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Classification of

Price – Invariant Preferences

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Abstract

The present paper contains a complete classification of price – invariant preference relations and thus solves a problem posed by Grandmont (1985). The classification theorem is then applied to characterize Cobb – Douglas representable preferences without taking recourse to any utility representation as the only price – invariant weakly monotone continuous preferences.

1. Introduction

In a recent article Grandmont (1985) used homothetic transformations of preferences to provide necessary structure for stating assumptions on the distribution of preferences allowing the derivation of the "law of demand". His homothetic transformations represent a special case of affine transformations which, as Grandmont asserts, can be considered equivalently as "a dynamical system acting on the space of preferences". This was used for the first time by Mas–Colell and Neuefeind (1977) as a technical tool to generate a family of preferences from a given one. The same tool, i.e. looking at the same preference when the different axes of the commodity space are stretched in different ways, had been used before by Barten (1964) (see also Deaton and Muellbauer (1980) and Jorgensen and Slesnick (1984)) to establish the notion of household equivalent scales. On the basis of the mathematical concept of a G–space a general version of this method was introduced by E. Dierker, H. Dierker and Trockel (1984) as the basic concept for the formulation of suitable dispersion of consumers' tastes and for deriving C^1 market demand functions even for non–convex consumers' preferences. A corresponding C^0 result was given on the same basis in Trockel (1984a). For a general treatment see Trockel (1984b).

Looking at all orbits generated by the acting group of prices from the different single preferences rather than only at one orbit as in the case of household equivalent scales endowes the space of preferences with the structure of a G–space and by this means provides a method to partition it in equivalence classes on each of which the group action is ergodic. This method might very well yield further progress in deriving specific structure of the market demand in different setups via distribution assumptions on preferences or individual demand correspondences.

Quite recently this concept has been used by H. Dierker (1986) to generate dispersion via the group of prices acting on demand functions and to prove on this basis an existence result for a pure strategy Nash equilibrium in an oligopoly with price setting firms.

While there are different classes of preferences which are invariant under the action of the group of prices, there are even invariant single preferences, namely the Cobb–Douglas representable ones. Whether these are the only ones is a question which was posed by Grandmont (1985). I shall answer this question in the following on the basis of a general classification of price–invariant preferences.

A partial answer has already been provided by H. Dierker (1986). She states that among all homogeneous demand functions which satisfy Walras law, the only ones which are invariant under the action of the group of prices are those generated by Cobb–Douglas utility functions. The specific context of demand functions has the advantage to allow for a quick elegant proof. Moreover, this result even covers demand functions which are not derivable from utility maximization. On the other hand it does not provide an answer for non–convex preferences. Also preferences which do not satisfy Walras law or those for which there does not even exist a demand correspondence are not covered by Dierker's result.

2. The Model

As in Grandmont (1985) I shall consider preferences which are complete, transitive binary relations on the consumption set $X = \mathbb{R}^e_{++}$, $e \geq 2$. Assuming the sets of better commodity bundles and of worse commodity bundles to be open amounts to make such a preference representable by a continuous utility function (cf. Debreu (1959)).

I shall denote commodity bundles in $\mathbb{R}^e_{++}$ by x,y etc, a preference by $\succsim$, and a utility function by u. Then $x*y$ means $(x_1y_1,\ldots,x_ey_e)$.
It will turn out useful to work in $\mathbb{R}^e$ rather than $\mathbb{R}^e_{++}$. Therefore I shall employ the group isomorphism
$L:(\mathbb{R}^e_{++},*) \rightarrow (\mathbb{R}^e,+) : x = (x_1,\ldots,x_e) \mapsto \bar{x}$, where
$\bar{x} = (\bar{x}_1,\ldots,\bar{x}_e) := (\ln x_1,\ldots,\ln x_e)$.
The inverse isomorphism is
$E: (\mathbb{R}^e,+) \rightarrow (\mathbb{R}^e_{++},*) : \bar{x} \mapsto x = (\exp\bar{x}_1,\ldots,\exp\bar{x}_e)$.

