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# Weighted majority games and the matrix of homogeneity

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## ABSTRACT

If a weighted majority game (not necessarily constant sum or super additive) is described by the weights (voting strength) of the players involved and a majority level, then it is desirable to know whether the game is in addition homogeneous. The paper provides a recursive procedure defining a test for homogeneity. This procedure involves the computation of a number theoretical function, the "matrix of homogeneity". If this matrix is known all majority levels with respect to which the given set of weights represents a homogeneous simple weighted majority game are known at once.

INTRODUCTION

Let $\Omega$ be a finite set ("the set of players") and $\mathfrak{P}(\Omega)$ the power set of $\Omega$ ("the system of coalitions"). A simple game is a mapping $v : \mathfrak{P}(\Omega) \to \{0,1\}$, $v(\emptyset) = 0$. A coalition $S \in \mathfrak{P}(\Omega)$ is winning if $v(S) = 1$ and loosing if $v(S) = 0$.

A measure $M$ (nonnegative additive set function) defined on $P(\Omega)$ (which is completely specified by the numbers $M(\{\omega\})_{\omega\in\Omega}$ ) together with a real number $\lambda, 0 < \lambda \leq M(\Omega)$, induces a simple game $v$ via

$$v(S) = \begin{cases} 1 & M(S) \geq \lambda \\ 0 & M(S) < \lambda \end{cases} \qquad (S \in \mathfrak{P}(\Omega))$$

or, for short $v = 1_{[\lambda,M(\Omega)]} \circ M$ ($1_F$ is the indicator function of a set F), and we call $v$ a weighted majority game if a representation by some $(M,\lambda)$ exists at all. In this case, $M(\{\omega\})$ is the voting strength or weight of player $\omega \in \Omega$ and $\lambda$ is the majority level: once a coalition can muster sufficient total voting strength in order to exceed the majority level $\lambda$ it is winning.

A weighted majority game is called homogeneous if there is a representation $(M,\lambda)$ such that all minimal winning coalitions have exactly weight $\lambda$.

These terms (with slight modifications) were introduced by von NEUMANN-MORGENSTERN [ 6 ]. Weighted majority games and in particular homogeneous games were studied by SHAPLEY (see e.g. [14], ISBELL (e.g. in [1] [ 2] [ 3]), and PELEG ([8 ] [9 ]).

If "dummies get zero voting strength" (see [ 9 ] [10 ]) and the game is superadditive and constant sum $(v(S) + v(S^c) = 1$ for all $S \in P(\Omega))$, then the game is uniquely represented by $M$ and $\lambda$ (up to a scaling factor, see [ 9 ] [10]). In this case, $M$ and $\lambda$ are necessarily rationals (or, by suitable scaling nonnegative integers).

In addition, M is nondegenerate with respect to $\lambda$ (i.e., uniquely determined by the minimal winning coalitions, see [12] [13]). Also, nondegeneracy may replace the constant sum property in order to ensure uniqueness; thus, a homogeneous nondegenerate representation, if it exists, is the only nondegenerate representation.

The above mentioned uniqueness theorems always yield nonnegative integer representations (after rescaling). As a nonnegative integer representation always exists, we shall restrict our discussion to the case that both, M and $\lambda$ take only nonnegative integer values.

More recently OSTMANN [ 7 ] has proved that there is a unique nonnegative integer representation for any homogeneous weighted majority game (constant sum or not) having minimal total weight $M(\Omega)$ (the smallest committee representing a homogeneous parlament). This author studies the incidence matrix of the minimal winning coalitions in order to derive his results.

The relation between homogeneity and nondegeneracy seems to be intricate. Both properties allow for a wide class of solution concepts, see e.g. [10], both seem to resemble properties that are usually attached to nonatomic measures on a larger measure space and both seem to have common number theoretical roots (see [11]). As to the solution concepts it should be observed that the intensively studied case of symmetric games (M is uniform distribution) (see e.g. LUCAS et.al. [4] , MUTO [5], can of course be summarized under the homogeneous as well as under the nondegenerate games.

The aim of this paper is to provide a recursive test for homogeneity of $(M,\lambda)$. In a sense, this implies a method of computing all homogeneous games. Frequently, given some committee or parlament, the weights of the players (members of the groop, parties of the parlament) are specified, thus a weighted majority game is given. However, in order to test whether the representation is homogeneous one has to find all minimal winning coalitions and check whether the voting strength exactly equals the majority level - a rather tedious procedure.

The present procedure is much shorter (and easily converted into a computer program). Let $M$ be an integer valued measure we first classify players according to <u>types</u> (some players may have equal weights). Depending on the distribution of players over the types, the "<u>matrix of homogeneity</u>" $C = C(M)$ is specified (section II). This is a number theoretical function which may be computed recursively (section III). If $C$ is known, we immediately obtain all majority levels $\lambda$ such that $(M,\lambda)$ represents a homogeneous game. In fact $C$ yields at once all homogeneous pairs that are obtained by restricting $M$ to a smaller set of types and specifying suitable majority levels. Thus, $C$ completely describes the homogeneity properties of $M$ .

A few notations: $r \wedge \beta$ and $r \vee \beta$ denote the min and max of rationals or integers $r,s$ respectively; $[r]$ is the greatest integer smaller than or equal to $r$ ; $|S|$ denotes the cardinality of a set $S$ and, finally, $\mathbb{N}$ of course denotes the set of natural numbers.

## § 1 Admissible levels

For $r \in \mathbb{N}$, let

$$\mathbb{N}_{\uparrow}^{r} := \{g = (g_1,\ldots,g_r) \in \mathbb{N}^r \mid g_1 < g_2 < \ldots < g_r\} .$$

If $g \in \mathbb{N}_{\uparrow}^{r}$ and $k = (k_1,\ldots,k_r) \in \mathbb{N}^r$, then the pair $(g,k)$ induces an integer valued measure $M$ as follows.

Fix a finite set $\Omega$ such that $|\Omega| = k_1 +\ldots+ k_r$ and let

$$\Omega = K_1 +\ldots+ K_r$$

be a decomposition of $\Omega$ into disjoint subsets $K_i$ such that $|K_i| = k_i$ $(i = 1,\ldots,r)$ ("+" replaces "$\cup$" in case of a "disjoint union"). For any $S \subseteq \Omega$ define

$$M(S) := \sum_{i=1}^{r} |S \cap K_i| \; g_i \quad ;$$

thus

$$M : \mathfrak{P}(\Omega) \to \mathbb{N} \cup \{0\}$$

is specified.

