INSTITUTE OF MATHEMATICAL ECONOMICS

WORKING PAPERS

Nr. 209

The Kernel of Homogeneous Games with Steps

by

Bezalel Peleg Joachim Rosenmüller Peter Sudhölter

January 1992

University of Bielefeld

4800 Bielefeld, Germany

The Kernel of Homogeneous Games with Steps

B.Peleg, J.Rosenmüller, P.Sudhölter

Abstract

The class of homogeneous n–person constant sum games was introduced by VON NEUMANN–MORGENSTERN [20]; as a solution concept they treated the v.N.–M.–solution (or main simple solution). PELEG [9] discussed the kernel and nucleolus within this framework. The general theory of non–constant sum homogeneous games was developed by OSTMANN [8], ROSENMÜLLER [14], [15], [16] and SUDHÖLTER [19]. Based on these results, PELEG–ROSENMÜLLER [12] discussed the kernel (and other solution concepts) for homogeneous games without steps while ROSENMÜLLER–SUDHÖLTER [18] proved a reduction theorem for the nucleolus. The present paper provides the reduction theorem for the kernel: it is shown that the kernel of a homogeneous game with steps equals the one of the game without steps obtained by reducing in a suitable way, that is, cutting off all players following the largest step.

0. Introduction, notations; homogeneous games

This section is of an introductory nature. We will present our notation as well as a short survey providing the necessary foundations from the theory of homogeneous games.

Sections 1 to 4 are organized as follows. Section 1 introduces the notations for treating the kernel and its relatives. Essentially we are dealing with the pre–kernel of a directed game and, by showing that its elements are nonnegative, we eventually convince ourselves that dealing with the kernel is sufficient in our present context.

In Section 2 we concentrate on a small problem that in most papers concerning the kernel (or the nucleolus for that matter) is not explicitly mentioned: how to treat the case that winning players are present. As one can expect, this case can be eliminated in a straightforward way; however, it takes some meticulous arguing. For the corresponding treatment with respect to the nucleolus the reader is referred to ROSENMÜLLER–SUDHÖLTER [18]. Now, as the existence of inevitable or veto–players can be excluded in a rather easy manner, the following sections deal only with the case that neither winning players nor veto–players, nor dummies are present.

Section 3 actually presents the main burden of proofs: the technique of reducing the kernel. More precisely, it is shown that in homogeneous games (with the above mentioned restrictions) every element of the kernel has zero coordinates for all players following the largest (first) step.

Finally, Section 4 collects the results and presents the main theorem. First we prove the converse of the main theorem of Section 3, that is, the kernel of the game obtained by properly reducing or truncating the game after the first step contains elements which, augmented by zero coordinates, yield kernel elements of the original game. Hence we come up with the final result that the kernel of a game with steps equals the kernel of the reduced game.

To come back to the purpose of the present section, let us first introduce some notation and then focus on the theory of homogeneous games.

The term homogeneous game goes back to VON NEUMANN and MORGENSTERN [20]; they were dealing with constant sum games only and the main purpose to introduce the topic seems to be the fact that homogeneous simple games have a nice VON NEUMANN–MORGENSTERN–solution (the so called "main simple solution").

In a series of papers ISBELL [2], [3], [4] treated several aspects of homogeneous games and later, PELEG [9], [10] studied the kernel and the nucleolus for this particular class of games. However, all these authors were concerned with constant sum games only. In 1975 one of the present authors noticed that homogeneity is a useful concept to describe extreme (convex or superadditive) games (cf. ROSENMÜLLER [13]) and subsequently he tried to obtain an overview over the structure of a set of weights admissible for constructing homogeneous games (cf. ROSENMÜLLER [14], [15]). However, it was OSTMANN [8] who first found out that a homogeneous game has a unique minimal representation. An alternative definition was given by ROSENMÜLLER [16] (this works also in the case of countably many players cf. [17]). In the next step SUDHÖLTER [19] introduced the theory of the "incidence vector" showing that a homogeneous game is equivalent to an integer vector describing the minimal lengths of a certain class of minimal winning coalitions.

This series of papers was dealing with a general and nonconstant sum case which, as it turned out, was much richer than the constant sum structure. In particular, it contained the unique representation theorem of VON NEUMANN– MORGENSTERN as a simple case of OSTMANN's general existence theorem for the minimal representation. So far, however, these papers were only concerned with describing the structure of homogeneous games.

The first attempt to capitalize on the new representation theory with respect to elaborating solution concepts was made by PELEG–ROSENMÜLLER [12]. For the general homogeneous weighted majority game "without steps" they discuss the least core, the nucleolus, and the kernel and show their close relationship with the unique minimal representation. At this stage it was clear that all attempts in this direction would necessarily call for some version of a reduction theorem: as SUDHÖLTER [19] has shown, homogeneous games in general and homogeneous games without steps can be brought into a one to one correspondence by adding just one player. On the other hand, the fact that steps play a very dominant role in the homogeneous game leads one to believe that concepts like the nucleolus and the kernel, which react very sensitive to the fact that "steps rule their followers", should somehow give a considerable advantage to players preceding the steps of a game.

Indeed, it turned out that the nucleolus vanishes after the first step and hence a computation of the nucleolus of an arbitrary homogeneous game can be reduced to the computation of the nucleolus of the reduced game which is obtained by cutting off the players after the first step ([18]).

The present paper is designed to show the same property for the kernel. Eventually we will come up with the result that the kernel of a homogeneous game with steps is essentially computed by the kernel of the homogeneous game without steps which is obtained by truncating the players following the first step. Note that the techniques of the present paper and those presented in [18] are quite different. In [18] it is shown that the system of minimal winning coalitions of a homogeneous game with steps cannot be weakly balanced and then provide an induction argument for the vanishing of nucleolus coordinates after the first step. This procedure rests heavily on SUDHÖLTER's theory of the incidence vector ([19]).

The present paper also rests on arguments implicitly producing the "incidence vector". But we do not make use of the balancedness of the system of minimal winning coalitions (see the results of [12] for the case of games without steps). Instead it is seen that sufficiently many minimal winning coalitions with maximal excess can be produced in order to determine the shape of the kernel after the largest step. Thus, the proofs provided in [18] and in the present version are completely independent; nevertheless it is of course clear that our present results induce that the nucleolus has coordinates vanishing after the first step of a homogeneous game.

We now start introducing our notation.

Let $\mathbb{N} = \{1,2,3,...\}$ denote the universe of *players*. Subsets of $\mathbb{N}$ are called *coalitions*; *intervals* are special types of coalitions of the form

$$[a,b] = \{i \in \mathbb{N} \mid a \leq i \leq b\}, \tag{1}$$

where $a,b \in \mathbb{N}$. The *grand coalition* is some distinguished interval Ω, predominantly we use $\Omega = [1,n]$. Next,

$$\mathbf{P} = \mathbf{P}(\Omega) = \{S \mid S \subseteq \Omega\} \tag{2}$$

and, if

$$v : \mathbf{P} \to \mathbb{R},\ v(\emptyset) = 0 \tag{3}$$

is a mapping (the *coalition* (or characteristic) *function*) then $(\Omega,\mathbf{P},v)$ is a *game*.

In most cases the nature of Ω and $\mathbf{P}$ is clear, so we call v a *game* as well. v is *simple* if $v : \mathbf{P} \to \{0,1\}$ holds true.

In order to simplify the notation, we write $S \cup i$ instead of $S \cup \{i\}$ etc. for $S \in \mathbf{P}$ and $i \in \Omega$. $S + T$ and $S + i$ denotes *disjoint* unions (i.e. $S + T = S \cup T$ iff $S \cap T = \emptyset$ etc.).

Given a game, the *desirability relation* is a binary relation on players (only, it can be considered on coalitions as well). Player $i \in \Omega$ is more *desirable* than player $j \in \Omega$ (written $i \succsim j$) iff

$$v(S+i) \geq v(S+j) \tag{4}$$

for all $S \in \mathbf{P}$, $S \cap \{i,j\} = \emptyset$. If $i \sim j$ (i.e. $i \succsim j$ and $j \succsim i$), then i and j are *equivalent* or *of the same type*.

A game is *ordered* if $\succsim$ is complete and *directed* if the ordering coincides with enumeration, i.e., if

$$i < j \text{ implies } i \succsim j. \tag{5}$$

Thus, in a directed game the "strong" "more desirable" or "large" players are first in enumeration (or "index"). If a game is obviously ordered, we will always tacitly assume that it is directed since this can be enforced by just renaming the players.

We write

$$\ell(S) = \max\{i \mid i \in S\} \tag{6}$$

for $S \in \mathbf{P}$, sometimes calling this the *length* of coalition S (counting starts at $1 \in \mathbb{N}$!) – but note that in a directed game, $\ell(S)$ is also the "weakest", "smallest" or "last" player of coalition S.

Let us now turn to simple games.

Given a simple game v, $W = W(v) = \{S \in \mathbf{P} \mid v(S) = 1\}$ is the system of *winning* coalitions while

$$W^m = W^m(v) = \{S \in W \mid v(T) = 0 \text{ for } T \subsetneq S\} \tag{7}$$

is the system of *minimal* winning coalitions ("*min–win coalitions*").

