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ABSTRACT

An informational extension of a given normal form game is the result of introducing a procedure (informational pattern) for choosing the players' strategies in the initial game. As an n-tuple of "new" strategies generates an "old" outcome, it is meaningful to talk about new equilibria. Sets of all equilibria under different informational patterns are studied; the importance of the possibility for a player to commit himself to a certain reaction rule is discussed.

This paper is composed on the basis of my lectures delivered at Bielefeld University in April 1991. The aim was to impart some ideas and mathematical framework developed in Moscow in 1970th and 80th: I followed partially my earlier Russian book Kukushkin, Morozov (1984). A reader interested in a more broad and detailed picture may try to read the book Germeier (1976).

1. Introduction

Let us imagine a finite society of players each of whom aims for a goal and has some range of possible actions. The degree to which anyone's goal is achieved depends, generally speaking, on actions of all the players. If I add that every player has to choose his action independently of his partners, we would have the standard concept of a normal form game; but I will not add this sacramental phrase. Instead I say that this situation may generate various normal form games in accordance with how the process of choosing the actions is organized. From this point of view simultaneous independent choice is just one (the simplest) of possible ways to organize the process.

Certainly, when such a way (or informational pattern) is chosen and fixed, i.e. the sequence of moves, information available for every player at every moment, possibilities to reveal one's decision or just send a message etc., we can include into the definition of one's strategies all possibilities to react with one's choice to all information available, and so have again a normal form game with extended strategy sets. But the point is that our attention here will be directed not to any

one of these possible normal form games as such but to their interconnections, to their variety as a whole. So I'll prefer not to forget the connection between the simplest, initial game and all these extended games which will be called its informational extensions.

In order not to be too abstract, let us consider an example.

Suppose a peculiar person has decided to organize a game. He has invited two players and given each of them four cards with the following inscriptions: North, South, East, West. Cards of one of them are red, of the other, blue. The rules of the game are these: each of the players has to put one of his cards on a table, and then the organizer is to pay the players. If they both have chosen the same direction, "Red" has a dollar, "Blue" nothing; if the directions are just opposite, Red has nothing, Blue the dollar; at last, if the directions are orthogonal to each other, the players receive 50 cents each.

The organizer is not interested in how they would make their choices, he does not object to their reaching an agreement before putting their cards on the table but neither does he guarantee any such agreement.

Under these circumstances it is rather worrying for the players that the game taken as a bimatrix 4×4 game has no equilibrium point. Moreover, in H. Moulin's terminology (see Moulin 1986), it is characterized by the "struggle for the followership". If anyone manages to make his choice after that of his partner, knowing his decision, then he will certainly win the dollar. But if everyone tries to be the last to put his card on the table, the most plausible outcome is that nobody will do so and the bored organizer will give up his idea and quit the room carrying his dollar with him.

So a natural question arises: with what advice could a game expert help the players?

A quite traditional recommendation would be that to use mixed strategies. The players may agree to put their cards on the table with the faces down (such an agreement would be self-enforcing), and then they may choose their cards at random. If both are risk-neutral, such a procedure would lead to fair division of the dollar. But suppose both of them are risk-averse; then this equilibrium is Pareto dominated by the division "50 - 50".

Now the game expert may advise one of the players (let him be Red) openly to tear into pieces any one or two of his cards. After this Blue can put on the table such a card which his partner is unable to match, and the outcome "50 - 50" with orthogonal directions becomes an equilibrium.

I would like to stress that the proposed solution is *not* to eliminate a strategy preventing the existence of an equilibrium. If, e.g., Red is especially proud of his right to choose North, he may well retain the card to the very end.

In the light of this example it may be stated that what is to be discussed here is a version of Nash's idea of modeling cooperative agreements with non-cooperative decision-making, see Nash (1951). To interpret the following, we may suppose that there is nobody to guarantee any cooperative agreement between the players but it is possible to enforce a chosen informational pattern, i.e. procedure for non-cooperative decision-making. Here lies a source of informal considerations as to how

plausible seems "enforcibility" of this or that informational pattern.

Remark. The paper Moulin (1976) is certainly relevant to our topic, but the definition of extension by information exchange ("prolongement par échange d'information") there seems too narrow: it does not include, e.g., voluntary revealing of one player's choice to another.

2. Equilibria of quasi-informational extensions

Now I'll introduce a general definition which may seem too general, too abstract so far, but, I hope, will not seem so later on.

Let a normal form game Γ be given, i.e., a (finite) set of players N ($|N|=n$), a set of strategies X_i for each $i \in N$, and preferences of each player over the set of outcomes $X = \times_{i \in N} X_i$. We shall regard the preferences as defined by the utility functions $u_i(x)$ permitting ordinal utilities, i.e. functions defined up to a strictly monotonic transformation.

Now a quasi-informational extension of the game Γ is a game $\tilde{\Gamma}$ with the same set of players N, and the mappings $\pi: \tilde{X} \to X$ and $c_i: \tilde{X}_i \to X_i$ for each $i \in N$ satisfying the following conditions:

1) $u_i(\tilde{x}) = u_i(\pi(\tilde{x}))$

2) $\pi(c_i(\tilde{x}_i), \tilde{x}_{-i}) = (x_i, x_{-i})$

for any $i \in N$, $x_i \in X_i$, $\tilde{x}_{-i} \in \tilde{X}_{-i}$ and some $x_{-i} \in X_{-i}$.

It means that an n-tuple of "new" strategies $\langle \tilde{x}_i \rangle_{i \in N}$ defines an outcome $\pi(\tilde{x}) \in X$ of the initial game, and each player retains every of his "old" strategies $c_i(x_i) \in \tilde{X}_i$. If the strategies $\tilde{x}_i$ are rules to choose a decision under a given informational pattern, then when everybody has decided on his strategy an outcome of the game Γ must emerge, hence the mapping π. Condition 1) reflects an essential assumption that the players' utilities are derived from outcomes, and not from procedures as such. A strategy $c_i(x_i)$ is a rule which prescribes to choose x_i regardless of any information available (it is not necessarily unique); condition 2) states just this. The prefix "quasi" means that this definition requires no interpretation for the new strategies as rules for choosing old strategies under a given informational pattern. When such interpretation is given, the extension will be called informational, but this latter concept is intrinsically informal. I hope to elucidate this point later.

For a given normal form game Γ denote NE(Γ) the set of all Nash equilibria of the game.

