \quad$ for every $c \in C_u$

The set of all local strategies at u for $\hat{\Gamma}$ is denoted by $\hat{B}_u$. For i = 1, 2 a behavior strategy b_i for Γ is a <u>behavior strategy</u> for $\hat{\Gamma}$ if it assigns local strategies for $\hat{\Gamma}$ to all information sets of player i. The set of all behavior strategies of player 1 for $\hat{\Gamma}$ is denoted by $\hat{B}$ and the symbol $\hat{B}'$ is used for the set of all behavior strategies of player 2 for $\hat{\Gamma}$.

<u>7.2 Comment</u>: For the purposes of defining a perfect equilibrium point it has been important to require positive minimum probabilities η_c everywhere (Selten 1975). In the present context it is convenient to relax this condition and to permit $\eta_c = 0$ for some choices or even everywhere. In this way, the unperturbed game (Γ,f) can be looked upon as a special perturbed game.

Condition (39) is introduced in order to make sure that the set of local strategies $\hat{B}_u$ will be non-empty. Condition (40) is imposed in order to preserve the natural symmetry in the perturbed game.

In the perturbed game the behavior strategies of players 1 and 2 are restricted by the minimum probability condition (41). This has to be taken into account in the definition of best replies and equilibrium points for perturbed games. A direct ESS for the perturbed game will be defined in the same way as in 6.1 for the unperturbed game.

In the following all definitions will refer to an arbitrarily fixed perturbed game $\hat{\Gamma} = (\Gamma,f,\eta)$ of a symmetric extensive 2-person game (Γ,f).

<u>7.3 Best replies</u>: $r \in \hat{B}$ is called a <u>best reply to</u> $b' \in B'$ <u>in</u> $\hat{\Gamma}$ if we have:

$$(43) \qquad E(r,b') = \max_{b \in \hat{B}} E(b,b')$$

Analogously $r' \in \hat{B}'$ is a <u>best reply to</u> $b \in B$ <u>in</u> $\hat{\Gamma}$ if we have:

$$(44) \qquad E'(b,r') = \max_{b' \in \hat{B}'} E(b,b')$$

A best reply $r \in \hat{B}$ to $b' \in \hat{B}'$ in $\hat{\Gamma}$ is called _strong_ if player 1 does not have another best reply to b' or, in other words, if any other strategy of player 1 yields a lower payoff against b'. A _strong_ best reply of player 2 is defined analogously.

The unperturbed game (Γ,f) is identified with the special perturbed game where all minimum probabilities are zero. For this special case the definition of a best reply given above is the same as in 4.8.

It is an immediate consequence of the symmetry properties (26) and (27) of expected payoffs (see 5.7) that r is a best reply to b' in $\hat{\Gamma}$ if and only if f(r) is a best reply to f(b') in $\hat{\Gamma}$.

7.4 Equilibrium point: A pair of behavior strategies (r,r') with $r \in \hat{B}$ is an _equilibrium point_ of $\hat{\Gamma}$ if r and r' are best replies to each other in $\hat{\Gamma}$. An equilibrium point (r,r') of $\hat{\Gamma}$ is called _strong_ if r and r' are strong best replies to each other in $\hat{\Gamma}$. It is called _symmetric_ if the following condition is satisfied:

$$(45) \qquad r' = f(r)$$

If (r,r') is a symmetric equilibrium point of $\hat{\Gamma}$ then r is called a _symmetric equilibrium strategy for_ $\hat{\Gamma}$.

7.5 Perturbed game direct ESS: A _direct evolutionarily stable strategy_ or shortly a _direct ESS_ for $\hat{\Gamma}$ is a behavior strategy $b \in \hat{B}$ for player 1 with the following two properties (a) and (b).

(a) _Equilibrium condition_: b is a symmetric equilibrium strategy for $\hat{\Gamma}$

(b) _Stability condition_: If r is an alternative best reply to f(b) in $\hat{\Gamma}$, i.e. a best reply with $r \neq b$, then we have:

(48) $E(b,f(r)) > E(r,f(r))$

<u>7.6 Test sequences</u>: Let (Γ,f) be a symmetric extensive 2-person game. A sequence $\hat{\Gamma}^1, \hat{\Gamma}^2,\ldots$ where for $k = 1, 2, \ldots$ the game $\hat{\Gamma}^k = (\Gamma,f,\eta^k)$ is a perturbed game of (Γ,f), is called a <u>test sequence</u> for (Γ,f) if for every choice c of player 1 or player 2 the sequence of the minimum probabilities η_c^k assigned to c by η^k converges to 0 for $k \to \infty$.

Let $\hat{\Gamma}_1,\hat{\Gamma}_2,\ldots$ be a test sequence for (Γ,f). A pair (r,r') of behavior strategies for Γ is called a <u>limit equilibrium point</u> of this test sequence, if for $k = 1,2,\ldots$ an equilibrium point (r_k,r_k') of $\hat{\Gamma}_k$ can be found such that for $k \to \infty$ the sequence of the (r_k,r_k') converges. Similarily a behavior strategy b is called a <u>limit ESS</u> for the test sequence $\hat{\Gamma}_1,\hat{\Gamma}_2,\ldots$ if for $k = 1,2,\ldots$ a direct ESS $\hat{b}_k$ of $\hat{\Gamma}$ can be found such that for $k \to \infty$ the sequence of the $\hat{b}_k$ converges to b .

<u>7.7 Remark</u>: It can be shown that a limit equilibrium point of a test sequence for (Γ,f) is an equilibrium point of (Γ,f). Essentially,the same fact has been proved elsewhere (Selten 1975, lemma 3, p. 37). Contrary to this, a limit ESS of a test sequence for (Γ,f) need not be a direct ESS of (Γ,f). The male desertion game of figure 9 with $a+s<1$ provides an obvious example. Consider a monotonically decreasing sequence of positive numbers $\varepsilon_1,\varepsilon_2,\ldots$ with $\varepsilon_k \to 0$ for $k \to \infty$ and define η^k as that perturbance which assigns the minimum probability ε_k to the choices L at u and u' and 0 to all other choices in figure 8. In the test sequence $\hat{\Gamma}^1,\hat{\Gamma}^2, \ldots$ obtained in this way, each perturbed game has a direct ESS, namely that behavior strategy b_k of player 1 which assigns $1-\varepsilon_k$ to R at u and 1 to R at v. The limit ESS of this test sequence is the pure strategy which selects R at u and v. As has been shown in 6.6 this limit ESS fails to be a direct ESS of the game of figure 9.