Let $\mathbb{R}^e_{++}$ be the space of (not normalized) prices. For $q,q' \in \mathbb{R}^e_{++}$ the group operation $*$ is defined as
$q*q' := (q_1q_1',\ldots,q_eq_e')$.
The space of prices acts on the commodity space $\mathbb{R}^e_{++}$ in the same way, i.e. by
$\mathbb{R}^e_{++} \times \mathbb{R}^e_{++} \rightarrow \mathbb{R}^e_{++} : (q,x) \mapsto x^q = q*x$.
If $\mathbb{P}$ is a space of preferences on $\mathbb{R}^e_{++}$ we get an action of $\mathbb{R}^e_{++}$ on $\mathbb{P}$ by
$\mathbb{R}^e_{++} \times \mathbb{P} \rightarrow \mathbb{P} : (q,\succsim) \mapsto \succsim_q$
where $\succsim_q$ is defined by
$x^q \succsim_q y^q :\Leftrightarrow x \succsim y$.
Clearly, via L we get a preference R on $\mathbb{R}^e$ associated with the original preference on $\mathbb{R}^e_{++}$.
$\succsim,\succ,\sim$ resp. R,P,I denote the "better or indifferent" – the "strictly better" – and the "indifferent"–relations on $\mathbb{R}^e_{++}$ resp. $\mathbb{R}^e$. Indifference

classes in $\mathbb{R}^e_{++}$ or $\mathbb{R}^e$ are denoted I_x and $I_{\bar{x}}$, respectively, $x \in \mathbb{R}^e_{++}$, $\bar{x} \in \mathbb{R}^e$.

A preference $\succsim$ on $\mathbb{R}^e_{++}$ is <u>price–invariant</u> iff
$\forall q \in \mathbb{R}^e_{++} : \succsim_q \equiv \succsim$.

The corresponding invariance for R on $\mathbb{R}^e$ is derived easily.
Let $L(\succsim) = R$ defined by
$L(x)\ R\ L(y) :\Leftrightarrow x \succsim y$
$L(\succsim) = R$ is called <u>invariant</u> iff $\succsim$ is price–invariant.
Hence, we get

$$L(x)\ R\ L(y) \Leftrightarrow L(x^q)\ R\ L(y^q)$$
$$\Leftrightarrow (\ln q_1x_1,\ldots,\ln q_ex_e)\ R\ (\ln q_1y_1,\ldots,\ln q_ey_e)$$
$$\Leftrightarrow (\ln q_1,\ldots,\ln q_e) + (\ln x_1,\ldots,\ln x_e)\ R\ (\ln q_1,\ldots,\ln q_e) + (\ln y_1,\ldots,\ln y_e)$$
$$\Leftrightarrow L(q) + L(x)\ R\ L(q) + L(y)$$
$$\Leftrightarrow (\bar{x} + \bar{q})\ R\ (\bar{y} + \bar{q})$$

Invariance of R is therefore invariance with respect to translations in $\mathbb{R}^e$.

3. Results

Theorem 1: Let R be an invariant preference on $\mathbb{R}^e$. Then exactly one of the three following statements holds true:

1) R is not representable by a continuous utility function.
2) R is the trivial preference on $\mathbb{R}^e$.
3) All indifference classes of R are hyperplanes in $\mathbb{R}^e$.

Proof: An example for the occurrence of case 1) is the lexicographic preference on $\mathbb{R}^e$, which is obviously invariant. Also case 2), i.e. the invariance of the trivial preference which is representable by a constant utility function is a trivial fact.
It remains to be shown that all non–trivial continuous preferences on $\mathbb{R}^e$ have indifference sets which are hyperplanes.

In the first step I shall assume that some indifference class of R, say $I_{\bar{x}}$, where $\bar{x} \in \mathbb{R}^e$, is a linear manifold. Under this assumption I shall show that all indifference classes $I_{\bar{y}}$ for R, $\bar{y} \in \mathbb{R}^e$, are linear manifolds of dimension $e-1$.