A vector $m = (m_i)_{i \in \Omega} \in \mathbb{R}_+^{\Omega}$ is tantamount to a function on $\mathbf{P}$ via $m(S) = \sum_{i \in S} m_i$ $(S \in \mathbf{P})$ (thus, it is a non-simple "game") and hence called a "*measure*" (m is additive). Games and in particular measures, may be restricted on subsets $T \subseteq \Omega$, the notation is $v_{|T}$ or $m_{|T}$; e.g.

$$\begin{aligned} v_{|T}(S) &= v(T \cap S) \quad (S \in \mathbf{P}(\Omega)) \\ v_{|T}(S) &= v(S) \quad (S \in \mathbf{P}(T)); \end{aligned} \tag{8}$$

the version living on $P(\Omega)$ and the one living on $P(T)$ are not distinguished. We tolerate $v_{|\emptyset}$.

If m is a measure and $\lambda > 0$, then $(m;\lambda)$ is a *representation* of v if

$$v(S) = \begin{cases} 1 & m(S) \geq \lambda \\ 0 & m(S) < \lambda \end{cases} \tag{9}$$

holds true, in this case we write $v = v^m_\lambda$. Of course, integer representations are of particular interest.

A measure m is said to be *homogeneous* w.r.t. $\lambda \in \mathbb{R}_{++}$ (written "m hom λ"), if, for any $T \in P$ with $m(T) > \lambda$, there is $S \subseteq T$ with $m(S) = \lambda$.

A game v is *homogeneous* if there exists a representation $(m;\lambda)$ with m hom λ and $v(\Omega) = 1$. (The definition is due to VON NEUMANN–MORGENSTERN [20]).

Clearly, a representable game is always ordered, thus in accordance with our remark above we tacitly assume that representable games are directed. That is, there exists a representation $(m;\lambda)$ such that $i < j$ implies $m_i \geq m_j$ $(i,j \in \Omega)$.

While players are ordered according to "size", coalitions are ordered lexicographically. In particular, the *lex–max min–win coalition* is the lexicographically first minimal winning coalition; in a homogeneous game with homogeneous representation $(m;\lambda)$ this coalition is denoted by $S^{(\lambda)}$ (an interval with measure $m(S^{(\lambda)}) = \lambda$). We write $\ell^{(\lambda)} = \ell(S^{(\lambda)})$.

Player $i \in \Omega$ is a *dummy* if $v(S \cup i) = v(S)$ for all $S \in P$; this notion is not restricted to simple games. All dummies are equivalent or of the same type).

Returning to simple games, player $i \in \Omega$ is *winning* if $v(\{i\}) = 1$ (i.e. $\{i\} \in W^m$); again all winning players are equivalent.

Thus dummies and winning players provide two particular types; of course types establish a decomposition of Ω.

There is a second decomposition of Ω which is fundamental in the case of a homogeneous game, this is the decomposition into sets of players of equal *character*. To explain the nature of this notion, let us focus on a homogeneous game.

There are three characters to be attached to players w.r.t. a homogeneous game called "dummy", "sum" and "step". The dummy character is the dummy type as explained above, so let us attempt to define the other two.

To this end, fix a non–dummy player $i \in \Omega = [1,n]$ and consider the minimal length of all min–win coalitions containing i, say

$$\ell^{(i)} := \min \{\ell(S) \mid S \ni i,\ S \in W^m\}. \tag{10}$$

The *domain* of i is

$$C^{(i)} := [\ell^{(i)} + 1,\ n]. \tag{11}$$

Now, player i is a sum ("his character is sum") if

$$m_i \leq m(C^{(i)}), \tag{12}$$

otherwise i is a *step*.

A sum may be replaced in at least one min–win coalition by a coalition of smaller players, his weight being exactly the sum of the weights of the smaller players (homogeneity is essential here).

On the other hand, "steps rule their followers", i.e., whenever a smaller player is a member of a min–win coalition, any preceding step is also a member.

Thus Ω decomposes into three subsets of characters: dummy, sum, and step. (Note that a winning player may be sum or step). A game may have no dummies or sums (e.g. the unanimous game of the grand coalition) but steps are always present.

The following remark collects some facts from the theory of homogeneous games. The details can be found e.g. in OSTMANN [8], ROSENMÜLLER [16] and SUDHÖLTER [19].

Remark 0.1:

1. The smallest non–dummy player is always a step. If v is constant sum, then this is the only step. To simplify matters, we say v is a (homogeneous) *game without steps* if the smallest non–dummy is the only step.

2. A homogeneous game v has a unique minimal representation. (i.e., an integer valued $(\bar{m};\bar{\lambda})$, representing v such that $\bar{m}(\Omega)$ is minimal among all integer representations –

this $(\bar{m};\bar{\lambda})$ is homogeneous.) For games without steps and dummies, every representation is a multiple of the minimal one.

3. Let again $S^{(\lambda)}$ be the lex–max min–win coalition, then, for $j \in S^{(\lambda)}$ the domain is $C^{(j)} = [\ell(S^{(\lambda)}) + 1, n]$. The steps in $S^{(\lambda)}$ are exactly the *inevitable* (or *veto–*) players. If all players in $S^{(\lambda)}$ are steps, then v is the unanimous game of $S^{(\lambda)}$ (with minimal representation $(\bar{m};\bar{\lambda}) = (1,...,1, 0,...,0; \lambda)$.

4. The sums in $S^{(\lambda)}$ determine the nature of the smaller players: the *satellite game* of a sum $j \in S^{(\lambda)}$ is

$$(C^{(j)}, P(C^{(j)}), v^{(j)})$$

where $v^{(j)}$ is represented by $(m_{|C^{(j)}}; m_j)$.

For every sum $j \in S^{(\lambda)}$, $v^{(j)}$ is a homogeneous game (The BASIC LEMMA). $i > \ell(S^{(\lambda)})$ is a sum w.r.t. v if i is a sum w.r.t. at least one $v^{(j)}$; i is a dummy if i is a dummy in every $v^{(j)}$, and i is a step in any other case.

5. In every homogeneous representation $(m;\lambda)$ of v, sums of the same type have the same weight. Steps of the same type may have different weights, but then they appear or do not appear simultaneously ("as a block") in every min–win coalition.

6. The procedure described in 3. suggests that sums may be replaced in the lex–max min–win coalition $S^{(\lambda)}$ if the players following $\ell^{(\lambda)} = \ell(S^{(\lambda)})$ can muster enough weight. This procedure may be generalized as follows. Let $(m;\lambda)$ be the minimal representation of a homogeneous game $v = v^m_\lambda$. Let $S \in W^m$ and let $\ell = \ell(S)$ again denote the last player in S. Suppose $j \in S$ is such that

$$[j,\ell] \subseteq S,\ S - j + [\ell+1,n] \in W. \tag{13}$$

Then j is *expendable*; we may replace him in S by an interval of smaller players, thus generating a coalition

$$\rho_j(S) := S - j + [\ell(S) + 1,t] \tag{14}$$

where t is uniquely defined by $m([\ell(S) + 1,t]) = m_j$. This procedure is based on the BASIC LEMMA (ROSENMÜLLER [16]), see SUDHÖLTER [19].

On the other hand, let $T \in W^m$ and suppose that $r \notin T$ satisfies

$$[r+1, \ell(T)] \subseteq T. \tag{15}$$

Then r is the *last dropout* (denoted by $r = r(T)$) and there is a unique $t' \in [r+1, \ell(T)]$ such that

$$\varphi(T) := T + r - [t',\ell(T)] \tag{16}$$

is a min–win coalition. That is, φ inserts the last dropout and cuts off an appropriate tail of T as to generate a min–win coalition. Thus, ρ_j renders j to be the last dropout if he is expendable in S.

Clearly, if r is the last dropout in T, then (he is expendable in $\varphi(T)$ and)

$$\rho_r(\varphi(T)) = T. \tag{17}$$

Similarly, if j is expendable in S then (he is the last dropout in $\rho_j(S)$ and)

$$\varphi(\rho_j(S)) = S \tag{18}$$

holds true.

We provide a few examples of homogeneous games which may be useful for the purpose of illustrating the general theory developed in the following sections.

Examples 0.2:

For simpler reading we omit the brackets in a representation $(m;\lambda)$ and indicate the semicolon by a *. The characters are indicated by s for sum, τ for step, and d for dummy.

For n = 11

22	11	11	9	2	2	2	2	2	1	1	* 33
s	s	s	s	s	s	s	s	s	s	τ	

is a constant sum game – hence a game without steps by convention, since technically the smallest player is the only step. By omitting this smallest player, we obtain for n = 10:

22	11	11	9	2	2	2	2	2	1	* 33
s	s	s	s	s	s	s	s	τ	τ	

here the last two players are steps, but of different type. Note that, on the other hand, of course all players with weight 2 are of the same type.

Next, for n = 8, consider

12	10	5	3	2	2	1	1	* 22
s	s	τ	s	s	s	s	τ	

were a step with weight 5 follows immediately behind the lex–max min–win coalition $S^{(\lambda)}$ – this example is treated in more detail in Example 3.7.

For $n = 6$ note the two very similar looking representations

5	4	3	2	1	1	* 12
T	s	s	T	s	T	

and

5	3	3	2	1	1	* 11
T	s	s	T	s	T	

which are nevertheless different – in both cases there is an inevitable (or veto–) player and further steps present.