On the other hand, an integer valued measure $M$ on a finite set $\Omega$ induces a pair $(g,k)$ for suitable $r$ (points of measure zero being ignored). As all properties if integer valued measures we want to deal with do not depend on the particular choice of the decomposition, we shall say that $(g,k)$ and $M$ correspond to each other and frequently use $M$ and $(g,k)$ synonymously. Sometimes we shall write $M \in \mathfrak{M}^r$ if $(g,k) \in \mathbb{N}_{\uparrow}^{r} \times \mathbb{N}^r$.

<u>Definition</u> 1.1. $M \in \mathfrak{M}^r$ is said to be homogeneous with respect to $\lambda \in \mathbb{N}$ (written "M hom $\lambda$" or "(g,k) hom $\lambda$") if

(1) $M(\Omega) \geq \lambda$ ,

(2) for $S \subseteq \Omega$, $M(S) > \lambda$, there is $T \subseteq S$ such that $M(T) = \lambda$.

In addition "$M \text{ hom}_0 \lambda$" means that either $M \text{ hom } \lambda$ or $M(\Omega) < \lambda$ .

In the framework of cooperative game theory, $\Omega$ is the "set of players", $K_i$ contains the "players of type i" and $g_i$ is the "weight" or "voting strength" of players of type i. If a coalition $S \subseteq \Omega$ has sufficient voting strength in order to exceed the "majority level" $\lambda$ then it is a winning coalition; the corresponding game is represented by the function

$$v : \mathcal{P}(\Omega) \to \{0,1\} , \quad v = 1_{[\lambda,M(\Omega)]} \circ M$$

($1_F$ is the indicator function of F) and a coalition $S \subseteq \Omega$ is winning or loosing according to whether $v(S) = 1$ or $v(S) = 0$ .

In this framework, $M \text{ hom } \lambda$ means that minimal winning coalitions have exactly a total weight of $\lambda$.

<u>Lemma</u> 1.2. Let $M \in \mathfrak{M}^r$ . If $M \text{ hom } \lambda$, then there is $i_0 \in \{1,\dots,r\}$ and $c \in \mathbb{N}$ , $1 \leq c \leq k_{i_0}$ such that

(3) $$\lambda = c\, g_{i_0} + \sum_{i=i_0+1}^{r} k_i\, g_i$$

Proof: Choose $i_0 \in \{1,\dots,r\}$ such that

(4) $$\sum_{i=i_0+1}^{r} k_i\, g_i < \lambda \leq \sum_{i=i_0}^{r} k_i\, g_i$$

and let

(5) $$c := \frac{\lambda - \sum_{i=i_0+1}^{r} k_i\, g_i}{g_{i_0}} .$$

If $[c] < c$ , then

$$(6)\qquad [c]\, g_{i_0} + \sum_{i=i_0+1}^{r} k_i\, g_i < \lambda < [c+1]\, g_{i_0} + \sum_{i=i_0+1}^{r} k_i\, g_i \ .$$

In this case, the right hand side of (6) defines a set $T \subseteq \Omega$ the elements of which have at least weight $g_{i_0}$ . Removing one of these elements will cause the measure to fall below of $\lambda$ , contradicting homogeneity. Thus, $[c] = c$ and (3) follows at once.

For $i_0, r, s \in \mathbf{N}$ , $i_0 \leq r \leq s$ and $(g,k) \in \mathbf{N}_{\uparrow}^{s} \times \mathbf{N}^{s}$ let us use the notation

$$(7)\qquad \lambda_{i_0}^{c} := \lambda_{i_0,r}^{c} := \Lambda_{i_0,r}^{c}(g,k) := c\, g_{i_0} + \sum_{i=i_0+1} k_i\, g_i$$

whenever $c \in \mathbf{N}$ , $c \leq k_{i_0}$ . The number $\lambda_{i_0}^{c}$ depends only on $(g_1,\ldots,g_r;\ k_1,\ldots,k_r)$, the first $r$ coordinates of $(g,k) \in \mathbf{N}_{\uparrow}^{s} \times \mathbf{N}^{s}$ (and of course not on the first $i_0-1$ coordinates).

Sometimes, however, it will be preferable to slightly change our viewpoint; thus $\Lambda_{i_0,r}^{c}$ is regarded as a function defined on a suitable subset $(k_{i_0} \geq c)$ of

$$\bigcup_{s=r}^{\infty} \mathbf{N}_{\uparrow}^{s} \times \mathbf{N}^{s} \ .$$

The numbers $\lambda_{i_0}^{c}$ are the only candidates for majority levels such that $M \in \mathfrak{M}^{r}$ is possibly homogeneous. Also, a number $\lambda_{i_0}^{c}$ suggests a minimal winning coalition ($c$ players of type $i_0$ and $k_i$ players of type $i$ ,

$i = i_0+1,\dots,r$). If we remove these players, then the remaining ones are commanding weights as represented by

(8) $$(g_1,\dots,g_{i_0} \; ; \; k_1,\dots,k_{i_0-1}, \, k_{i_0}-c) \; .$$

The measure corresponding to (8) is denoted by $M^c_{i_0}$ ; this quantity may be visualized as a restriction of $M$ onto $K_1 + \dots + K_{i_0-1} + D$ where $D \subseteq K_{i_0}$ is a subset of size $|D| = (k_{i_0}-c)$. Thus

(9) $$M^c_{i_0}(S) = \sum_{i=1}^{i_0-1} |K_i \cap S| \; g_i + |D \cap S| \; g_{i_0}$$

In particular

$$M_{i_0-1} := M_{i_0}^{k_{i_0}} \cong (g_1,\dots, g_{i_0-1} \; ; \; k_1,\dots,k_{i_0-1}) \; .$$

As a notational convenience, $m$ will always denote the total mass of $M$ ; thus

$$m = M(\Omega) = \sum_{i=1}^{r} k_i \, g_i \; ,$$

(10) $$m^c_{i_0} = \sum_{i=1}^{i_0-1} k_i \, g_i + (k_{i_0}-c) \, g_{i_0} \; ,$$

$$m_{i_0-1} = \sum_{i=1}^{i_0-1} k_i \, g_i \; ,$$

etc.