In the second step I shall establish then that this assumption is indeed fulfilled.

So let $\bar{x} \in \mathbb{R}^e$ and assume the indifference set $I_{\bar{x}}$ to be a linear manifold in $\mathbb{R}^e$ with dimension smaller than e. Let v a utility representation of R and let $\bar{y} \in I_{\bar{x}}$. Since R is invariant $v(\bar{x}) = v(\bar{y})$ implies for any $\bar{z} \notin I_{\bar{x}}$
$v(\bar{z}) = v(\bar{x} + (\bar{z} - \bar{x})) = v(\bar{y} + (\bar{z} - \bar{x})) = v(\bar{z} + (\bar{y} - \bar{x}))$. Hence, $I_{\bar{z}} = I_{\bar{x}} + (\bar{z} - \bar{x})$. Therefore all indifference classes can be generated from any specific one by translation and hence are of the same dimension.
Now, assume to the contrary that $\dim I_{\bar{x}} = \dim I_0 < e-1$. Now, consider the quotient space $\mathbb{R}^e/I_0$ which is a linear space of dimension

$k := e - \dim I_0 > 1$. The utility function v on $\mathbb{R}^e$ generates in a canonical way a utility function

$v: \mathbb{R}^e/I_0 \rightarrow \mathbb{R}$. Let K the closed unit ball around the origin in $\mathbb{R}^e/I_0$. Then $v|_K: K \rightarrow v(K)$ is continuous and surjective. Would $v|_K$ be bijective, then it would be a homoemorphism, which is impossible by the different dimensions of K and v(K). Hence, v cannot be injective. Therefore there exist different points in $\mathbb{R}^e/I_0$ given the same value by v. Equivalently, there are disjoint manifolds, say $I_{\bar{x}}$, $I_{\bar{y}}$ on which v takes the same value. But this contradicts the assumption that $I_{\bar{x}} = v^{-1}(v(\bar{x}))$. Hence, $\dim I_{\bar{x}} = e-1$.

It remains to be shown in the second step that for any $\bar{x} \in \mathbb{R}^e$, the indifference set $I_{\bar{x}}$ is a linear manifold. Would $I_{\bar{x}}$ be a singleton then the same would hold true for all $I_{\bar{y}}$, $\bar{y} \in \mathbb{R}^e$. Then every point in $\mathbb{R}^e$ would be mapped by v to a different real number, which is impossible for a real−valued utility function. (Again, the continuous utility function v would define local homoemorphisms from $\mathbb{R}^e$ to subsets of $\mathbb{R}$,which is impossible.).

Let $\bar{y} \in I_{\bar{x}}$, $\bar{y} \neq \bar{x}$. Then for the segment $\{\bar{z} \in \mathbb{R}^e | \exists \lambda \in [0,1]:$
$\bar{z} = \lambda\bar{x} + (1 - \lambda)\,\bar{y}\} =: [\bar{x},\bar{y}]$ we get $[\bar{x},\bar{y}] \subset I_{\bar{x}}$. To see this consider the set $D := [\bar{x},\bar{y}] \cap I_{\bar{x}}$. In the case that D is dense in $[\bar{x},\bar{y}]$, i.e. $\text{cl}\, D = [\bar{x},\bar{y}]$, we get by the closedness of $I_{\bar{x}}$ that $\text{cl}\, D = [\bar{x},\bar{y}] = D \subset I_{\bar{x}}$.

