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A Discrete Approach to General Equilibrium and NTU-Games

by

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Abstract

We consider a pure exchange economy or "market" such that every player/agent has piecewise linear utilities. The resulting NTU-market game is a piecewise linear correspondence. Using a version of "nondegeneracy" for games in characteristic function form, we exhibit conditions to ensure the finite convergence of the core towards the Walrasian equilibrium.

SECTION 0 :

Introduction and Notation

Equivalence theorems in General Equilibrium Theory and Game Theory state that for increasing or large sets of players/agents some solution concepts like the Walrasian equilibrium, the Core, or the Shapley value approximately coincide. Earlier, these results where formulated within the framework of replicated markets or games (e.g. DEBREU – SCARF [3]). Beginning with AUMANN's paper [1], the measure space of agents/players turned out to be a very fruitful concept, see also HILDENBRAND [4] and MAS–COLELL [5].

However, to some extent there is a third approach based on combinatorial or number–theoretical considerations related to the concept of a "non–degenerate" characteristic function in the sense of Cooperative Game Theory. Models of this type deal with a finite framework. There is a fixed number r of *types* (similar to the replica–model) of players and integer vectors of the grid $\mathbb{N}^r$ are interpreted as distributions of players over the types. The task is to describe certain areas in $\mathbb{N}^r$ and certain subgrids such that, within these areas the elements of the subgrids yield distributions of players such that a certain equivalence theorem holds true.

An overview over some applications of *nondegeneracy* and *homogeneity* as "surrogates" for nonatomicity is presented in [9]. More recent evidence is provided in PELEG–ROSENMÜLLER [6] and ROSENMÜLLER–SUDHÖLTER [10], where it is shown that the *nucleolus* is as well a suitable object for an equivalence theorem in as much as, for sufficiently large sets of players in a homogeneous simple game, it coincides with the unique representation of this game.

The above–mentioned methods appear to work smoothly mainly in models which exhibit a "side–payment" or "transferable utility" character. Clearly, this is so since some version of "optimization" or "linear production" is always involved, at least implicitly. The present paper is meant to provide a first approach to the NTU–case. We want to study a pure exchange economy and the NTU–game it generates in a framework with finitely many types and piecewise linear utility functions. Can we exhibit areas in $\mathbb{N}^r$ such that the Core and the Walrasian Equilibrium coincide by suitably extending the definition of a *nondegenerate* game to the NTU–case and solving the appropriate combinatorial problem?

As yet the result is only a partial one: for certain classes of pure exchange economies the approach is successful, thus, we may formulate a "finite convergence"–theorem. However, since *nondegeneracy* and *piecewise linearity* of an exchange economy and the corresponding NTU–game viewed as a correspondence on distributions of players over the types can be established as consistent concepts, it may be that a departure point for more insight into the combinatorial structures of equivalence theorems has been reached.

The paper is organized as follows. SECTION 1 studies the NTU–market–game resulting from a pure exchange economy viewed as a correspondence from generalized profiles of coalitions into utility–space.It turns out that piecewise linearity of the utility functions of types renders this correspondence also to be piecewise linear. The regions of linearity are cones in $\mathbb{R}^r$ – the space of idealized distributions of players. Hence, in SECTION 2, we may study the behavior of a face of the correspondence within a region of linearity. Such a face is of course a polyhedron of lower dimension and the behavior of its normal as a function of profiles of coalitions is crucial.The extremals of the faces again can be identified as piecewise linear function (SECTION 3). Thus , gradually we approach the behavior of prices corresponding to faces of the NTU–game–correspondence. Eventually, in SECTION 4, we link Equilibrium–prices and Core–payoffs. SECTION 5 provides an extended example.

We now start out to provide the setup for our discussion. the piecewise linear pure exchange economy and the NTU–game derived from its data.

Let us consider a pure exchange economy with finitely many ***types*** of agents or players, for short such an economy shall be called a ***market***. Types are indicated by $\rho \in R = \{1,...,r\}$. The commodity space is $\mathbb{R}^m_+$, thus $j \in J = \{1,...,m\}$ refers to a commodity. Each player of type ρ commands an initial allocation $a^\rho \in \mathbb{R}^m_{++}$ (strictly positive).

We assume the preferences to be represented by a piecewise linear utility function which, for type ρ, is denoted by

$$u^\rho : \mathbb{R}^m_+ \longrightarrow \mathbb{R}$$

More precisely, let us assume that there is a finite set L (not depending on $\rho \in R$ without loss of generality) and vectors $c^{\rho l} \in \mathbb{R}^m_+$ ($\rho \in R$, $l \in L$) such that, for any $x \in \mathbb{R}^m_+$,

(1) $$u^{\rho}(x) = \min_{l \in L} c^{\rho l}x + d^{\rho l} = \min_{l \in L} h^{\rho l}(x)$$

(where $h^{\rho l}(x) := c^{\rho l}x + d^{\rho l}$ for $x \in \mathbb{R}^m_+$).

We assume $c^{\rho l} > 0$, $d^{\rho l} > 0$ ($\rho \in R$, $l \in L$) thus, each $u^{\rho l}$ is positive, strictly monotone and, of course, continuous.

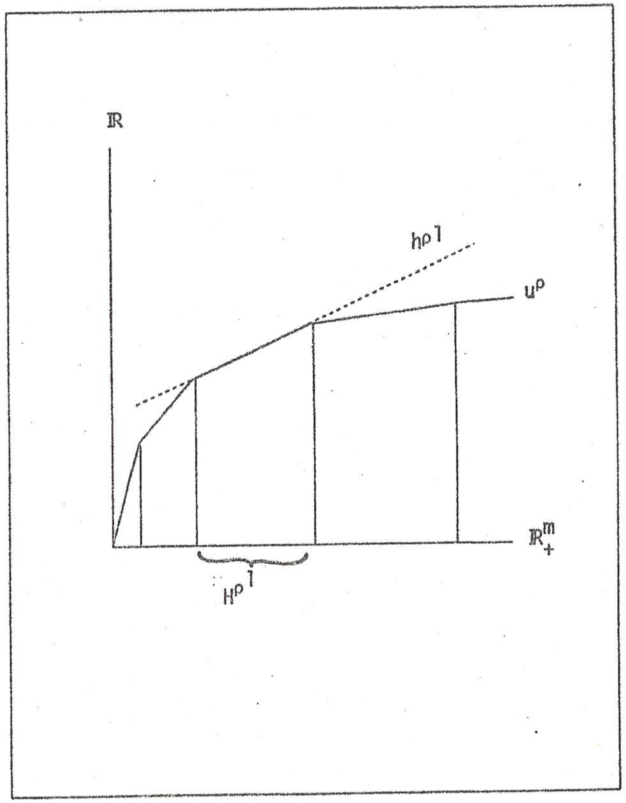

Figure 1

If we introduce the convex closed polyhedron

$$H^{\rho l} = \{x \in \mathbb{R}^m_+ \mid h^{\rho l}(x) \leq h^{\rho l'}(x)\ (l' \in L)\}$$

for some $\rho \in R$ and $l \in L$, then $u^{\rho l}$ equals $h^{\rho l}$ within $H^{\rho l}$ and $\mathbb{R}^m_+$ is covered via

$$\mathbb{R}^m_+ = \bigcup_{l \in L} H^{\rho l}$$

for each $\rho \in R$. There is no loss of generality involved in assuming that each $H^{\rho l}$ has nonempty interior.

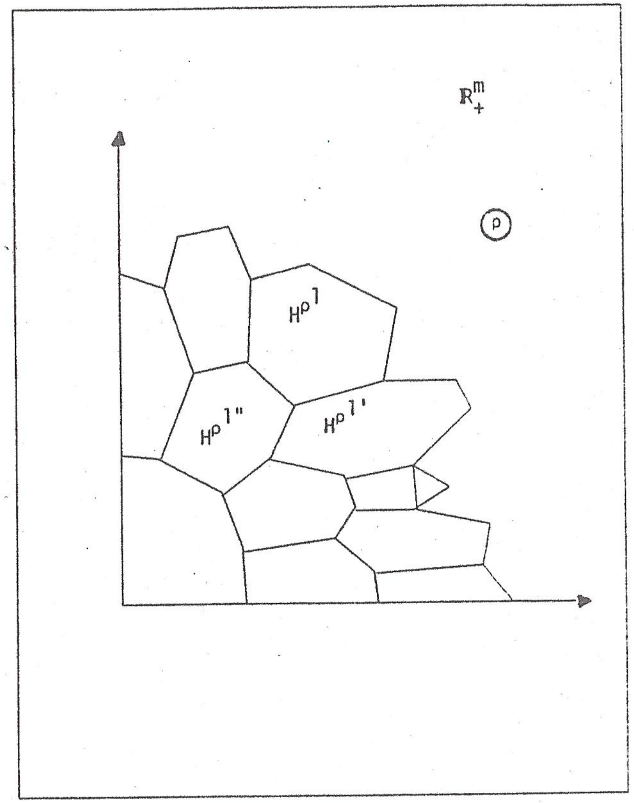

Figure 2

Let $k = (k_1,...,k_r) \in \mathbb{N}^r$ be a vector of integers. Then the resulting market $\bar{u}^k$ consists of k_1 players with utility u^1 and initial allocation a^1,..., k_r players with utility u^r and initial allocation a^r; that is, $\bar{u}^k$ is the "k–replica" of $\bar{u}^{(1,\dots,1)} = \bar{u}^e$ $(e = (1,..,1) \in \mathbb{N}^r)$.

Let $s = (s_1,\dots,s_r) \in \mathbb{N}_0^r := \mathbb{N}^r \cup \{0\}$ (some coordinates allowed to be zero) be an integer (or 0) vector such that $s \leq k$. Then s is the ***profile*** of a coalition having s_1 players of type 1,..., s_r players of type r.

Without offering an interpretation, we consider also "generalized profiles", i.e., vectors $t \in \mathbb{R}_+^r$. For any $t \in \mathbb{R}_+^r$ (and in particular for profiles), the (***aggregate***) ***initial allocation*** is

(2) $$a^t := \sum_{\rho \in R} t_\rho a^\rho$$

and the ***feasible*** allocations (assuming equal treatment) are given by

(3) $$\mathfrak{A}_t := \{X = (x^\rho)_{\rho \in R} \mid x^\rho \in \mathbb{R}_+^m, \sum_{\rho \in R} t_\rho x^\rho = a^t\}.$$

Switching to utility space we introduce the notation

$$u(X) = (u^1(x^1),\dots,u^r(x^r))$$

$$t \otimes u(X) = (t_1 u^1(x^1),\dots,t_r u^r(x^r))$$

for $t \in \mathbb{R}_{++}$, $X \in \mathfrak{A}_t$. Finally consider, for $t \in \mathbb{R}_+$

(4) $$F(t) = \{u \in \mathbb{R}_+^r \mid u \leq t \otimes u(X) \text{ for some } X \in \mathfrak{A}_t\},$$

then F is a correspondence that maps generalized profiles into subsets of $\mathbb{R}_+^r$, the element of which can be considered as feasible utility vectors to a (generalized) coalition with profile t, coordinate ρ denoting the joint utility for all players of type ρ.