<u>7.8 Definition of a limit ESS</u>: A behavior strategy b of player 1 in a symmetric extensive 2-person game (Γ,f) is a <u>limit ESS</u> of (Γ,f) if b is a limit ESS of at least one test

sequence of (Γ,f).

In order to be able to describe a limit ESS in a slightly different way we introduce some auxiliary definitions. For every pair of two behavior strategies b and r of player 1 let $|b-r|$ be the maximum of the absolute difference between the probabilities assigned by b and r to the same choice. Similarily, for every perturbance η let $|\eta|$ be the maximum of all minimum probabilities assigned by η to a choice in Γ.

7.9 Lemma 1 (alternative limit ESS description): A behavior strategy b for a symmetric extensive 2-person game (Γ,f) is a limit ESS of (Γ,f) if and only if for every $\varepsilon>0$ at least one perturbed game $\hat{\Gamma} = (\Gamma,f,\eta)$ with $|\eta|<\varepsilon$ has a direct ESS $\hat{b}$ with $|\hat{b}-b|<\varepsilon$.

Proof: Assume that $\hat{\Gamma}^1,\hat{\Gamma}^2,...$ is a test sequence which has b as a limit ESS. Obviously, for every ε a member of the sequence can be found which satisfies the conditions of the lemma. It is also clear that under these conditions a test sequence with b as a limit ESS can be constructed.

7.10 Interpretation and comment: A limit ESS of an extensive game can be interpreted as an approximation of a direct ESS of a perturbed game whose otherwise unspecified perturbance is assumed to be small. This is made precise by lemma 1. The words "at least one" are of special significance in this description as well as in 7.8. It is possible that a test sequence whose existence is required by 7.8 must be selected with care. It may happen that the perturbances η^k must have very special properties in order to generate a test sequence which has a specific ESS as one of its limit ESS. This is a desirable feature of the definition. Since generally very little is known about the relative size of various small mistake probabilities it seems to be advisable not to exclude any special structure.

It may be of interest to distinguish different kinds of ESS which differ with respect to implied assumptions on small mistake probabilities. If plausible assumptions on the relative size of mistake probabilities can be made they may sometimes serve as a basis for the selection among several ESS of the same game.

7.11 Remark: A direct ESS of an unperturbed symmetric extensive game (Γ,f) is also a limit ESS. This is due to the fact that the definition of a perturbance does not exclude the trivial case of zero minimum probabilities everywhere. Obviously, a direct ESS of Γ is also a direct ESS of $\hat{\Gamma}=(\Gamma,f,\eta)$ with $\eta_c = 0$ everywhere. Moreover, with this game $\hat{\Gamma}$ one can form the test sequence $\hat{\Gamma},\hat{\Gamma},\ldots$ which yields a direct ESS of $\hat{\Gamma}$ as a limit ESS.

7.12 Remark: If b is a limit ESS of (Γ,f) then b is a symmetric equilibrium strategy of (Γ,f). This follows by definition 7.8 together with the fact that a limit equilibrium point of a test sequence is an equilibrium point (see remark 7.7).

8. Local optimality and pervasiveness

The definition of an ESS as a limit ESS of at least one test sequence has the consequence that one has to look at the perturbed games of an extensive game in order to find its evolutionarily stable strategies. Therefore, it is important to investigate the properties of a direct ESS of a perturbed game. For this purpose it will be useful to define local payoffs at information sets and local best replies.

Local payoffs cannot be defined in a meaningful way in cases where an information set of a player cannot be reached no matter what he does, since his opponent uses a strategy which excludes the possibility of reaching this information set. It is a remarkable fact that this difficulty does not arise if the players play a direct ESS and its symmetric image. Of course, it is not necessary to define local payoffs for information sets with only one choice. It will be shown that all other information sets of player 1 and 2 are reached with positive probability if the players play a direct ESS and its symmetric image. The name "pervasiveness will be introduced for this property of a direct ESS of a perturbed or unperturbed game.

The pervasiveness of a direct ESS has the consequence that the first condition in the ESS definition can be reduced to a local optimality condition. This local optimality condition requires that local best replies to the symmetric image are prescribed everywhere.

Since local conditions can be checked more easily than global ones, it is desirable to reduce global definitions to local characterizations. Unfortunately, this goal cannot be fully attained for the direct ESS definition. This will become clear in section 9.

All definitions will refer to a fixed symmetric extensive 2-person game (Γ,f) and to its perturbed games $\hat{\Gamma} = (\Gamma,f,\eta)$. In view of the fact that for $\eta_c=0$ everywhere, there is no difference between a direct ESS for (Γ,f) and for (Γ,f,η), the identification of both games does not pose any difficulties (see 7.3).

8.1 Realization probabilities of information sets: For every information set u define

$$(47) \qquad \gamma(u,b,b') = \sum_{x \in u} \gamma(x,b,b')$$

(see 4.5 for the definition of the realization probability $\gamma(x,b,b')$ of a vertex). We call $\gamma(u,b,b)$ the realization probability of u under b and b'. An endpoint which comes after a vertex in an information set u is called an endpoint after u. The set of all endpoints after u is denoted by Z(u). The realization probability $\gamma(u,b,b')$ can be interpreted as the probability that an endpoint after u is reached:

$$(48) \qquad \gamma(u,b,b') = \sum_{z \in Z(u)} \gamma(z,b,b')$$

Equation (25) in 5.7 shows that the realization probability of an endpoint z under b and b' is equal to the realization probability of f(z) under f(b') and f(b). Therefore, we have:

$$(49) \qquad \gamma(u,b,b') = \gamma(f(u),f(b'),f(b))$$

for every information set u and for every pair (b,b') of behavior strategies for players 1 and 2.

We say that an information set u is blocked by a behavior strategy b of player 1 if $\gamma(u,b,b')$ vanishes for every $b' \in B'$. Similarly, we say that u is blocked by a behavior strategy b' of player 2 if $\gamma(u,b,b')$ vanishes for every $b \in B$.

8.2 Posterior strategies: Let b be a behavior strategy of player 1 and let r_u be a local strategy at an information set u of player 1. The notation b/r_u is used for that behavior strategy s of player 1 which agrees with b at information sets v with $v \neq u$ and with r_u at u. Analogously, for a behavior strategy b' of player 2 and a local strategy

r'_u at one of his information sets b'/r'_u denotes that behavior strategy s' of player 2 which agrees with b' at information sets v with $v \neq u$ and with r'_u at u.