Now assume to the contrary that D is not dense in $[\bar{x},\bar{y}]$. Then there exist a point

$\bar{\bar{z}} = \bar{\lambda}\bar{x} + (1 - \bar{\lambda})\,\bar{y} \in [\bar{x},\bar{y}],\ \bar{\lambda} \in\ ]0,1[$

and a relative neighborhood $U(\bar{\bar{z}})$ of $\bar{\bar{z}}$ in $[\bar{x},\bar{y}]$ such that $U(\bar{\bar{z}}) \cap I_{\bar{x}} = \emptyset$.
Consider all $\bar{x}'' = \mu\bar{x} + (1 - \mu)\bar{y}$, $1 \geq \mu > \bar{\lambda}$ and
$\bar{y}'' = \nu\bar{x} + (1 - \nu)\bar{y}$, $0 \leq \nu < \bar{\lambda}$, $\bar{x}'',\bar{y}'' \in I_{\bar{x}}$. Among all pairs of these $\bar{x}'',\bar{y}''$ chose that one whose Euclidean distance is minimal. Denote these

points by $\bar{x}'$ and $\bar{y}'$. Now we have either

$v(\bar{z}') > v(\bar{x})$ for all $\bar{z}' \in]\bar{x}',\bar{y}'[$ or $v(\bar{z}') < v(\bar{x})$ for all $\bar{z}' \in]\bar{x}',\bar{y}'[$,

since otherwise by the continuity of v there would exist a point in $]\bar{x}',\bar{y}'[$ with the same utility as $\bar{x}'$ in contradiction to the assumptions about $\bar{x}'$ and $\bar{y}'$. Without loss of generality assume

$v(\bar{z}') > v(\bar{x}')$ for all $\bar{z}' \in]\bar{x}',\bar{y}'[$. Since v is continuous it takes a maximum on $[\bar{x}',\bar{y}']$ say at $\bar{z}''$. Hence there exist points $\bar{x}_0 \in]\bar{x}',\bar{z}''[$ and $\bar{y}_0 \in]\bar{z}'',\bar{y}'[$ with $v(\bar{x}_0) = v(\bar{y}_0)$ by the intermediate value theorem. Then we get

$\bar{x}' + (\bar{y}_0 - \bar{x}_0) \in]\bar{x}',\bar{y}'[$ and by the invariance of R

$v(\bar{x}' + (\bar{y}_0 - \bar{x}_0)) = v(\bar{y}_0 + (\bar{x}' - \bar{x}_0)) = v(\bar{x}_0 + (\bar{x}' - \bar{x}_0)) = v(\bar{x}')$.

This again contradicts the assumption that there is no point between $\bar{x}'$ and $\bar{y}'$ in $[\bar{x},\bar{y}]$ having the same utility as $\bar{x}'$ and $\bar{y}'$. So the assumption that D is not dense in $[\bar{x},\bar{y}]$ cannot be true.

With any two points $\bar{x},\bar{y}$ also the points $\bar{x} + n(\bar{y} - \bar{x})$, $n \in \mathbb{Z}$ are elements of $I_{\bar{x}}$. As was just proved, all points between them are also elements of $I_{\bar{x}}$. Hence, for $\bar{x},\bar{y} \in I_{\bar{x}}$ the whole line through $\bar{x}$ and $\bar{y}$ is a subset of $I_{\bar{x}}$. Hence, $I_{\bar{x}}$ is the affine hull of its elements, or equivalently, a linear manifold.

q.e.d.

Since according to Theorem 1 all non–trivial continuous invariant preferences have indifference surfaces which are hyperplanes, they have utility representations which are linear functionals on $\mathbb{R}^e$. So each of those preferences is representable by an element of the dual space of $\mathbb{R}^e$. Moreover, since multiplication with a positive real number changes only the representation rather than the represented preference, the set of non–trivial continuous invariant preferences may be identified with the (e–1)–simplex Δ. Hence, our preferences can be looked at as (not necessarily positive) normalized price vectors on $\mathbb{R}^e$.

Let us consider now the consequences of Theorem 1 for the price–invariant preferences on $\mathbb{R}^e_{++}$.