The restriction of F to all profiles

$$\{s \in \mathbb{N}_0^r \mid s \leq k\} \text{ for some } k \in \mathbb{N}^r$$

is denoted by $V^k = V^{k,u}$ and called the ***characteristic function*** of (the nonside–payment– or NTU–game corresponding to) $\bar{u}^k$.

SECTION 1:

The Market Game as a Piecewise Linear Correspondence

Since all utilities are piecewise linear, the correspondence F should exhibit a similarly simple shape. We expect F(t) to be a convex compact polyhedron for fixed t; thus our first aim should be to describe the nature of the extremals. Next, with varying t, it turns out that F also behaves "piecewise linear" in the sense that within the interior of certain well defined cones in $\mathbb{R}^m$, the extremals are linear functions of t. This exposition is the aim of the first section.

For fixed type ρ the utility function u^ρ is linear within each of the convex polyhedra $H^{\rho l}$ as depicted in Figure 2., and these polyhedra provide a polyhedral covering of $\mathbb{R}^m$. If we consider an (equal treatment) allocation $X = (x^\rho)_{\rho \in R}$, then each x^ρ is somehow located in some $H^{\rho l}$, thus refers to the system of polyhedra generated by u^ρ as explained above.

Next, consider a utility vector $\bar{u}$ which is an extremepoint of F(t) for some $t \in \mathbb{R}_{++}$. Suppose that this vector is generated by some $\bar{X} \in \mathfrak{A}_t$ via $\bar{u} = t \otimes u(\bar{X})$. Then we expect a tendency of the $\bar{x}^\rho$ $(\rho \in R)$ to move towards the boundaries of the $H^{\rho l}$, roughly speaking. That is, next to the inevitable equation $\Sigma t_\rho x^\rho = a^t$, there will be "many" equations of the types $x_j^\rho = 0$ and $t_\rho h^{\rho l}(x^\rho) = u^\rho$ to be satisfied by $\bar{X}$ and $\bar{u}$ respectively; say for $j \in J_\rho$ and $l \in L_\rho$.

More precisely, let us first formulate the appropriate property for a system of sets of indices $\mathfrak{J} = (J_1,\dots,J_r,L_1,\dots,L_r)$.

Definition 1.1. Let $\mathfrak{J} = (J_1,\dots,J_r,L_1,\dots,L_r)$ be such that $J_\rho \subseteq J$ $(\rho \in R)$ and $L_\rho \subset L$ $(\rho \in R)$. $\mathfrak{J}$ is said to be an ***admissible system*** if the following equations are satisfied.

$$\sum_{\rho=1}^{r} |J_\rho| \leq r(m-1)$$

(1) $$|L_\rho| \geq 1 \quad (\rho \in R)$$

$$\sum_{\rho=1}^{r} |J_\rho| + \sum_{\rho=1}^{r} |L_\rho| = mr + r - (m - \min_{\rho \in R} |J_\rho|).$$

Now we may establish the connections between extreme points of F(t), admissible systems, and generating elements of $\mathfrak{A}_t$ as follows.

Lemma 1.2: Let $t \in \mathbb{R}^r_{++}$ and let $\bar{u} \in F(t)$ be a Pareto efficient ("P.E.") extreme point of F(t). Then there exists $\bar{X} \in \mathfrak{A}_t$ and an admissible system $\mathfrak{J} = (J_1,\ldots,J_r,L_1,\ldots,L_r)$ such that $(\bar{X},\bar{u})$ is the unique solution of the linear system of rm + r equations (in variables $(X,u) \in \mathbb{R}^{m \times r} \times \mathbb{R}^r$) given by

$$x^\rho_j = 0 \qquad (\rho \in R, j \in J^\rho)$$

(2) $$t_\rho h^{\rho l}(x^\rho) - u_\rho = 0 \qquad (\rho \in R, l \in L^\rho)$$

$$\sum_{\rho \in R} t_\rho x^\rho = a^t$$

Proof: As $\bar{u}$ is Pareto optimal, it is easy to show that

(3) $$\mathfrak{A}^{\bar{u}}_t := \{X \in \mathfrak{A}_t, t \otimes u(X) = \bar{u}\}$$

is a nonempty, compact, convex polyhedron. Let $\bar{X}$ be an extreme point of $\mathfrak{A}^{\bar{u}}_t$ and put

$$J_\rho := \{j \in J \mid \bar{x}^\rho_j = 0\} \qquad (\rho \in R)$$

$$L'_\rho := \{l \in L \mid t_\rho h^{\rho l}(\bar{x}^\rho) = \bar{u}_\rho\} \quad (\rho \in R)$$

It is then verified by standard arguments that $(\bar{X},\bar{u})$ is the unique solution of the corresponding system (2) (– where L_ρ has been replaced by L'_ρ)

Now inspect the coefficient matrix of (2) (note that $h^{\rho l}(x) = c^{\rho l}x + d^{\rho l}$). Observe that the first and last group of equations yield a submatrix of rank

$$|J^1| + \ldots + |J^r| + m - \min_{\rho \in R} |J_\rho| \leq r\,m.$$

As the rank of the original matrix is mr + r, it is possible to complete this submatrix in order to obtain a square submatrix by choosing (at least r) rows from the middle group of (2) in such a way that for each ρ at least on row corresponding to a variable u_ρ appears. This procedure defines index sets $L_\rho \subseteq L'_\rho$ ($\rho = 1,\ldots,r$), $|L_\rho| \geq 1$. Note, that for each j there is ρ such that $j \notin J_\rho$; this justifies the first equation (1). The last equation (1) is obtained by counting rows. q.e.d.

Note that $\bar{X}$ is a P.E. allocation which is extreme in $\mathfrak{A}^{\bar{u}}_t$.

Henceforth, a P.O. extreme point $\bar{u}$ of F(t) is called a *t–vertex*, an extreme point $\bar{X} \in \mathfrak{A}^{\bar{u}}_t$ is called a ***corresponding node.***

Corollary 1.3:

For every $t \in \mathbb{R}^r_{++}$, F(t) is a compact, convex polyhedron.

For, the number of admissible systems $\mathfrak{J}$ is finite and so is the number of vertices. Given an admissible system $\mathfrak{J}$, consider for some $t \in \mathbb{R}^r_{++}$ the system of linear equations in variables η_ρ ($\rho \in R$), ξ^ρ_j ($\rho \in R$, $j \in J$) given by

$$\xi^\rho_j = 0 \qquad \rho \in R, j \in J^\rho$$

(4) $$c^{\rho l} \xi^\rho + \eta_\rho = -t_\rho d^{\rho l} \qquad \rho \in R, l \in L^\rho$$

$$\sum_{\rho \in R} \xi^\rho = a^t,$$

we have

Lemma 1.4:

Let $\mathfrak{J}$ be admissible. Then the system (4) has a unique solution if and only if (2) has a unique solution.

Proof: Clearly, $(\bar{\xi}, \bar{\eta})$ is a solution of (4) if and only if $(\ldots, \bar{x}^{\rho}, \ldots, \ldots, \bar{u}) = (\ldots, \frac{\bar{\xi}^{\rho}}{t_{\rho}}, \ldots, -\bar{\eta})$ is a solution of (2).

Obviously, the square coefficient matrix of (4) does not depend on $t \in \mathbb{R}_{++}$. Hence, it is or is not nonsingular independently of t.

Definition 1.5:

Let us call a system $\mathfrak{J} = (J_1, \ldots, J_r, L_1, \ldots, L_r)$ ***nonsingular*** if it is admissible and the coefficient matrix of (4) is nonsingular.

A nonsingular system obtained by a vertex $\bar{u}$ via some corresponding $\bar{X}$ node by means of the procedure indicated in the proof of Lemma 1.2 is also called ***corresponding*** (to $\bar{u}$ or to $(\bar{X}, \bar{u})$).

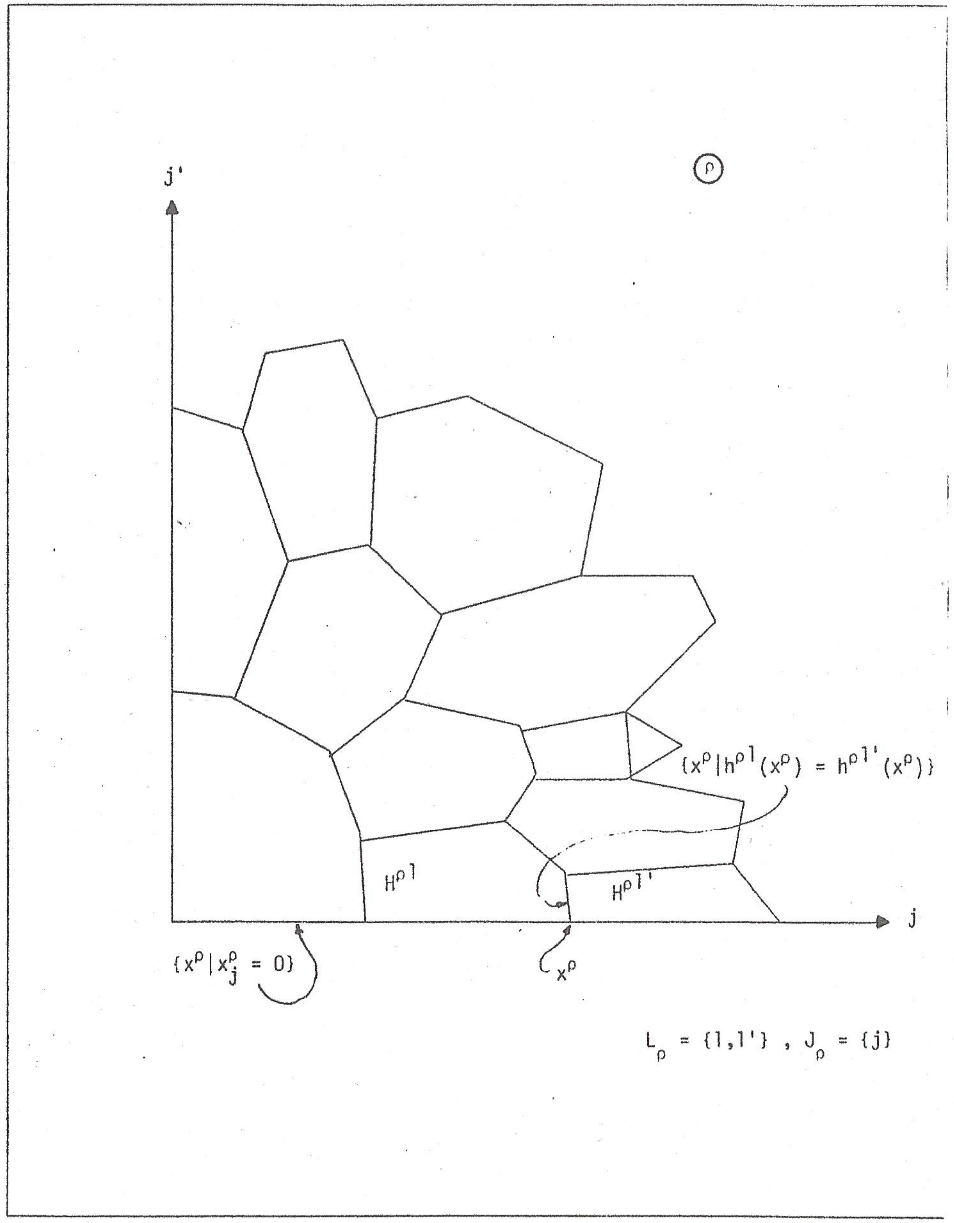

Figure 3

Lemma 1.6:

Let $\mathfrak{J}$ be a nonsingular system. Then there exist an $r \times r$ matrix $A = A^{\mathfrak{J}}$ and r $m \times r$ matrices $B^{\rho} = B^{\rho\mathfrak{J}}$ ($\rho = 1,\dots,r$) such that, for $t \in \mathbb{R}_{++}$, the unique solution of (4) is given by

(5) $$\bar{\eta} = -A\,t \quad , \quad \bar{\xi}^{\rho} = B^{\rho}\,t \qquad (\rho \in R)$$

Thus, in particular, $\bar{\eta}$ and $\bar{\xi}^{\rho}$ are linear in t. Of course, the unique solution of (2) is equivalently given by

(6) $$\bar{u} = A\,t \quad , \quad \bar{x}^{\rho} = \frac{1}{t_{\rho}} B^{\rho}\,t \qquad (\rho \in R)$$

Proof: This follows by Cramers rule. Indeed, the typical representation of the solution of (4) (say, for coordinate $\bar{\eta}_{\rho}$) is

$$\bar{\eta}_{\rho} = \frac{\begin{vmatrix} * & \cdots & 0 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & -t_{\sigma} d^{\sigma 1} & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & \sum_{\tau} t_{\tau}\, a_j^{\tau} & \cdots & * \end{vmatrix}}{\begin{vmatrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{vmatrix}}$$

which, by expansion according to the ρ'th column, is linear in t.

We denote by int T the ***interior*** of a subset T of $\mathbb{R}^r_+$ and by CVCPH T its *convex comprehensive hull*

Theorem 1.7:

There exists a finite collection $\mathfrak{T} = \{T,\dots,T'\}$ of subsets of $\mathbb{R}^r_+$ with the following property.

1. $T \in \mathfrak{T}$ is a closed, convex polyhedral cone with apex $0 \in \mathbb{R}^r_+$ and nonempty interior.

2. $\mathbb{R}^r_+ = \bigcup \{T \mid T \in \mathfrak{T}\}$.
3. For $T, T' \in \mathfrak{T}$ the cone $T \cap T'$ is lower–dimensional.
4. For $T \in \mathfrak{T}$ there exists a (finite) family $Q = Q_T$ of nonsingular systems such that, whenever $t \in T$, $t > 0$

(7) $$F(t) = \text{CVCPH}\,\{A^{\mathfrak{J}}t \mid \mathfrak{J} \in Q\}$$

i.e., F(t) is the *convex comprehensive hull* of vectors $A^{\mathfrak{J}}t$, $\mathfrak{J} \in Q$.

Proof:

1st Step:

Consider a nonsingular system $\mathfrak{J} = (J_1,\dots,J_r, L_1,\dots,L_r)$ and the corresponding solution $(\bar{X}, \bar{u})$ of (3). Clearly, $\bar{u} \in F(t)$ if $\bar{X} \geq 0$ and $h^{\rho l}(\bar{x}^{\rho}) \geq h^{\rho l'}(\bar{x}^{\rho})$ ($\rho \in R$, $l \in L_{\rho}$, $l' \in L–L_{\rho}$).

This is equivalent to $\bar{\xi} \geq 0$ and $c^{\rho l}\,\bar{\xi}^{\rho} + t_{\rho}\, d^{\rho l} \geq c^{\rho l'}\,\bar{\xi}^{\rho} + t^{\rho}\, d^{\rho l'}$ ($\rho \in R$, $l \in L_{\rho}$, $l' \in L–L_{\rho'}$). Again, if $A = A^{\mathfrak{J}}$ and $B^{\rho} = B^{\rho\mathfrak{J}}$ ($\rho \in R$) are determined by Lemma 1.6, we conclude that $\bar{u} \in F(t)$ if

(8)
$$B^{\rho}t \geq 0 \qquad (\rho \in R)$$
$$(c^{\rho l} - c^{\rho l'})\,B^{\rho}t \geq t_{\rho}\,(d^{\rho l} - d^{\rho l'}) \qquad (\rho \in R, l \in L_{\rho}, l' \in L–L_{\rho})$$

Now, let us regard the rows of this system (which is linear in t) as to be represented by a matrix $D = D^{\mathfrak{J}}$ – which depends on $\mathfrak{J}$ only (and can be computed by means of the $c^{\rho l}$ and $B^{\mathfrak{J}}$).

From this, it follows that

(9) $$F(t) = \text{CVCPH}\,\{A^{\mathfrak{J}}t \mid \mathfrak{J}\text{ nonsingular}, D^{\mathfrak{J}}t \geq 0\}$$

holds true for $t \in R_{++}$.

2nd Step:

Fix a (finite) family of nonsingular systems, say Q. For any nonsingular system $\mathfrak{J} \notin Q$ let $D^{\mathfrak{J}} = (D^{\mathfrak{J}}_{k.})_{k \in K^{\mathfrak{J}}}$ be the notation indicating the *rows* $D^{\mathfrak{J}}_{k.}$ of $D^{\mathfrak{J}}$; thus $K^{\mathfrak{J}}$ is an index set counting the rows of $D^{\mathfrak{J}}$.

Let $\emptyset \neq K^{\mathfrak{J}}_0 \subseteq K^{\mathfrak{J}}$ and $K^{\mathfrak{J}}_1 = K^{\mathfrak{J}} - K^{\mathfrak{J}}_0$. There are only finitely many choices of systems

$$\sigma = (Q, (K^{\mathfrak{J}}_0)_{\mathfrak{J} \notin Q})$$

Consider the convex closed cone with apex 0 given by

$$T^{\sigma} = \{t \in \mathbb{R}^r_+ \mid D^{\mathfrak{J}}t \geq 0 \ (\mathfrak{J} \in Q), D^{\mathfrak{J}}_{k.}\ t \leq 0$$
$$(\mathfrak{J} \notin Q, k \in K^{\mathfrak{J}}_0), D^{\mathfrak{J}}_{k.}\ t \geq 0 \ (\mathfrak{J} \notin Q, k \in K^{\mathfrak{J}}_1)\}$$

Then clearly

$$\mathbb{R}^r_+ = \bigcup_{\sigma} T^{\sigma}$$

By omitting those T^{σ} that are empty or of lower dimension, we obtain easily a covering

$$\mathbb{R}^r_+ = \bigcup_{\sigma \in \Sigma} T^{\sigma}$$

such that, for $\sigma \in \Sigma$, T^{σ} has nonempty interior, $\emptyset \neq \operatorname{int} T^{\sigma}$.

This construction has been arranged in a way such that

$$\operatorname{int} T^{\sigma} \cap \operatorname{int} T^{\sigma'} = \emptyset \quad (\sigma \neq \sigma').$$

Moreover, for $t \in \operatorname{int} T^{\sigma}$

$$\{\mathfrak{J} \mid D^{\mathfrak{J}} t \geq 0\} = Q$$

and hence

$$F(t) = \text{CVCPH}\ \{A^{\mathfrak{J}} t \mid \mathfrak{J} \in Q\}.$$

q.e.d.

SECTION 2 :

t-Faces

The previous section explains the behavior of the correspondence F(.): essentially it can be seen as the convex hull of finitely many linear functions in t – provided we restrict our observation to an appropriate subcone of $\mathbb{R}^r_{++}$. We would like to eventually obtain the same picture with respect to a certain face (r–1 – dimensional subpolyhedron in the boundary) of F(t). If necessary, we shall of course restrict ourselves to further subcones, however, it would be desirable to obtain a boundary face again as the convex hull of a finite number of linear functions.

First of all we shall have to clear up the relation of extremals in utility space $\mathbb{R}^r_{++}$ and the polyhedral decomposition in $\mathbb{R}^m_+$ – which is specified for each type ρ (cf. SEC. 1). We should expect that the nodes corresponding to extremals of the same face are located within the same $H^{\rho l}$.

Lemma 2.1.: Let $t \in \mathbb{R}^r_{++}$ and let $(\bar{u}^q)_{q \in Q}$ be a finite set of vertices of F(t) belonging to the same P.E. face. Then, for $\rho \in R$, there is $\ell = \ell(\rho) \in L$ with the following property: if, for each $q \in Q$, X^q is a corresponding node then, for $\rho \in R$,

$$\bar{x}^{q\rho} \in H^{\rho \ell} \ (q \in Q).$$

That is, any choice of extreme points of $\mathfrak{A}^{\bar{u}^q}_t$ $(q \in Q)$ yields points situated in a common $H^{\rho \ell}$.

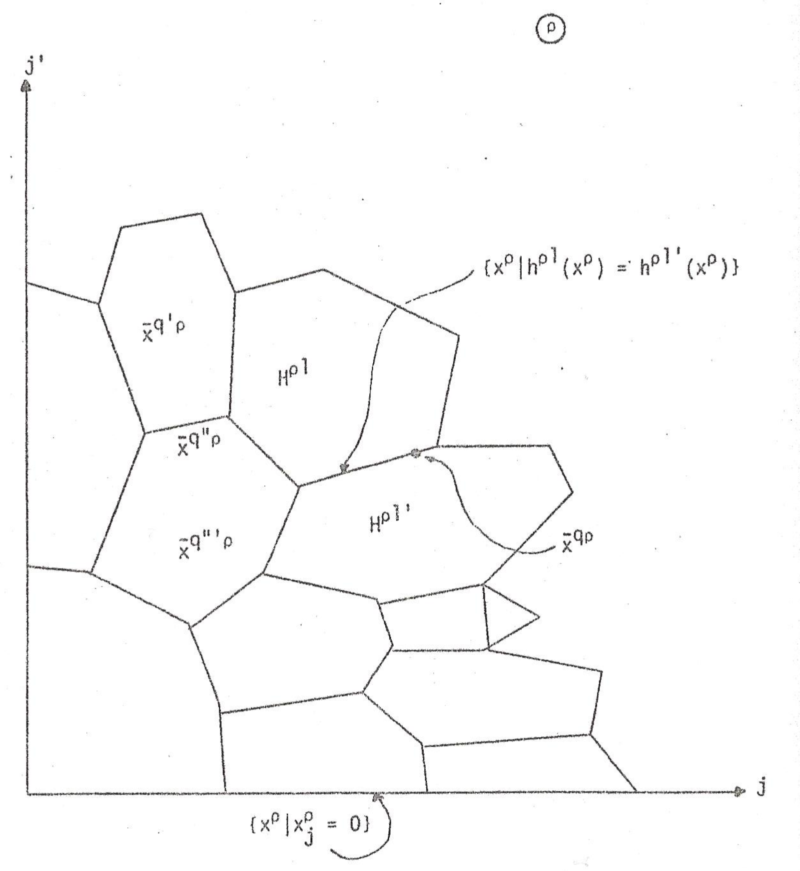

Figure 4

It is sufficient to just sketch the following

<u>Proof:</u> If, for some $\rho \in R$, the $\bar{x}^{q\rho}$ are not adjacent to at least one $H^{\rho l}$, then consider for $\alpha_q > 0$, $(q \in Q)$, $\sum_{q \in Q} \alpha_q = 1$, the utility vector

$$\bar{u} = \sum_{q \in Q} \alpha_q \bar{u}^q$$

which is an element of the face in consideration. Also, $\bar{X} := \Sigma \alpha_q \bar{x}^q$ satisfies $\bar{X} \in \mathfrak{A}_t$.