Suppose that u has been reached and player 1 intends to play b in the future. Since the game has perfect recall he knows that he already has taken those of his choices which precede u. In order to describe his intention and his knowledge of his own past choices at this point we introduce the notion of a posterior strategy. The <u>prosterior strategy of</u> b <u>at</u> u is that behavior strategy t of player 1 whose local strategies t_v satisfy the following conditions (i) and (ii):

(i) If player 1's choice c at v necessarily precedes u, then we have $t_v(c) = 1$

(ii) For information sets v without a choice which necessarily precedes u, the local strategy t_v agrees with the local strategy b_v assigned by b to v.

The posterior strategy of b at u is denoted by v//u. Instead of $b/r_u//u$ we use the shorter notation $b//r_u$. The same notations are used analogously for behavior strategies of player 2.

<u>8.3 Local payoffs</u>: Consider an information set u of player 1. The local payoffs at u to be defined below can be interpreted as conditional payoff expectations of player 1 under the condition that u has been reached. Suppose that b and b' are behavior strategies of players 1 and 2, respectively, such that the realization probability $\gamma(u,b,b')$ of u is positive. For every local strategy s_u at u the <u>local payoff</u> $E_u(s_u,b,b')$ <u>for</u> s_u <u>under</u> b <u>and</u> b' is defined as follows:

$$E_u(s_u,b,b') = \frac{1}{\gamma(u,b,b')} \sum_{z \in Z(u)} \gamma(z,b/s_u,b')h(z) \tag{50}$$

All realization probabilities in (50) are changed by the same factor if b is replaced by b//u and b/s_u by $b//s_u$; this factor is reciprocal to the product of all probabilities assigned by b to choices which necessarily precede u. Consequently, we have:

$$(51) \qquad E_u(s_u,b,b') = E_u(s_u,b//u,b')$$

This equation is used to extend the local payoff definition to cases where $\gamma(u,b,b')$ vanishes, but $\gamma(u,b//u,b')$ is positive. It can be seen immediately that $\gamma(u,b//u,b')$ vanishes if and only if u is blocked by b'. For every $s_u \in B_u$, every $b \in B$ and every $b' \in B'$ such that b' does not block u, the <u>local payoff</u> $E_u(s_u,b,b')$ <u>for</u> s_u <u>under</u> b <u>and</u> b' is as follows:

$$(52) \qquad E_u(s_u,b,b') = \frac{1}{\gamma(u,b//u,b')} \sum_{z \in Z(u)} \gamma(z,b//s_u,b')h(z)$$

At information sets u of player 2 local payoffs are defined analogously as conditional payoff expectations of player 2; if b does not block u we have:

$$(53) \qquad E_u(s'_u,b,b') = \frac{1}{\gamma(u,b,b'//u)} \sum_{z \in Z(u)} \gamma(z,b,b'//s'_u)h'(z)$$

for information sets u of player 2.

It follows by (25) in 5.7 together with (49) and by (e) in 5.1 that the following symmetry property holds wherever local payoffs are well defined:

$$(54) \qquad E_u(s_u,b,b') = E_{f(u)}(f(s_u),f(b'),f(b))$$

<u>8.4 Local and global payoffs</u>: The expected payoffs defined in 4.5 will sometimes be called <u>global</u> payoffs in order to be able to make a clear distinction between both kinds of payoffs. Let u be an information set of player 1 and let b and b' be behavior strategies of players 1 and 2, respectively.

Assume that u is not blocked by b'. Let s_u and t_u be two local strategies at u. As we shall see, the following statements (a) to (d) on the connection between local and global payoffs hold:

(a) $E(b/s_u,b') = E(b/t_u,b')$ if $\gamma(u,b,b') = 0$

(b) $E(b/s_u,b') = E(b/t_u,b')$ if $E_u(s_u,b,b') = E_u(t_u,b,b')$

(c) $E(b/s_u,b') \geq E(b/t_u,b')$ if $E_u(s_u,b,b') \geq E_u(t_u,b,b')$

(d) $E(b/s_u,b') > E(b/t_u,b')$ if $E_u(s_u,b,b') > E_u(t_u,b,b')$ and $\gamma(u,b,b') > 0$

A local strategy change from s_u to t_u does not influence the realization probabilities of u and of the endpoints which do not come after u. Therefore, an influence on global payoffs, if it is exerted at all, must be exerted in the same direction as on the local payoffs. For $\gamma(u,b,b') > 0$ an improvement of local payoffs results in an improvement of global payoffs. Obviously, global payoffs do not change for $\gamma(u,b,b') = 0$. From what has been said it is clear that (a) to (d) hold. In view of (54) analogious statements for player 2's local and global payoffs can be derived.

<u>8.5 Local best replies</u>: Let u be an information set of player 1 and let b and b' be behavior strategies of players 1 and 2, respectively. Assume that u is not blocked by b'. The local strategy r_u at u is a <u>local best reply to</u> b <u>and</u> b' <u>in</u> $\hat{\Gamma} = (\Gamma,f,\eta)$ if we have:

(55) $$E_u(r_u,b,b') = \max_{s_u \in \hat{B}_u} E_u(s_u,b,b')$$

A local best reply at an information set of player 2 is defined analogously.

If there is only one choice c at an information set u of player 1 or 2, then this choice c is always called a <u>local best reply to</u> b <u>and</u> b', regardless of whether local payoffs at u are well defined or not.

A local best reply is called <u>strong</u> if it is the only best reply at the concerning information set or, in other words, if every other local strategy at this information set (if there is one) yields a lower local payoff.

It follows by (54) that r_u is a local best reply to b and b' in $\hat{\Gamma}$ if and only if $f(r_u)$ is a local best reply to f(b') and f(b) in $\hat{\Gamma}$.

A local strategy $r_u \in \hat{B}_u$ is called extreme in $\hat{B}_u$ if there is one choice d at u, such that for every choice c at u with $c \neq d$ we have $r_u(c) = \eta_c$. We call this choice d the intended choice of the extreme local strategy r_u.

It can be seen immediately that r_u is a strong best reply to b and b' in $\hat{\Gamma} = (\Gamma, f, \eta)$ if and only if the following condition is satisfied: r_u is extreme in $\hat{B}_u$ and the intended choice d of r_u is a strong local best reply to b and b' in $\Gamma = (\Gamma, f)$.