<u>Lemma</u> 1.3. Let $M \in \mathfrak{M}^r$ and $M$ hom $\lambda \in \mathbb{N}$. Also, let $i_0 \in \{1,\dots,r\}$ and $c \in \{1,\dots,k_{i_0}\}$ be such that $\lambda = \lambda^c_{i_0}$ . Then

$$M_{i_o}^c \ \mathrm{hom}_o \ g_i$$

for all $i \in \{i_o,\ldots,r\}$ .

Proof: Assume $c < k_{i_o}$ , the case $c = k_{i_o}$ is treated in the same way.

Pick $i \in \{i_o,\ldots,r\}$ such that $m_{i_o}^c > g_i$ and let

$$T = A + K_{i_o+1} + \ldots + K_r \ , \ A \subseteq K_{i_o} \ , \ |A| = c \ ,$$

be such that $M_{i_o}^c$ lives on $\Omega$-T .

Assume that there is $S \subseteq \Omega - T$ ,

$$M_{i_o}^c(S) > g_i \ ,$$

then w.l.g.

$$M_{i_o}^c(S) - g_i \leq g_{i_o} \tag{11}$$

Next pick $\omega \in K_i$ (i.e. $M(\{\omega\}) = g_i$) then

$$M(S+T-\{\omega\}) = M_{i_o}^c(S) + M(T) - g_i > \lambda$$

and as $M$ hom $\lambda$, there is $R \subseteq S+T - \{\omega\}$, $M(R) = \lambda$. In view of (11) we may assume that $R = S'+T - \{\omega\}$, $S' \subseteq S$ .
Therefore

$$M(T) = \lambda = M(R) = M_{i_o}^c(S') + M(T) - g_i$$

which implies $M_{i_o}^c(S') = g_i$ , q.e.d.

<u>Theorem</u> 1.4. Let $(g,k) \in \mathbb{N}_{\uparrow}^{r} \times \mathbb{N}^{r}$ and $M = \sum_{i=1}^{r} |\cdot \cap K_i| \; g_i$ be the corresponding integer valued measure on $\Omega = K_1 + \ldots + K_r$. Also, let $\lambda \in \mathbb{N}$, $\lambda \leq M(\Omega) = \sum_{i=1}^{r} k_i g_i$. Then

$$M \text{ hom } \lambda$$

if and only if there is $i_0 \in \{1,\ldots,r\}$ and $c \in \{1,\ldots,k_{i_0}\}$ such that

(I)
$$\lambda = \lambda_{i_0}^{c} = cg_{i_0} + \sum_{i=i_0+1}^{r} k_i g_i$$

(II)
$$M_{i_0}^{c} \text{ hom}_0 \; g_i \quad (i_0 \leq i \leq r)$$

Proof: It is sufficient to verify that conditions (I) and (II) are sufficient in order to obtain $M$ hom $\lambda$.

Suppose, there is $T \subseteq \Omega$ such that

(12)
$$M(T) > \lambda = \lambda_{i_0}^{c} = cg_{i_0} + \sum_{i=i_0+1}^{r} k_i g_i$$

Now, remove all players of type $r$ from $T$ (say $t_r$), then remove all players of type $r-1$ (say $t_{r-1}$),... etc. Finally, remove all players of type $i_0$ <u>but at most $c$</u> from $T$ (say $t_{i_0} \leq c$). The remaining set is called $T'$ and the procedure induces an inequality

(13)
$$M(T') > (c-t_{i_0}) \, g_{i_0} + \sum_{i=i_0+1}^{r} (k_i - t_i) \, g_i =: \lambda'$$

Now because either all players of type $i_0$ or $c$ of them were removed from $T$, there are at most $k_{i_0} - c$ players of type $i_0$ left in $T'$.

Therefore (13) reads in fact

$$M^c_{i_o}(T') > \lambda' \quad .$$

Because of (II), there is $S' \subseteq T'$ such that

$$M^c_{i_o}(S') = \lambda' \quad .$$

If we now add $t_{i_o}$ players of type $i_o$ to $S',\ldots,t_r$ players of type $r$ to $S'$, we obtain a set $S$ of exactly measure $\lambda$, q.e.d.

Remark 1.5. The following interpretation of Theorem 1.5. is offered.

$M$ hom $\lambda$ is equivalent to the existence of a minimal winning coalition consisting of all "big players" ($i \geq i_o+1$), some "medium players" ($i=i_o$) and no "small players" ($i \leq i_o-1$), this coalition having exactly weight $\lambda$, thus

$$\lambda = \lambda^c_{i_o} = cg_{i_o} + \sum_{i=i_o+1}^{r} k_i \, g_i \quad .$$

The distribution of voting strength or weight over the remaining players is indicated by $M^c_{i_o}$ .

Whenever the total weight of all the remaining players exceeds the weight of a player already involved ($m^c_{i_o} > g_i \ \ (i \geq i_o)$), then $M^c_{i_o}$ hom $g_i$ , i.e. any coalition mustering more weight than a player of type $i$ may exactly replace him within the original minimal winning coalition.

## § 2 The matrix of homogeneity

Theorem 1.4. allows for a closed description of levels $\lambda$ with respect to which a measure $M \in \mathfrak{M}^r$ is homogeneous. We know that $\lambda$ has to be of the form $\lambda^c_{i_o}$ and we want to specify the range of integers $c$ such that (for fixed $i_o$) $M$ hom $\lambda^c_{i_o}$ .

<u>Lemma</u> 2.1. Let $\mathfrak{M}^r \ni M$ hom $\lambda^c_{i_o}$ for some $i_o \leq r$ . If $c < k_{i_o}$ then $M$ hom $\lambda^{c+1}_{i_o}$ .