Theorem 1. If $\tilde{\Gamma}$ is a quasi-informational extension of Γ then $NE(\Gamma) \subseteq \pi(NE(\tilde{\Gamma}))$.

Let $x^o \in NE(\Gamma)$, pick $\tilde{x}^o = \langle c_i(x_i^o) \rangle_{i \in N}$ and show $\tilde{x}^o \in NE(\tilde{\Gamma})$. Indeed, for any $i \in N$, $\tilde{y}_i \in \tilde{X}_i$ we have $\tilde{u}_i(\tilde{y}_i, \tilde{x}^o_{-i}) = u_i(\pi(\tilde{y}_i, \tilde{x}^o_{-i})) = u_i(y_i, x^o_{-i}) \le u_i(x^o) = u_i(\pi(\tilde{x}^o)) = \tilde{u}_i(\tilde{x}^o)$. ■

Denote for every normal form game Γ and every $i \in N$

$$\alpha_i(\Gamma) = \sup_{x_i \in X_i} \inf_{x_{-i} \in X_{-i}} u_i(x), \tag{1}$$

$$\beta_i(\Gamma) = \inf_{x_{-i} \in X_{-i}} \sup_{x_i \in X_i} u_i(x). \tag{2}$$

Proposition 1. For every normal form game Γ, if $x \in NE(\Gamma)$ then for every $i \in N$ the inequality $u_i(x) \ge \beta_i(\Gamma)$ holds.

The proof of the proposition is quite straightforward. ■

Proposition 2. If $\tilde{\Gamma}$ is a quasi-informational extension of Γ then for every $i \in N$ the following inequalities hold:

$$\alpha_i(\tilde{\Gamma}) \ge \alpha_i(\Gamma), \qquad \beta_i(\tilde{\Gamma}) \le \beta_i(\Gamma).$$

Indeed, for any $i \in N$, $\varepsilon > 0$ there exists $x_i^\varepsilon \in X_i$ such that $u_i(x_i^\varepsilon, x_{-i}) \ge \alpha_i(\Gamma) - \varepsilon$ for every $x_{-i} \in X_{-i}$. Pick $\tilde{x}_i^\varepsilon = c_i(x_i^\varepsilon) \in \tilde{X}_i$; for any $\tilde{x}_{-i} \in \tilde{X}_{-i}$ we have $\tilde{u}_i(\tilde{x}_i^\varepsilon, \tilde{x}_{-i}) = u_i(\pi(\tilde{x}_i^\varepsilon, \tilde{x}_{-i})) = u_i(x_i^\varepsilon, x_{-i}) \ge \alpha_i(\Gamma) - \varepsilon$. So $\alpha_i(\tilde{\Gamma}) \ge \alpha_i(\Gamma) - \varepsilon$; as ε is arbitrarily small, the first inequality is proved. Validity of the second inequality is proved quite similarly. ■

From now on we suppose that in the initial game Γ every strategy set X_i is a compact and every utility function u_i is continuous, then every sup and inf in (1) and (2) may be replaced with max or min, respectively. (This may be untrue with respect to a quasi-informational extension of Γ)

Proposition 3. If $\tilde{\Gamma}$ is a quasi-informational extension of Γ then for every $i \in N$ the inequality $\beta_i(\tilde{\Gamma}) \ge \alpha_i(\Gamma)$ holds.

This follows immediately from the well-known inequality $\beta_i(\tilde{\Gamma}) \ge \alpha_i(\tilde{\Gamma})$ and Proposition 2. ■

Denote IR(Γ) the subset of X defined by the system of inequalities $u_i(x) \ge \alpha_i$.

Theorem 2. If $\tilde{\Gamma}$ is a quasi-informational extension of Γ then $\pi(NE(\tilde{\Gamma})) \subseteq IR(\Gamma)$.

This is an obvious corollary of Propositions 1 and 3. ■

Theorem 3. For every $x^o \in IR(\Gamma)$ there exists a quasi-informational extension $\tilde{\Gamma}$ of Γ such that $x^o \in \pi(NE(\tilde{\Gamma}))$.

Define $\tilde{X}_i = X_i \cup \{*_i\}$. For every $i \in N$, $x_i \in X_i$ define $\varphi_i(x_i) \in X_{-i}$ by the condition

$$\varphi_i(x_i) \in \operatorname{Arg} \min_{x_{-i} \in X_{-i}} u_i(x).$$

Now let $\pi(\langle *_i \rangle_{i \in N}) = x^o$, $\pi(x_j, \langle *_i \rangle_{i \in N \setminus \{j\}}) = (x_j, \varphi_j(x_j))$, and for every other $\tilde{x} \in \tilde{X}$ let $\pi(\tilde{x})$ be any outcome from X satisfying $\pi(\ldots, x_i, \ldots) = (\ldots, x_i, \ldots)$.

It is straightforward to see that $\bar{\Gamma}$ is a quasi-informational extension of Γ (with c_i defined by the natural inclusion $X_i \subseteq \bar{X}_i$). Let us show that $\langle *_i \rangle_{i\in N} \in NE(\bar{\Gamma})$. Indeed, for any $i\in N$, $x_i \in X_i$ we have

$$\bar{u}_i(x_j, \langle *_i \rangle_{i\in N\setminus\{j\}}) = u_i(\pi(x_j, \langle *_i \rangle_{i\in N\setminus\{j\}})) = u_i(x_j, \rho_j(x_j)) \leq$$

$$\leq \alpha_j(\Gamma) \leq u_i(x^0) = u_i(\pi(\langle *_i \rangle_{i\in N})) = \bar{u}_i(\langle *_i \rangle_{i\in N}). \blacksquare$$

This extension has an obvious interpretation. Suppose that besides the players there is a mediator who has no objectives of his own and is ready to implement any agreed upon decision rule. Every player may either choose his strategy himself (x_i) or transfer the right to choose his strategy to the mediator $(*_i)$. The mediator makes his choice after all the players, knowing all their choices. His decision rule is described by the mapping π.

Now we have an example of the informal considerations mentioned above. If we agree to regard this extension as informational, our theory is completed (Theorems 2 and 3 describe all possible equilibria). But one may argue that the difference between an arbiter who guarantees an agreement and a mediator who is ready to choose for the players their strategies in accordance with a prescribed rule is intangible. So the only meaningful interpretation of the concept of informational extension is that each player should choose his strategy himself and the rules of the game may only define information available to each player. I'll prefer to take this second position which allows me to develop further constructions.