8.6 Pervasiveness: An information set u is called essential if it is an information set of either player 1 or player 2 and if there are at least two choices at u. A behavior strategy b of player 1 is called dispersed if none of his own essential information sets is blocked by b and permeable, if none of the other player's essential information sets is blocked by b. In the same sense we speak also of dispersed and permeable behavior strategies of player 2. A behavior strategy is called pervasive, if it is both dispersed and permeable.

It follows by (49) that u is blocked by a behavior strategy b of player 1,if and only if f(u) is blocked by f(b). Therefore, the following is true for every behavior strategy b of player 1:

(i) : b is dispersed, if and only if f(b) is dispersed
(ii): b is permeable, if and only if f(b) is permeable
(iii): b is pervasive, if and only if f(b) is pervasive

8.7 Lemma 2 (pervasiveness lemma): A behavior strategy b of player 1 is pervasive,if and only if for every essential information set u of player 1 we have:

(56) $\gamma(u,b,f(b)) > 0$

Proof: Suppose that b is pervasive. Consider an essential information set u of player 1. Those choices of player 1 which necessarily precede u must be selected with positive probabilities by b. Otherwise u would be blocked by b and b could not be dispersed. For at least one vertex x in u player 2's choices on the path to x must be selected with positive probabilities by f(b). Otherwise u would be blocked by f(b) and f(b) would not be permeable, contrary to (ii) in 8.6. Therefore $\gamma(u,b,f(b))$ is positive. (56) holds for every essential information set of play 1, if b is pervasive.

Now suppose that (56) holds for every essential information set u of player 1. Then no such information set is blocked by b or by f(b). Consequently, b is dispersed and f(b) is permeable. It follows by (ii) in 8.6 that b is pervasive.

8.8 Theorem 2 (pervasiveness theorem): Let $\hat{\Gamma} = (\Gamma,f,\eta)$ be a perturbed game of a symmetric extensive 2-person game (Γ,f) and let b be a direct ESS for $\hat{\Gamma}$. Then b is pervasive.

Proof: Assume that b is a symmetric equilibrium strategy for $\hat{\Gamma}$. It is sufficient to show that b cannot be a direct ESS for $\hat{\Gamma}$ unless b is pervasive.

Suppose that b is not pervasive. In view of lemma 2 an essential information set u of player 1 with $\gamma(u,b,f(b)) = 0$ can be found. Let u be an information set of this kind. In view of (49) the realization probability $\gamma(f(u),b,f(b))$ vanishes, too.

Since u is essential, a local strategy r_u at u can be found which is different from the local strategy b_u assigned to u by b. Let r_u be a local strategy of this kind. Consider the behavior strategy $r = b/r_u$. We shall show that r is an alternative best reply which does not satisfy the second condition in the direct ESS definition.

Both u and f(u) are not reached by b and f(b). Moreover, u and f(u) remain unreached if b or f(b) or both are locally changed at u or f(u). Therefore, global payoffs are not influenced by such changes (see (a) in 8.4). It follows that r is an alternative best reply to f(b) and that the following is true:

(57) $E(b,f(b)) = E(b,f(r)) = E(r,f(r))$

This is a contradiction to the second condition in the definition of a direct ESS.

8.9 Comment: Since the special case of zero minimum probabilities everywhere is not excluded, the theorem also covers the case of a direct ESS of an unperturbed game (Γ,f). In view of the pervasiveness property it must be expected that many extensive game models of animal fights will not have a direct ESS for the unperturbed game. The male desertion game with $a+s<1$ is not an isolated case. Imagine a model where two animals can fight for up to 20 periods; as long as fighting goes on decisions have to be made and, therefore, some essential information sets will represent situations in period 20. These information sets must be reached with positive probability by a direct ESS. There cannot be a direct ESS which always results in a shorter fight or in no fight at all.

A direct ESS for a perturbed game must be pervasive, too, but there pervasiveness may be due to positive minimum probabilities. A limit ESS of a test sequence need not be pervasive.

A direct ESS generates a symmetric equilibrium point in pervasive strategies. The pervasiveness property has the consequence that, at equilibrium, local payoffs are defined at every essential information set. As we shall see, it is necessary and sufficient for the equilibrium properties of a pervasive symmetric equilibrium strategy that local best replies are prescribed everywhere.

8.10 Lemma 3 (necessary local optimality conditions): Let $\hat{\Gamma} = (\Gamma,f,\eta)$ be a perturbed game of a symmetric extensive 2-person game (Γ,f). Let b' be a behavior strategy of player 2 in $\hat{\Gamma}$. Let r be a behavior strategy of player 1 in $\hat{\Gamma}$. If r is a best reply to b' in $\hat{\Gamma}$, then for every information set u of player 1 with $\gamma(u,r,b') > 0$ the local strategy r_u assigned by r to u is a local best reply to r and b' in $\hat{\Gamma}$.

Proof: Assume that r is a best reply to b' in $\hat{\Gamma}$ and that nevertheless for some information set u of player 1 with $\gamma(u,r,b') > 0$ the local strategy r_u assigned to u by r is not a local best reply to r and b' in $\hat{\Gamma}$. Let u be an information set of this kind. Since the local payoff at u is continous function of the local strategy at u and since $\hat{B}_u$ is compact a local best reply at u always exists. Let s_u be a local best reply to r and b' at u.

Consider the strategy $s = r/s_u$. The local change from r_u to s_u improves the local payoff at u. Moreover, $\gamma(u,r,b')$ is positive. Therefore, player 1's global payoff is improved by this local change (see (d) in 8.4). Consequently, we have:

(58) $$E(s,b') > E(r,b')$$

This shows that r cannot be a best reply to b'. It follows that the assertion of the lemma is true.

8.11 Lemma 4 (sufficient local optimality conditions): Let $\hat{\Gamma} = (\Gamma,f,\eta)$ be a perturbed game of a symmetric extensive 2-person game (Γ,f). Let b' be a permeable behavior strategy of player 2 in $\hat{\Gamma}$. Let r be a behavior strategy of player 1 in $\hat{\Gamma}$. If for every information set u of player 1 the local strategy r_u assigned to u by r is a local best reply to r and b' in $\hat{\Gamma}$, then r is a best reply to b' in $\hat{\Gamma}$.

Proof: Assume that every local strategy r_u prescribed by r is a local best reply to r and b' in $\hat{\Gamma}$ and that nevertheless r is not a best reply to b' in $\hat{\Gamma}$. A behavior strategy $s \in \hat{B}$ with

(59) $E(s,b') > E(r,b')$

can be found. Let S be the set of all strategies $s \in \hat{B}$ with (58). For every $s \in S$ let k(s) be the number of information sets u of player 1 for which r_u is different from the local strategy s_u asigned to u by s. Assume that s is minimally different from r in the sense that $k(t) < k(s)$ holds for no $t \in S$.