Proof: $M^{c+1}_{i_o}$ is a restriction of $M^c_{i_o}$ and we have

$$m^{c+1}_{i_o} = m^c_{i_o} - g_{i_o} \quad .$$

Hence, if $m^{c+1}_{i_o} \geq g_i$ for some $i \in \{i_o,\ldots,r\}$ , then $m^c_{i_o} > g_i$ and $M^c_{i_o}$ hom $g_i$ by Theorem 1.4.. Clearly $M^{c+1}_{i_o}$ hom $g_i$ holds true a fortiori. Thus, the lemma is a consequence of Theorem 1.4..

<u>Definition</u> 2.2. Let $i_o, r, s \in \mathbb{N}$ , $i_o \leq r \leq s$ .

For $(g,k) \in \mathbb{N}^s_+ \times \mathbb{N}^s$ let $M_r \in \mathfrak{M}^r$ correspond to $(g_1,\ldots,g_r ; k_1,\ldots,k_r)$, the first $r$ coordinates of $(g,k)$, and put $\lambda^c_{i_o} = \Lambda^c_{i_o,r}(g_1,\ldots,g_r ; k_1,\ldots,k_r) = \Lambda^c_{i_o,r}(g,k)$ $(1 \leq c \leq k_{i_o})$ .

Define

$$(1) \qquad c^r_{i_o} := C^r_{i_o}(g,k) := \min \{c \in \mathbb{N} \mid 1 \leq c \leq k_{i_o} , M_r \text{ hom } \lambda^c_{i_o}\} .$$

The quantity $c^r_{i_o}$ depends only on the first $r$ coordinates of $(g,k)$ but is frequently appropriate to consider $C^r_{i_o}$ as a function defined on

$$\bigcup_{s=r}^{\infty} \mathbb{N}_{\uparrow}^{s} \times \mathbb{N}^{s}$$

and taking values in $\mathbb{N} \cup \{\infty\}$ (min $\emptyset = \infty$!). Clearly, if $c_{i_0}^r$ is finite, then $M_r \in \mathfrak{M}^r$ satisfies

$$M_r \text{ hom } \lambda_{i_0}^c$$

for $c_{i_0}^r \leq c \leq k_{i_0}$; thus the numbers $c_{i_0}^r$ $(1 \leq i_0 \leq r)$ completely describe the homogeneity properties of $M_r$. Changing our viewpoint slightly, for $(g,k) \in \mathbb{N}_{\uparrow}^s \times \mathbb{N}^s$ the (triangular) matrix

$$(2) \qquad C = (c_{i_0}^r)_{\substack{1 \leq r \leq s \\ 1 \leq i_0 \leq r}} = C\,(g,k)$$

describes the hom-properties of $(g,k)$ and of all initial sequences $(g_1,\ldots,g_r\,;\,k_1,\ldots,k_r)$ of $(g,k)$; thus (2) is called the <u>matrix of homogeneity</u> (the hom-matrix) of $(g,k)$.

The following are simple properties of the hom-matrix.

<u>Lemma</u> 2.3. For $(g,k) \in \mathbb{N}_{\uparrow}^s \times \mathbb{N}^s$, we have

$$(3) \qquad c_{i_0}^r < \infty \quad \text{iff} \quad M_{i_0-1} \text{ hom}_0\, g_i \ (i=i_0,\ldots,r)\,,$$

$$(4) \qquad c_r^r = \begin{matrix} 1 \\ \infty \end{matrix} \quad \text{according to whether } M_{r-1} \text{ hom}_0\, g_r \text{ or not.}$$

Proof: (3) is obvious as $c_{i_0}^r < \infty$ is equivalent to $M_{i_0}^{k_{i_0}} \text{ hom}_0\, g_i$ $(i=1,\ldots,r)$.
(4) follows for $i_0 = r$ since $c_r^r$ is finite iff $M_{r-1} \text{ hom}_0\, g_r$ which is easily seen to <u>imply</u> $M_r^c \text{ hom}_0\, g_r$ for <u>all</u> $c$, $1 \leq c \leq k_r$.

<u>Lemma</u> 2.4. For $(g,k) \in \mathbb{N}^s_\uparrow \times \mathbb{N}^s$, $i_0 \leq r \leq s$ and $t \in \mathbb{N}$ :

(5) $$\Lambda^c_{i_0,r}(tg,k) = t\,\Lambda^c_{i_0,r}(g,k) \quad (1 \leq c \leq k_{i_0}) ,$$

(6) $$C^r_{i_0}(tg,k) = C^r_{i_0}(g,k) .$$

Proof: Obvious. The lemma conveys the meaning that we may always assume $g_1,\dots,g_r$ (or $g_1,\dots,g_s$) to have greatest common divisor 1.

<u>Lemma</u> 2.5. For $(g,k) \in \mathbb{N}^s_\uparrow \times \mathbb{N}^s$, $i_0 \leq r \leq s$

(7) $$C^r_{i_0}(g,k) = C^r_{i_0}(g;\, k_1,\dots,k_{i_0}, 1,\dots,1)$$

Proof: Observe that statement II of Theorem 1.4. does not refer to the numbers $k_{i_0+1},\dots,k_r$.

<u>Lemma</u> 2.6. Let $(g,k) \in \mathbb{N}^s_\uparrow \times \mathbb{N}^s$ and $i_0 < j \leq r \leq s$ . If $m_{i_0} < g_j$ , then

(8) $$C^r_{i_0}(g,k) = C^{j-1}_{i_0}(g_1,\dots,g_{j-1}\,;\, k_1,\dots,k_{j-1})$$

Proof: Follows again from Theorem 1.4., since $m_{i_0} < g_j$ implies $m^c_{i_0} < g_j$ $(1 \leq c \leq k_{i_0})$ and thus $M^c_{i_0}$ $\text{hom}_0$ $g_j$ trivially.

The simple properties of the functions $C^\cdot_\cdot$ as indicated by our previous lemmata of course reflect certain features of the corresponding (homogeneous) games. For instance, in the situation of Lemma 2.4., assuming $C^r_{i_0}(g,k)$ is finite, pick $c \in \mathbb{N}$, $c^r_{i_0} \leq c \leq k_{i_0}$ and $M_r \in \mathfrak{M}^r$ and consider the

the game represented by $v^r = 1_{[\lambda^c_{i_0}, m_r]} \circ M_r$ . Now in this game, a player of type $j$ (and those with larger weights) is <u>inevitable</u>; he is a member of every winning coalition.