From this point of view Theorem 2 establishes an upper bound for possible extension of the set of equilibria. In the following sections we shall consider some specific ways to extend the initial game which are certainly informational, thereby establishing lower bounds for the set of all informational equilibria.

3. Equilibria of meta-games

The first such example is provided by the so called meta-games of Nigel Howard. The idea was first introduced in Howard (1966), but this paper was mathematically incorrect; in his later book Howard (1971) there was no attempt to describe the set of outcomes which can be made equilibrium in a meta-game. The following results were obtained in Kukushkin (1974).

The basic definition is inductive, and the principal step is as follows. Suppose that one of the players, let him be player i, is to choose his strategy after all his partners, knowing their strategies; then we have a new normal form game where a strategy of player i is a mapping $\varphi_i: X_{-i} \to X_i$ while the strategy sets of other players are the same as before. This game will be denoted ${}_i\Gamma$; it is easy to see that it constitutes a quasi-informational extension of the initial game Γ with the following mapping π: $\pi(\varphi_i, x_{-i}) = (\varphi_i(x_{-i}), x_{-i})$.

These games ${}_i\Gamma$ for $i\in N$ are called meta-games of rank 1 over the game Γ; meta-games of rank 2 over Γ are meta-games of rank 1 over meta-games of rank 1 over the game Γ and so on. A meta-game of rank m over Γ is defined by a sequence $i_1,\ldots,i_m$ and will be usually denoted as ${}_{i_m\ldots i_1}\Gamma$.

Let us discuss informally the "implementability" of meta-game procedures. So far as information on x_i is concerned, the interpretation is rather straightforward: choice of x_i implies some material actions which can be observed from outside. But information on reaction rules φ_i is a different matter – how could player i observe what player j intends to do when it is his turn to move?

One possible approach is to restrict ourselves to situations where players are not persons but organizations; then implementation of a chosen rule for reaction requires certain material steps to be taken which can be observed by other players. Certainly, it is not so easy to imagine complete information derived from such observations.

Another approach is to regard a meta-game as a, so to speak, compression of a repeated game; during the repetitions the reaction rules used by the players are, in a sense, observable (though getting complete information is again dubious). In fact, there exists an unquestionable similarity between the results to follow and the "Folk Theorem" on repeated games, see Aumann (1989), Theorem 8.14. On the other hand, this similarity should not be overestimated.

Further, we may give more freedom to our imagination, just supposing that the whole decision-making process is organized via a computer controlled by the game organizer (whose duty is to enforce the procedure). Every player submits a diskette with a program implementing his decision rule (programming language is fixed beforehand). All these programs are put on the hard disk, and the organizer allows one player's program to analyze other players' programs whenever this is required by the rules of the meta-game.

At last, I would like to argue that "implementability" of the meta-game procedures would seem much more plausible, if transmission of information were voluntary, i.e. if each player

just had the possibility to commit himself to a certain reaction rule. The present definition allows no such interpretation. Later on we shall return to this possibility.

Proposition 4. Every meta-game ${}_{i_m \ldots i_1}\Gamma$ over Γ is a quasi-informational extension of Γ.

Proposition 5. Every meta-game ${}_{i_{m+p} \ldots i_{m+1} i_m \ldots i_1}\Gamma$ is a meta-game over ${}_{i_m \ldots i_1}\Gamma$ (therefore, a quasi-informational extension of ${}_{i_m \ldots i_1}\Gamma$).

Both propositions follow immediately from the inductive character of the meta-game definition. ∎

Proposition 6. For every sequence $i_1, \ldots, i_m, i_{m+1}, \ldots i_{m+p}$ the following inclusions hold:

$$NE(\Gamma) \subseteq \pi(NE({}_{i_m \ldots i_1}\Gamma)) \subseteq \pi(NE({}_{i_{m+p} \ldots i_{m+1} i_m \ldots i_1}\Gamma)).$$

This follows immediately from Theorem 1 and Propositions 4 and 5. ∎

Now we are beginning to describe all meta-game equilibria. First we'll get necessary conditions for an outcome of the initial game to be a meta-game equilibrium, then prove their sufficiency for the simplest case of $n=2$ and, afterwards, for the general case which needs more sophisticated reasoning. From now on we suppose $N=\{1, \ldots n\}$.

For a given sequence $i_1, \ldots, i_m$ and $i \in N$ denote $N^-(i)$ the set of $j \in N$ which enter the sequence earlier than i does for the first time (if $i \notin \{i_1, \ldots, i_m\}$ then $N^-(i) = \{i_1, \ldots, i_m\}$), $N^+(i) = N \setminus (N^-(i) \cup \{i\})$.

Proposition 7. For any $i_1, \ldots, i_m$ and $i \in N$ the inequality

$$\beta_i({}_{i_m \ldots i_1}\Gamma) \geq \min_{\langle x_j \rangle_{j \in N^+(i)}} \max_{x_i} \min_{\langle x_j \rangle_{j \in N^-(i)}} u_i(x) \qquad (3)$$

holds.

Proof. Suppose for the simplicity of notation that $\{i_1,\dots,i_m\}=\{1,\dots,s\}$ $(s\le n)$ and $N^-(1)=\emptyset$, $N^-(i)=\{1,\dots,i-1\}$ for all $i\in\{1,\dots,s\}$. Consider two alternatives: either $i\le s$ or $i>s$.

In the first case let k be the number of the first entry of i into the sequence $i_1,\dots,i_m$ $(i_k=i,\ i_l\ne i$ for $l<k)$. In the meta-game ${}_{i_k\dots i_1}\Gamma$ player i has as his strategies mappings $\varphi_i\colon \bar X_1\times\dots\times\bar X_{i-1}\times X_{i+1}\dots\times X_n\to X_i$ (it is unessential now what exactly are the strategy sets $\bar X_j$ for $j<i$). There is among them (at least one) mapping φ_i^+ such that

$$\varphi_i^+(\bar x_1,\dots,\bar x_{i-1},x_{i+1},\dots,x_n)\in \operatorname{Argmax}_{x_i}\min_{x_{i-1}}\dots\min_{x_1} u_i(x),$$

hence $\alpha_i({}_{i_k\dots i_1}\Gamma)\ge \min_{x_n}\dots\max_{x_i}\dots\min_{x_1} u_i(x)$. In accordance with Propositions 3 and 5 we have the needed inequality (3).