For the purposes of this proof an information set u will be called critical, if it is an essential information set of player 1 with $s_u \neq r_u$ such that u does not precede any other essential information set v of player 1 with $s_v \neq r_v$. Since in view of (59) we have $k(s) \geq 1$ it is clear that a critical information set can be found. Let u be a critical information set.

Since u does not precede information sets of player 1 where r and s prescribe different local strategies, (51) has the follwing consequence

(60) $E_u(t_u,s,b') = E_u(t_u,r,b')$

for every local strategy t_u at u. Since r_u is a local best reply to r and b' in $\hat{\Gamma}$ the local strategy s_u does not yield a higher local payoff under r and b'. This together with (60) yields

(61) $E_u(s_u,s,b') \leq E_u(r_u,s,b')$

It follows by (c) in 8.4 that a local change at u from s to $t = s/r_u$ does not decrease player 1's global payoff. We have:

(62) $E(t,b') \geq E(s,b') > E(r,b')$

This shows that t belongs to S. Moreover, the number k(t) of information sets with different local strategies is equal to k(s) - 1, contrary to the assumption that s is minimal in S with respect to k(s). Consequently, the assertion of the lemma is true.

<u>8.12 Theorem 3 (decentralization theorem)</u>: Let $\hat{\Gamma} = (\Gamma,f,\eta)$ be a perturbed game of a symmetric extensive 2-person game (Γ,f) and let b be a pervasive behavior strategy of player 1 in Γ. Then b is a symmetric equilibrium strategy for $\hat{\Gamma}$, if and only if for every information set u of player 1, the local strategy b_u assigned by b to u is a local best reply to f(b) in $\hat{\Gamma}$.

<u>Proof</u>: It has been pointed out in 7.3 that $r \in \hat{B}$ is a best reply to $b' \in \hat{B}'$ in $\hat{\Gamma}$, if and only if f(r) is a best reply to f(b'). Therefore, (b,f(b)) is a symmetric equilibrium point of $\hat{\Gamma}$, if and only if b is a best reply to f(b) in $\hat{\Gamma}$.

It follows by lemma 2 that $\gamma(u,b,f(b))$ is positive for every essential information set u. Therefore, lemma 3 permits the conclusion that b prescribes local best replies to f(b) in $\hat{\Gamma}$ at every information set of player 1, if b is a best reply to f(b). Lemma 4 shows that b is a best reply to f(b), if this condition on the local strategies prescribed by b is satisfied.

9. Image confrontation and detachment

The second condition in the definition of a direct ESS for perturbed games permits some further conclusions on local properties of a direct ESS for a perturbed game. The conclusions rely on an important distinction between image confronted and image separated information sets. An information set is image confronted if at least one play intersects this information set and its symmetric image. Otherwise, it is image detached.

Local payoffs at an image detached information set do not depend on the opponent's local strategy at the symmetric image of this information set. If a mutant differs from the population strategy at an image detached information set only, then it makes no difference for the fitness of an animal contestant whether his opponent is a mutant or not. Therefore, a local strategy change to an alternative best reply at an image detached information set always violates the second condition. Stability requires that a strong local best reply is played at an image detached information set. Of course, this is only a heuristic argument which needs to be worked out in detail.

The local properties of a direct ESS for a perturbed game have the consequence that a limit ESS must be a symmetric equilibrium strategy which prescribes pure local strategies at image detached information sets. One can expect that in many cases the analysis of biological game models is greatly simplified by the use of this fact.

In models with distinguishing initial random decisions (see 5.4) all information sets of players 1 and 2 are image detached and a limit ESS must be a pure symmetric equilibrium strategy.

Unfortunately, the necessary local conditions for a direct ESS of a perturbed game are not sufficient; a counterexample indicates that there is not much hope for purely local sufficient conditions. However, as we shall see in section 11 such conditions can be derived for a special class of games.

9.1 Image confronted and image detached information sets: An information set u of a symmetric extensive 2-person game (Γ,f) is called image confronted if at least one play of Γ intersects both u and f(u). If this is not the case, then u is called image detached.

Examples are provided by figures 7 and 9. In figure 7 the information sets u and u' are image confronted. In figure 9 the information sets u and u' are image detached.

The following statements (i) and (ii) are an immediate consequence of properties (d) and (e) of a symmetry (see 5.1):

(i) An information set u is image confronted, if and only if f(u) is image confronted

(ii) An information set u is image detached, if and only if f(u) is image detached

9.2 Theorem 4: (image confrontation and detachment): Let $\hat{\Gamma} = (\Gamma,f,\eta)$ be a perturbed game of a symmetric extensive 2-person game (Γ,f) and let b be a direct ESS for $\hat{\Gamma}$. Then the following conditions (a) and (b) are satisfied for the local strategies b_u assigned by b to information sets of players 1:

(a) If u is image detached, then b_u is a strong local best reply to b and f(b) in $\hat{\Gamma}$

(b) If u is image confronted, then the following inequality holds for every alternative local best reply r_u at u, i.e. for every local best reply r_u at u with $r_u \neq b_u$ in $\tilde{\Gamma}$:

(63) $\quad E_u(b_u,b,f(b/r_u)) > E_u(r_u,b,f(b/r_u))$

Proof: It will first be shown that (b) holds for every information set of player 1, not only for image confronted ones. Then (a) will be derived as a consequence of (b) for image detached information sets.

Consider an alternative local best reply r_u at u in $\hat{\Gamma}$. Local payoffs for r_u under b and f(b) are the same as for b_u. It follows by (b) in 8.4 that $r=b/r_u$ is an alternative best reply to b and f(b) in $\hat{\Gamma}$. Therefore, the second condition in the definition of a direct ESS requires:

(64) $$E_u(b_u,b/r_u,f(b/r_u)) > E_u(r_u,b/r_u,f(b/r_u))$$

This is nothing else than inequality (63); it does not matter whether b or b/r_u is the second argument in the local payoff, since the first argument is the relevant local strategy at u. This shows that (b) holds for every information set of player 1.