We may obviously remove all these inevitable players, that is consider $M_{j-1}$ , take $\Lambda^c_{i_0,j-1}(g,k) = \lambda'^c_{i_0}$ as the majority level and study the function $v^{j-1} = 1_{[\lambda'^c_{i_0},\, m^{j-1}]} \circ M_{j-1}$ .

All winning coalitions of the first game ($v^r$) are obtained by taking all winning coalitions of the second one ($v^{j-1}$) and adding all inevitable players.

Of course the same game is obtained by <u>reducing</u> the weight of every inevitable play to, say, $m_{i_0} + 1$ . This fact may be reflected by a formula which, analogously to (8), reads

(9) $$c^r_{i_0}(g,k) = c^j_{i_0}(g_1,\ldots,g_{j-1}, m_{i_0}+1 \; ; k_1,\ldots,k_{j-1}, k_j+\ldots+k_r)$$

<u>Remark</u> 2.6. Let us write for $(g,k) \in \mathbb{N}^s_+ \quad \mathbb{N}^s$ , $i \leq j \leq s$ :

(10) $$j \in I_i \quad \text{iff} \quad g_i \mid g_j \quad \text{or} \quad k_i\, g_i < g_j \ .$$

It is then verified by 1.4. and 2.3. that the case $r = 2$ is extensively treated by the following formula.

$$c^2_2 = \begin{cases} 1 & 2 \in I_1 \\ \infty & \text{otherwise} \end{cases}$$

$$c^2_1 = \begin{cases} 1 & 2 \in I_1 \\ k_1 - [\frac{g_1}{g_2}] & \text{otherwise} \end{cases}$$

(the case $r = 1$ is trivial; $c^1_1 = 1$).

## § 3 Rekursive computation of C

During this section we shall tacitly assume that

$$(g,k) \in \mathbb{N}_{\uparrow}^{s} \times \mathbb{N}^{s} \quad \text{and} \quad i_0 \leq r \leq s \; .$$

For $i_0 < r$ , put

(1) $$\gamma_{i_0}^{r} = \Gamma_{i_0}^{r} (g,k) = \min \{c \mid 1 \leq c \leq k_{i_0} , M_{i_0}^{c} \; \hom_0 \; g_r\}$$

such that

(2) $$c_{i_0}^{r} = c_{i_0}^{r-1} \; \vee \; \gamma_{i_0}^{r}$$

follows immediately from Theorem 1.4. and Definition 2.2.; this suggests a recursive computation of the matrix (or function) C. (For $i_0 = r$ , $c_r^r$ is recursively specified by Lemma 2.3.).

Let us use the notation

(3) $$l_{ij} := [\frac{g_j}{g_i}]$$

(4) $$l_j^i := \begin{cases} 1 & i \in I_1 \\ k_1 - l_{1i} & \text{otherwise} \end{cases}$$

$(i \leq j \leq s)$ .

<u>Lemma</u> 3.1. For $r > 1$

(5) $$\begin{aligned} c_1^r &= c_1^{r-1} \; \vee \; l_1^r \\ &= \max_{i=1,\dots,r} l_1^i = \\ &= (\max_{i \notin I_1} k_1 - l_{1i}) \; \vee \; 1 \end{aligned}$$

Proof: The lemma follows from (2) since, for $r > 1$,

$$\gamma_1^r = \min\{c \mid 1 \le c \le k_1,\ M_1^c \text{ hom}_0\ g_r\}$$

$$= \min\{c \mid 1 \le c \le k_1,\ g_1 \mid g_r \text{ or } (k_1-c)\, g_1 < g_r\}$$

$$= 1_1^r \quad .$$

Lemma 3.2. Let $2 \le i_0 \le r-1$. If $m_{i_0} < g_r$, then $c_{i_0}^r = c_{i_0}^{r-1}$.

The proof is trivial: for $1 \le c \le k_{i_0}$ we have $m_{i_0}^c < g_r$ and $M_{i_0}^c$ hom$_0$ $g_r$, hence $\gamma_{i_0}^r = 1$.

Lemma 3.3. Let $2 \le i_0$.

1. If $M_{i_0-1}$ h$\not{o}$m$_0$ $g_{i_0}$, then

$$c_{i_0}^r = \ldots = c_{i_0}^{i_0} = \infty \quad .$$

2. If $M_{i_0-1}$ hom$_0$ $g_{i_0}$ and $g_{i_0} \mid g_r$, then

$$c_{i_0}^r = c_{i_0}^{r-1} \qquad (i_0 \le r-1)$$

$$c_{i_0}^r = 1 \qquad (i_0 = r)$$

Proof: The first statement follows immediately from Lemma 2.3. and the recursive formula (2).

As to the second statement: the case $i_o = r$ is again treated by means of Lemma 2.3. while the case $i_o \leq r - 1$ is delt with by observing that $M_{i_o-1}$ $\text{hom}_o$ $g_{i_o}$ and $g_{i_o} \mid g_r$ implies that $M^c_{i_o}$ $\text{hom}_o$ $g_r$ , $(1 \leq c \leq k_{i_o})$ , and thus $\gamma^r_{i_o} = 1$ .

In order to state the next lemma let us introduce the quantity

$$\text{(6)} \qquad \Delta^r_{i_o} := g_r - l_{i_o r}\, g_{i_o} = g_r - \left[\frac{g_r}{g_{i_o}}\right] g_{i_o}$$

<u>Lemma</u> 3.4. Let $2 \leq i_o \leq r - 1$ .

Assume, in addition

$$M_{i_o-1} \text{ hom}_o\, g_{i_o} \;,\; g_{i_o} \nmid g_r ,\; m_{i_o} \geq g_r \;.$$

Put

$$\text{(7)} \qquad \tau^r_{i_o} := \min \{t \mid 0 \leq t \leq l_{i_o r},\; l_{i_o r} - k_{i_o} + 1 \leq t ,$$

$$M_{i_o-1} \text{ hom}_o\, (t\, g_{i_o} + \Delta^r_{i_o})\} \;.$$

Then

$$\text{(8)} \qquad c^r_{i_o} = c^{r-1}_{i_o} \vee \tau^r_{i_o} + (k_{i_o} - l_{i_o r}) \;.$$

(Note that $\tau^r_{i_o} = \infty$ is feasible).