In the second case we have $N^+(i)=\{s+1,\dots,n\}\setminus\{i\}$; so while evaluating $\beta_i({}_{i_m\dots i_1}\Gamma)$ we can pick $x_i\in\operatorname{Argmax}_{x_i}\min_{\langle x_j\rangle_{j\in N^-(i)}} u_i(x)$ and get inequality (3). ∎

Let σ be a permutation of the set N. Denote for $i\in N$

$$v^\sigma_{\sigma(i)}=\min_{x_{\sigma(n)}}\dots\min_{x_{\sigma(i+1)}}\max_{x_{\sigma(i)}}\min_{x_{\sigma(i-1)}}\dots\min_{x_{\sigma(1)}} u_i(x).$$

Theorem 4. For every $i_1,\dots,i_m$ there exists a permutation σ such that for every $x\in\pi(NE({}_{i_m\dots i_1}\Gamma))$ and every $i\in N$ the inequality

$$u_i(x)\ge v_i^\sigma \tag{4}$$

holds.

Let $j_1,\dots,j_s$ be the result of excluding from the sequence $i_1,\dots,i_m$ any repetition ($j_1=i_1$, if $i_2\ne i_1$ then $j_2=i_2$, and so on); define $\sigma(k)=j_k$ for $k\in\{1,\dots,s\}$; for $k>s$ define $\sigma(k)$ arbitrarily (but in such a way that σ be a permutation). Now the theorem follows from Proposition 7 and the well-known inequality "minmax ≥ maxmin". ∎

The values v_i^σ on the right-hand side of inequalities (4) have straightforward meaning: if all the players make their choices in the order defined by the permutation σ (starting with $\sigma(n)$ and ending with $\sigma(1)$), the best guaranteed utility level of each player i is just v_i^σ. So we may try, at least, to make an equilibrium of any outcome satisfying (4) by organizing "punishment" of a deviating player by everybody else; in which case his utility level can not exceed the level v_i^σ. The problem is that everybody's equilibrium strategy must combine the readiness to punish everybody else, should he deviate, with signaling his own good intentions. This task is much easier for $n=2$ where there is no need for signaling than for the general case.

If $n=2$, Theorem 4 states that at any equilibrium of any meta-game the player who is the first in the sequence defining the meta-game gets at least his minmax, while his partner gets at least his maxmin.

Theorem 5. If $N=\{1,2\}$ and $x^\circ=(x_1^\circ,x_2^\circ)\in X$ is such that $u_1(x^\circ)\ge\beta_1(\Gamma)$, $u_2(x^\circ)\ge\alpha_2(\Gamma)$, then $x^\circ\in NE({}_{21}\Gamma)$.

Pick $x_2^{(1)}\in\operatorname{Argmin}_{x_2}\max_{x_1} u_1(x)$ and define the mappings $\varphi_1^\circ\colon X_2\to X_1$, $\varphi_2^\circ\colon\Phi_1\to X_2$ constituting the needed Nash equilibrium:

$$\varphi_1^\circ(x_2)=\begin{cases}x_1^\circ, & \text{if } x_2^\circ=x_2,\\ \chi_1^{(2)}(x_2), & \text{if } x_2^\circ\ne x_2,\end{cases}\qquad \varphi_2^\circ(\varphi_1)=\begin{cases}x_2^\circ, & \text{if } \varphi_1=\varphi_1^\circ,\\ x_2^{(1)}, & \text{if } \varphi_1\ne\varphi_1^\circ,\end{cases}$$

where $\chi_1^{(2)}(x_2)\in\operatorname{Argmin}_{x_1} u_2(x)$.

Indeed, if $\varphi_1\ne\varphi_1^\circ$ then $\pi(\varphi_1,\varphi_2^\circ)=(\varphi_1(x_2^{(1)}),x_2^{(1)})$, so $\bar u_1(\varphi_1,\varphi_2^\circ)\le\beta_1(\Gamma)\le u_1(x^\circ)$. If $\varphi_2(\varphi_1^\circ)=x_2^\circ$ then $\pi(\varphi_1^\circ,\varphi_2)=x^\circ$. At last, if $\varphi_2(\varphi_1^\circ)\neq x_2^\circ$ then $\pi(\varphi_1^\circ,\varphi_2)=(\chi_1^{(2)}(x_2),x_2)$ for some $x_2\in X_2$; so $u_2(\varphi_1^\circ,\varphi_2)\leq \alpha_2(\Gamma)\leq u_2(x^\circ)$. ∎

These strategies have a rather simple interpretation: every player is ready to choose x°, but if his partner deviates he is ready to punish him. The conditions of the theorem guarantee deterring effect.

So for $n=2$ we have necessary and sufficient conditions for an outcome to be a meta-game equilibrium. It appears natural to expect that any outcome satisfying inequalities (4) becomes an equilibrium in the meta-game ${}_{\sigma(n)\dots\sigma(1)}\Gamma$ for which the systems of inequalities (3) and (4) coincide. In his original paper N. Howard stated just this, but was wrong.

Consider the following finite three-person game Γ where $X_1=\{0,1,2\}$, $X_2=X_3=\{0,1\}$, and the utilities are as follows: $u_1(0,0,0)=20$, $u_1(x)=15$ for any other $x\in X$; $u_2(0,0,0)=10$, $u_2(1,0,1)=u_2(2,1,1)=5$, $u_2(x)=15$ for any other $x\in X$; $u_3(0,0,0)=10$, $u_3(1,1,1)=u_3(1,0,0)=5$, $u_3(x)=15$ for any other $x\in X$. It is easy to see that $v_1^e=15$, $v_2^e=5$, $v_3^e=5$ for the identity permutation e: $e(i)=i$ $\forall i\in N$, so the outcome (0,0,0) satisfies conditions (4).

Proposition 8. The outcome (0,0,0) does not belong to the set $\pi(NE({}_{321}\Gamma))$.

Note that in the meta-game the strategy set of player 1 is $\Phi_1=\{\varphi_1:X_2\times X_3\to X_1\}$, the strategy set of player 2 is $\Phi_2=\{\varphi_2:\Phi_1\times X_3\to X_2\}$, the strategy set of player 3 is $\Phi_3=\{\varphi_3:\Phi_1\times\Phi_2\to X_3\}$. Suppose, to the contrary, that there exist $(\varphi_1^\circ,\varphi_2^\circ,\varphi_3^\circ)\in NE({}_{321}\Gamma)$ such that $\pi(\varphi_1^\circ,\varphi_2^\circ,\varphi_3^\circ)=(0,0,0)$ which means $\varphi_3^\circ(\varphi_1^\circ,\varphi_2^\circ)=0$, $\varphi_2^\circ(\varphi_1^\circ,0)=0$, $\varphi_1^\circ(0,0)=0$. Consider two possibilities.