Now, suppose that u is image detached. Then a local change at f(u) does not influence the local payoff at u, since no play through u intersects f(u); the realization probabilities of u and of endpoints after u remain unchanged. Therefore, for every local strategy s_u at u the following must be true:

(65) $$E_u(s_u,b,f(b/r_u)) = E_u(s_u,b,f(b))$$

In view of the fact that r_u is a local best reply to b and f(b) in $\hat{\Gamma}$ it follows that both local payoffs in (63) must be equal. Consequently, (63) cannot be satisfied unless b_u is the only best reply to b and f(b) at u in $\hat{\Gamma}$. Condition (a) holds for image detached information sets.

<u>9.3 Remark</u>: It has been pointed out in 8.5 that r_u is a local best reply to b and b' in $\hat{\Gamma}$, if and only if $f(r_u)$ is a local best reply to f(b') and f(b) in $\hat{\Gamma}$. This remains true if the word "local best reply" are replaced by "strong local best reply". This together with (i) and (ii) in 9.1 and the symmetry property (54) of local payoffs yields the conclusion that under the assumptions of theorem 4 the following conditions (a') and (b') hold for the local strategies b'_u assigned by b' = f(b) to information sets u of player 2:

(a') If u is image detached then b'_u is a strong local best reply to b and f(b) in $\hat{\Gamma}$

(b') If u is image confronted then the following inequality holds for every alternative local best reply r'_u at u, i.e. for every local best reply r'_u with $r'_u \neq b'_u$

(66) $$E'_u(b'_u, b/f(r'_u), b') > E'_u(r'_u, b/f(r'_u), b')$$

9.4 Theorem 5 (properties of a limit ESS): Let (Γ,f) be a symmetric extensive 2-person game and let b be a limit ESS of (Γ,f). Moreover, for every information set u of player 1 let b_u be the local strategy assigned to u by b. The following statements (i), (ii) and (iii) hold:

(i) b is a symmetric equilibrium strategy of (Γ,f)

(ii) For every image detached information set u of player 1 the local strategy b_u is pure

(iii) At every image confronted information set u of player 1 which is not blocked by f(b) the following condition is satisfied for every alternative local best reply r_u to b and f(b), i.e. for every local best reply r_u with $r_u \neq b_u$

(67) $$E_u(b_u, b, f(b/r_u)) \geq E_u(r_u, b, f(b/r_u))$$

Proof: Statement (i) simply repeats the content of remark 7.12. It has been included in the assertion for systematic reasons only.

As we have seen in 8.5 a strong local best reply in a perturbed game is extreme. Statement (a) in theorem 4 has the consequence that a direct ESS of a perturbed game of (Γ,f) assigns an extreme local strategy to every image detached information set. Consider a test sequence $\hat{\Gamma}^1, \hat{\Gamma}^2, \ldots$ such that b is a limit equilibrium point of this test sequence and let $b^1, b^2, \ldots$ be a sequence of direct ESS's for the corresponding members of the test sequence which converges to b. Clearly, the sequence of extreme local strategies b_u^1, b_u^2,... assigned to an image detached information set u of player 1 by $b^1, b^2, \ldots$, respectively cannot converge to the local strategy b_u unless b_u is a pure local strategy. Therefore, (ii) holds.

Consider an image confronted information set u of player 1 and let $b_u^1, b_u^2, \ldots$ be defined as above. Each of the b_u^k must satisfy (i) of theorem 4 in Γ^k. It follows by (63) that (iii) in the assertion of the theoren holds.

9.5 The example of figure 3: The full power of theoretical tools cannot be revealed by extremely simple examples, but even there the analysis is facilitated by the application of the results obtained up to now. The game of figure 3 with the natural symmetry described in 5.3 begins with a random decision which is distinguishing in the sense of 5.4. This has the consequence that all information sets of player 1 in figure 3 are image detached. It follows by (i) and (ii) in theorem 5 that a limit ESS of this game must be a pure symmetric equilibrium strategy. If player 2 chooses e at u_8 then player 1's only local best reply at u_4 and u_5 is d. Similarly, if player 2 chooses d at u_8, then player 1's only local best reply at u_4 and u_5 is e. This shows that the game has exactly two symmetric equilibrium strategies in pure strategies, namely the following pure strategies φ and ψ

(68) $\varphi(u_4) = e$

(69) $\varphi(u_5) = e$

(70) $\varphi(u_9) = d$

(71) $\psi(u_4) = d$

(72) $\psi(u_5) = d$

(73) $\psi(u_9) = e$

In the literature it is customary to refer to strategies like φ as "bourgeois strategies" since they favor the possessor of the territory whereas strategies like ψ are called "paradoxical" (see for example Maynard Smith 1982).

Player 1 does not have any other best reply to $f(\varphi)$ than φ;

similarily, he does not have any other best reply to $f(\psi)$ than ψ. This shows that in both cases the second condition in the definition 6.1 of a direct ESS is satisfied. We can conclude that each of both strategies φ and ψ is a direct ESS and that the game of figure 3 has no further limit ESS.

The game has a third symmetric equilibrium strategy with mixed local strategies at some of the information sets of player 1. As we have seen it is not necessary to compute this strategy in order to exclude the possibility that it is a limit ESS.

9.6 Comment: Theorem 2 shows that a direct ESS of a perturbed game is pervasive. On the basis of this result theorem 3 gives a local characterization of the global equilibrium properties expressed by condition (a) in definition 7.5 of a direct ESS for a perturbed game.Theorem 4 shows that further local conditions are imposed by the second condition (b) in 7.5.

Since the local conditions for image detached and image confronted information sets are quite strong, the conjecture suggests itself that these conditions are not only necessary but also sufficient in the sense that every pervasive symmetric equilibrium strategy for a perturbed game with the properties (a) and (b) in theorem 4 is a direct ESS of this perturbed game. A counterexample will show that this conjecture is wrong but there are special classes of games for which it is true.

In order to have a convenient way of speaking about behavior strategies which have the local properties of an ESS, the notion of a locally stable strategy, abbreviated LSS, will be introduced. For this purpose we need some auxiliary definitions. For each image confronted information set a "local game" will be defined. The local games are "perturbed symmetric bimatrix games"; perturbed games of this kind are completely analogous to perturbed extensive games.