However, since the $\bar{x}^{q\rho}$ are not adjacent to a common $H^{\rho l}$, we have

$$t_\rho u^\rho(\bar{x}^\rho) = t_\rho u^\rho\Big(\sum_{q \in Q} \alpha_q \bar{x}^{q\rho}\Big) > t_\rho \sum_{q \in Q} \alpha_q u^\rho(\bar{x}^{q\rho}) = \sum_{q \in Q} \alpha_q \bar{u}^q_\rho = \bar{u}_\rho,$$

while, for all remaining $\rho' \in R$ we have at least $\geq$. This contradicts the fact that $\bar{u}$ is P.E.

Let us use the notation *t–face* for a P.E. face of F(t) which has dimension r–1. A t–face is, of course, the convex hull of all the t–vertices it contains. F(t) is the comprehensive convex hull of all its t–vertices. We say that x, $y \in \mathbb{R}^m_+$ are *adjacent* (referring to some $\rho \in R$) if there is some $l \in L$ such that $U^{\rho l}$ contains both points.

Corollary 2.2.:

1. If $\bar{u}$ is a t–vertex, then all corresponding nodes are adjacent.

2. Let $(\bar{u}^q)_{q \in Q}$ be (all) t–vertices belonging to the same face, say $F^Q(t)$, and let $\bar{X}^q$ be a corresponding node $(q \in Q)$. Suppose, for each q, $L^q_\rho \subseteq L$ is given by Lemma 1.1. $(\rho = 1,\ldots,r)$. Then, for every $\rho \in R$,

(1) $$\bigcap_{q \in Q} L^q_\rho =: L^Q_\rho \neq \emptyset.$$

Thus, the $\bar{x}^{q\rho}$ $(\rho \in R)$ are adjacent.

3. $\mathcal{A}_t^Q := \{X \in \mathcal{A}_t \mid u(X) \in F^Q(t)\}$ is a compact convex polyhedron; the extreme points are the extreme points of $\mathcal{A}_t^{\bar{u}^q}$ $(q \in Q)$, hence corresponding nodes. The ρ–coordinates of these nodes are adjacent.

4. If, for $\rho \in R$, we choose $\ell = \ell(\rho) \in L_\rho^Q$, according to Lemma 2.1., then

$$F^Q(t) = \{t \otimes \bar{h}(X) := (t_1 h^{1\ell(1)}(x^1), \ldots, t_r h^{r\ell(r)}(x^r)) \mid X \in \mathcal{A}_t^Q\} \tag{2}$$

The choice of an index set Q in order to identify a face may seem to be an arbitrary one. We shall make an attempt to justify this procedure in SEC. 3.

The next remark shortly describes the way, the normal of a face depends on the parameter t.

Remark 2.3.:

Let $t \in \mathbb{R}^r_{++}$ and let $(\bar{u}^q)_{q \in Q}$ be a set of r t–vertices spanning a t–face, say $F^Q(t)$, of $F(t)$. In view of Lemma 1.1 and 1.5 there is, for any $q \in Q$, an $r{\times}r$–matrix A^q such that

$$\bar{u}^q = A^q t \qquad (q \in Q)$$

holds true. Let $\bar{\lambda} = \bar{\lambda}^{Q,t}$ denote the *normal* of the t–face $F^Q(t)$; this normal can be chosen to be nonnegative and hence can be normalized to satisfy

$$\bar{\lambda}_1 + \ldots + \bar{\lambda}_r = 1$$

If so, then $\bar{\lambda}$ may be obtained as the solution of the linear system of equations

$$\lambda(\bar{u}^q - \bar{u}^p) = 0 \qquad (q \in Q,\ q \neq p)$$
$$\sum_{\rho \in R} \lambda_\rho = 1$$

where p is some fixed element of Q. That is

$$\lambda(A^q - A^p)t = 0$$
$$\sum_{\rho \in R} \lambda_\rho = 1 \tag{3}$$

Using Cramers rule and expanding determinants we find at once r polynomials in $t_1, \ldots, t_r$ of degree r–1, say D_ρ $(\rho = 1, \ldots, r)$ such that

$$\bar{\lambda}_\rho(t) = \frac{D_\rho(t)}{D_1(t) + \ldots + D_r(t)}$$

Next we would like to discuss the behavior of *price vectors* corresponding to some face of $F(t)$.

Pick $t \in \mathbb{R}_{++}$ and consider some face, say $F^Q(t)$. Let $\bar{u} \in F^Q(t)$ and let $\bar{X} \in \mathcal{A}_t$ be a corresponding node, thus $\bar{u} = t \otimes u(\bar{X})$ holds true. Let $\bar{\lambda}$ denote the normal at $F^Q(t)$ in $\bar{u}$.

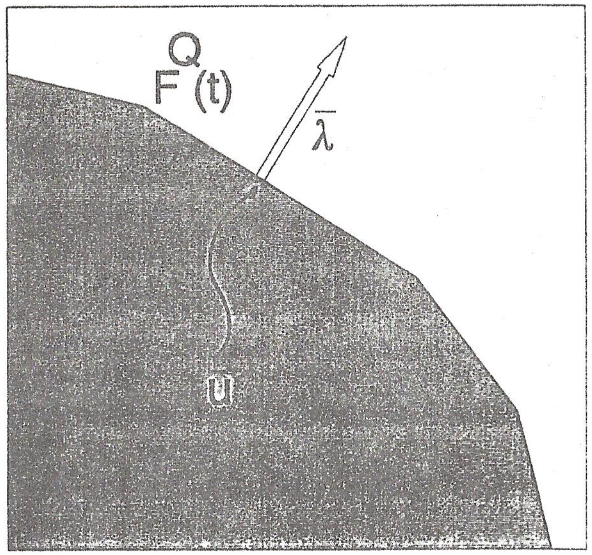

Figure 5

Since $\bar{u}$ is Pareto efficient, we know that

$$\Sigma \bar{\lambda}_\rho \bar{u}_\rho = \Sigma \bar{\lambda}_\rho t_\rho u^\rho(\bar{x}^\rho) = \max \{ \Sigma \bar{\lambda}_\rho t_\rho u^\rho(x^\rho) \mid X \in \mathcal{O}_t \},$$

and by a standard Kuhn – Tucker – argument there is a joint tangential hyperplane for the graphs of all functions $\bar{\lambda}_\rho u^\rho(\ )$ at $\bar{x}^\rho$. (The constraints defining $\mathcal{O}_t$ include the t_ρ – therefore the t_ρ do not appear in the description of these hyperplanes.)

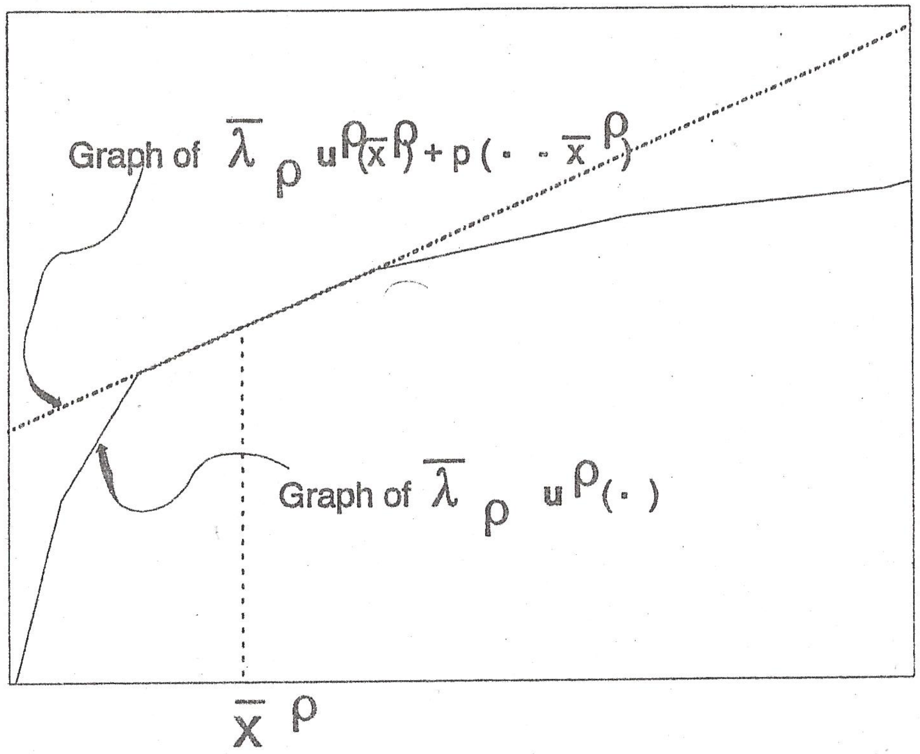

Figure 6

As gradients of these hyperplanes all gradients of $\bar{\lambda}_\rho u^\rho(\ )$ at $\bar{x}^\rho$ are feasible. With $\bar{x}^\rho$ in the interior of some $H^{\rho l}$, they are uniquely defined (and equal $\lambda_\rho c^{\rho l}$) – and at the boundaries of the $H^{\rho l}$ we may take the corresponding convex combinations.

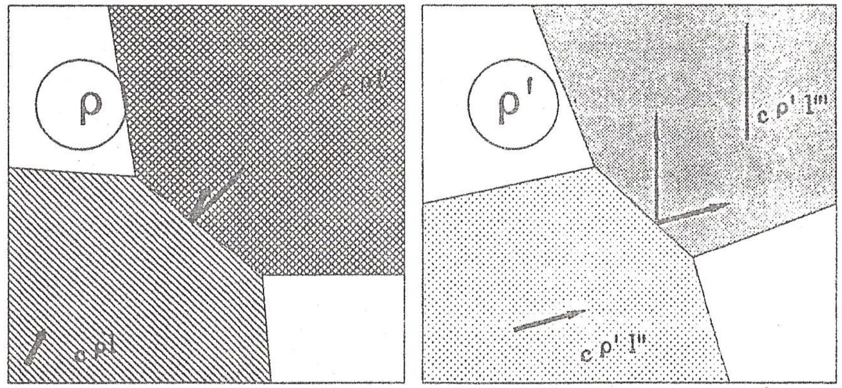

Figure 7

The joint tangential hyperplanes may be viewed as graphs of the linear functions

$$\bar{\lambda}_\rho u^\rho(\bar{x}^\rho) + \bar{p}(y - \bar{x}^\rho) \qquad (y \in \mathbb{R}^m_+)$$

satisfying

$$\bar{\lambda}_\rho u^\rho(\bar{x}^\rho) + \bar{p}(y - \bar{x}^\rho) \geq \bar{\lambda}_\rho u^\rho(y) \qquad (y \in \mathbb{R}^m_+).$$

Because of the above–mentioned situation, the *price vector* $\bar{p}$ for each ρ has to be a convex combination of certain $\lambda_\rho c^{\rho l}$ – and the set of indices l depends on Q only.