Let $\varphi_1^\circ(1,1)\neq 2$; denote $\varphi_2^*\in\Phi_2$ the mapping for which $\varphi_2^*(\varphi_1,x_3)\equiv 1$. Now we have $u_2(\pi(\varphi_1^\circ,\varphi_2^*,\varphi_3^\circ))=u_2(x_1^*,1,x_3^*)=15$ as $x_1^*=\varphi_1^\circ(1,x_3^*)$. This means that using φ_2^* instead of φ_2° player 2 would raise his utility level.

Let $\varphi_1^\circ(1,1)\neq 1$; denote $\varphi_3^*\in\Phi_3$ the mapping for which $\varphi_3^*(\varphi_1,\varphi_2)\equiv 1$. Now we have $u_3(\pi(\varphi_1^\circ,\varphi_2^\circ,\varphi_3^*))=u_3(x_1^*,x_2^*,1)=15$ as $x_1^*=\varphi_1^\circ(x_2^*,1)$. This means that using φ_3^* instead of φ_3° player 3 would raise his utility level. ∎

It is worthwhile to discuss this example in more details. Player 1 may be regarded as a party interested in the outcome (0,0,0). The previous proof shows that in the meta-game his knowledge of his partners' strategies is insufficient to make this outcome an equilibrium. The situation $x_2=1$, $x_3=1$ may arise when player 2 has deviated and player 3 began to punish him as well as when player 2 has begun punishment of deviating player 3. Player 1 can not distinguish between these two possibilities and he can not punish both his partners simultaneously. Hence, whatever strategy of player 1 be adopted, at least one of his partners can profit by deviating.

So to make any outcome satisfying (4) an equilibrium it is necessary that every player before making his final decision know about everyone of his partners whether he intends to deviate or not.

Theorem 6. For any permutation σ and any outcome x° of the game Γ satisfying (4) the following inclusion holds:

$$x^\circ \in {}_{\sigma(n-2)\dots\sigma(1)\sigma(n)\dots\sigma(1)}\Gamma.$$

Without any restriction on generality suppose $\sigma=e$ and introduce some denotations. Let $\Gamma^*={}_{n\dots1}\Gamma$, $\tilde\Gamma^*={}_{n-2\dots1n\dots1}\Gamma={}_{n\dots1}\Gamma^*$; the projection from the set of the outcomes of $\tilde\Gamma^*$ to that of Γ^* existing according to Proposition 5 will be denoted

π^x: $\varphi_1,\dots,\varphi_n$ will denote mappings $\varphi_1: X_2\times\dots\times X_n\to X_1$, ..., $\varphi_i:\Phi_1\times\dots\times\Phi_{i-1}\times X_{i+1}\times\dots\times X_n\to X_i,\dots,\varphi_n:\Phi_1\times\dots\times\Phi_{n-1}\to X_n$; $\psi_1,\dots,\psi_{n-2}$ will denote mappings $\psi_1:\Phi_2\times\dots\times\Phi_n\to\Phi_1$, ..., $\psi_i:\Psi_1\times\dots\times\Psi_{i-1}\times\Phi_{i+1}\times\dots\times\Phi_n\to\Phi_i,\dots,\psi_{n-2}:\Psi_1\times\dots\times\Psi_{n-3}\times\Phi_{n-1}\times\Phi_n\to\Phi_{n-2}$. In the meta-game Γ^x the strategies of the players are mappings $\varphi_1,\dots,\varphi_n$; in the meta-game $\bar{\Gamma}^x$ they are $\psi_1,\dots\psi_{n-2},\varphi_{n-1},\varphi_n$. For every pair $i,j\in N$ $(i\neq j)$ fix a mapping $\chi_i^{(j)}:X_{i+1}\times\dots\times X_n\to X_i$, satisfying for all $(x_{i+1},\dots,x_n)$ the following condition:

$$\chi_i^{(j)}(x_{i+1},\dots,x_n)\in\begin{cases}\underset{x_i}{\text{Argmin}}\ \underset{x_{i-1}}{\min}\dots\underset{x_1}{\min}\ u_j(x), & \text{if } i<j,\\ \underset{x_i}{\text{Argmin}}\ \underset{x_{i-1}}{\min}\dots\underset{x_j}{\max}\dots\underset{x_1}{\min}\ u_j(x), & \text{if } i>j,\end{cases}$$

Now define inductively the sets $\Phi_i^o\subseteq\Phi_i$ by the following equalities:

$\Phi_1^o = \{\varphi_1^o\in\Phi_1 | \varphi_1^o(x_2^o,\dots,x_n^o)=x_1^o\}$,

$\Phi_i^o = \{\varphi_i^o\in\Phi_i | \varphi_i^o(\varphi_1^o,\dots,\varphi_{i-1}^o,x_{i+1}^o,\dots,x_n^o)=x_i^o\ \forall\langle\varphi_j^o\in\Phi_j^o\rangle_{j<i}\}$.

The strategies from Φ_i^o may be called "well-meant". Now for every pair $i,j\in N$ $(i\neq j)$ introduce the "strategy of deterrence" $\varphi_i^{(j)}\in\Phi_i^o$:

$$\varphi_i^{(j)}(\varphi_1,\dots,\varphi_{i-1},x_{i+1},\dots,x_n)=\begin{cases}x_i^o, \text{ if } \varphi_s\in\Phi_s^o\ \forall s<i \text{ and}\\ \qquad x_s=x_s^o\ \forall s>i,\\ \chi_i^{(j)}(x_{i+1},\dots,x_n) \text{ else.}\end{cases}$$

For $i\neq n-1$ define the mappings $\varphi_i^o\in\Phi_i^o$:

$$\varphi_i^o(\varphi_1,\dots,\varphi_{i-1},x_{i+1},\dots,x_n)=\begin{cases}x_i^o, \text{ if } \varphi_s\in\Phi_s^o\ \forall s<i,\\ \chi_i^{(j)}(x_{i+1},\dots,x_n) \text{ if}\\ \qquad j=\min\{s|\varphi_s\notin\Phi_s^o\}.\end{cases}$$