So far all the above examples are given by their minimal representation. For $n = 15$

252	252	56	56	56	56	28	28	9	9	5	5	5	2	2	* 672
T	T	s	s	s	s	s	s	s	s	s	s	T	T	T	.

is a homogeneous representation but not the minimal one, which is

86	86	32	32	32	32	16	16	5	5	3	3	3	1	1	* 268 .
T	T	s	s	s	s	s	s	s	s	s	s	T	T	T	

Similarly,

170	70	70	31	29	10	4	4	2	2	2	1	* 240
T	s	s	T	T	s	s	s	s	s	T	d	

is minimally represented by

50	21	21	8	8	5	2	2	1	1	1	0	* 71
T	s	s	T	T	s	s	s	s	s	T	d	

– note that the two steps (weight 31 and 29) in the first representation are of equal type, hence get the same weight in the minimal representation.

The largest homogeneous game appearing in reality we know of is the Thai parliament of 1983, given by

92	73	56	24	18	15	4	3	2	1	* 145

which is a non–homogeneous representation of a homogeneous game represented minimally by

23	17	17	6	6	5	1	1	1	1	* 40 .
s	s	s	s	s	T	s	s	s	T	

1. Preliminary results about pre–kernel and kernel

We consider only games without dummies. This section serves to recall some definitions and to prove some elementary properties of the pre–kernel and its relatives.

Definition 1.1:

$\mathscr{X}^* = \{x \in \mathbb{R}^n \mid x(\Omega) = 1\}$ is the set of *pre–imputations*, and $\mathscr{X} = \{x \in \mathscr{X}^* \mid x \geq 0\}$ is the set of *pseudo–imputations*.

We write $T_{ij} = \{S \in P \mid i \in S, j \notin S\}$. Next, for $S \in P$ and $x \in \mathbb{R}^n$, $e(S,x) = v(S) - x(S)$ is the *excess* of x at S (with respect to some game v).

Definition 1.2:

The *maximal* excess of $x \in \mathbb{R}^n$ with respect to a game v is

$$\mu(x) = \mu(x,v) = \max_{S \in P} e(S,x); \tag{1}$$

also, we use for $i,j \in \Omega$, $i \neq j$:

$$s_{ij}(x) = s_{ij}(x,v) = \max \{e(S,x) \mid S \in T_{ij}\} . \tag{2}$$

The corresponding systems of coalitions reaching maximal excess are given by

$$\mathscr{D}(x) = \mathscr{D}(x,v) = \{S \in P \mid e(S,x) = \mu(x)\} \tag{3}$$

and

$$\mathscr{D}_{ij}(x) = \mathscr{D}_{ij}(x,v) = \{S \in T_{ij} \mid e(S,x) = s_{ij}(x)\} . \tag{4}$$

Next, let us recall the definitions for the pre-kernel and its relatives.

Definition 1.3:

Let v be a game.

1. The *pre–kernel* of v is given by

$$\mathscr{P\!K}(v) = \{x \in \mathscr{X}^* \mid s_{ij}(x) = s_{ji}(x) \ (i,j \in \Omega, i \neq j)\} . \tag{5}$$

2. The *pseudo–kernel* of v is given by

$$\mathscr{P\!s\!K}(v) = \{x \in \mathscr{X} \mid s_{ij}(x) \leq s_{ji}(x) \text{ or } x_j = 0 \ \ (i,j \in \Omega, i \neq j)\}. \tag{6}$$

3. The *kernel* of v is given by

$$\mathscr{K}(v) = \{x \in \mathscr{X}^* \mid x_j \geq v_j \ (j \in \Omega),\ s_{ij}(x) \leq s_{ji}(x) \text{ or } x_j = v_j \ (i,j \in \Omega, i \neq j)\} . \tag{7}$$

The kernel was introduced in DAVIS–MASCHLER [1], see also MASCHLER–PELEG–SHAPLEY [5], MASCHLER–PELEG [6], [7] and PELEG [9].

The pseudo–kernel has no satisfying game–theoretical interpretation. We will only use it temporarily to formulate the results of this section – this concept will not be mentioned in the following section. In fact, the aim of this section is to show that in all cases to be treated essentially all three concepts coincide.

Lemma 1.4: ("Equal treatment – monotonicity")

Let v be a directed game and $x \in \mathcal{P}\!\mathcal{K}(v)$. Then the following holds true:

1. If $i \sim j$, then $x_i = x_j$.
2. $x_1 \geq x_2 \geq \cdots \geq x_n$.

Proof:

The pre–kernel respects the desirability relation; see e.g. PELEG [11], CH.5, Theorem 5.3.5 and Remark 5.3.10. q.e.d.

Lemma 1.5:

Let v be a directed simple game (without dummies!). If $x \in \mathcal{P}\!\mathcal{K}(v)$, then $x \geq 0$.

Proof:

1st STEP:

By Lemma 1.4., we have $x_1 \geq \cdots \geq x_n$. Assuming $x_n < 0$ we would like to end up with a contradiction.

To this end let i_0 be the last player with nonnegative coordinate of x, thus

$$x_{i_0} \geq 0 > x_{i_0+1}. \tag{8}$$

Note that player n cannot be a winning player for otherwise (the game is directed!) *all* players would be winning, hence all would be of the same type – in which case $x_n < 0$ and Lemma 1.4. are not compatible.

Now, fix a coalition $\bar{S}$ such that $n \in \bar{S}$ and

$$e(\bar{S},x) = \max \{e(S,x) \mid n \in S\}.$$

The next two steps distinguish two cases according to whether $\bar{S}$ is winning or losing – and in both cases we will eventually end up with a contradiction.

2nd STEP:

Assume $\bar{S}$ is winning. Necessarily we have

$$[i_0+1, n] \subseteq \bar{S} \tag{9}$$

for if some player of the interval is missing, adding him to $\bar{S}$ would increase the excess.

Next, $\bar{S} = \Omega$ cannot occur: Ω has excess 0 and $\{n\}$ has excess $-x_n > 0$ ($\{n\}$ is not winning by the first step!), contradicting the definition of $\bar{S}$.

Thus, there is $i \in \Omega - \bar{S}$ and by (8) and (9):

$$x_i \geq 0. \tag{10}$$

We have now $\bar{S} \in T_{ni}$ and $e(\bar{S},x) = s_{ni}(x)$. Since $x \in \mathcal{P}\!\mathcal{K}(v)$, we must be able to find $\hat{S} \in T_{in}$ such that

$$e(\hat{S},x) = s_{in}(x) = s_{ni}(x) = e(\bar{S},x) \tag{11}$$

holds true.

However, $\hat{S}$ is losing (because n is not a member and adding him increases the excess for any winning coalition), and in addition $(\hat{S} - i) + n$ is *a fortiori* losing (i is stronger than n, as our game is directed). Hence

$$e((\hat{S} - i) + n,x) = e(\hat{S},x) + x_i - x_n > e(\hat{S},x) \tag{12}$$

since $x_n < 0$ and $x_i \geq 0$ by (10).

Now, we have the desired contradiction as (12) is opposed to the choice of $\hat{S}$, manifested in (11).

This finishes the 2nd STEP.

3rd STEP:

Assume now that $\bar{S}$ is losing. Clearly, $\bar{S} \neq \Omega$. But we can state much more, namely

$$[1,i_0] \not\subseteq \bar{S}. \tag{13}$$

Indeed, if (13) is not true, then pick $i \in \Omega - \bar{S}$. By (8) clearly $x_i < 0$. But

$$e((\bar{S}-1) + i, x) = e(\bar{S},x) + x_1 - x_i > e(\bar{S},x) \tag{14}$$

again using (8) this contradicts the definition of $\bar{S}$. Hence, (13) is verified.

Next, pick $j \in [1, i_0] - \bar{S}$ and let $\tilde{S} \in T_{jn}$ be such that

$$e(\tilde{S},x) = s_{jn}(x) = s_{nj}(x) = e(\bar{S},x) \tag{15}$$

(this corresponds to (11)).

Again, $\tilde{S}$ has to be losing by the same argument as above (n is no member!) and once more the fact that $(\tilde{S}-j) + n$ is losing (j is stronger than n) yields a contradiction, since

$$e((\tilde{S}-j) + n,x) = e(\tilde{S},x) + x_j - x_n > e(\bar{S},x) \tag{16}$$

is incompatible with the definition of $\bar{S}$ in view of (15). q.e.d.

Lemma 1.6:

Let v be a directed simple game (without dummies). If $x \in \mathcal{PsK}(v)$ and $i \neq j$, then $s_{ij} = s_{ji}$.

Proof:

Let $x \in \mathcal{PsK}(v)$ and let $i \neq j$. Assume, *per absurdum*, that $s_{ij}(x) > s_{ji}(x)$. Now, as $x \in \mathcal{PsK}(v)$ we have $x_j = 0$. Choose $\bar{S} \in T_{ij}$ such that

$$e(\bar{S},x) = s_{ij}(x) \tag{17}$$

and $t \in \Omega$ be given by

$$t = \max \{r \mid x_r > 0\} < n. \tag{18}$$

We are going to discuss two cases, both of which end up with a contradiction, proving our Lemma.

1st CASE: $[1,t] \not\subseteq \bar{S}$.

In this case, pick $k \in [1,t] - \bar{S}$ and $\tilde{S} \in T_{ki}$ with

$$e(\tilde{S},x) = s_{ki}(x).$$

Note that $x_k > 0$ implies $s_{ik}(x) \leq s_{ki}(x)$. Thus

$$e(\tilde{S},x) = s_{ki}(x) \geq s_{ik}(x) \geq e(\bar{S},x) \tag{19}$$

as $\bar{S} \in T_{ik}$.