Because these reflections are still vague, we supply a more formal description as follows.

Remark 2.4.:

1. For $x \in \mathbb{R}^m_+$ let P_ρ denote the set of "feasible price vectors for type ρ", i.e. the set of gradients of all linear functions, the graph of which defines a tangential hyperplane at the graph of u^ρ in x. Now, if $L_\rho = \{l \mid x \in H^l_\rho\}$ (and hence $x \in \bigcap_{l \in L_\rho} H^{\rho l}$) and $j = \{i \mid x_i = 0\}$, then it is verified at once that

(4) $$P_\rho = \mathrm{CVH}\,\{c^{\rho l} + \sum_{i \in J_\rho} t_i e^i \mid t \geq 0, l \in L_\rho\};$$

hence P_ρ depends on L_ρ and J_ρ only.

2. Next, let $(\bar{u}^q)_{q \in Q}$ be the t-vertices of a t-face $F^Q(t)$. Consider a point $\bar{u}$ within the relative interior of $F^Q(t)$ and let $\bar{X}$ be a corresponding allocation, i.e., $\bar{u} = t \circledast u(\bar{X})$. Also, suppose that $\bar{\lambda} = \lambda^{Q,t}$ be the normal of this face (cf. Remark 2.3). Then $\bar{X}$ maximizes the total (weighted) utility of t, i.e.,

$$\Sigma \bar{\lambda}_\rho \bar{u}_\rho = \Sigma \bar{\lambda}_\rho t_\rho u^\rho(\bar{x}^\rho) = \max\,\{\Sigma \bar{\lambda}_\rho t_\rho u^\rho(x^\rho) \mid X \in \mathfrak{A}_t\}$$

for any choice of $I(\rho) \in L^Q_\rho \neq \emptyset$ (cf. 2.2.4)

According to a standard Kuhn-Tucker Argument, there is a common supporting hyperplane for the graphs of all (weighted) utilities $\bar{\lambda}_\rho u^\rho(\cdot)$ at $\bar{x}^\rho$, i.e., a price vector

$$\bar{p} \in \bar{\lambda}_\rho P_\rho$$

where $P_\rho = P^Q_\rho$ (in view of (4)) depends on $L^Q_\rho = \bigcap_{q \in Q} L^Q_\rho$ and $J^Q_\rho = \bigcap_{q \in Q} I^q_\rho$ only. In other words, P_ρ depends only on the nonsingular systems $\mathfrak{J}^q$ $(q \in Q)$ that define the face $F^Q(t)$. Thus

(5) $$P^Q(\bar{\lambda}) := \bigcap_{\rho \in R} \bar{\lambda}_\rho P^Q_\rho \neq \emptyset,$$

and any vector

(6) $$\bar{p} \in P^Q(\bar{\lambda})$$

serves as a price vector. That is, for $\rho \in R$ and $y \in \mathbb{R}^m_+$ we have

(7) $$\bar{p}(y - \bar{x}^\rho) \geq \bar{\lambda}_\rho (u^\rho(y) - u^\rho(\bar{x}^\rho)).$$

3. Note that (6) reads as well as follows:

$$\bar{p} = \bar{\lambda}_\rho \bar{c}^\rho,$$

(8) $$\bar{c}^\rho \in P^Q_\rho = \mathrm{CVH}\,\{c^{\rho l} + \sum_{i \in J_\rho} t_i e^i \mid t \geq 0, l \in L^Q_\rho\}.$$

In particular, if one of the $\bar{x}^\rho$, say $\bar{x}^{\bar{\rho}}$, is located within the interior of some $H^{\bar{\rho}\bar{l}}$, (i.e.), $L^Q_{\bar{\rho}} = \{\bar{l}\}$, $J^Q_{\bar{\rho}} = \emptyset$, then

(9) $$P^Q_{\bar{\rho}} = \{c^{\bar{\rho}\bar{l}}\},\ P^Q(\bar{\lambda}) = \{\bar{\lambda}_{\bar{\rho}} c^{\bar{\rho}\bar{l}}\}$$

i.e.

(10) $$\bar{c}^{\bar{\rho}} = c^{\bar{\rho}\bar{l}},\ \bar{p} = \bar{\lambda}_{\bar{\rho}} c^{\bar{\rho}\bar{l}} = \bar{\lambda}_{\bar{\rho}} \bar{c}^{\bar{\rho}}$$

and

(11) $$\bar{c}^\rho = \frac{\bar{p}}{\bar{\lambda}_\rho} = \frac{\bar{\lambda}_{\bar{\rho}}}{\bar{\lambda}_\rho} \bar{c}^{\bar{\rho}} \quad (\rho \in R).$$

At this state of affairs the choice of an index set Q to identify a face $F^Q(t)$ is still not quite justified. Of course we cannot expect that this is possible quite independently on the parameter t. However, it is not unlikely that the faces of F keep certain identifying systems of equations that determine their extreme elements within certain cones in $\mathbb{R}^r_{++}$. This we will be shortly exhibited within the next section.

SECTION 3 :

Q - Faces

Consider a cone T as specified by Theorem 1.6. With T, for any t we know that F(t) is the convex, comprehensive hull of finitely many points of the form $A^{\mathfrak{I}}t$; here $\mathfrak{I}$ denotes a nonsingular system in the sense of SEC. 1. However, if $\mathfrak{I}$ is nonsingular and $A^{\mathfrak{I}}t$ is feasible then, nevertheless, the latter is not necessarily Pareto efficient. Also, if $(\mathfrak{I}^q)_{q\in Q}$ is a family of nonsingular systems, then the points $A^q t = A^{\mathfrak{I}q}\ t$ do not necessarily constitute a t–Face.

In order to specify a family of nonsingular systems $Q \subseteq \{\mathfrak{I} \mid \mathfrak{I}$ is nonsingular$\}$ we identify indices $q \in Q$ and systems $\mathfrak{I} = \mathfrak{I}^q$.

Theorem 3.1.:

Let $T \subseteq \mathbb{R}^r_+$ be a cone as given by Theorem 1.6, and let Q be the corresponding family of nonsingular systems. Then there is a finite collation $\hat{\mathfrak{T}} = \{\hat{T}, \hat{T}', ...\}$ of subsets of T with the following property.

1. $\hat{T} \in \hat{\mathfrak{T}}$ is a closed cone with apex 0 and nonempty interior.
2. $T = \bigcup\ \{\hat{T} \mid \hat{T} \in \mathfrak{T}\}$
3. For $\overset{\circ}{T}$, $\hat{T}' \in \hat{\mathfrak{T}}$ the cone $\hat{T} \cap \hat{T}'$ is lower–dimensional.
4. For any $\hat{T} \in \hat{\mathfrak{T}}$ there exists a finite family $\underline{\hat{Q}} = \{\hat{Q}\}$ of subsets of Q such that, for $t \in \hat{T}$, the sets
$$F^{\hat{Q}}(t)\ \mathrm{CVH}\ \{A^q t \mid q \in \hat{Q}\}$$
are exactly the t–faces of F(t).

In other words, within every $\hat{T}$ the t–faces depend essentially only on the choice of a system $\hat{Q} \subseteq Q$.

Hence, if we pick $\hat{Q} \in \underline{\hat{Q}}$, then for any $t \in \hat{T}$ we know that $F_{\hat{Q}}(t)$ is a t–face.

Proof:

1st Step:

There are only finitely many families Q of nonsingular systems. Therefore, it is sufficient to prove the following:

Let Q be a family of nonsingular systems. Then the cone

(1) $$\{t \mid A^q t\ (q \in Q) \text{ form exactly the extremes of a t–face of } F(t)\}$$

is either open or contained in an algebraic manifold of lower dimension.

Indeed, if (1) holds true then, with the exception of finitely many lower dimensional cones the Faces $F^Q(t)$ of F(t) for any $t \in \hat{T}$ are also Faces $F^Q(s)$ whenever s is sufficiently close and vice versa. Thus T is decomposed into a family of open cones, within the interior of which the t–faces are defined by a finitely family of nonsingular systems.

2nd Step:

In order to prove the claim of the first step, proceed as follows:

$$\{t \mid A^q t\ (q \in Q) \text{ form exactly the extremes of a t–face of } F(t)\}$$

(2) $$= \quad \{t \mid \lambda\,(A^q t - A^p t) = 0\ (q,p \in Q)$$

has a one–dimensional linear subspace of solutions $\lambda \in \mathbb{R}^r$ such that for some solution we have $\lambda > 0$ and

$$\lambda\, A^r t < \lambda\, A^p t\ (r \notin Q)\}$$

Consider first

$$\{t \mid \lambda\,(A^q t - A^p t) = 0\ (q,p \in Q)$$
has a one–dimensional linear subspace of solutions}

$$= \quad \{t \mid \mathrm{rank}\ (A^q t - A^p t)_{q,p\in Q} = r{-}1\}$$

which, for short, is written

$$= \quad \{t \mid \mathrm{rank}\ (B^{\cdot}\, t) = r - 1\} = \ldots$$

If ${}^{rr}B't$ denotes an arbitrary r x r submatrix (abbreviated SM) of the matrix $B't$, then

$$\dots = \left(\bigcup_{\text{all}(r-1)\times(r-1)\text{SM's}} \{t \mid \det({}^{r-1,r-1}B't) \neq 0\} \right)$$

(3)

$$\cap \left(\bigcap_{\text{all } r\times r \text{ SM's}} \{t \mid \det({}^{rr}B't) = 0\} \right)$$

Now, for every r×r submatrix of $B't$

$$\{t \mid \det({}^{rr}B't) = 0\}$$

is an algebraic manifold to which a dimension is assigned ([11], pp 161/163). If this dimension is strictly less than n, then (2) is contained in a lower dimensional manifold.

If the dimension is n, then the polynomial $\det({}^{rr}B't)$ is the null–polynomial and the corresponding set in (3) can be omitted. Thus, unless one of the corresponding sets cannot be omitted, we may continue in (3)

$$\dots = \bigcup_{\text{all } r-1\times r-1 \text{ SM's.}} \{t \mid \det({}^{r-1,r-1}B't) = 0\}$$

which is an open set.

The additional conditions to be imposed on λ as explained by (2) obviously define an open set.

q.e.d.

In the following we want to deal with those situations only that yield a unique price vector for every face of the correspondence F at varying t. Clearly, inside a cone T as specified by Theorem 3.1, faces may be identified with index–sets Q, and hence the required property is a property of the cone – and not of varying t.

Definition 3.2:

$\bar{u}^e$ has *finite character* if, for every T as given by 3.1. and every Q specifying a face $F^Q(t)$ for all $t \in T$, the price vector is uniquely defined, i.e., $P^Q(\bar{\lambda})$ is a singleton, depending on Q only.