Proof: <u>1st Step:</u> We are going to show that the following holds true:

" for $1 \leq c \leq k_{i_o}$ and $g_r > (k_{i_o} - c)\, g_{i_o}$ the statements

$$\text{(9)} \qquad M^c_{i_o} \text{ hom}_o\, g_r$$

and

(10) $$M_{i_0-1} \text{ hom}_0 \; g_r - (k_{i_0} - c)\, g_{i_0}$$

are equivalent".

Obviously, it is sufficient to prove this under the additional assumption that $m_{i_0-1} \geq g_r - (k_{i_0} - c)\, g_{i_0}$, in other words, we may replace "$\text{hom}_0$" by "hom" in (9) and (10) and then show that both versions are equivalent.

Assume first that (10) holds true. We want to establish (9).

To this end, let $S \subseteq \Omega$ be such that $M^c_{i_0}(S) > g_r$, i.e.

$$M_{i_0-1}(S_0) + t\, g_{i_0} > g_r ,$$

where $S_0$ contains only types $\leq i_0 - 1$ and $t \leq k_{i_0} - c$. Clearly

$$M_{i_0-1}(S_0) > g_r - t\, g_{i_0} \geq g_r - (k_{i_0} - c)\, g_{i_0} .$$

Therefore we may apply (10), thus finding $T_0 \subseteq S_0$ such that $M_{i_0-1}(T_0) = g_r - (k_{i_0} - c)\, g_{i_0}$.

Because of

$$M_{i_0-1}(S_0 - T_0) > g_r - t\, g_{i_0} - (g_r - (k_{i_0} - c)\, g_{i_0}) = ((k_{i_0} - c) - t)\, g_{i_0} ,$$

we may use the fact that $M_{i_0-1} \text{ hom}_0 \; g_{i_0}$ holds also true (by hypothesis of the present lemma), thus, there is $R_0 \subseteq S_0 - T_0$ such that $M_{i_0-1}(R_0) = ((k_{i_0} - c) - t)\, g_{i_0}$.

As $R_0 + T_0 \subseteq S_0$ and as, by construction, $M^c_{i_0}(S - S_0) = t\, g_{i_0}$, the set

$$T : R_0 + T_0 + (S - S_0)$$

satisfies

$$\begin{aligned} M^c_{i_0}(T) &= M_{i_0-1}(R_0) + M_{i_0-1}(T_0) + t\, g_{i_0} \\ &= ((k_{i_0} - c) - t)\, g_{i_0} + g_r - (k_{i_0} - c)\, g_{i_0} + t\, g_{i_0} \\ &= g_r , \end{aligned}$$

which proves (9).

In order to prove the converse direction assume now that (9) holds true.

Let $S_0 \subseteq \Omega$ (only types $\leq i_0 - 1$) be such that $M_{i_0-1}(S_0) > g_r - (k_{i_0} - c)\, g_{i_0}$.

Using the fact that $M_{i_0-1}$ $\text{hom}_0$ $g_{i_0}$, we find $T_0 \subseteq S_0$ such that

(11) $\quad g_r - (k_{i_0} - c)\, g_{i_0} < M_{i_0-1}(T_0) < g_r - (k_{i_0} - c - 1)\, g_{i_0}$

(remove appropriate multiples of $g_{i_0}$ from $S_0$; the first inequality in (11) is w.l.o.g. strict for otherwise the proof is already completed).

Now

$$M_{i_0-1}(T_0) + (k_{i_0} - c)\, g_{i_0} > g_r$$

and this suggests a set

$$T := T_0 + \{k_{i_0} - c \text{ elements of type } i_0\} \subseteq \Omega$$

having measure $M^c_{i_0}(T) > g_r$. In view of (9) we may construct $R \subseteq T$ such that

(12) $$M^c_{i_o}(R) = g_r ,$$

that is ,

$$M_{i_o-1}(R_o) + t\, g_{i_o} = g_r ,$$

where $R_o \subseteq T_o$ and $t \leq k_{i_o} - c$ . Now, $t < k_{i_o} - c$ cannot occur, for this would necessarily imply that in view of (11)

$$M^c_{i_o}(R) \leq M_{i_o-1}(T_o) + (k_{i_o} - c - 1)\, g_{i_o} < g_r ,$$

a contradiction to (12). Hence $t = (k_{i_o} - c)$ and

$$M_{i_o-1}(R_o) = g_r - (k_{i_o} - c)\, g_{i_o} .$$

As $R_o \subseteq T_o \subseteq S_o$ . we have verified (10) and the first step is completed.

2nd Step: By definition we have

$$\gamma^r_{i_o} = \min \{1 \leq c \leq k_{i_o} , M^c_{i_o} \text{ hom}_o\, g_r\} .$$

Now, in case that $k_{i_o} > l_{i_o r}$ holds true, it follows that

$$M^c_{i_o} \text{ h}\phi\text{m } g_r$$

whenever $1 \leq c \leq k_{i_o} - l_{i_o r}$ ; this is seen by converting the last inequality to $(k_{i_o} - c)\, g_{i_o} > g_r$ and recalling the definition of $M^c_{i_o}$ .
Hence

$$\gamma^r_{i_o} = \min \{c \mid 1 \leq c \leq k_{i_o} , c \geq k_{i_o} - l_{i_o r} , M^c_{i_o} \text{ hom}_o\, g_r\} .$$