Now, if $\tilde{S}$ is winning, then so is $\tilde{S} \cup j$, hence

$$e(\tilde{S} \cup j,x) = e(\tilde{S},x) \geq e(\bar{S},x) = s_{ij}(x) > s_{ji}(x) \tag{20}$$

contradicting the fact that $\tilde{S} \cup j \in T_{ji}$. On the other hand, if $\tilde{S}$ is losing, then $e(\tilde{S},x) < 0$ as $k \in \tilde{S}$, thus

$$0 > e(\tilde{S},x) \geq e(\bar{S},x) = s_{ij}(x) \geq e(\{j\},x) = -x_j = 0. \tag{21}$$

2nd CASE: $[1,t] \subseteq \bar{S}$.

In this case $x(\bar{S}) = 1$ and

$$e(\bar{S},x) = s_{ij}(x) > s_{ji}(x) \geq e(\{j\},x) \geq 0 = 1 - x(\bar{S}) \geq e(\bar{S},x) \tag{22}$$

which yields a contradiction. q.e.d.

Corollary 1.7:

Let v be a directed simple game (without dummies). Then

1. $\mathcal{PsK}(v) = \mathcal{PnK}(v)$.

2. If v has no winning players, then

   $\mathcal{PsK}(v) = \mathcal{PnK}(v) = \mathcal{K}(v)$.

3. If v has no winning players but k inevitable (or veto–) players, then

   $\mathcal{PsK}(v) = \mathcal{PnK}(v) = \mathcal{K}(v) = (\underbrace{\tfrac{1}{k}, \ldots, \tfrac{1}{k}}_{k}, 0, \ldots, 0).$

2. How to avoid winning players

Within this section we shortly deal with the case that winning players are present. *A priori*, this is not excluded for a homogeneous game – we may have several players i with weight $m_i = \lambda$. However, as we are going to show, the pre–kernel of such a game and the pre–kernel of a game obtained by "omitting" the winning players in a suitable way are closely related. Of course, the advantage of the second game is that it is trivially "weakly superadditive" and hence the kernel and the pre–kernel coincide.

Let v be a directed simple game and let

$$\kappa = \max\{i \in \Omega \mid \{i\} \in W^m\} \tag{1}$$

be the smallest winning player ($\max \emptyset = 0$).
Consider the decomposition of Ω into the winning players and the rest

$$\Omega = [1,\kappa] + [\kappa+1,n] =: \Omega_0 + \Omega_1 \tag{2}$$

and the corresponding restrictions

$$v^0 = v|_{\Omega_0} \quad \text{and} \quad v^1 = v|_{\Omega_1}. \tag{3}$$

It is easily verified that $v(\cdot) = \max(v^0(\cdot), v^1(\cdot))$.

In order to exhibit the relationship between the pre–kernels of v, v^0 and v^1, some trivial cases may be excluded.

E.g., if Ω_1 is losing (w.r.to v), then all members of Ω_1 are dummies, v^1 is identically 0 and v^0 equals v (up to omitting the dummies – that is, depending on which version of the restriction is favored).

Thus, the pre–kernel of v is $(\frac{1}{\kappa},\dots,\frac{1}{\kappa},0,\dots,0)$. Similar observations are due if $\Omega_0 = \emptyset$ or $\Omega_1 = \emptyset$.

Theorem 2.1:

Let v be a directed simple game (on $\Omega = [1,n]$). Let κ denote the smallest winning player and let v^0 and v^1 be defined by (2) and (3). Assume $\Omega_0 \neq \emptyset \neq \Omega_1$.
If $\bar{x} \in \mathcal{PK}(v^1)$ and

$$\bar{\alpha} := \min\{\bar{x}(S) \mid S \in W(v^1)\} \tag{4}$$

then

$$\bar{z} := \frac{1}{\kappa\bar{\alpha}+1}(\bar{\alpha},\dots,\bar{\alpha},\bar{x}_{\kappa+1},\dots,\bar{x}_n) \tag{5}$$

is an element of $\mathcal{PK}(v)$.

Proof:

We have to show that $s_{ij}(\bar{z}) = s_{ji}(\bar{z})$ for all i, $j \in \Omega$. Naturally, this goes by distinguishing all cases according to whether i and j are winning players or not.

1st CASE:

If i, $j \in \Omega_0$, then $\bar{z}_i = \bar{z}_j$; hence $s_{ij}(\bar{z}) = s_{ji}(\bar{z})$ is trivial.

2nd CASE:

Let $i \in \Omega_0$ and $j \in \Omega_1$. First we have

$$s_{ij}(\bar{z}) = 1 - \bar{z}_i = 1 - \frac{\bar{\alpha}}{\kappa\bar{\alpha}+1}. \tag{6}$$

On the other hand, as $\bar{x} \in \mathcal{PK}(v^1)$, there is $S \in W(v^1)$ such that $j \in S$ and $\bar{x}(S) = \bar{\alpha}$.
Clearly

$$s_{ji}(\bar{z}) = 1 - \frac{\bar{x}(S)}{\kappa\bar{\alpha}+1} = 1 - \frac{\bar{\alpha}}{\kappa\bar{\alpha}+1} = s_{ij}(\bar{z}). \tag{7}$$

3rd CASE:

Now consider the case that i, $j \in \Omega_1$.
Now, if v^1 has no inevitable (or "veto"–) players, then $s_{ij}(\bar{z}) = e(\bar{S},\bar{z})$ and $s_{ji}(\bar{z}) = e(\bar{T},\bar{z})$ for some $\bar{S}, \bar{T} \in W(v^1)$. Hence $s_{ij}(\bar{z}) = s_{ji}(\bar{z})$.

If v^1 has inevitable players, then the same argument holds true if i and j both are not inevitable.

If both are inevitable, then $\bar{x}_i = \bar{x}_j$ (Theorem 1.1.) and hence $\bar{z}_i = \bar{z}_j$ and $s_{ij}(\bar{z}) = s_{ji}(\bar{z})$.

Finally, if i is inevitable and j is not, then $\bar{x}_j = 0 = \bar{z}_j$. In this case actually $s_{ij}(\bar{z}) = 1 - \frac{\bar{\alpha}}{\kappa\bar{\alpha}+1} = s_{ji}(\bar{z})$,

q.e.d.

Theorem 2.2:

Let v be a directed simple game with smallest winning player κ and v^0 and v^1 be defined by (2) and (3). Assume $\Omega_0 \neq \emptyset \neq \Omega_1$. Let $\bar{z} \in \mathcal{PK}(v)$. Then $\bar{z}(\Omega_1) > 0$ and

$$\bar{x} := \frac{\bar{z}|\Omega_1}{\bar{z}(\Omega_1)} \in \mathcal{PK}(v^1) \tag{8}$$

Proof:

By the absence of dummies Ω_1 contains a winning coalition, thus $\bar{z}(\Omega_1) > 0$.
Let $\bar{\mathscr{D}} = \{S \subseteq \Omega_1 \mid e(S,\bar{z}) \geq e(T,\bar{z})\ (T \subseteq \Omega_1)\}$ and $\bar{s} = e(S,\bar{z})$ for $S \in \bar{\mathscr{D}}$. Let $i \in S \in \bar{\mathscr{D}}$ be such that $\bar{z}_i > 0$. Then

$$\bar{s} = 1 - \bar{z}_1 \tag{9}$$

since $s_{1i}(\bar{z}) = s_{i1}(\bar{z})$. We distinguish the following cases:

(i) v^1 has no inevitable (or veto–) player.
In this case $\cap \{S \mid S \in \bar{\mathscr{D}}\} = \emptyset$ since there cannot be a coalition $S \in \bar{\mathscr{D}}$ with $S \supseteq \{i \in \Omega_1 \mid \bar{z}_i > 0\}$. Hence, if $i,j \in \Omega_1$, $i \neq j$, then both $s_{ij}(\bar{z})$ and $s_{ji}(\bar{z})$ are attained by winning coalitions w.r.t. v^1, thus $s_{ij}(\bar{x}) = s_{ji}(\bar{x})$.

(ii) v^1 has veto players.
In this case $\cap \{S \mid S \in \bar{\mathscr{D}}\} \subseteq \{i \in \Omega_1 \mid i \text{ is inevitable w.r.t. } v^1\}$, since there is $S \in \bar{\mathscr{D}}$ with $\emptyset \neq S \neq \Omega_1$ as long as not all players of v^1 are inevitable. Consequently $\bar{z}_i = 0$ if $i \in \Omega_1$ is not inevitable and $\bar{z}_j = \bar{z}_k$ for any inevitable players j,k; thus $\bar{x}$ is the unique member of $\mathcal{PK}(v^1)$, q.e.d.

Remark 2.3:

For the discussion in the following sections we will now always assume that no winning players are present.

3. The kernel for games with steps

This section is devoted to the task of reducing the (pre–) kernel of a homogeneous game with steps to the one of a "smaller" game without steps. More precisely, we show that coordinates of the (pre–) kernel vanish behind the largest step, thus one can restrict oneself to computing the (pre–) kernel of the reduced game obtained by neglecting the players following the first step.

By the results of SECTION 2, we may restrict ourselves to the case that no winning players are present. Also we can clearly dispose of the case in which dummies are present. And finally, in view of Corollary 1.7., we will assume that there are no inevitable players.