This definition is very much "ad hoc" and it will have to be enhanced by pointing to appropriate requirements for $\bar{u}^e$ to satisfy such that finite character is ensured. This we will postpone to some other treatment. However, *there are* classes of markets satisfying this condition. An obvious candidate is the class of *side–payment* or *TU–markets* as the normal λ equals $e = (1,\dots,1)$ constantly – and indeed, this class has been shown to yield a finite convergence theorem based on nondegeneracy – see [8].

Another class (not very rich though) can be obtained by "generically" requiring that, for any T and Q the resulting face $F^Q(t)$ enjoys a (relatively) interior point, say $\bar{u}$, such that a corresponding element $\bar{X} \in \mathfrak{A}_t$ provides at least one $\bar{x}^\rho$ within the interior of some $H^{\rho l}$.

Indeed, by counting equations, it is easily seen that $mr \geq m + r$ is sufficient to ensure that generically every face of F has dimension r–1. (The mapping

$$\{ X \in \mathbb{R}^{mr} \mid X = (x^\rho)_{\rho \in R}, \sum_\rho t_\rho x^\rho = \sum_\rho t_\rho a^\rho \} \rightarrow \{ t_1 c^1 x^1, \dots, t_r c^r x^r \}$$

must have full rank, which is generically ensured by $mr - m \geq r$).

On the other hand, if all elements of a face $F^Q(t)$ yield corresponding $X \in \mathfrak{A}_t$ such that *every* x^ρ in at least *two* $H^{\rho l}$, then these elements satisfy r + m equations; thus if $mr - (m + r) < r - 1$, then the Pareto face cannot have full dimension. Therefore, by asking for

(4) $$r + m + (r-1) > mr > m + r$$

we ensure the finite character of the market.

With finite character we will essentially be able to demonstrate finite convergence of the core towards the Walrasian equilibrium – this is the topic of the next section.

SECTION 4 :

Finite Convergence of the Core

Within this section we are now going to collect our structural results in order to prove a "finite convergence" property of the core towards the Walrasian equilibrium.

Presently, we can only succeed for core–payoffs that are within the relative ϵ–interior of some Pareto–Face of F(t), say $F_Q(t)$. However, similar to the situation in [7] and [8], it is possible to exhibit regions in $\mathbb{N}^r$ (to be interpreted as areas of distributions of players over the types) such that the core and the Walrasian equilibrium coincide.

Within some cone $\hat{T}$ as defined by Theorem 3.1 a face $F^Q(T)$ is essentially defined by a system Q (of index sets) and by the resulting extremals of the form $A^q t$ $(q \in Q)$. Denote by $K^Q(t)$ the *cone with vertex 0 that is spanned by* $A^q t$, i.e.

(1) $$K^Q(t) = \{ \sum_{q \in Q} \alpha_q A^q t \mid \alpha_q \geq 0 \; (q \in Q) \}.$$

If $u \in \mathbb{R}^r_+$ is a vector satisfying $u \in K^Q(t)$, $u \notin F(t)$, then u is located in front of $F^Q(t)$, this can be separated from F(t) by the hyperplane containing $F^Q(t)$ – which is defined by the normal λ.

Lemma 4.1:

Let $\bar{u}$ be of finite character. Fix Q and consider $t \in \hat{T}$ (cf. 3.1) such that $F^Q(t)$ is a face of F(t), let $\bar{p}$ be the corresponding price vector (Definition 3.2).
Let $\bar{X} \in \mathfrak{A}_t(\bar{u})$ be such that $\bar{u} = t \circledast u(\bar{X}) \in F^Q(t)$.
Then, for any $s \in \mathbb{R}^r_+$ such that $s \circledast u(\bar{X}) \in K_Q(s)$, $s \circledast u(\bar{X}) \notin F(s)$, it follows that

$$\bar{p} \sum_{\rho \in R} s_\rho (\bar{x}^\rho - a^\rho) \geq 0.$$

<u>Proof:</u>

Let $\bar{\lambda}$ be the normal to $F^Q(t)$ and $\tilde{\lambda}$ the one to $F^Q(s)$. Then, for suitable $\rho \in \mathbb{R}$ and $\bar{c} \in \mathbb{R}^m_{++}$ the corresponding prices $\bar{p}$ and $\tilde{p}$ satisfy

$$\bar{p} = \bar{\lambda}_\rho \bar{c}, \; \tilde{p} = \tilde{\lambda}_\rho \bar{c}$$

(Definition 3.2), hence

(2) $$\tilde{p} = \text{const } \bar{p}$$

with a certain positive constant. From this we deduce the following line of inequalities for any $\bar{X} \in \mathfrak{A}_s(\bar{u})$ with $s \circledast u(\bar{X}) \in F^Q(s)$:

$$\sum_{\rho \in R} \tilde{\lambda}_\rho s_\rho u^\rho(\bar{x}^\rho) \geq \sum_{\rho \in R} \tilde{\lambda}_\rho s_\rho u^\rho(\tilde{x}^\rho) \ldots$$

(since $\tilde{\lambda}$ separates $s \circledast u(\bar{X})$ from $F^Q(s)$)

$$\ldots = \sum_{\rho \in R} s_\rho (\tilde{\lambda}_\rho u^\rho(\tilde{x}^\rho) - \tilde{p}(\tilde{x}^\rho - a^\rho)) \ldots$$

(since $\tilde{X} \in \mathfrak{A}_s(\bar{u})$)

(3) $$\ldots \geq \sum_{\rho \in R} s_\rho (\tilde{\lambda}_\rho u^\rho(\bar{x}^\rho) - \tilde{p}(\bar{x}^\rho - a^\rho))$$

(by the Kuhn–Tucker–argument, see (7), SECTION (2))

$$= \sum_{\rho \in R} \tilde{\lambda}_\rho s_\rho u^\rho(\bar{x}^\rho) - \tilde{p} \sum_{\rho \in R} s_\rho (\bar{x}^\rho - a^\rho).$$

Hence, using (2) and (3):

$$\bar{p} \sum_{\rho \in R} s_\rho (\bar{x}^\rho - a^\rho) = \text{const } \bar{p} \sum_{\rho \in R} s_\rho (\bar{x}^\rho - a^\rho) \geq 0.$$ q.e.d.

Next, for small $\epsilon > 0$, let $F^Q_\epsilon(t)$ denote the (closed) "ϵ–interior" of $F^Q(t)$; more precisely

(4) $$F^Q_\epsilon(t) = \{ \sum_{\rho \in Q} \alpha_q A^q t \mid \sum_{q \in Q} \alpha_q = 1, \alpha_q \geq \epsilon \quad (q \in Q) \}.$$

This is a compact convex polyhedron with extremals, say,

$$\bar{u}^{q\epsilon} \in F^Q_\epsilon(t) \quad (q \in Q).$$

Lemma 4.2:

Fix $\hat{T}$ according to Theorem 3.1.

Let $\epsilon > 0$. For every $t \in \hat{T}$, there is a closed cone $C^Q_\epsilon(t)$ such that for all $\bar{u} = t \circledast u(\bar{X}) \in F^Q_\epsilon(t)$ it follows that

(a) t is located within the interior $C^Q_\epsilon(t)$;

(b) if $s \in C^Q_\epsilon(t)$, then $s \circledast u(\bar{X}) \in K^Q(s)$.

Proof:

Let $t \in \hat{T}$. First of all, fix $\bar{u} = t \circledast u(\bar{X}) \in F^Q_\epsilon(t)$.

Note that

$$t \circledast u(\bar{X}) = \sum_{q \in Q} \alpha_q A^q t$$

with suitable "convexifying" coefficients α. The ρ'th coordinate is

$$t_\rho u^\rho(\bar{x}^\rho) = \sum_{q \in Q} \alpha_q A^q_{\rho \cdot} t .$$

($A_{\cdot \rho}$ of the ρ'th row of a matrix A); and hence the ρ'th coordinate of $s \circledast u(\bar{X})$ is

(6) $$s_\rho u^\rho(\bar{x}^\rho) = \frac{s_\rho}{t_\rho} \sum_{q \in Q} \alpha_q A^q_{\rho \cdot} t .$$

(Note that this is linear in s!). Consider

$$C := \{ s \in \mathbb{R}^r_{++} \mid s \circledast u(\bar{X}) \in K^Q(s) \}.$$

Clearly, $t \in C$ and as $t \circledast u(\bar{X})$ is in the interior of $F^Q(t)$ (hence in the interior of $K^Q(t)$), C is seen to be a (closed) cone containing t in it's interior. (In fact, C is described by finitely many algebraic hypersurfaces.)

Now, let C^q ($q \in Q$) denote the cones generated by taking the extreme points $\bar{u}^{q\epsilon}$ ($q \in Q$) of $F^Q_\epsilon(t)$ and performing the above procedure. Put

(7) $$C^Q_\epsilon(t) := \bigcap_{q \in Q} C^q.$$

Then again, t is within the interior of the closed cone $C^Q_\epsilon(t)$.

Finally, let again $\bar{u} = t \circledast u(\bar{X})$ be arbitrary in $F^Q_\epsilon(t)$; we know that

$$t \circledast u(\bar{X}) = \bar{u} = \sum_{q \in Q} \alpha_q \bar{u}^{q\epsilon} = \sum_{q \in Q} \alpha_q \, t \circledast u(\bar{X}^{q\epsilon})$$

with suitable α_q and $\bar{X}^{q\epsilon}$ ($q \in Q$). The ρ'th coordinate of $s \circledast u(\bar{X})$ is, therefore,

$$s_\rho u^\rho(\bar{x}^\rho) = \frac{s_\rho}{t_\rho} \sum_q \alpha_q t_\rho u(\bar{x}^{q\epsilon\rho})$$

$$= \sum_q \alpha_q s_\rho u(\bar{x}^{q\epsilon\rho}),$$

i.e.,

(8) $$s \circledast u(\bar{X}) = \sum_{q \in Q} \alpha_q \, s \circledast u(\bar{X}^{q\epsilon})$$

(essentially the linearity of (6) is s is used!).

Therefore, if $s \in C^Q_\epsilon(t)$, then $s \circledast u(\bar{X}^{q\epsilon}) \in K^Q(s)$ (by (7)), and, as $K^Q(s)$ is convex, (8) implies that $s \circledast u(\bar{X}) \in K^Q(s)$, q.e.d.

Remark 4.3:

1. Note that $C^Q_\epsilon(t)$ is "algebraic", i.e., its boundary is constituted by a finite number of algebraic hypersurfaces.

2. Clearly, $C^Q_\epsilon(t)$ can be chosen to exhibit some property of being "positively homogeneous", i.e., if $\alpha > 0$ then

$$C^Q_{\frac{\epsilon}{\alpha}}(\alpha t) = C^Q_\epsilon(t).$$

Definition 4.4:

Let $\mathbb{N}^r \ni k \in \hat{T}$. k *generates a full cone* (w.r.t. ϵ, Q) if there are r linearly independent vectors ("profiles") $s \in \mathbb{N}^r$ such that

(a) $s \leq k$

(b) $s \in C^Q_\epsilon(k)$

(c) $k - s \in C^Q_\epsilon(k)$

is satisfied.