The mapping $\varphi_{n-1}^o\in\Phi_{n-1}^o$ is defined in a specific way:

$$\varphi_{n-1}^o(\varphi_1,\dots,\varphi_{n-2},x_n)=\begin{cases}x_{n-1}^o, & \text{if } \varphi_s\in\Phi_s^o\ \forall s<i \text{ and } x_n^o=x_n,\\ \chi_{n-1}^{(n)}(x_n), & \text{if } \varphi_s\in\Phi_s^o\ \forall s<i \text{ and } x_n^o\neq x_n,\\ \chi_{n-1}^{(j)}(x_n) & \text{if } j=\min\{s|\varphi_s\notin\Phi_s^o\}.\end{cases}$$

At last, define the mappings $\psi_i^o\in\Psi_i$ for $i\leq n-2$:

$$\psi_i^o(\psi_1,\dots,\psi_{i-1},\varphi_{i+1},\dots,\varphi_n)=\begin{cases}\varphi_i^o, & \text{if } \varphi_s\in\Phi_s^o\ \forall s>i,\\ \varphi_i^{(j)}, & \text{if } j=\min\{s|\varphi_s\notin\Phi_s^o\}.\end{cases}$$

Now denote $\tilde{x}^o=(\psi_1^o,\dots,\psi_{n-2}^o,\varphi_{n-1}^o,\varphi_n^o)$; verification of the equality $\pi(\tilde{x}^o)=x^o$ is straightforward. Let us show $\tilde{x}^o\in NE(\tilde{\Gamma}^x)$, i.e. that no player can profit by deviating.

Consider the consequences of a deviation by a player $j\in\{1,\dots,n-2\}$. Let $\psi_j^*\in\Psi_j$, $\tilde{\psi}^*=(\psi_1^o,\dots,\psi_{j-1}^o,\psi_j^*,\psi_{j+1}^o,\dots,\psi_{n-2}^o,\varphi_{n-1}^o,\varphi_n^o)$; denote $x^*=\pi(\tilde{\psi}^*)$, $(\varphi_1^*,\dots,\varphi_{n-2}^*,\varphi_{n-1}^o,\varphi_n^o)=\pi^x(\tilde{\psi}^*)$. It follows immediately from the definition of ψ_i^o that $\varphi_i^*=\varphi_i^o$ for $i\in\{j+1,\dots,n-3\}$. Now we have just two alternatives: either $\varphi_j^*\in\Phi_j^o$ or not. If $\varphi_j^*\in\Phi_j^o$ then we have $\varphi_i^*=\varphi_i^o$ for $i\in\{1,\dots,j-1\}$ hence $x^*=x^o$ which means the deviation has brought player j neither profit nor loss. If $\varphi_j^*\notin\Phi_j^o$, we have $\varphi_i^*=\varphi_i^{(j)}$ for $i\in\{1,\dots,j-1\}$ and $x_i^*=\chi_i^{(j)}(x_{i+1}^*,\dots,x_n^*)$ for $i\in\{j+1,\dots,n\}$. If it occurs that $x_i^*=x_i^o$ for $i\in\{j,\dots,n\}$, then, in accordance with the definition of $\varphi_i^{(j)}$, the equality $x_i^*=x_i^o$ holds for $i\in\{1,\dots,j-1\}$, too; hence $x^*=x^o$ again. At last, if $x_i^*\neq x_i^o$ for at least one $i\in\{j,\dots,n\}$, then we have $x_i^*=\chi_i^{(j)}(x_{i+1}^*,\dots,x_n^*)$ for every $i\in\{1,\dots,j-1\}$, hence $u_j(x^*)\leq v_j^e$ and the deviation is unprofitable.

Now suppose player n has deviated choosing φ_n^* instead of φ_n^o. If $\varphi_n^*\in\Phi_n^o$, nobody will notice anything, so $x^*=x^o$. If $\varphi_n^*\notin\Phi_n^o$ then every player $i<n-1$ will use $\varphi_i^{(n)}$; if it occurs that

$x^*_n=\varphi^*_n(\varphi_1^{(n)},\dots,\varphi_{n-2}^{(n)},\varphi^o_{n-1})=x^o_n$, then every player $i<n$ will choose x^o_i and nothing will change; if $x^*_n\neq x^o_n$, then every player $i<n$ will choose $x^*_i=\chi_i^{(n)}(x^*_{i+1},\dots,x^*_n)$, hence player n's utility can not exceed the level v^e_n.

For a player n-1's deviation the reasoning is essentially the same. If $\varphi^*_{n-1}\neq\varphi^o_{n-1}$ then every player $i<n-1$ will use $\varphi_i^{(n-1)}$ while player n will choose $x^*_n=\chi_n^{(n-1)}$; if it occurs that $x^*_n=x^o_n$ and $x^*_{n-1}=\varphi^*_{n-1}(\varphi_1^{(n-1)},\dots,\varphi_{n-2}^{(n-1)},x^o_n)=x^o_{n-1}$ then everybody else will choose $x^*_i=x^o_i$ and nothing will change; if $x^*_n\neq x^o_n$ or $x^*_{n-1}\neq x^o_{n-1}$, then every player $i<n-1$ will choose $x^*_i=\chi_i^{(n-1)}(x^*_{i+1},\dots,x^*_n)$, hence player n-1's utility can not exceed the level v^e_{n-1}. ■

Remark 1. The strategies ψ^o_i do not use any information about the chosen strategies ψ^o_j for $j<i$.

Remark 2. The system (4) is always consistent because in the meta-game $\sigma(n)\dots\sigma(1)\Gamma$ (in fact, even in $\sigma(n-1)\dots\sigma(1)\Gamma$) every player i has a strategy guaranteeing him the utility level v^σ_i.

4. Filling the hole in the Pareto border for n=2

Let us return to the case n=2. It is rather easy to see that if the system $u_i(x)\geq\beta_i$ (i=1,2) is consistent then any Pareto optimal outcome can be made equilibrium in an appropriate meta-game. The problem is that the system may very well be inconsistent, in which case the most "symmetrical" Pareto optimal outcomes of the game are *not* meta-game equilibria. What shall we (or the players) do in this case?