A further reduction seems useful for notational convenience. Let v be a homogeneous game with steps and let $\underline{\tau} = \underline{\tau}(v)$ be first (largest step). Next, let τ be the smallest player of $\underline{\tau}$'s type. Then $[\underline{\tau},\tau]$ appear in every min–win coalition either simultaneously or not at all – they form a "block". Now, we shall assume that $\underline{\tau} = \tau$, i.e., the first type of steps consist of one player only. Indeed, the following proofs have to be altered only in an obvious way in order to be carried over to the case of an existing "block".

To simplify matters, we begin with

Definition 3.1:

1. A *standard step game* is a homogeneous game *with* steps, having no dummies, no winning and no inevitable players, such that the first step is the only step of his type.

2. If v is a standard step game then $\tau = \tau(v)$ denotes the first step.

3. If v is a standard step game, $\bar{x} \in \mathcal{K}(v)$ and $\bar{x}_\tau > 0$, then $(v,\bar{x})$ will be called a "standard situation".

4. If $(v,\bar{x})$ is a standard situation then $t = t(v) = \max \{i \mid \bar{x}_i > 0\}$ denotes the smallest player with positive coordinate at $\bar{x}$.

Of course, for any standard step game the kernel and the pre–kernel coincide, so we will mention the kernel only. Clearly, if v is a standard step game, $\bar{x} \in \mathcal{K}(v)$ and $x_\tau = 0$, then $x_i = 0$ for $i \geq \tau$ by Lemma 1.4.

Thus our aim is to show that for any standard situation $(v,\bar{x})$ it follows that $\bar{x}_i = 0$ $(i > \tau)$.

Let us start out with some preparations.

Definition 3.2:

Let $(v, \bar{x})$ be a standard situation. Let $\tau = \tau(v)$ and $t = t(v)$. Define

$$\begin{aligned} \mathscr{M} &= \mathscr{M}(v,\bar{x}) = W^m(v) \cap \mathscr{D}(\bar{x}), \\ \mathscr{M}^+ &= \{S \in \mathscr{M} \mid S \supseteq [\tau,t]\}, \\ \mathscr{M}^- &= \{S \in \mathscr{M} \mid S \cap [\tau,t] = \emptyset\}. \end{aligned} \tag{1}$$

If we refer to a standard situation, then we will sometimes omit the argument $\bar{x}$, then writing $\mathscr{D}$, $\mathscr{M}$, $\mathscr{M}^+$ etc.

Remark 3.3:

Let $(v,\bar{x})$ be a standard situation. If $S \in \mathscr{D}$, then there is $i \in S$ such that $S \cap [1,i] \in \mathscr{M}$. I.e., homogeneity ensures that dropping the smallest players (of largest index) results in a min–win coalition and, as this procedure can but increase the excess, it will result in an $\mathscr{M}$–coalition.

The principle of "dropping the smallest players in a winning coalition yields to hitting λ" is part of the BASIC LEMMA (see [15]), thus we shall refer to it as to the BASIC PRINCIPLE.

Theorem 3.4:

Let $(v,\bar{x})$ be a standard situation. Then $\mathscr{M} = \mathscr{M}^+ + \mathscr{M}^-$ and $\mathscr{M}^+ \neq \emptyset \neq \mathscr{M}^-$.

Proof:

First of all, let us show that $S \in \mathscr{M}$ cannot cut properly into $[\tau,t]$. Indeed, let $S \cap [\tau,t] \neq \emptyset$. As $S \in W^m$ and "steps rule their followers" (SEC.0), we have $\tau \in S$.

If, for some $i \in [\tau+1,t]$, $i \notin S$, then $S \in \mathscr{M} \cap T_{\tau i}$ and (as $\bar{x}_i > 0$) there is $S' \in \mathscr{D} \cap T_{i\tau}$, clearly $S' \in W$. Now, S' contains some $S'' \in W^m$ and, if $i \notin S''$, then $e(S'') > e(S')$, which is impossible. Thus, $S'' \in W^m \cap T_{i\tau}$ – contradicting the fact that τ is a step and "rules his followers".

Thus, $S \in \mathscr{M}$ will either contain $[\tau,t]$ or be disjoint to this interval.

Next, since no veto players are present (and $\bar{x} \geq 0$), it is clear that

$$\bigcap_{S \in \mathscr{M}} S = \emptyset, \tag{2}$$

thus $\mathscr{M}^- \neq \emptyset$. On the other hand, any $i \in S \in \mathscr{M}^-$ yields $S \in \mathscr{D} \cap T_{i\tau}$. A min–win coalition $T^0 \in T_{\tau i}$ yields $T^0 \in \mathscr{M} \cap T_{\tau i}$, since $\bar{x}_\tau > 0$ and $\tau \in T^0$, thus $T^0 \in \mathscr{M}^+$, q.e.d.

Lemma 3.5:

There exists $S \in \mathscr{M}^+$ such that $[\tau,\ell(S)] \subseteq S$.

Proof:

Pick $S \in \mathscr{M}^+$ and let $r = r(S)$ be the last dropout of S. If $r \in [\tau,\ell(S)]$, then $r \notin [\tau,t]$ (since $S \supseteq [\tau,t]$), thus $\bar{x}_r = 0$ and $\bar{x}_i = 0$ for all i with $r < i$.

Applying the operation φ to S which inserts r and omits smaller players (cf. SEC.0) does, therefore, not change the excess. Also, $\varphi(S) \in \mathscr{M}^+$. We may continue this procedure, until no dropout behind τ is missing in S, q.e.d.

Remark 3.6:

In particular, the shortest coalition (the one of minimal length) in $\mathscr{M}^+$ satisfies the condition specified by Lemma 3.5. Clearly, the procedure applied in 3.5 will work in a different context. I.e., if we take $S \in \mathscr{M}^+$ with certain additional properties, then we may frequently assume or conclude that $[\tau,\ell(S)] \subseteq S$, using φ to replace dropouts if necessary. To this we will refer to as the FULL TAIL PRINCIPLE.

Now, in order to begin the discussion, it should be noted that we have already treated one important, though not very enlightening case. This is the one in which inevitable players are present, and of course Corollary 1.7 shows that elements of the kernel have zero coordinates for all players following τ.

For, indeed, in a standard step game with inevitable players, τ is an element of $S^{(\lambda)} = [1,\ell^{(\lambda)}]$, τ is the only step in $S^{(\lambda)}$ and the only inevitable player (of course $\tau = 1$ in this case).

For didactical reasons we shall now treat the case that τ follows immediately behind the lex–max min–win coalition $S^{(\lambda)}$, i.e., $\tau = \ell^{(\lambda)}+1$. Although this case is subsumed under the general proof provided for Theorem 3.14, studying this particular situation is very enlightening. The proof of our main result is much simpler if $\tau = \ell^{(\lambda)}+1$ and still exhibits some structure of the general situation. We believe that it will help the reader to follow the main exposition and hence we shall treat it in advance.

Example 3.7:

Let $(m;\lambda) = (12,10,5,3,2,2,1,1;22)$.

We identify coalitions with 0–1–vectors ("characteristic functions of coalitions") thus clearly

$$S^{(\lambda)} = \underset{12}{1}\ \underset{10}{1}\ \underset{0}{0}\ \underset{0}{0}\ \underset{0}{0}\ \underset{0}{0}\ \underset{0}{0}\ \underset{0}{0}$$

and since both of them can be replaced, players 1 and 2 are sums. However, the coalition of minimal length containing player 3 (weight 5) is

$$\underset{12}{1}\ \underset{0}{0}\ \underset{5}{1}\ \underset{3}{1}\ \underset{2}{1}\ \underset{0}{0}\ \underset{0}{0}\ \underset{0}{0}$$

and there is not enough weight among the smaller players in order to replace player 3 – thus player 3 is a step, that is $\tau = \tau(v) = 3 = 2+1 = \ell^{(\lambda)} + 1$.

The reader may want to follow the proof of Lemma 3.8 viewing the above example.

Lemma 3.8:

Let $(v,\bar{x})$ be a standard situation. Assume

$$\tau = \ell^{(\lambda)} + 1 \tag{3}$$

Then $\bar{x}_i = 0$ for all $i > \tau$.

Proof:

1st STEP:

In this particular case $S^{(\lambda)}$ is the only min–win coalition not containing τ, thus

$$\mathscr{M}^{-} = \{S^{(\lambda)}\}. \tag{4}$$

2nd STEP:

The shortest min–win coalition containing τ is

$$S^{\tau} := (S^{(\lambda)} - \ell^{(\lambda)}) + [\tau,s] \tag{5}$$

with suitable $s > \tau$. All elements in $[\tau+1,s]$ are sums, for otherwise they would be of the same type as τ. Since S^{τ} is shortest containing them, they may be replaced by players following s – without disturbing the min–win property, that is.

3rd STEP:

By (4), $S^{(\lambda)} \in T_{\ell^{(\lambda)},\tau} \cap \mathscr{D}$, consequently, there is $\bar{S} \in T_{\tau,\ell^{(\lambda)}} \cap \mathscr{D}$.