Using profiles $k \in \mathbb{N}^r$ we now switch to markets u^k with a certain distribution $k = (k_1, \ldots, k_r)$ of players over the types; this setup we started out with in SECTION 0. We want to compare the Core $\mathcal{C}(\bar{u}^k)$ and the Walrasian allocations of $\bar{u}^k$.

Theorem 4.5:

Let $k \in \mathbb{N}^r$ generate a full cone. If $\bar{X} \in \mathcal{C}(\bar{u}^k)$ is such that $k \circledast u(\bar{X}) \in F^Q_\epsilon(k)$, then $\bar{X}$ is Walrasian.

<u>Proof:</u>

Fix $\bar{X}$ with the required properties. Since $\bar{X} \in \mathcal{C}(\bar{u}^k)$, we know that $s \circledast u(\bar{X})$ is not in the interior of $F(s) = V(s)$. If, in addition, $s \in C^Q_\epsilon(k)$ holds true, then we have all the conditions of Lemma 4.1.

Therefore, there are r linearly independent and integer vectors $s \in \mathbb{N}^r$ satisfying the conclusion of Lemma 4.1., i.e.,

(10) $$\bar{p} \sum_{\rho \in R} s_\rho(\bar{x}^\rho - a^\rho) \geq 0,$$

here $\bar{p}$ is the price–vector corresponding to Q.

Since for any such s, k – s has the same property ("full cone"–definition) we have as well

(11) $$\bar{p} \sum_{\rho \in R} (k_\rho - s_\rho)(\bar{x}^\rho - a^\rho) \geq 0.$$

However, as $\bar{X} \in \mathcal{O}_k(\bar{u})$, we have

(12) $$\sum_{\rho \in R} k_\rho(\bar{x}^\rho - a^\rho) = 0$$

and hence it follows from (10) and (11) that

(13) $$\bar{p} \sum_{\rho \in R} s_\rho(\bar{x}^\rho - a^\rho) = 0.$$

Since our assumption is that there are r linearly independent vectors s satisfying (13), we conclude that

(14) $$\bar{p}(\bar{x}^\rho - a^\rho) = 0 \qquad (\rho \in R).$$

It is well known that this implies that $(\bar{p}, \bar{X})$ is a Walrasian equilibrium. q.e.d.

Theorem 4.6:

Let $\bar{u}^e$ be of finite character and let $\hat{T} \subseteq \mathbb{R}^r_{++}$ be defined by Theorem 3.1.

Then, for any $\epsilon > 0$ there exists a set $H_\epsilon \subseteq \hat{T}$ with the following properties:

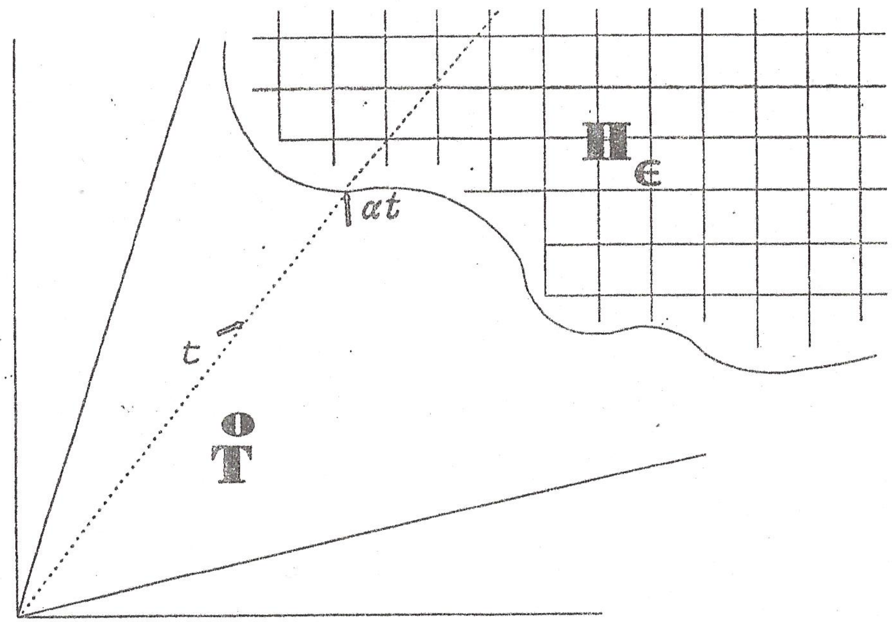

Figure 7a

(a) If $t \in \hat{T}$, then there is $\alpha > 0$ such that $\alpha t \in H_\epsilon$.

(b) ∂H_ϵ is an algebraic hypersurface.

(c) If $k \in \mathbb{N}^r$, $k \in H_\epsilon$, and $\bar{X} \in \mathcal{C}(\bar{u}^k)$ is such that $k \circledast u(\bar{X}) \in F^Q_\epsilon(k)$, then $\bar{X}$ is Walrasian.

Thus, for sufficiently large $k \in \mathbb{N}^r$, all core allocations that yield payoffs which are in the ϵ–Pareto efficient face, are Walrasian.

Proof:

For $t \in \hat{T}$, $C^Q_\epsilon(t)$ is a closed cone with nonempty interior (and "algebraic"). In view of Remark 4.3, $C^Q_\epsilon(t)$ increases ("linearly") with t. Therefore the set

(15) $$E^Q_\epsilon(t) := C^Q_\epsilon(t) \cap \{s \mid s \leq t\} \subseteq \mathbb{R}^r_{++}$$

contains a convex closed body, the volume of which increases ("linearly") in t.

By MINKOWSKI's ("second") Theorem (see CASSELS [2], and compare the argument in ROSENMÜLLER [7], [8],) the set $E^Q_\epsilon(t)$ described by (15) contains r linearly independent integer vectors s once the volume exceeds a certain constant, say c_Q. Define

$$H^\epsilon = \{t \in \hat{T} \mid \text{volume}\,(E^Q_\epsilon) \geq c_Q\}.$$

Then H^ϵ has properties (a) and (b). Property (c) follows from Theorem 4.5., since $k \in H^\epsilon$ implies that k generates a full cone. q.e.d.

SECTION 5 :

Examples

Example 5.1:

Let m = 2, r = 2 and consider the utility functions

$$u^1 : \mathbb{R}^2_+ \to \mathbb{R}, \quad u^1(x) = \min\{x_1+2x_2, 2x_1+x_2\} \; (x \in \mathbb{R}^2_+)$$

for type 1

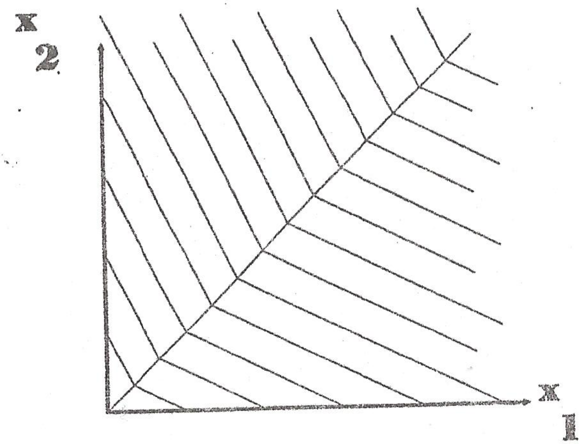

Figure 8

as well as

$$u^2 : \mathbb{R}^2_+ \to \mathbb{R}, \quad u^2(x) = x_1+x_2 \; (x \in \mathbb{R}^2_+)$$

for type 2.

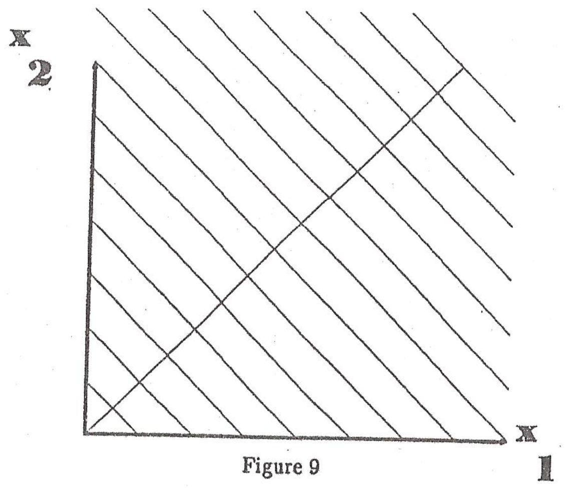

Figure 9

Also, we fix initial allocations

$$a^1 = (2,1),\ a^2 = (1,1)$$

for both types; hence, for $t \in \mathbb{R}^2_{++}$ the "ideal" initial allocation is

(1) $$t \circledast a = a^t = t_1 a^1 + t_2 a^2 = (2t_1+t_2,\ t_1+t_2).$$

The Pareto optimal allocations are represented by the following sketch

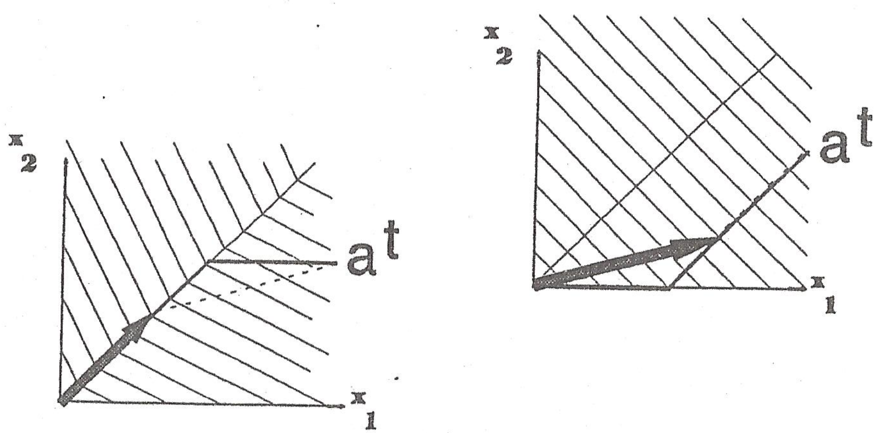

Figure 10

It follows that we have three allocations yielding extreme Pareto efficiant points in utility space, to wit:

$$t \circledast \mathbb{X}^R = (t_1 \bar{x}^{R1}, t_2 \bar{x}^{R2}) = (a^t, 0)$$

$$t \circledast \mathbb{X}^M = (t_1 \bar{x}^{M1}, t_2 \bar{x}^{M2}) = ((a_2^t, a_2^t); (a_1^t - a_2^t, 0))$$

$$t \circledast \mathbb{X}^T = (t_1 \bar{x}^{T1}, t_2 \bar{x}^{T2}) = (0, a^t)$$

((R,M,T) for "right, middle, top"); inserting (1) yields

$$t \circledast \mathbb{X}^R = ((2t_1+t_2,\ t_1+t_2),\ 0)$$

$$t \circledast \mathbb{X}^M = ((t_1+t_2,\ t_1+t_2);\ (t_1, 0))$$

$$t \circledast \mathbb{X}^T = (0;\ (2t_1+t_2,\ t_1+t_2))$$

The corresponding points in utility space are easily computed; observe that $u(t \circledast x) = t \circledast u(x)$ (since $u = (u^1, u^2)$ is homogeneous). Also, we may substitute $u^1(x) = h^{11}(x) = x_1 + 2x_2$ as any P.E. extreme point yields $h^{11}(x) \le h^{12}(x)$.