9.7 Perturbed symmetric bimatrix games: A perturbance η for a symmetric bimatrix game $G = (\Pi, E)$ is a function which assigns a non-negative minimum probability η_π to every $\pi \in \Pi$ such that the sum of all η_π with $\pi \in \Pi$ is smaller than 1. The symmetric bimatrix game $G = (\Pi, E)$ together with a perturbance η for G forms a perturbed game $\hat{G} = (\Pi, E, \eta)$ of G. The set of all mixed strategies q for G with $q(\pi) \geq \eta_\pi$ is denoted by $\hat{Q}$. In $\hat{G}$ only the mixed strategies in $\hat{Q}$ are permissible. A mixed strategy $r \in \hat{Q}$ is a best reply in $\hat{G}$ to $q \in \hat{Q}$ if we have:

$$(74) \qquad E(r,q) = \max_{s \in \hat{Q}} E(s,q)$$

Moreover, r is a strong best reply to q in $\hat{G}$ if there is no other best reply to q in $\hat{G}$. An equilibrium point of $\hat{G}$ is a pair (q,r) such that q and r are best replies to each other; an equilibrium point of the form (q,q) is called symmetric; q is a symmetric equilibrium strategy if (q,q) is a symmetric equilibrium point. An ESS of $\hat{G}$ is a symmetric equilibrium strategy q of $\hat{G}$ with

$$(75) \qquad E(q,r) > E(r,r)$$

for every best reply r with $r \neq q$ to q in $\hat{G}$. An equilibrium point (q,r) of $\hat{G}$ is called strong if q and r are strong best replies to each other. r is a strong symmetric equilibrium strategy if (r,r) is a strong equilibrium point.

9.8 Local games: Let $\hat{\Gamma} = (\Gamma, f, \eta)$ be a perturbed game of a symmetric extensive 2-person game (Γ, f) and let b be a pervasive behavior strategy for $\hat{\Gamma}$. Let u be an image confronted information set u of (Γ, f). The local game of $\hat{\Gamma}$ at u is a perturbed symmetric bimatrix game $\hat{G}_{ub} = (C_u, E_{ub}, \eta_u)$ whose pure strategy set is the choice set at u and whose perturbance η_u is the restriction of η to C_u and whose payoff function E_{ub} is defined as follows:

$$(76) \qquad E_{ub}(s_u, t_u) = E_u(s_u, b, f(b/t_u))$$

for every pair (s_u,t_u) of local strategies at u in $\hat{\Gamma}$. Here E_u is the local payoff function defined by (50) in 8.3. It is clear that $\hat{G}_{ub}$ is a perturbed game of $G_{ub} = (C_u,E_{ub})$. This game G_{ub} is the <u>local game of</u> (Γ,f) at u.

In the case that η_u assigns zero minimum probabilities to all choices at u, the local game $\hat{G}_{ub} = (C_u,E_{ub},\eta_u)$ is identified with $G_{ub} = (C_u,E_{ub})$.

<u>9.9 Locally stable strategies</u>: Let $\hat{\Gamma}=(\Gamma,f,\eta)$ be a perturbed game of a symmetric extensive 2-person game (Γ,f). Let b be a behavior strategy for $\hat{\Gamma}$ and for every information set u of player 1, let b_u be the local strategy assigned to u by b. The behavior strategy b is a <u>locally stable strategy</u> or shortly an LSS for $\hat{\Gamma}$, if b has the following properties (i), (ii) and (iii).

(i) b is pervasive.

(ii) If u is image detached, then b_u is a strong local best reply to b and f(b) in $\hat{\Gamma}$.

(iii) If u is image confronted, then b_u is an ESS of the local game $\hat{G}_{ub}$ of $\hat{\Gamma}$ at u.

It is clear that conditions (i), (ii) and (iii) summarize the local properties of a direct ESS of $\hat{\Gamma}$ expressed by theorems 3 and 4. A direct ESS of $\hat{\Gamma}$ must be an LSS of $\hat{\Gamma}$.

<u>9.10 A counterexample</u>: An LSS of a perturbed game is not necessarily a direct ESS of this perturbed game. An example is provided by figure 10. This game is more complicated than the examples considered up to now. Therefore, the conventions of graphical representation have been supplemented by double arrowed connecting lines above the payoff vectors; these lines show which pairs of endpoints are symmetric images of each other and which endpoints are symmetric images of themselves.

The game under consideration is a perturbed game $\hat{\Gamma}=(\Gamma,f,\eta)$ of the game (Γ,f) represented by the figure. As indicated