By the BASIC PRINCIPLE, $\bar{S} \in \mathscr{M}$, thus $\bar{S} \in \mathscr{M}^{+}$. By the FULL TAIL PRINCIPLE $\bar{S} \supseteq [\tau,\ell(\bar{S})]$. Now, $\ell^{(\lambda)}$ is the *only* player from $S^{(\lambda)}$ missing in $\bar{S}$ (since τ cannot be replaced by players following s, no player $i \in [1,\tau-1]$ can be replaced by players following s) – thus $\bar{S}$ has the form

$$\bar{S} = (S^{(\lambda)} - \ell^{(\lambda)}) + [\tau,\ell(S)]$$

and (since both are min–win), $\bar{S} = S^{\tau}$.

In view of the 2nd STEP any $i \in [\tau+1, \ell(\bar{S})]$ can be replaced by players to the right of $\ell(\bar{S})$ – but since $t \in \bar{S}$, these have zero $\bar{x}$–coordinates. Hence $\bar{x}_i = 0$ (otherwise the excess increases). Thus $t = \tau$, q.e.d.

The general case, to be tackled by a series of auxiliary statements, is of course more involved – but some flavor of the simple proof offered in 3.8 is always present.

For instance, note that $S^{\tau} = \bar{S}$ as constructed in the third step has the largest dropout possible for $S \in \mathscr{M}^{+}$ and S obeying the FULL TAIL PRINCIPLE. As a result $\varphi(\bar{S}) = S^{(\lambda)}$.

Our first aim is to imitate this idea on a more general basis. This is attempted by the following Definition 3.9 and Theorem 3.10.

Definition 3.9:

Let $(v,\bar{x})$ be a standard situation. Define

$$\mathscr{M}^{++} := \{ S \in \mathscr{M}^{+} \mid [\tau,\ell(S)] \subseteq S\}, \tag{6}$$

$$\bar{r} := \max \{r(S) \mid S \in \mathscr{M}^{++}\}. \tag{7}$$

Note that $\mathscr{M}^{++} \neq \emptyset$ by Lemma 3.5!

Theorem 3.10:

Let $\bar{S} \in \mathscr{M}^{++}$ be such that $r(\bar{S}) = \bar{r}$. Then $\varphi(\bar{S}) \in \mathscr{M}^{-}$ and $\ell(\varphi(\bar{S})) = \bar{r}$.
As a consequence, we have

$$m_{\bar{r}} = m([\bar{r}+1, \ell(\bar{S})]). \tag{8}$$

Proof:

1st STEP:

Since $\bar{S} \in T_{\tau,\bar{r}}$, there is $\bar{T} \in T_{\bar{r},\tau}$ such that $\bar{T} \in \mathscr{M}$: use the BASIC PRINCIPLE. In view of Theorem 3.4,

$$\bar{T} \in \mathscr{M}^{-}. \tag{9}$$

2nd STEP:

We want to show that

$$\ell(\bar{T}) = \bar{r} \tag{10}$$

holds true. To this end, we are going to show that $[\bar{r}+1, \tau-1] \cap \bar{T} = \emptyset$. Assume on the contrary that $i \in [\bar{r}+1, \tau-1] \cap \bar{T}$, then i separates τ via $\bar{T}$. Hence, there exists $\tilde{T} \in T_{\tau i} \cap \mathscr{D}$. By the BASIC PRINCIPLE, we may assume that $\tilde{T} \in \mathscr{M}$ holds true. Moreover $\tilde{T} \in \mathscr{M}^{+}$ as $\tau \in \tilde{T}$ and w.l.o.g. $\tilde{T} \in \mathscr{M}^{++}$ by the FULL TAIL PRINCIPLE. But then the fact that $\bar{r} < i \leq r(\tilde{T})$ contradicts the maximality of $\bar{r}$ (i.e. (7)).
Thus, we have indeed verified (10).

3rd STEP:

Define now

$$\bar{I} := [\ell(\varphi(\bar{S})) + 1, \ell(\bar{S})] \ . \tag{11}$$

This interval serves to replace $\bar{r}$ in $\bar{S}$, thus

$$m_{\bar{r}} = m(\bar{I}). \tag{12}$$

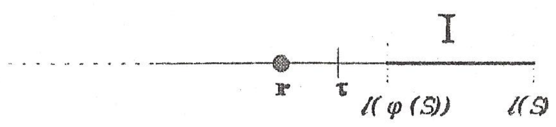

Fig.1 Coalition $\bar{S}$

Therefore, if we put

$$\hat{T} := (\bar{T} - \bar{r}) + \bar{I} \ ,$$

then clearly $\hat{T} \in W^m$ in view of (10) and (12).

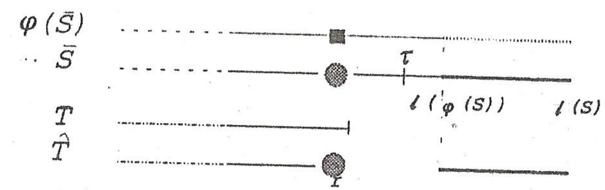

Fig.2 $\bar{S}$, $\bar{T}$ and their derivates

4th STEP:
Next, it is our aim to show that $\hat{T}$ as well as $\varphi(\bar{S})$ have maximal excess.

Indeed, because $\bar{S}, \bar{T} \in \mathscr{M}$, we have

$$\begin{aligned} e(\bar{S}) &\geq e(\varphi(\bar{S})) = e(\bar{S}) - \bar{x}_{\bar{r}} + x(\bar{I}) \ , \\ e(\bar{T}) &\geq e(\hat{T}) \quad = e(\bar{T}) + \bar{x}_{\bar{r}} - x(\bar{I}) \ . \end{aligned} \tag{13}$$

It follows that necessarily all inequalities in (13) must be equations, that is

$$\varphi(\bar{S}), \hat{T} \in \mathscr{M} \tag{14}$$

(and $\bar{x}_{\bar{r}} = x(\bar{I})$).

5th STEP:

Now $\hat{T}$

- ·) has maximal excess,
- ·) contains $\ell(S)$, hence τ;
- ·) is a member of $\mathscr{M}^{+}$,
- ·) contains $[\tau, \ell(\hat{T})]$,
- ·) hence is in $\mathscr{M}^{++}$.

Therefore, $\hat{T}$ cannot have dropouts in $[\bar{r}+1, \tau-1]$, for this would contradict the maximality of $\bar{r}$, i.e., (7).

We conclude that necessarily

$$\bar{I} = [\bar{r}+1, \ell(S)] \qquad (15)$$

must be satisfied. In view of (11), this means $\ell(\varphi(\bar{S})) = \bar{r}$; this is what we wanted to prove. Clearly, $\tau \notin \varphi(\bar{S})$ and thus $\varphi(\bar{S}) \in \mathscr{M}^-$ (by Theorem 3.4), q.e.d.

In order to proceed with our exposition we will now draw on the theory of homogeneous games. The simple shape of the situation as presented in Lemma 3.8 is no longer prevailing, nevertheless our development so far exhibits "coalition $\bar{S}$" – which also appears in 3.8. What is the natural generalization of S^τ as defined in (5)?

Theorem 3.11:

Let v be a standard step game. There exists a sequence of min–win coalitions

$$S_1,\dots,S_\tau \in W^m$$

with the following properties:

1. $S_1 = S^\lambda$.
2. For every $k \in [1,\dots,\tau]$, the system
$$\mathcal{S}_k := \{S_i \mid i \in [1,k],\ k \in S_i\}$$
is nonempty.
3. For all $k \in [1,\dots,\tau-1]$, among all $S_i \in \mathcal{S}_k$ with minimal length, let S_{i_0} be the one with minimal index, then
$$S_{k+1} = \rho_k(S_{i_0}).$$

Proof:

This follows from SUDHÖLTER [19], Theorem 2.3, Definition 2.4; see also the formulation in [18].

Definition 3.12:

1. Among all coalitions of S_τ with minimal length, let S^τ be the one which has minimal index, i.e., in particular
$$\ell(S^\tau) \geq \tau. \qquad (16)$$

2. Let $r_a < \dots < r_1$ denote the dropouts of S^τ (the last dropout is enumerated first since $\varphi = \varphi^1$ will reestablish him; thus $r_1 = r(S^\tau)$). Write $0 =: r_{a+1}$, $r_0 := \tau$ for convenience.

Remark 3.13:

1. If $\varphi^k(S^\tau)$ denotes the k'th iterate of φ applied to S^τ, $\varphi^k(S^\tau) \ni r_k$ and $\varphi^k(S^\tau)$ has smallest length among *all* min–win coalitions containing r_k. Note that $\varphi^a(S^\tau)$ is a min–win coalition without dropouts. (See SUDHÖLTER [19]).

2. If $\ell(S^\tau) > \tau$, then $\tau+1$ is expendable in S^τ. For, S^τ is then certainly a shortest min–win coalition containing $\tau+1$. If $\tau+1$ were a step then τ and $\tau+1$ would never be separated ("steps rule their followers"), hence – as $\tau+1$ differs in type from τ – $\tau+1$ has to be sum. As such he has to be expendable in the shortest coalition containing him.

3. Consider a standard situation $(v,\bar{x})$. Suppose that we can find $\hat{S} \in \mathscr{M}^+$ such that $\ell(\hat{S}) = \ell(S^\tau)$. We know that $t = t(v) \in \hat{S}$ and – thus either $t = \tau$ or $t > \tau$, $\tau+1 \in \hat{S}$, *and* $\tau+1$ is expendable in $\hat{S}$ by the argument presented above in 2. Replacing $\tau+1$ by players behind t will, however, increase the excess properly – thus we have necessarily $\tau = t$.