First of all, remark that the constructions of Theorem 5 can be applied to any extension $\bar\Gamma$ of the game Γ, not necessarily to Γ itself (provided it be possible to write min instead of inf in equalities (1) and (2) for $\bar\Gamma$). So we can organize the game $\bar\Gamma$ in such a way that nobody should have his $\beta_i(\bar\Gamma)$ as $\alpha_i(\bar\Gamma)$, our first example demonstrates how this can be achieved, and then apply Theorem 5 to the game $\bar\Gamma$.

Suppose one of the strategy sets (let it be X_2) is provided with a metric ρ; denote R=diam(X_2). For any r, $0\leq r\leq R$, define the following game Γ^r. First, player 2 chooses $y_2\in X_2$ (a preliminary choice), then player 1, knowing y_2, chooses $x_1\in X_1$, at last, player 2 chooses $x_2\in B(y_2,r)=\{x_2\in X_2|\rho(x_2,y_2)\leq r\}$. In this informational extension of the game Γ strategies of player 1 are just mappings $\varphi_1:X_2\to X_1$, while strategies of player 2 are pairs of a choice $y_2\in X_2$ and a mapping $\varphi_2:X_1\to B(y_2,r)$; the projection π runs as follows: $\pi(\varphi_1,(y_2,\varphi_2))=(\varphi_1(y_2),\varphi_2(\varphi_1(y_2)))$.

It is easy to see that

$$\beta_1(\Gamma^r) = \alpha_1(\Gamma^r) = \min_{y_2}\ \max_{x_1}\ \min_{x_2\in B(y_2,r)} u_1(x),$$

$$\alpha_2(\Gamma^r) = \beta_2(\Gamma^r) = \max_{y_2}\ \min_{x_1}\ \max_{x_2\in B(y_2,r)} u_1(x).$$

Denote $v_1(r)=\beta_1(\Gamma^r)$, $v_2(r)=\alpha_2(\Gamma^r)$.

Call a metric compact X essentially connected if the mapping $X\times[0,R]\mapsto B(x,r)$ is continuous w.r.t. Hausdorff's metric at the right side.

Proposition 9. If the set X_2 is essentially connected and x^o is a Pareto optimal outcome of the game Γ satisfying

$\beta_1(\Gamma)\geq u_1(x^o)\geq\alpha_1(\Gamma)$, $\beta_2(\Gamma)\geq u_2(x^o)\geq\alpha_2(\Gamma)$,

then there exists $r\in[0,R]$ such that

$u_i(x^o)\geq v_i(r)$ for both i=1,2.

It follows immediately from the supposed essential connectedness of X_2 that $v_1(r)$ is continuous on $[0,R]$. As $v_1(0)=\beta_1(\Gamma)$ and $v_1(R)=\alpha_1(\Gamma)$, there exists $r\in[0,R]$ such that $u_1(x^\circ)=v_1(r)$. As $v_i(r)=\alpha_i(\Gamma^r)$ for both i, there exist strategies ρ_1°, $(y_2^\circ,\varphi_2^\circ)$ for which $u_i(\pi(\varphi_1^\circ,(y_2^\circ,\varphi_2^\circ)))\geq v_i(r)$. So the inequality $u_2(x^\circ)<v_2(r)$ would contradict the Pareto optimality of the outcome x°. ∎

Theorem 7. If the conditions of Proposition 9 hold then there exists $r\in[0,R]$ such that $x^\circ\in\pi(\mathrm{NE}({}_{21}\Gamma^r))$.

It follows immediately from Proposition 9 and Theorem 5. ∎

Remark. The theorem remains quite meaningful for a zero-sum game Γ, if the players are risk-averse and, consequently, their use of mixed strategies could destroy Pareto optimality.

Let us discuss the essential connectedness property without going too deep into topological details.

Proposition 10. If X is a convex compact subset of a Banach space then X is essentially connected w.r.t. the metric induced by the norm.

I will omit the simple and rather tedious proof. ∎

Proposition 11. If X is a connected compact polyhedron (see, e.g., Spanier 1966) then there exists an equivalent metric on X under which it is essentially connected.

Let us fix a triangulation of X. For any two points belonging to the same simplex define our metric as the distance in the pre-image of the simplex. For any other pair of points x,y we consider all sequences $x_0,\dots,x_T$ where $x_0=x$, $x_T=y$ and every two points x_i,x_{i+1} belong to a simplex; now we define the distance between x and y as the minimum of the sums of distances between the neighboring points over the set of all such sequences.

I'll skip over all further details. ∎

Remark. We could redefine the meta-games supposing on each step that the corresponding player can reveal his strategy to his partner but is not obliged to do so (though no cheating is possible). Under this modified definition it is possible to show that any outcome giving each player at least β_i is a meta-game equilibrium. As in the game Γ^r the equalities $\alpha_i=\beta_i$ hold, we may state that passing of information about x_2 must be compulsory, while revealing the reaction functions φ_i may be voluntary.

Certainly, if X is disconnected, then it can not be essentially connected w.r.t. any equivalent metric. So our theorem is not applicable to (bi)matrix games. Indeed, some time ago A.Vasin constructed (but had never published) the following peculiar example of an antagonistic 3×3 game Γ:

0	-1	-1
1	1	-1
1	-1	1

It is easy to see that $\beta_1(\Gamma)=1$, $\alpha_1(\Gamma)=-1$; so the game has no saddle-point. Certainly, the value of Γ in mixed strategies is 0, but the players are risk-averse and would prefer to get 0 in a pure-strategy equilibrium.

Suppose there is a procedure for sequential decision-making leading to choosing the outcome 0 as an equilibrium. As we are considering pure strategies only, the trajectory is unique, and one of the players has to be the last to fix his final choice. Suppose, it is player 1; it means that at some moment player 1 knows that x_2 is fixed, while having some range of possibilities for himself. Obviously, he can then ensure the utility level of 1, and our supposed strategies do not constitute an equilibrium. Quite similarly, if player 2 is the last, he can ensure the utility level of −1 (for player 1); equilibrium is impossible again.

If we drop the idea of voluntary provision of information on reaction functions, it becomes possible to make an equilibrium of any individually rational outcome (see Howard 1974), but a vague philosophical question remains whether this extension is better interpretable than that of Theorem 3.

5. "Vasin - Gurvich"'s Meta-Games

Now consider a very peculiar construction due to A.Vasin and V.Gurvich (never published properly, all but just announced in Vasin, Gurvich 1980).