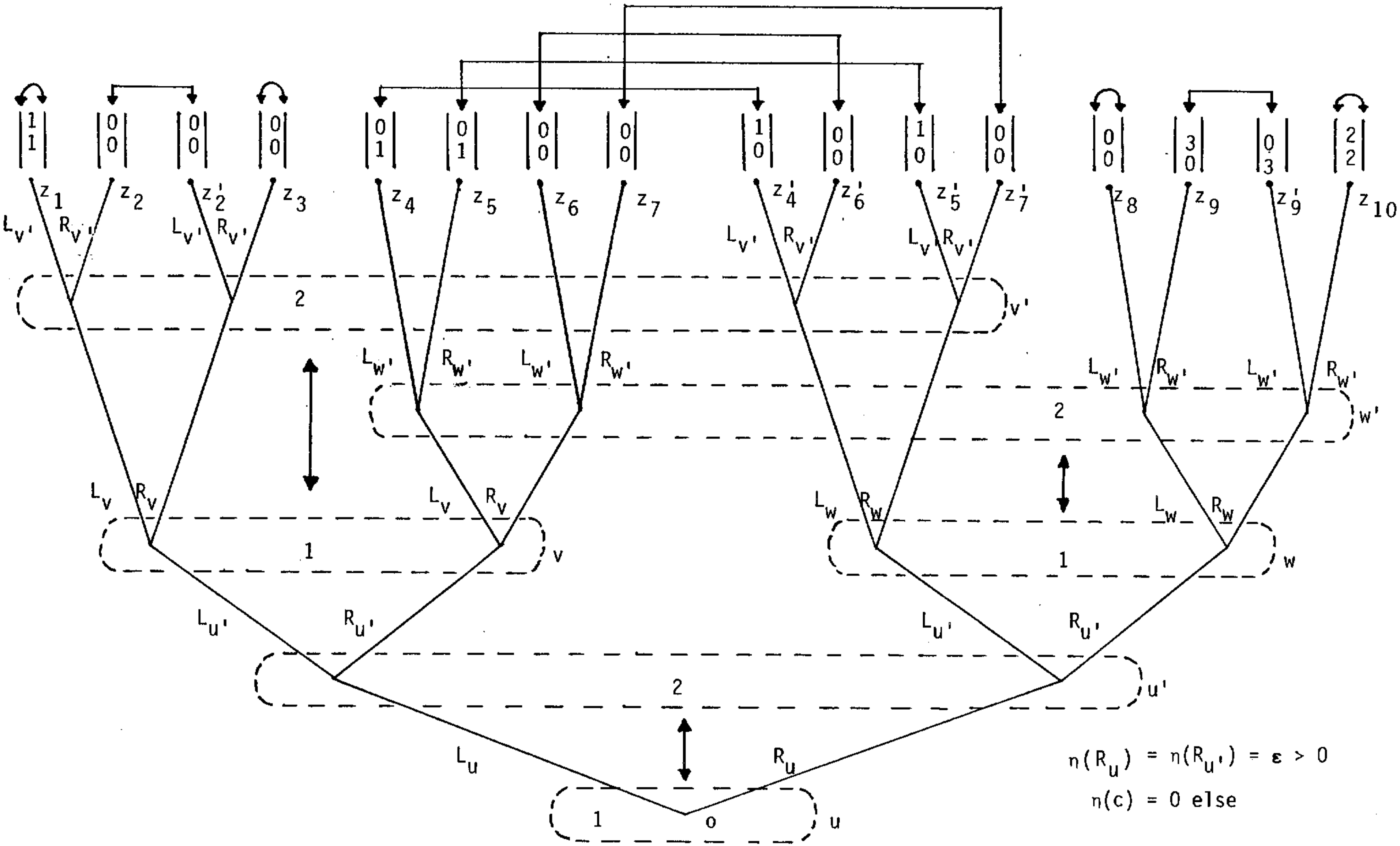


Figure 10: A counterexample. b with $b_u(L_u) = 1-\varepsilon$ and $b_v(L_v) = b_w(L_w) = 1$ is an LSS but not a direct ESS of the perturbed game. The alternative best reply $r = R_u L_v R_w$ violates the second condition.

in the figure the minimum probabilities are ε with $0 < \varepsilon < 1/2$ for the choices R_u and $R_{u'}$ at u and u', respectively, and zero everywhere else.

The strategy b which is an LSS of $\hat{\Gamma}$ but, nevertheless, fails to be a direct ESS of $\hat{\Gamma}$ is described in the explanation below the figure.

Obviously, b is pervasive and, therefore, satisfies (i) in the definition 9.9 of an LSS. All information sets in figure 10 are image confronted. Therefore, (ii) is trivally satisfied. We have to check whether (iii) holds at player 1's information sets u, v and w in figure 10.

Figure 11, 12 and 13 show the local games of $\hat{\Gamma}$ under b at u, v and w, respectively. The graphical conventions are those of figure 1. Wherever a choice has a positive minimum probability the extreme local strategy with this intended choice corresponds to a row and column instead of the choice itself. The bimatrix representation obtained in this way can be analyzed in the same way as an ordinary symmetric bimatrix game, since every permissible mixed strategy in a local game is a convex linear combination of the extreme local strategies.

In figure 11 every mixed strategy r_u with $r_u \neq b_u$ is an alternative best reply to b_u. It can be seen immediately that

(77) $$E_{ub}(b_u,r_u) > E_{ub}(r_u,r_u)$$

holds for all these alternative best replies r_u. Therefore, (iii) in definition 9.9 is satisfied at u.

In figure 12 the local strategy L_v assigned to v by b is a strong best reply to L_v. There is no alternative best reply. (iii) is satisfied at v.

	b_u	R_u
b_u	$1-\varepsilon$, $1-\varepsilon$	0, $1-\varepsilon$
R_u	$1-\varepsilon$, 0	0, 0

$b_u(L_u) = 1-\varepsilon$

$b_u(R_u) = \varepsilon$

Figure 11: The local game $\hat{G}_{ub}$ of $\hat{\Gamma}$ at u under b

	L_v	R_v
L_v	$1-\varepsilon$, $1-\varepsilon$	0, 0
R_v	0, 0	0, 0

Figure 12: The local game $\hat{G}_{vb}$ of $\hat{\Gamma}$ at v and b

	L_w	R_w
L_w	$1-\varepsilon$, $1-\varepsilon$	3ε, $1-\varepsilon$
R_w	$1-\varepsilon$, 3ε	2ε, 2ε

Figure 13: The local game $\hat{G}_{wb}$ of $\hat{\Gamma}$ at w under b

	b	r
b	$1-\varepsilon$, $1-\varepsilon$	3ε, $1-\varepsilon$
r	$1-\varepsilon$, 3ε	2, 2

Figure 14: Comparison of b with the alternative best reply $r = R_u L_v R_w$

In figure 13 the situation is essentially the same as in figure 11. All mixed strategies r_w with $r_w \neq L_w$ are alternative best replies to the local strategy L_w assigned to w by b and

(78) $$E_{wb}(L_w,r_w) > E_{wb}(r_w,r_w)$$

holds for all these alternative best replies r_w. Therefore, (iii) is satisfied at w.

It is now clear that b is an LSS of $\hat{\Gamma}$. Nevertheless, we can find an alternative best reply r to b in $\hat{\Gamma}$ which violates the second condition in definition 7.5 of a direct ESS for a perturbed game. This strategy r is a pure strategy which assigns R_u to u, and L_v to v, and R_w to w. This is expressed by the notation $r = R_u L_v R_w$.

Figure 14 does not represent a local game. It describes the payoffs obtained by the four strategy pairs which can be formed with b and r. The graphical conventions are the same as for bimatrix games. It can be seen immediately that r is an alternative best reply to b and that we have:

(79) $$E(b,r) < E(r,r)$$

This shows that b fails to be a direct ESS of the perturbed game $\hat{\Gamma}$ in spite of the fact that it is an LSS of $\hat{\Gamma}$.

9.11 Comment: The local properties of an LSS do not secure global stability in the sense of the direct ESS definition 7.5 but they do provide stability against mutants whose behavior differs from the population strategy at one information set only. Suppose that local strategies at different information sets are always controlled by different genes. If this is the case one can be satisfied with the limited stability of an LSS.

Unfortunately, the assumption of separate genetic control for each information set is less plausible than it may seem at first glance. Animal fighting behavior is often described as controlled by the relative intensities of drives like fear and aggressiveness. (Baerends 1975, Leyhausen 1965, Lorenz 1965, 1978). If this theory is correct, it is reasonable to expect that there are genes which influence the overall strength of such drives rather than their intensities in specific fighting situations. A mutation may increase the general level of aggressiveness and thereby lead to more aggressive behavior at many information sets at the same time.

It is probably true that some mutations can be excluded as practically impossible. However, plausible assumptions of this kind seem to require a detailed picture of the internal organization of animal behavior.

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