That is: if there is $\hat{S} \in \mathscr{M}^+$ with $\ell(\hat{S}) = \ell(S^\tau)$, then $t = \tau$!

This reasoning again is quite analogous to the one offered in the 3rd STEP of the proof of Lemma 3.8.

Theorem 3.14:

Let $(v,\bar{x})$ be a standard situation. Then $t(v) = \tau(v)$. That is, for any standard step game, the coordinates of kernel payoffs vanish behind the first step.

Proof:

Let S^τ and $0 = r_{a+1} < r_a < \dots < r_1 < r_0 = \tau$ be defined by Definition 3.12; thus in particular S^τ is a shortest coalition containing τ and has smallest index in the family defined by Theorem 3.11 among all coalitions with this property.

Next, choose $\bar{S} \in \mathscr{M}^{++}$ such that $r(\bar{S}) = \bar{r}$ (cf. (6), (7)), thus by Theorem 3.10 we know that $\varphi(S) \in \mathscr{M}^-$ and $\ell(\varphi(\bar{S})) = \bar{r}$.

Now, fix $k \in [0,a]$ such that

$$r_{k+1} < \bar{r} \leq r_k. \qquad (17)$$

Accordingly, we are going to distinguish several cases.

1st CASE:

If $k = a$, i.e., $\bar{r} \leq r_a$, then we argue as follows:

In this case in view of $\varphi^a(S^\tau) = S^\lambda$ (Remark 3.13.1) and $\ell(\varphi(\bar{S})) = \bar{r}$ (Theorem 3.10), it follows that $\varphi(\bar{S}) = S^\lambda$.
Then clearly $\bar{S} = S^\tau$; and Remark 3.13.3 shows that $t = \tau$.

So here, the argument is quite direct and the attentive reader will have observed that Lemma 3.8 actually deals with this case.
In the following we may now assume that $k < a$.

2nd CASE:

Consider now the situation in which

$$r_{k+1} \notin \bar{S} \qquad (18)$$

holds true. Observe that $r_{k+1} \notin \varphi(\bar{S})$, since $r_{k+1} < \bar{r} \notin \bar{S}$; moreover

$$\ell(\varphi^k(S^\tau)) \geq r_k \geq \bar{r} = \ell(\varphi(\bar{S})).$$

Thus we may draw the following sketch.

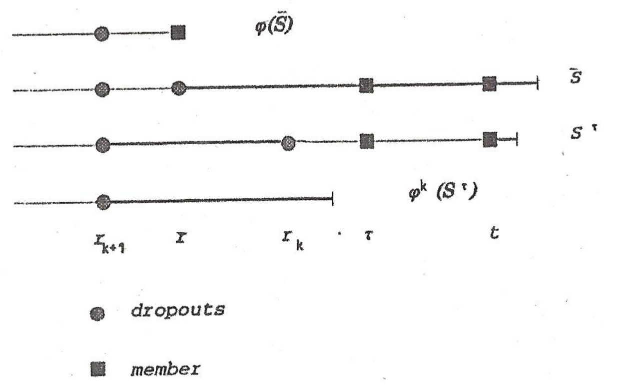

Fig. 3 $\bar{S}$, S^τ and their derivates

Now, take $\varphi(\bar{S})$ and $\varphi^k(S^\tau)$ into consideration. Both do not contain r_{k+1}. If we apply φ to $\varphi^k(S^\tau)$, then we obtain a shortest coalition such that r_{k+1} is expendable (Theorem 3.11). Hence, $\varphi(\bar{S})$ cannot be shorter than $\varphi^k(S^\tau)$, that is

$$\ell(\varphi^k(S^\tau)) = \bar{r}. \qquad (19)$$

Since $\bar{S} \ni \tau$, this can only mean that $k = 1$ and $\ell(\bar{S}) = \ell(S^\tau)$. Again we fall back on Remark 3.13.3 in order to conclude that $t = \tau$.

3rd CASE:

Suppose that $r_k = \bar{r}$, then $\ell(\varphi^k(S^\tau)) = \bar{r}$. Again, as $\ell(\varphi(\bar{S})) = \bar{r}$, we must have $k = 1$ and $\ell(\bar{S}) = \ell(S^\tau)$. The argument now proceeds analogously to the 2nd CASE.

4th CASE:

It remains to consider the case that $r_{k+1} \in \bar{S}$ and $\bar{r} < r_k$. This is the only one that requires a somewhat more elaborate argument.

Now, since $\bar{S} \in T_{r_{k+1},\bar{r}}$, there is $\bar{T} \in \mathcal{M} \cap T_{\bar{r}r_{k+1}}$. Since $\bar{T}$ is minimally winning, this coalition contains a "tail" $[i, \ell(\bar{T})] \cap \bar{T}$ exactly replacing r_{k+1}, thus

$$(\bar{T} + r_{k+1}) \cap [1,i-1] \in W^m(v)$$

and

$$\ell(\bar{T}) \geq \ell(\varphi^k(S^\tau)) \geq r_k \qquad (20)$$

since $\varphi^k(S^\tau)$ is shortest such that r_k is contained. (Theorem 3.11.)

We claim that $\bar{T} \in \mathcal{M}^+$. Indeed, otherwise (i.e., if $\bar{T} \in \mathcal{M}^-$) $\ell(\bar{T}) < \tau$ – since "steps rule their followers" – and thus $s_{\ell(\bar{T})\tau} = s_{\tau\ell(\bar{T})}$, a contradiction to the maximality of $\bar{r}$ (i.e., (7)).

Thus $\bar{T} \in \mathcal{M}^+$. By the FULL TAIL PRINCIPLE we may immediately assume that

$$\bar{T} \in \mathcal{M}^{++}, \qquad (21)$$

i.e., $[\tau,\ell(\bar{T})] \subseteq \bar{T}$. That is, $\bar{T}$ has dropouts only to the left of τ.

Next, the definition of $\bar{r}$ together with the fact that $\bar{T}$ is a member of $\mathcal{M}^{++}$ directly implies $r(\bar{T}) < \bar{r}$ and (since $m([\bar{r}+2,n]) < m_{\bar{r}} - m_{\bar{r}+1} + m_1 \leq m_{\bar{r}} \leq m_{r(\bar{T})}$)

$$\ell(\varphi(\bar{T})) \leq \bar{r}. \qquad (22)$$

Hence $r(\bar{T}) = r_{k+1}$, for otherwise

$$\ell(\varphi(\bar{T})) \geq \ell(\varphi^k(S^\tau)) \geq r_k > \bar{r}.$$

Therefore, we have only two alternatives: either

$$\ell(\varphi(\bar{T})) = \ell(\varphi^{k+1}(S^\tau)),$$

then $k = 0$ and $\ell(\bar{T}) = \ell(S^\tau)$ – a situation we know how to deal with via Remark 3.13.3, or

$$\ell(\varphi(\bar{T})) > \ell(\varphi^{k+1}(S^\tau)).$$

Then we come up with

$$m([\bar{r}+1,n]) \geq m([\bar{r}+1, \ell(\bar{T})]$$

$$\geq m(S^\tau \cap [\bar{r}+1, \ell(\bar{T})]) + m_{\ell(\varphi(\bar{T}))}$$

(by (22))

$$\geq m(S^\tau \cap [\bar{r}+1, \ell(\bar{T})]) + m_{\bar{r}}$$

(again by (22))

$$\geq m_\tau + m_{\bar{r}}$$

$$= m_\tau + m_{\bar{r}+1} + \dots + m_{\ell(\bar{S})}$$

(by Theorem 3.10),
i.e.

$$m([\ell(\bar{S}) + 1, n]) \geq m_\tau. \qquad (23)$$

But (23) contradicts the fact that τ is a step, q.e.d.

4. The Reduction Theorem

Within this section we draw the conclusions and collect results in order to finally formulate Theorem 4.5, the main theorem of this paper.

Definition 4.1:

Let v be a standard step game, $\tau = \tau(v)$. Let

$$\tilde{m} := (m_1,\dots,m_\tau) = m\Big|_{[1,\tau]},$$

$$\tilde{\lambda} := \lambda - m[\tau+1,n],$$

$$\tilde{v}^{(\tau)} := v^{\tilde{m}}_{\tilde{\lambda}}.$$

$\tilde{v}^{(\tau)}$ is the *truncation* of v at τ or for short the *truncated game.*

If $(v,\bar{x})$ is a standard situation, then the truncated game equals the *reduced game* (see PELEG [11]. Definition 3.6.8; note that the pre-kernel has the "reduced game property") – more precisely and more general:

Theorem 4.2:

Let v be a standard step game and $\bar{x} \in \mathscr{K}(v)$. Then

1. $\bar{x}_i = 0 \quad (i > \tau = \tau(v))$,
2. $\bar{x}\Big|_{[1,\tau]} \in \mathscr{K}(\tilde{v}^{(\tau)})$.

This follows immediately from Theorem 3.14.

Remark 4.3:

It is important to note that the truncated game $\tilde{v}^{(\tau)}$ is a homogeneous game without dummies; this has been shown in ROSENMÜLLER–SUDHÖLTER [18], Lemma 3.7 (see also Corollary 3.9). In addition $\tilde{v}^{(\tau)}$ is a game without steps. Note that truncation is not always the same as reduction, in particular if we reduce "behind a sum", homogeneity may be destroyed! Note the different version of a truncation presented in [18] for sums.