We just have to see what happens to hyperplanes in $\mathbb{R}^e$ under the map $E: \mathbb{R}^e \to \mathbb{R}^e_{++}: \bar{x} \mapsto (\exp \bar{x}_1,\ldots,\exp \bar{x}_e)$. Consider for $p \in \Delta$ the hyperplane

$$H_{p\bar{c}}: = \{\bar{x}|p \cdot \bar{x} = \bar{c}\} \equiv \{\bar{x}|p_1\bar{x}_1 + \ldots + p_e\bar{x}_e = \bar{c}\}.$$

$$E(H_{p\bar{c}}) = \{x|p \cdot (\ln x_1,\ldots,\ln x_e) = \bar{c}\}$$

$$= \{x|\ln(x_1^{p_1}\cdot\ldots\cdot x_e^{p_e}) = \ln c\}, \text{ where } c = \exp \bar{c}$$

$$= \{x|x_1^{p_1}\cdot\ldots\cdot x_e^{p_e} = c\}.$$

This is the typical indifference class of a non–trivial continuous price–invariant preference relation on $\mathbb{R}^e_{++}$. It is immediate that these preferences are locally non–satiated. A non–trivial continuous price–invariant preference $\succsim$ on $\mathbb{R}^e_{++}$ and the associated invariant preference R on $\mathbb{R}^e$ are monotone ($x > y \Rightarrow x \succ y$) if and only if the representing vector $p \in \Delta$ is strictly positive (i.e. $p >> 0$). They are weakly monotone ($x >> y \Rightarrow x \succ y$) if and only if p is positive (i.e. $p > 0$).In general there are price–invariant preferences which do not generate demand correspondences. Take $e = 2$, $p_1 = 1/3$, $p_2 = -2/3$. Then a typical indifference curve in $\mathbb{R}^e_{++}$ has the form

$\{x \in \mathbb{R}^e_{++}|x_1 = cx_2^2\}$, $c \in \mathbb{R}_+$.

If the representing utility function u increases with c then the preference is strictly convex, but for any $p \in \Delta_{++} = \Delta \cap \mathbb{R}^e_{++}$ and $w \in \mathbb{R}_{++}$ fails to have a maximal element in the budget set $\{x \in \mathbb{R}^e_{++}|p \cdot x \leq w\}$. But very weak monotony of the preference (i.e. $x >> y \Rightarrow x \succsim y$) suffices to guarantee Cobb–Douglas representability of non–trivial continuous price–invariant preferences. I state this answer to the problem posed by Grandmont (1985) as

Theorem 2: A preference $\succsim$ on $\mathbb{R}^e_{++}$ is representable by a Cobb–Douglas utility function

$u: \mathbb{R}^e_{++} \to \mathbb{R}: x \mapsto x_1^{p_1} \cdot \ldots \cdot x_e^{p_e}$

with $0 < p_i < 1$ for all $i \in \{1,\ldots,e\}$ and $\Sigma\ p_i = 1$

if and only if it is non–trivial, continuous, price–invariant, and weakly monotone.

4. Concluding Remarks

Obviously, it is in the first instance a purely mathematical problem to classify objects which are invariant under a certain group action. There are two aspects, however, which give the present analysis economic significance. The first one is that preferences, the objects of classification, build a fundamental micro–economic concept on which analysis of individual and of aggregate demand is based. The second aspect is the role which Cobb–Douglas utility functions play in the invariance problem. Under the reasonable assumption of weak monotony it is indeed the class of Cobb–Douglas representable preferences which coincides with the non–trivial continuous price–invariant preferences. There are good reasons for the omnipresence of Cobb–Douglas functions in economic textbooks as well as in the literature. These functions and the derived demand functions combine almost all nice properties one could think of. In particular the derived demand functions enjoy essentially all those properties one would like to derive for aggregate demand functions in different setups. While other types of preferences allow via the group action the generation of orbits of related similar preferences from a single original one and by these means to disperse bad demand behavior over many different budgets at which it occurs, a price–invariant preference resists the endeavor to create a multitude of slightly different preferences via the action of the group of prices or budgets. But this fact turns out to be completely harmless for projects of aggregation of demand, since the only (reasonable) price–invariant preferences are the Cobb–Douglas representable ones which already guarantee the nice behavior of the derived demand functions.

Let me conclude with drawing attention to the amusing fact that the subclass of those preferences in the huge space of preferences which are price–invariant, turns out to be isomorphic to the space of normalized prices.

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