Applying the result of the first step, we may write

$$\gamma_{i_o}^r = \min \{c \mid 1 \le c \le k_{i_o},\ c \ge k_{i_o} - l_{i_o r},$$

$$M_{i_o - 1} \hom_o g_r - (k_{i_o} - c)\, g_{i_o}\} .$$

Introducing $t := c - (k_{i_o} - l_{i_o r})$ we obtain

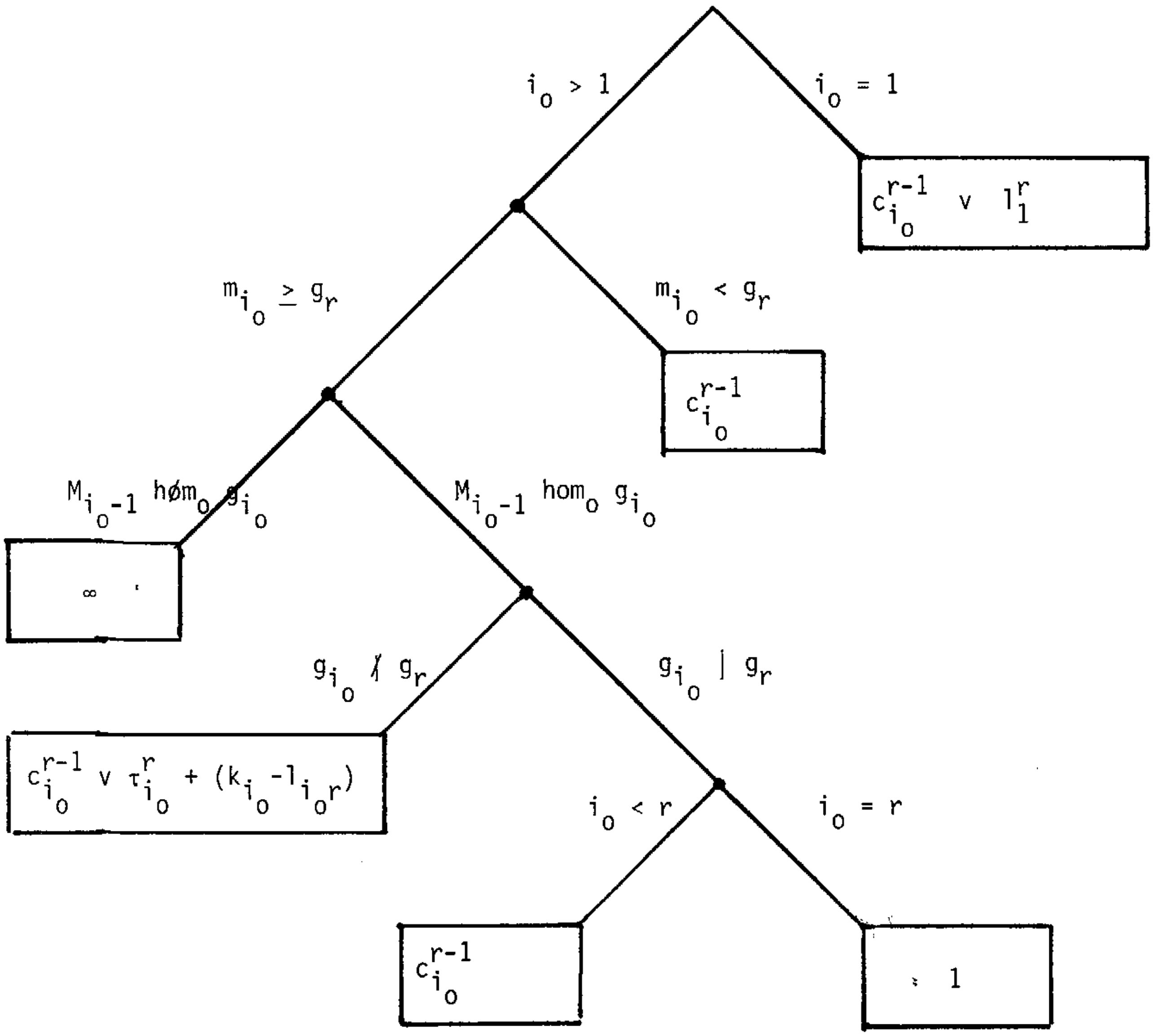
$$\gamma_{i_o}^r = \min \{t \mid l_{i_o r} - k_{i_o} + 1 \le t \le l_{i_o r},\ t \ge 0 ,$$

$$M_{i_o - 1} \hom_o g_r + (t - l_{i_o r})\, g_{i_o}\} + (k_{i_o} - l_{i_o r})$$

which implies (8) by definition of $\Delta_{i_o}^r$ , q.e.d.

<u>Theorem</u> 3.5. Let $(g,k) \in \mathbb{N}_{\uparrow}^s \times \mathbb{N}^s$ . Then $c_1^1 = 1$ and for $i_o \le r \le s$ , $r > 1$ , $c_{i_o}^r$ is recursively obtained by the following diagram:

$c^{r}_{i_o}$ =


$i_o > 1$
$i_o = 1$
$c^{r-1}_{i_o} \vee 1^r_1$
$m_{i_o} \geq g_r$
$m_{i_o} < g_r$
$c^{r-1}_{i_o}$
$M_{i_o-1}$ høm$_o$ $g_{i_o}$
$M_{i_o-1}$ hom$_o$ $g_{i_o}$
$\infty$
$g_{i_o} \nmid g_r$
$g_{i_o} \mid g_r$
$c^{r-1}_{i_o} \vee \tau^r_{i_o} + (k_{i_o} - 1_{i_o r})$
$i_o < r$
$i_o = r$
$c^{r-1}_{i_o}$
1

Remark 3.6. The iterative procedure indicated by Theorem 3.5. involves several tests of the type

$$\text{"}M_{i_0-1} \ \hom_0 \ g_{i_0} \ \text{ or not"}$$

where the matrix elements $(c_i^{i_0-1})_{1 \le i \le i_0 - 1}$ are known by induction.

Generally speaking, if the matrix elements $(c_i^r)_{1 \le i \le r}$ are known, a test

$$\text{"}M_r \ \hom \lambda \ \text{ or not"}$$

for $0 < \lambda < M_r(\Omega) = \sum_{i=1}^{r} k_r g_r$ is performed as follows (of Lemma 2.2.):

1. Choose $i_0 \in \{1,\dots,r\}$ such that

$$\sum_{i=i_0+1}^{r} k_i g_i < \lambda \le \sum_{i=i_0}^{r} k_i g_i$$

2. Put

$$c := \frac{\lambda - \sum_{i=i_0+1}^{r} k_i g_i}{g_{i_0}}$$

If $c \notin \mathbb{N}$, then $M_r$ h$\emptyset$m $\lambda$ .

3. If $c \in \mathbb{N}$, then $M_r \hom \lambda$ if and only if

$$c \ge c_{i_0}^r \quad .$$

Remark 3.7. Theorem 3.5. together with Remark 3.6. provide the required test for homogeneity.