Thus, we come up with the following three P.E. extreme points of $V(t)$:

$$\bar{u}^R = (4t_1+3t_2, 0)$$

$$\bar{u}^M = (3t_1+3t_2, t_1)$$

$$\bar{u}^T = (0, 3t_1+2t_2).$$

For $t_1 < 2t_2$, the situation is represented by the following sketch. Note that the normal vectors of the P.O. surface of $V(t)$ are $\lambda^T = (2,3)$ and $\lambda^R = (1,1)$ (independent on t and without normalization.)

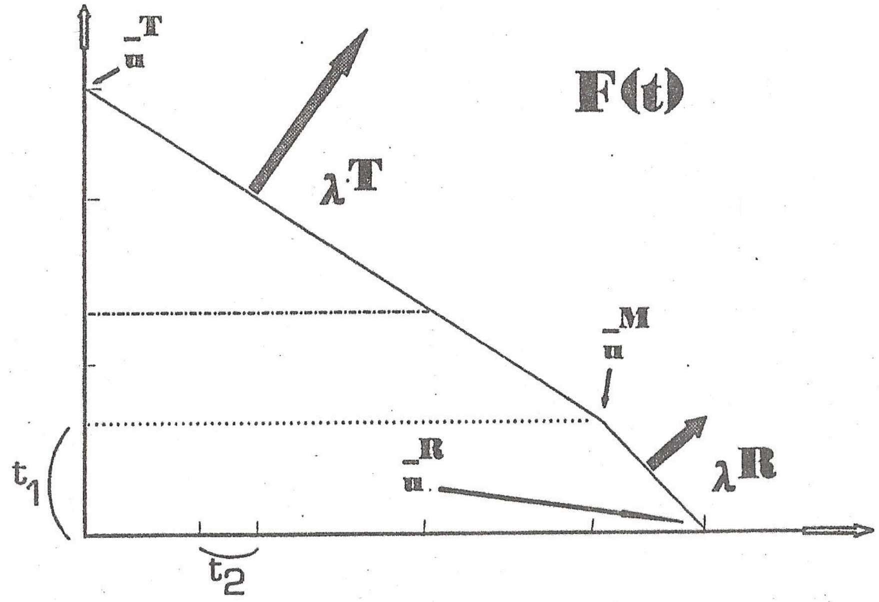

Figure 11

The Walrasian equilibrium is obtained in slightly modified "Edgeworth–Box" (featuring the aggregated allocations for types)

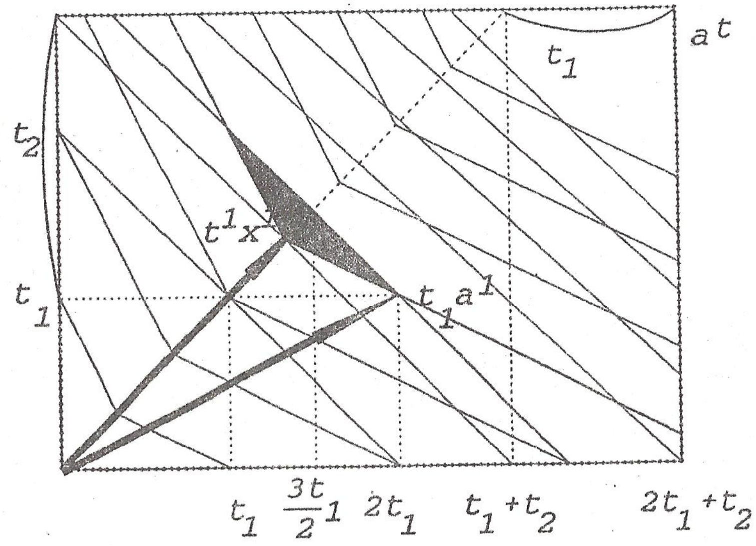

Figure 12

The equilibrium price is $(\frac{1}{2}, \frac{1}{2})$ and the corresponding allocations are

$$_{t_1}\bar{x}^1 = (\frac{3t_1}{2}, \frac{3t_1}{2}) \quad _{t_2}\bar{x}^2 = (t_2, t_2) + (\frac{t_1}{2}, -\frac{t_1}{2}).$$

The corresponding utilities are $^{(t)}\bar{u} = (\frac{9}{2}t_1, 2t_2)$

For $t_1 > 3t_2$ the situation is similar (although the "type–oriented Edgeworth Box" is not quite suitable) and the Equilibrium is given by

$$_{t_1}\bar{x}^1 = (2t_1 - 2t_2, t_1 + t_2)$$

$$_{t_2}\bar{x}^2 = (3t_2, 0)$$

with price $(\frac{1}{3}, \frac{2}{3})$ and utility vector

$$^{(t)}\bar{u} = (4t_1, 3t_2).$$

Finally, if $2t_2 < t_1 < 3t_2$ then the price depends on t, $p = (\frac{t_1 - t_2, t_2}{t_1})$ and we have

$$t_1 \bar{x}^1 = (t_1 + t_2, t_1 + t_2)$$

$$t_2 \bar{x}^2 = (t_1, 0)$$

with utility vector

$$^{(t)}\bar{u} = (3t_1 + 3t_2, t_1).$$

Note that $(\bar{x}^1, \bar{x}^2)$ is a (rational) function of t while $^{(t)}\bar{u}$ is used for

$$^{(t)}\bar{u} = (t_1\, u^1(\bar{x}^1), t_2\, u^2(\bar{x}^2))$$

$$= (u^1(t_1\, \bar{x}^1), u^2(t_2\, \bar{x}^2)).$$

Now for the Core. We are to consider only $\mathcal{C}(V(t))$ (i.e. a concept in utility space). Hence, we are interested in vectors $\hat{u} \in \mathbb{R}^2$ such that $t \otimes \hat{u} = (t_1\, u_1, t_2\, u_2) \in V(t)$ while $s \otimes \hat{u}$ is not in the relative interior of V(s).

Let us first discuss the continuous case $(s, t \in \mathbb{R}^2_{++})$. Assume for the moment that $t \otimes \hat{u}$ is on the upper part of the P.O.–surface; thus

$$\lambda^T\, t \otimes \hat{u} = \lambda^T\, {}^{(t)}b$$

where $^{(t)}b$ is the "corner point" $(3t_1 + 3t_2, t_1)$. Now, if $0 < s < t$ satisfies

$$\lambda^T\, s \otimes \hat{u} < \lambda^T\, {}^{(s)}b$$

then the profile $0 < t-s <$ satisfies

$$\lambda^T (t-s) \otimes \hat{u} > \lambda^T (t-s) \otimes \hat{u}.$$

In a small neighborhood of $\frac{t}{2}$ we shall find continuously many such profiles s for which, in addition, $\lambda^T\, s \otimes \hat{u} \leq \lambda^T\, {}^{(s)}b$ is equivalent to $\lambda^T\, s \otimes \hat{u} \in V(t)$.

It follows that, for such t we must have

$$\lambda^T\, s \otimes \hat{u} = \lambda^T\, {}^{(s)}b$$

for continuously many s; i.e.

$$\lambda_1^T\, s_1\, \hat{u}_1 + \lambda_2^T\, s_2\, \hat{u} = \lambda_1^T\, (3s_1 + 3s_2) + \lambda_2^T\, s_1.$$

In view of the prevailing linearity we may actually insert s = (1,0) and s = (0,1) in order to find the unique solution

$$\hat{u} = \left[\frac{\lambda^T\, {}^{(1,0)}b}{\lambda_1^T}, \frac{\lambda^T\, {}^{(0,1)}b}{\lambda_2^T}\right]$$

$$= (\tfrac{9}{2}, \tfrac{6}{3}).$$

Now, for $t_1 < 2t_2$, $t \otimes \hat{u} = (\frac{9t_1}{2}, \frac{6t_2}{3})$ is in fact an element of V(t) and equals the equilibrium payoff.

Now we switch to the discrete case.

Let

$$T^T = \{t \in \mathbb{R}^2_{++} \mid t_1 < 2t_2\}$$

and

$$E_t^T = \{s \in \mathbb{R}^2_{++} \mid s \leq t,\, s \in T^T,\, t-s \in T^T\}.$$

We should say that the payoff $^{(t)}\hat{u}$ is *nondegenerate* with respect to E_t^T, if it is the unique solution of the linear system in variables y_1, y_2 given by

$$\lambda^T\, s \otimes y = \lambda^T\, {}^{(s)}b \quad (s \in E_t^T \cap \mathbb{N}^2)$$

that is, if there are 2 linear independent profiles in E_t^T.

In this case, $\mathcal{C}(V(t)) = \{t \otimes \bar{u}\}$ and the core collapses towards the equilibrium.

Clearly the set

$$\{t \in \mathbb{N}^2 \mid \hat{u} \text{ n.d.w.r.t. } E_t^T\}$$

is the system of distributions such that $\mathscr{W} = \mathscr{C}$ is true and, as the number of linearly independent profiles depends on the volume of the compact convex polyhedron E_t^T we expect the region where $\mathscr{W} = \mathscr{C}$ (and $t_1 < 2t_2$) to be of the following shape.

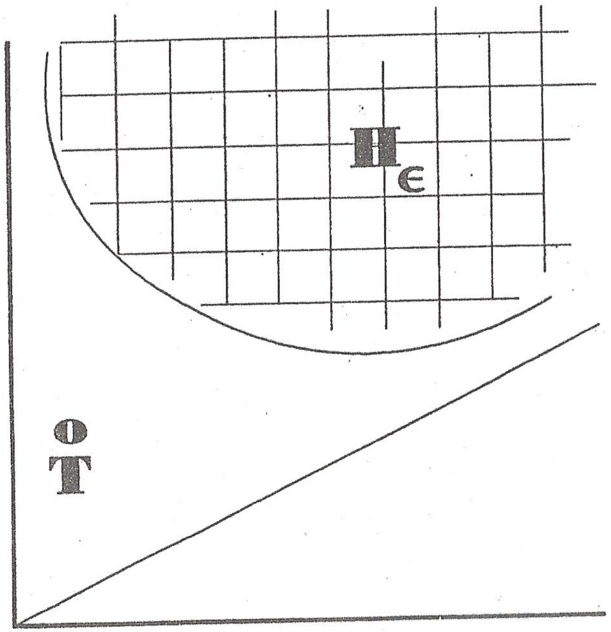

Figure 13

For $t_1 > 3t_2$, the argument is quite similar, except that

$$\hat{u} = (\tfrac{4}{1}, \tfrac{3}{1})$$

and

$$t \circledast \hat{u} = (4t_1, 3t_1)$$

is a point on the south–east part of the Pareto surface of V(t).

For $2t_2 < t_1 < 3t_3$ the argument is slightly different.

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