Naturally, we should set out for the "converse" of Theorem 4.2., i.e., we want to prove

Theorem 4.4:

Let v be a standard step game and $\bar{v}^{(\tau)}$ the truncated game. If $\bar{x} = (\bar{x}_1,\ldots,\bar{x}_\tau) \in \mathscr{K}(\bar{v}^{(\tau)})$, then $\bar{x} = (\bar{x}_1,\ldots,\bar{x}_\tau, 0,\ldots,0) \in \mathscr{K}(v)$.

Proof:

1st STEP:

We use $\tilde{}$ to indicate the quantities belonging to $\bar{v}^{(\tau)}$, e.g.,

$$\tilde{\mathscr{D}} = \{S \in W^m(\bar{v}^{(\tau)}) = \tilde{W}^m \mid \tilde{e}(S,\bar{x}) = \mu(\bar{x},\bar{v}) = \tilde{\mu}\}$$

etc. It is not hard to see that

$$\bigcap\{S \mid S \in \tilde{\mathscr{D}}\} = \emptyset, \bigcap\{S \mid S \in \mathscr{D}\} = \emptyset \tag{1}$$

$$\mu = \tilde{\mu} > 0, \tag{2}$$

holds true. Hence, for $i \neq j$, $i \in [\tau+1,n]$, $j \in [1,n]$

$$s_{ij}(\bar{x}) = \mu = \mu(\bar{x}). \tag{3}$$

Using the fact that $\bar{x} \in \mathscr{K}(\bar{v}^{(\tau)})$, we have immediately

$$s_{ij}(\bar{x}) = s_{ji}(\bar{x}) \tag{4}$$

for $i \neq j$, $i,j \in [1,\tau]$. Therefore, it remains to show that

$$s_{ij}(\bar{x}) = \mu(\bar{x}) \qquad (i \in [1,\tau], j \in [\tau+1,n]). \tag{5}$$

A further reduction is obtained by observing that

$$s_{ik}(\bar{x}) \geq s_{ik-1}(\bar{x}) \qquad (i \in [1,\tau], k \in [\tau+2,n]) \tag{6}$$

so that presently all that remains to be shown is that

$$s_{i\tau+1}(\bar{x}) = \mu(\bar{x}) \quad i \in [1,\tau] \tag{7}$$

holds true.

2nd STEP:

Next we claim that it suffices to show that

$$s_{\tau\tau+1} = \mu = \mu(\bar{x}) \tag{8}$$

holds true.

To this end assume that, for some $i < \tau$ we have $s_{i\tau+1}(\bar{x}) < \mu(\bar{x})$. Then we will immediately show that $s_{\tau\tau+1} < \mu$ is a consequence.

Indeed, $s_{i\tau+1}(\bar{x}) < \mu$ and $s_{\tau\tau+1} = \mu$ is incompatible as follows:

Pick $\bar{S} \in \mathscr{D} \cap T_{\tau\tau+1}$. Clearly, $i \notin \bar{S}$, hence $\tilde{s}_{\tau i}(\bar{x}) = s_{\tau i}(\bar{x}) = e(\bar{S}) = \mu = \tilde{\mu}$. Since $\bar{x} \in \mathscr{K}(\bar{v}^{(\tau)})$, we find $T \in \mathscr{D} \cap T_{i\tau}$. Eliminate players to the right of τ (smaller ones) from T. We obtain a winning coalition, for $\tau \notin T$ and steps rule their followers. Hence, we obtain a coalition, say T', with $T' \in \mathscr{D} \cap T_{i\tau+1}$ – contradicting $s_{i\tau+1} < \mu$.

This shows that it suffices to prove (8).

3rd STEP:

There is no problem with the proof of (8) if $x_\tau = 0$.For in this case

$$\tilde{s}_{\tau j}(\bar{x}) = \tilde{\mu} \qquad (j \in [1,\tau-1]) \tag{9}$$

and, consequently for all $j \in [1,\tau-1]$ it follows that

$$s_{\tau\tau+1} \geq s_{j\tau+1} \geq s_{j\tau} = \tilde{s}_{j\tau} = \tilde{\mu} = \mu \tag{10}$$

holds true.

4th STEP:

Hence, we will now set out to prove (8) assuming $\bar{x}_\tau > 0$.

Analogously to Definition 3.2, define

$$\mathscr{M}^+ := \{S \in \mathscr{M} \mid \tau \in S\}, \mathscr{M}^- := \{S \in \mathscr{M} \mid \tau \notin S\}, \tag{11}$$

such that $\mathscr{M} = \mathscr{M}^+ + \mathscr{M}^-$ and, as is easily verified, $\mathscr{M}^+ \neq \emptyset$ and $\mathscr{M}^- \neq \emptyset$. Similarly, copying 3.9, we put

$$\mathscr{M}^{++} := \{S \in \mathscr{M}^+ \mid [\tau,\ell(S)] \subseteq S\}, \tag{12}$$

$$\bar{r} := \max\{r(S) \mid S \in \mathscr{M}^{++}\}.$$

Of course, the FULL TAIL PRINCIPLE ensures that $\mathscr{M}^{++} \neq \emptyset$.

Referring to Theorem 3.11 and Remark 3.13 (in particular 3.13.3), we may now complete the proof by producing some $\hat{S} \in \mathscr{M}^+$ with $\ell(\hat{S}) = \ell(S^\tau)$. For again for such an $\hat{S}$ either $\ell(\hat{S}) = \tau$ or $\tau+1$ will be expendable without decreasing the excess.

The construction can, however, be completed quite analogously to the one presented in the proof of Theorem 3.14. Whenever within the course of this proof reference is made to $\bar{x}$ being an element of $\mathscr{K}(v)$, then, for our present problem the coalition being produced by this argument is already available by $\bar{x} \in \mathscr{K}(\bar{v}^{(\tau)})$. (Note that in the course of 3.14 and the previous theorems, essentially only players preceding τ are manipulated in their role as dropouts.)

Theorem 4.5:

Let v be a standard step game. Then

$$\mathscr{K}(v) = \{(\bar{x},0) \mid \bar{x} \in K(\bar{v}^{(\tau)})\}.$$

Literature

[1] Davis, M. and Maschler, M.:
The kernel of a cooperative game.
Naval Research Logist. Quarterly 12 (1965), pp.223 – 259

[2] Isbell, J.R.:
A class of majority games.
Quarterly Journal Math. 7 (1956), pp.183 – 187

[3] Isbell, J.R.:
A class of simple games.
Duke Math. Journal 25 (1958), pp.423 – 439

[4] Isbell, J.R.:
On the enumeration of majority games.
Math Tables Aids Comput, 13 (1959) pp.21 – 28

[5] Maschler, M., and Peleg, B.:
A characterization, existence proof, and dimension bounds for the kernel of a game.
Pacific Journal of Mathematics 18 (2), (1966), pp.289 – 328.

[6] Maschler, M. and Peleg, B.:
The structure of the kernel of a cooperative game.
SIAM Journal on Applied Mathematics 15 (3), (1967), pp.569 – 604.

[7] Maschler, M., Peleg, B., and Shapley, L.S.:
Geometric properties of the kernel, nucleolus, and related solution concepts.
Math. of Operations Research 4 (1979), pp.303 – 338

[8] Ostmann A.:
On the minimal representation of homogeneous games.
Int. Journal of Game Theory 16 (1987), pp.69 – 81

[9] Peleg,B.:
On the kernel of constant–sum simple games with homogeneous weights.
Ill. Journal Math. 10 (1966), pp.39 – 48

[10] Peleg, B.:
On weights of constant–sum majority games.
SIAM Journal Appl. Math. 16 (1968), pp.527 – 532

[11] Peleg, B:
Introduction to the theory of cooperative games.
Research Memoranda No. 81 – 88
Center for Research in Math. Economics and Game Theory.
The Hebrew University, Jerusalem, Israel

[12] Peleg, B. and Rosenmüller, J.:
The least–core, nucleolus, and kernel of homogeneous weighted majority games.
Working paper No. 193, Institute of Mathematical Economics, University of Bielefeld (1990), 26 pp

[13] Rosenmüller, J.:
Extreme games and their solutions.
Lecture Notes in Economics and Math.Systems 145.
Springer Verlag (1977), 126 pp

[14] Rosenmüller, J.:
On homogeneous weights for simple games.
Working paper No.115. Institute of Mathematical Economics. University of Bielefeld (1982), 44 pp

[15] Rosenmüller, J.:
Weighted majority games and the matrix of homogeneity.
Zeitschrift für Operations Research 28 (1984), pp.123 – 141

[16] Rosenmüller, J.:
Homogeneous games : recursive structure and computation.
Math. of Operations Research 12 (1987), pp.309 – 330

[17] Rosenmüller, J.:
Homogeneous games with countably many players.
Math. Social Sciences 17 (1989), pp.131 – 159

[18] Rosenmüller, J. and Sudhölter, P.:
The nucleolus of homogeneous games with steps.
Working Paper No.202. Insitute of Mathematical Economics. University of Bielefeld. (1991) 41 pp

[19] Sudhölter, P.:
Homogeneous games as anti step functions.
Intern. Journal of Game Theory 18 (1989), pp.433 – 469

[20] von Neumann, J. and Morgenstern, O.:
Theory of games and economic behavior.
Princeton Univ. Press, NJ., (1944)