In addition, if M (or (g,k)) ranges through some bounded subset of $\mathfrak{M}^r$, then all homogeneous games generated by these subsets may be described by listing (g,k), C(g,k), and $\Lambda_{i_0,r}^r$ (g,k) $(1 \le i_0 \le r \ , \ c_{i_0}^r \le c \le k_{i_0})$.

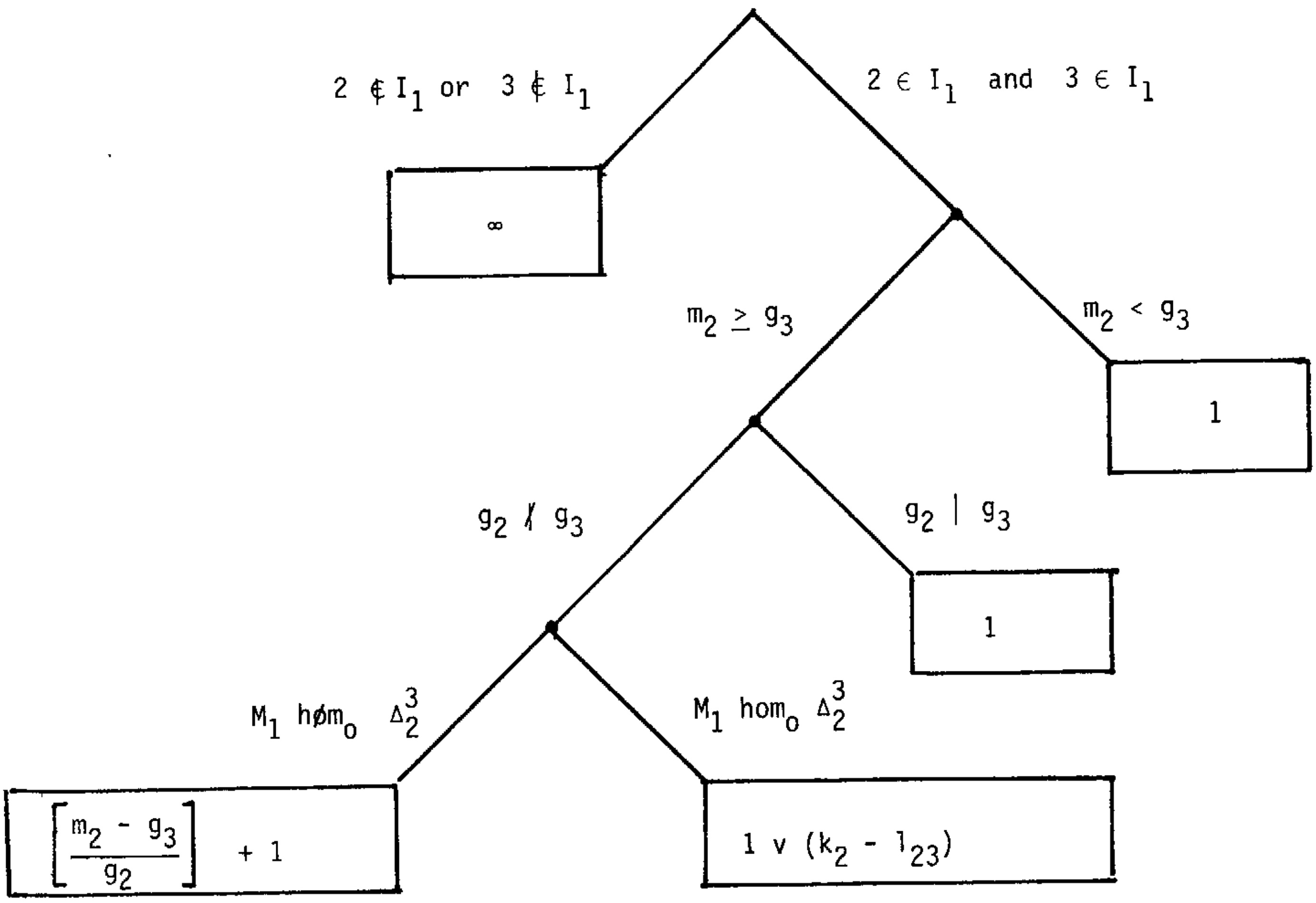
There remains, however, the following problem. A homogeneous weighted majority game may have several representations, Thus, if $M$ is given and test that $M$ hom $\lambda$ has been performed with a positive result - how do we test whether $(M,\lambda)$ is the minimal representation as studied by OSTMANN [ 7] ? What is a procedure to generate this minimal representation? Once this gap can be closed, a complete and unique description of all homogeneous games is at hand.

<u>Example</u> 3.8. For $r = 3$ , the matrix $C$ can be described in a rather closed form. We have

$$c_1^\rho = \max_{i=1\ldots\rho} l_1^i \qquad \rho = 1,2,3 \;;$$

$$c_2^2 = \begin{cases} 1 & 2 \in I_1 \\ \infty & \text{otherwise} \end{cases}$$

$$c_2^3 =$$



$$c_3^3 = 1 \quad \text{if} \quad \begin{cases} 1. \quad m_2 < g_3 \\ \text{or} \\ 2. \quad k_2g_2 < g_3 \leq m_2,\ g_1 \mid g_3 - k_2g_2\ ,\ (2 \in I_1 \text{ or } g_3 \geq m_2 - l_{12}g_1) \\ \text{or} \\ 3. \quad g_3 \leq k_2g_2,\ g_2 \mid g_3,\ 2 \in I_1 \end{cases}$$

and $c_3^3 = 0$ otherwise. E.g., if $g = (2,5,11)$ and $k = (2,k_2,k_3)$ with $k_2 \geq 3$ and $k_3 \in \mathbb{N}$ arbitrarily chosen, then

$$C = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & k_2-1 & \infty \end{pmatrix}$$

Thus, for $c = k_2-1,\ k_2$ and $\lambda_2^c = 5c + 11k_3$ the pair $(M, \lambda_2^c)$ $= (2,5,11;\ 2,\ k_2,k_3;\ 5c + 11k_3)$ yields a homogeneous game.

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