Their definition differs from the Howard's one just in one point: the basic step. For $i,j\in N, i\neq j$ denote ${}^{j}_{i}\Gamma$ the game where player i makes his choice of x_i knowing x_j (and everybody else has no information about the partners' choices). An arbitrary Vasin-Gurvich's meta-game is obtained by iterating this transformation. It is quite obvious that any Howard's meta-game can be obtained in this way, so the only question is whether this construction leads to a more expanded set of equilibria. And the answer is that it does.

Theorem 8. If $i,j\in N, i\neq j$ and an outcome $x^{\circ}\in X$ of a game Γ satisfies the inequalities:

$$u_i(x^{\circ})\geq \min_{x_j}\max_{x_i}\min_{\langle x_k\rangle k\neq i,j} u_i(x),$$

$$u_k(x^{\circ})\geq\alpha(\Gamma) \quad \text{for every } k\neq i,$$

then there exists a (Vasin-Gurvich's) meta-game $\dot{\Gamma}^{*}$ over Γ for which $x^{\circ}\in\pi(NE(\tilde{\Gamma}^{*}))$.

I will give a proof only for $n=3$. Denote $\tilde{\Gamma} = {}^{13212}_{21321}\Gamma$, $\Phi_1=\{\varphi_1:X_2\to X_1\}$, $\Phi_2=\{\varphi_2:\Phi_1\to X_2\}$, $\Phi_3=\{\varphi_3:\Phi_2\to X_3\}$, $\Psi_1=\{\psi_1:\Phi_3\to\Phi_1\}$, $\Psi_2=\{\psi_2:\Psi_1\to\Phi_2\}$. The sets Ψ_1, Ψ_2, Φ_3 are the strategy sets of the game $\tilde{\Gamma}$.

Lemma. The following equalities hold:

$$\beta_1(\tilde{\Gamma})= \min_{x_2}\max_{x_1}\min_{x_3} u_1(x), \tag{5}$$

$$\beta_2(\tilde{\Gamma})=\alpha_2(\Gamma), \tag{6}$$

$$\beta_3(\tilde{\Gamma})=\alpha_3(\Gamma). \tag{7}$$

These equalities together with Theorem 6 imply Theorem 8 (for $N=\{1,2,3\}$, $i=1$, $j=2$). So we have just to prove the lemma.

Prove equality (5). Here players 2 and 3 are punishing player 1, and the problem is how is player 3 to know x_1? The punishment strategy φ_3^{*} being fixed, player 2 knowing ψ_1 knows, in fact, φ_1; so the punishment mapping ψ_2 outside this φ_1 may be quite arbitrary, in particular, it may be used to convey complete information on φ_1 to player 3. The latter, knowing φ_1, is able to derive the choice x_1, as x_2 may be fixed beforehand.

To express this idea more technically, fix an $x_2^{*}\in \operatorname{Argmin}_{x_2}\max_{x_1}\min_{x_3} u_1(x)$ and denote $\Phi_2^{*}=\{\varphi_2^{*}\in\Phi_2 \mid \varphi_2^{*}(\varphi_1^{*})=x_2^{*}$ for a unique $\varphi_1^{*}\in\Phi_1\}$. Let $\varphi_3^{*}\in\Phi_3$ be such that for every $\varphi_2^{*}\in\Phi_2^{*}$ there holds $\varphi_3^{*}(\varphi_2^{*}) \in \operatorname{Argmin}_{x_3} u_1(\varphi_1^{*}(x_2^{*}),x_2^{*},x_3)$, where φ_1^{*} is the unique solution for $\varphi_2^{*}(\varphi_1^{*})=x_2^{*}$. At last, define $\psi_2^{*}\in\Psi_2$ by the equality $\psi_2^{*}(\psi_1)=\varphi_2^{*}$ for any $\psi_1\in\Psi_1$, where $\varphi_2^{*}(\psi_1(\varphi_3^{*}))=x_2^{*}$, $\varphi_2^{*}(\varphi_1)\neq x_2^{*}$ whenever $\varphi_1\neq\psi_1(\varphi_3^{*})$. Now it is easy to see that for any $\psi_1\in\Psi_1$ there holds $\pi(\psi_1,\psi_2^{*},\varphi_3^{*})=(x_1,x_2^{*},x_3^{*})$ where $x_3^{*}\in \operatorname{Argmin}_{x_3} u_1(x_1,x_2^{*},x_3)$.

Hence the equality (5).

Prove equality (6). Now players 1 and 3 are punishing player 2. As the punishment strategy φ_1^* may be chosen beforehand, player 3 knowing φ_2 is able to derive x_2.

In more precise words, fix a $\varphi_1^* \in \Phi_1$ for which $\varphi_1^*(x_2) \in \operatorname{Argmin}_{x_1} \min_{x_3} u_2(x)$, define ψ_1^* by the equality $\psi_1^*(\varphi_3)=\varphi_1^*$ for every $\varphi_3 \in \Phi_3$, and pick a $\varphi_3^* \in \Phi_3$ satisfying $\varphi_3^*(\varphi_2) \in \operatorname{Argmin}_{x_3} u_2(\varphi_1^*(\varphi_2(\varphi_1^*)), \varphi_2(\varphi_1^*), x_3)$ for every $\varphi_2 \in \Phi_2$. Now it is easy to see that for any $\psi_2 \in \Psi_2$ there holds $\pi(\psi_1^*, \psi_2, \varphi_3^*)=(x_1^*, x_2, x_3^*)$ where $(x_1^*, x_3^*) \in \operatorname{Argmin}_{x_1, x_3} u_2(x)$. Hence the equality (6).

At last, to the equality (7); now players 1 and 2 are punishing player 3. When the strategy $\varphi_2^* \in \Phi_2$ is fixed, player 1 knowing φ_3 is able to derive $x_3=\varphi_3(\varphi_2^*)$ and, consequently, to choose a proper punishment pair (x_1, x_2). Now he is to choose a $\varphi_1 \in \Phi_1$ communicating according to some (adopted beforehand) coding system the proper x_2 to be chosen by player 2 to him. And the latter's mapping φ_2^* is to decipher this message in accordance with the same coding system.

The reader is invited to produce the necessary technicalities him(her)self. ∎

Remark. It is rather instructive to compare the equilibrium strategies of Theorems 6 and 8. In the first case the use of information available to each player was in a sense straightforward: everybody tried to recognize the deviator and to punish him properly. Here the players are much more sophisticated: the possibility to commit himself to a certain reaction rule which becomes known to one's partner is used to communicate information on another partner's decision.

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